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The role of oscillations in some nonlinear problems

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Abstract *The still image compression standard which is being developed under the name of JPEG-2000 (Section 2) is a technological challenge which relies on some advances in pure mathematics. This interaction between image processing and functional analysis also benefits partial differential equations. Indeed new estimates on wavelet coefficients of functions with bounded variation (Theorem 4 and 8) imply new Gagliardo-Nirenberg inequalities (Section 7) and lead to a better understanding of blowup phenomena for solutions of some nonlinear evolution equations (Sections 10 to 13).*

1 Introduction

Explaining the performances of JPEG-2000 requires a model for still images. Among several models, the one on which this discussion is based was proposed by Stan Osher and Leonid Rudin (Section 3). In this model, an image f is decomposed into a sum of two pieces u and v. The first piece is aimed to model the main features in f. The second one takes care of the textured components (Section 5), of the noise, and of what is unorganized.

In the Osher-Rudin model, the first component u is assumed to be a function with bounded variation (Section 4). Then the efficiency of wavelet based algorithms will be related to new estimates on wavelet coefficients of functions with bounded variations (BV). These estimates were discovered by Albert Cohen, Ronald DeVore, Ingrid Daubechies, Wofgang Dahmen, Pencho Petrushev and Hong Xu (Theorem 4 in Section 6 and Theorem 8 in Section 7). New Gagliardo-Nirenberg inequalities will then be obtained in Section 7.

Albert Cohen, Ron DeVore and Guergana Petrova went one step further (Section 9). They proved that wavelet coefficients of functions in $L^1(\mathbb{R}^n)$ had some remarkable properties. This was much against the general feeling that wavelet analysis would be inefficient if it was used for Banach spaces which did not admit an unconditional basis. Let me confess that it was my own belief. Moreover Albert Cohen, Ron DeVore and Guergana Petrova found a spectacular application of their theorem. This concerns the Boltzmann equation and more precisely the "averaging lemma" of P.L. Lions and Ron DiPerna (Section 9).

Gagliardo-Nirenberg inequalities (Section 7) will manifest again in studying nonlinear evolution equations. Our first example will be the nonlinear heat equation for which blow up in finite time has been established for some smooth and compactly supported initial conditions. However there is no blowup when this inital condition is sufficiently oscillating and some Gagliardo-Nirenberg estimates will tell us why it is so and how these oscillations should be measured.

These successes led us to believe that the same was true for Navier-Stokes equations. We guessed that an oscillating initial condition should provide us with a solution which is global in time. In other words, the solution should not blow up and moreover the qualitative properties of the initial condition should be preserved under the evolution. This line of research started with some preliminary results by M. Cannone and F. Planchon and culminated with a beautiful theorem by Herbert Koch and Daniel Tataru. The Banach space which is used by Koch and Tataru for modeling the oscillations of the initial condition is exactly the same as the one we introduced for modeling textured components of images (Section 12).

2 Wavelets and still image compression

Let us begin with some examples of technological applications of wavelets. The first example is extracted from the web page of the Pegasus company (http://www.jpg.com). It reads the following:

> Pegasus Imaging Corporation has partnered with Fast Mathematical Algorithms & Hardware Corporation and Digital Diagnostic Corporation to

> develop new wavelet compression technologies designed for applications including medical imaging, fingerprint compression, video compression, radar imaging, satellite imaging and color imaging.
>
> Pegasus provides wavelet compression technology for both medical and non-medical application. Pegasus' wavelet implementation has received FDA market clearance for medical devices.
>
> This software is the only FDA-approved lossy compression software for image processing. Recent clinical studies have shown that the algorithm is comprehensively superior to other similar compression methods. It is licensed to multiple teleradiology developers and medical clinics including the Dutch software vendor Applicare Medical Imaging and the UK telecom giant British Telecom.

The second advertisement comes from a company named "Analog Devices". It reads:

> Wavelet compression technology is the choice for video capture and editing. The ADV601 video compression IC is based on a mathematical breakthrough known as wavelet theory...This compression technology has many advantages over other schemes. Common discrete schemes, like JPEG and MPEG, must break an image into rectangular sub-blocks in order to compress it... Natural images highly compressed with DCT schemes take on unnatural blocky artifacts...Wavelet filtering yields a very robust digital representation of a picture, which maintains its natural look even under fairly extreme compression. In sum ADV601 provides breakthrough image compression technology in a single affordable integrated circuit.

A third success story tells us about the FBI and fingerprints. It says:

> The new mathematical field of wavelet transforms has achieved a major success, specifically, the Federal Bureau of Investigation's decision to adopt a wavelet-based image coding algorithm as the national standard for digitized finger-print records... "

The interested reader is referred to a paper by Christopher Brislawn in the Notices of the AMS, November 1995, Vol 42, Number 11, pages 1278-1283 or to the remarkable web site of Christopher Brislawn [3].

Our next advertisement for wavelet-based image compression is coming from the celebrated Sarnoff Research Center. It reads:

A simple, yet remarkably effective, image compression algorithm has been developed, which provides the capability to generate the bits in the bit stream in order of importance, yielding fully hierarchical image compression suitable for embedded coding or progressive transmission...

Finally the last example will concern the upcoming JPEG-2000 still image compression standard. While the JPEG committee is still actively working, il is very likely that the JPEG-2000 standard will be based on a combination of wavelet expansion (the choice of the filter is not fixed, and could include biorthogonal filters such as the 9/7, as well as 2-10 integer filters) and trellis coding quantization. Applications range from Medical imagery, client/server application for the world wide web, to electronic photography and photo and art digital libraries.

These examples are showing that still image compression is a rapidly developing technology with far reaching applications.

A last remark concerns denoising by soft thresholding. This technique has been created and analyzed by David Donoho and his collaborators [16]. Donoho explains in IEEE spectrum (October 1996, pp. 26-35) what he is doing:

Ridding signals and images of noise is often much easier in the wavelet domain than in the original domain... The procedure works by taking the wavelet coefficients of the signal, setting to zero the coefficients below a certain threshold... Wavelet noise removal has been shown to work well for geophysical signals, astronomical data, synthetic aperture radar, digital communications, acoustic data, infrared images and biomedical signals...

3 Some $u + v$ models for still images

Why do wavelet algorithms perform better than Fourier methods in image compression? One answer to this problem relies on an axiomatic model proposed by Osher and Rudin (among others). This model is named a $u + v$ model.

We start with the superficial approach that a black and white analog image on a domain Ω can be viewed as a function $f(x_1, x_2) = f(x)$ belonging to the Hilbert space $H = L^2(\Omega)$. The grey level of our image at a given pixel x is precisely $f(x)$. The energy of such an image is, by definition, $\int_\Omega |f(x)|^2 dx$. It is obvious that an arbitrary such function $f(x)$ in H is far from being a natural image or something looking similar but this hot issue will be clarified now. Indeed our main problem will be to try to understand how an image differs from an arbitrary L^2 function.

In a $u + v$ model, images $f(x) \in H$ are assumed to be a sum of two components $u(x)$ and $v(x)$. The first component $u(x)$ is modeling the objects or features which are present in the given image while the $v(x)$ term is responsible for the texture and the noise. But the textures are often limited by the contours of the objects and $u(x)$ and $v(x)$ should be coupled by some geometrical constraints. These constraints are absent from most of the $u(x) + v(x)$ models.

In the Osher-Rudin model, the $u(x)$ component is assumed to be a function with bounded variation. We want to detect objects delimitated by contours. Then these objects can be modeled by some planar domains $D_1, ..., D_n$ and the corresponding contours or edges will be modeled by their boundaries $\partial D_1, ..., \partial D_n$.

In this model, the function $u(x)$ is assumed to be smooth inside $D_1, ..., D_n$ with jump discontinuities across the boundaries $\partial D_1, ..., \partial D_n$. However we do not want to break an image into too many pieces and the penalty for a domain decomposition of a given image will be the sum of the lengths of these edges $\partial D_1, ..., \partial D_n$.

But this sum of lengths is indeed one of the two terms which appear in the BV norm of $u(x)$. The other one is the L^1 norm of the gradient of the restriction of u to the interior of the domains $D_1, ..., D_n$. The BV norm of a function $f(x)$ is defined as the total mass of the distributional gradient of $f(x)$ and we will return to this definition in the next section.

In the Osher-Rudin model, $v(x)$ will be measured by a simply minded energy criterion which says that $\|v\|_2$ is sufficiently small. In Section 5, a new model which is taking care of the textured components will we proposed. In this model v can have a large energy but needs to be oscillating.

For the reader's convenience, some basic facts about functions with bounded variations are listed in the following section.

4 Functions with bounded variations

Assuming $n \geq 2$, we say that a function $f(x)$ defined on $\mathbb{R}^n$ belongs to BV if (a) $f(x)$ vanishes at infinity in a weak sense and (b) the distributional gradient of $f(x)$ is a bounded Radon measure. The BV norm of f is denoted by $\|f\|_{BV}$ and defined as the total mass of the distributional gradient of $f(x)$. The condition at infinity says that the convolution product $f \star \varphi$ should tend to 0 at infinity whenever φ is a function in the Schwartz class.

A second and equivalent definition reads the following:

Definition 1 *A function $f(x)$ belongs to $BV(\mathbb{R}^n)$ if it vanishes at infinity in the weak sense and if there exists a constant C such that*

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|dx \leq C|y|, \qquad y \in \mathbb{R}^n \tag{4.1}$$

We now return to functions defined on the plane. An example of a function in BV is given by the indicator (or characteristic) function χ_E of a domain E delimited by a rectifiable Jordan curve ∂E. The BV-norm of χ_E is the length l_E of the Jordan curve ∂E.

The co-area identity tells us that any positive function in BV can be written as a convex combination of some normalized indicator functions. These normalized indicator functions should belong to the unit ball of BV and are therefore defined as $(l_E)^{-1}\chi_E$. They are called "atoms".

This remarkable "atomic decomposition" clarifies the relevance of BV in modeling geometrical features: the atoms are the objects to be detected.

5 Modeling textures

The goal of this section is to address the issue of modeling textures by function spaces. We return to the Osher-Rudin model for representing images and we want to discuss the v component of our image. This v component both contains the textured components of our image and an additive noise. We will offer three choices for modeling these textured components.

Our first choice will be the Besov space $E = B_\infty^{-1,\infty}$ (see Definition 2, Section 7). If wavelet analysis [27] is being used, then this Besov space admits a trivial characterization which reads as follows.

Lemma 1 *Let* $2^j\psi(2^jx - k), j \in \mathbb{Z}, k \in \mathbb{Z}^2, \psi \in F,$ *be an orthonormal wavelet basis of* $L^2(\mathbb{R}^2)$ *where* F *is a finite set consisting of three analyzing wavelets belonging to* C^2 *and compactly supported. Then a generalized function* f *belongs to* $B_\infty^{-1,\infty}$ *if and only if its wavelet coefficients belong to* $l^\infty(\mathbb{Z}^3 \times F)$.

This lemma is extremely attractive since it nicely relates the functional norm in $B_\infty^{-1,\infty}$ to D. Donoho's wavelet shrinkage (see Section 2).

Wavelet shrinkage is a denoising algorithm which consists in putting to zero all wavelet coefficients which are less than a given threshold. Wavelet shrinkage will wipe out the v component of an image whenever its $B_\infty^{-1,\infty}$-norm is less than the threshold. Both the textured component and the noise are meeting this requirement. They will disappear in a wavelet shrinkage. In other words, Donoho's algorithm will treat the textured components of an image as being noise. We will return to this point at the end of this section.

Some slightly smaller Banach spaces F and G are also providing some efficient modeling for textures or oscillating patterns in an image.

The space F consists of generalized functions f which can be written as

$$f = \partial_1 g_1 + \partial_2 g_2, \qquad g_j \in BMO, \quad j = 1, 2 \tag{5.1}$$

The norm of f in F is defined as the infimum of the sums of the BMO norms of g_1 and g_2 and this infimum is computed over all possible decompositions of f.
This Banach space F will be met again when Navier-Stokes equations will be studied. Let us observe that BMO is a space of locally integrable functions, modulo constant functions. These floating constants will disappear in (5.1).

The Banach space G has a similar definition where BMO is replaced by L^∞.

Lemma 2 *We have*

$$BV \subset L^2(\mathbb{R}^2) \subset G \subset F \subset B_\infty^{-1,\infty} \tag{5.2}$$

We are now coming to the heart of this section and we will study the relevance of our function spaces in texture modeling. It is given by the following remark

Lemma 3 *Let f_n, $n \geq 1$, be a sequence of functions with the following three properties*

(a) there exists a compact set K such that the supports of f_n, $n \geq 1$ are contained in K

(b) there exists an exponent $q > 2$ and a constant C such that $\|f_n\|_q \leq C$

(c) the sequence f_n tends to 0 in the distributional sense.

Then $\|f_n\|_G$ tends to 0 as n tends to infinity.

This lemma tells the following. If our sequence f_n is developing important oscillations, then $\|f_n\|_G$ tends to 0 (which obviously implies the same property for the two other norms).

Let us observe that Lemma 3 is wrong if $q = 2$. Indeed if $\varphi(x)$ is any smooth and compactly suppported function, then $f_n(x) = n\varphi(nx)$ is an obvious counter-example.

Lemma 3 can be quantified. The following theorem describes a collection of functions $f(x)$ such that the $\boldsymbol{G}$–norm of $\exp(i\omega.x)f(x)$ decays as $|\omega|^{-1}$ when $|\omega|$ tends to infinity.

Theorem 1 *Let us assume that $f \in L^\infty$ and that a constant C exists such that the BV-norm of $f(x)$ on any ball B of radius R does not exceeds CR. Then*

$$\|\exp(i\omega.x)f(x)\|_{\boldsymbol{G}} \leq C/|\omega|. \tag{5.3}$$

The second requirement on f means that the two measures $\mu_j = \partial f/\partial x_j, j = 1, 2$ should satisfy the celebrated Guy David condition saying that $|\mu|(B) \leq CR$ for any ball B with radius R. This space M of measures will be met again in Section 12.

If textures are modeled as above, then Donoho's denoising algorithm named "wavelet shrinkage" will both erase the textures and the noise. This is exactly what Lemma 3 and Theorem 1 are telling. Let us now challenge Donoho's algorithm and define a "Fourier shrinkage" as the following nonlinear algorithm. One is given a small positive threshold η and one writes the Fourier expansion of a given function f. Then one only retains the terms for which $|c_k| \geq \eta$ in this expansion and this provides f_η.

If f represents an image which contains geometrical features, textured elements and some additive noise and if a "Fourier shrinkage" is applied to f, then this noise will be wiped out while must of the textured components will be kept. Extracting texture from noise is wished in image processing. Does that mean that "Fourier shrinkage" performs better than "wavelet shrinkage"? It is not clear since a "Fourier shrinkage" would more seriously damage the BV component u than a "wavelet shrinkage" does.

6 Fourier series vs. wavelet series: expansions of BV functions

For the sake of simplicity, let us first study periodic functions in BV. Let $f(x_1, x_2)$ be a function of two real variables which is 2π-periodic in each variable. We then abbreviate in saying that $f(x)$ is 2π-periodic. Let us write the Fourier series of $f(x)$ as $f(x) = \sum_{k \in \mathbb{Z}^2} c(k_1, k_2)\exp(ik.x)$ with $k = (k_1, k_2)$. Let us assume that $f(x)$ belongs to BV on $[0, 2\pi]^2$. Then we already know that $c(k)$ belongs to l^2. For such functions, Jean Bourgain proved the following

Theorem 2 *There exists a constant C such that for any 2π-periodic function $f(x)$ in $BV(\mathbb{R}^2)$, we have*

$$\sum_{k \in \mathbb{Z}^2} |c(k)|(|k| + 1)^{-1} \leq C\|f\|_{BV} \tag{6.1}$$

This estimate complements $\sum |c(k)|^2 < \infty$ and these two results obviously follow from a sharper estimate given by

$$\sum_{j=0}^{\infty} s_j \le C\|f\|_{BV} \tag{6.2}$$

where $s_j = (\sum_{2^j \le |k| < 2^{j+1}} |c(k)|^2)^{1/2}$.

This is a mixed $l^1(l^2)$ estimate on Fourier coefficients of a BV function. It is optimal in the sense that there exists a function in BV for which $\sum |c(k)|^p = \infty$ for any $p < 2$. An example is given by $f(x) = |x|^{-1}(\log|x|)^{-2}\varphi(x)$ where $\varphi(x)$ is any smooth function which vanishes when $|x| > 1/2$ and is identically 1 around the origin. Then the Fourier coefficients $c(k)$ of $f(x)$ can be estimated by $|c(k)| \simeq |k|^{-1}(\log|k|)^{-2}$ which obviously implies $\sum |c(k)|^p = \infty$ as announced. The sorted Fourier coefficients of this function behave as $n^{-1/2}(\log n)^{-2}$. This counter-example shows that nothing better than l^2 can be expected inside the dyadic blocks of the Fourier series expansion of a function $f(x)$ in BV.

Now (6.2) can be rewritten as a Besov norm estimate. Indeed let $\Delta_j(f)$ denote the dyadic blocks of the Fourier series expansion of $f(x)$. For defining $\Delta_j(f)$ we only retain the frequencies $k \in \Gamma_j$ in the Fourier expansion of f where Γ_j is the dyadic annulus defined as $\{k \,|2^j < |k| \le 2^{j+1}\}$. We then obviously have $f(x) = c_0 + \sum_0^\infty \Delta_j(f)$ and our next theorem reads:

$$\sum_{0}^{\infty} \|\Delta_j(f)\|_2 \le C\|f\|_{BV} \tag{6.3}$$

This theorem will be further improved. This improved version is not using a Fourier series expansion any more and we can therefore give up the periodic setting and switch to the space $BV(\mathbb{R}^2)$ and to a Littlewood-Paley analysis.

Let us start with a compactly supported smooth function ψ with enough vanishing moments such that the Fourier transform Ψ of ψ satisfies

$$\sum_{0}^{\infty} \Psi(2^{-j}\xi) = 1, \quad |\xi| \ge 1 \tag{6.4}$$

Next we write $\psi_j = 2^{2j}\psi(2^j x)$. Finally $\Delta_j(f)$ is the convolution product $f * \psi_j$.

With these notations (6.3) can be generalized to all exponents p in $(1,2]$. Indeed the following theorem is an easy consequence of the co-area identity.

Theorem 3 *There exists a constant C such that, for every function f in $BV(\mathbb{R}^2)$, and for every exponent p with $1 < p \leq 2$, we have*

$$\sum_{-\infty}^{+\infty} 2^{js} \|\Delta_j(f)\|_p \leq C_p \|f\|_{BV} \tag{6.5}$$

with $s = -1 + (2/p)$ and $C_p \leq C/(p-1)$.

Corollary 1 *If $f(x)$ belongs to $BV(\mathbb{R}^2)$, and if $\psi_{j,k}(x) = 2^j \psi(2^j x - k)$, $j \in \mathbb{Z}, k \in \mathbb{Z}^2$, is an orthonormal wavelet basis of $L^2(\mathbb{R}^2)$, where the three wavelets ψ are smooth and localized as in Lemma 1, then the corresponding wavelet coefficients $c(j,k) =< f, \psi_{j,k} >$ satisfy*

$$\sum_j \Big(\sum_k |c(j,k)|^p \Big)^{1/p} \leq C/(p-1) \|f\|_{BV}, \ 1 < p < 2 \tag{6.6}$$

Corollary 2 *With the same notations as above, we have*

$$\Big(\sum_j \sum_k |c(j,k)|^p \Big)^{1/p} \leq C/(p-1) \|f\|_{BV} \tag{6.7}$$

Corollary 3 *With the same notations, let us assume $\|f\|_{BV} \leq 1$. For each integer m, let N_m be the cardinality of the set on indices (j,k) such that $|c(j,k)| > 2^{-m}$. Then*

$$N_0 + \dots + 2^{-m} N_m \leq C(m+1) \tag{6.8}$$

It means that for most m's we have $N_m \leq C2^m$ since the average of $2^{-m} N_m$ is $O(1)$.

Indeed one has $N_m \leq C2^m$ for all m. Keeping the notation of Theorem 3, the sharp estimate $N_m \leq C2^m$ will be rephrased in the following theorem

Theorem 4 *Let $\psi_\lambda, \lambda \in \Lambda$, be a two-dimensional orthonormal wavelet basis of class C^2 with a rapid decay at infinity. Then for every f in $BV(\mathbb{R}^2)$, the wavelet coefficients $c_\lambda =< f, \psi_\lambda >, \lambda \in \Lambda$ belong to weak $l^1(\Lambda)$.*

This theorem was proved by A.Cohen et al. [9] in the Haar system case. The general case was obtained by the author and the best reference is [36].

In other words, if $c_\lambda =< f, \psi_\lambda >$ and if the $|c_\lambda|, \lambda \in \Lambda$, are sorted out by decreasing size, we obtain a non-increasing sequence c_n^* which satifies $c_n^* \leq C/n$ for $1 \leq n$. This decay of the sorted wavelet coefficients was announced by S. Mallat in his book [25].

If f is the indicator function of any smooth domain, an easy calculation shows that $c_n^* \geq C/n$ which led me to believe that Theorem 4 was optimal. This issue will be addressed in the next section.

7 Improved Gagliardo-Nirenberg inequalities

New Gagliardo-Nirenberg inequalities will now be proved using theorem 4 and wavelet methods. This is the first outstanding application of wavelet techniques inside mathematics. This success story was so encouraging that we thought that better estimates might exist.

Then A. Cohen and his collaborators met our challenge and improved on Theorem 4. This will lead us to Theorem 8 and to more refined Gagliardo-Nirenberg inequalities.

But let us return to the Sobolev embedding of BV into $L^2(\mathbb{R}^2)$.

The estimate $\|f\|_2 \leq C\|f\|_{BV}$ is obviously consistent with translations and dilations. Indeed, for any positive a and $f_a(x) = af(ax)$, we have $\|f_a\|_2 = \|f\|_2$ and similarly $\|f_a\|_{BV} = \|f\|_{BV}$. But this estimate is not consistent with modulations: if M_ω denotes the pointwise multiplication operator with $\exp(i\omega x)$, then M_ω acts isometrically on L^2 while $\|M_\omega f\|_{BV}$ blows up as $|\omega|$ when $|\omega|$ tends to infinity.

For addressing this invariance through modulations, let us introduce an adapted Besov norm.

Definition 2 *Let B be the Banach space of all tempered distributions $f(x)$ for which a constant C exists such that*

$$| < f, g_{a,b} > | < C \tag{7.1}$$

when $g(x) = \exp(-|x|^2)$, $g_{a,b} = ag(a(x-b))$, $a > 0$, $b \in \mathbb{R}^2$.

The infimum of these constants C *is the norm of* f *in* B *and is denoted by* $\|f\|_\epsilon$.

It is easily proved that this Banach space coincides with the space of second derivatives of functions in the Zygmund class. Therefore B is the homogeneous Besov space $B_\infty^{-1,\infty}$ of regularity index -1 which was already used for modeling textures.

We then have

Theorem 5 *There exists a constant* C *such that for any* f *in* $BV(\mathbb{R}^2)$ *we have*

$$\|f\|_2 \le C[\|f\|_{BV}\|f\|_\epsilon]^{1/2} \tag{7.2}$$

and $\|f\|_\epsilon$ *is the weakest norm obeying the same scaling laws as the* L^2 *or* BV *norm for which (7.2) is valid.*

To better understand this theorem, let us stress that we always have $\|f\|_\epsilon \le \|f\|_{BV}$ and the ratio $\|f\|_\epsilon/\|f\|_{BV}$ between these norms is denoted by β and is expected to be small in general. Then (7.2) reads

$$\|f\|_2 \le C\beta^{1/2}\|f\|_{BV} \tag{7.3}$$

which yields a sharp estimate of the ratio between the L^2 norm and the BV norm of f. Moreover $\beta^{1/2}$ in (7.3) is sharp as the example of $f(x)=\exp(i\omega x)w(x)$ shows. Indeed if $|\omega|$ tends to infinity and $w(x)$ belongs to the Schwartz class, then $\|f\|_2$ is constant, $\|f\|_\epsilon \simeq |\omega|^{-1}\|f\|_\infty$, and finally $\|f\|_{BV} \simeq |\omega|\|w\|_1$. In this example β is of the order of magnitude of $|\omega|^{-2}$ which corresponds to $\beta^{1/2} \simeq |\omega|^{-1}$.

The proof of (7.2) is straightforward. One uses the following trivial estimate on sequences

$$\sum_{n=1}^{\infty} |c_n|^2 \le 2\|c_n\|_\infty \|nc_n\|_\infty \tag{7.4}$$

Then one applies Theorem 4 to an orthonormal wavelet basis of class C^2. For concluding the proof, it suffices to make the following observation: if c_n^* denotes the non increasing rearrangement of the wavelet coefficients $|c(\lambda)|, \lambda \in \Lambda$, then $\|nc_n^*\|_\infty$ is precisely the norm of $c(\lambda)$ in the weak l^1 space.

Let us observe that (7.2) is an interesting improvement on the celebrated Gagliardo-Nirenberg estimates. These estimates read in the two-dimensional case

$$\|D^j f\|_p \leq C(\|D^m f\|_r)^\sigma (\|f\|_q)^{1-\sigma} \tag{7.5}$$

where $1 < p, q, r < \infty$, $j/m < \sigma < 1$ and $1/p - j/2 = \sigma(1/r - m/2) + (1-\sigma)/q$.

The notation $\|D^j f\|_p$ means $\sup\{\|\partial^\alpha f\|_p; |\alpha| = j\}$.

For comparing our new estimate to (7.5), we will assume $m = 2, j = 1, p = 2$ and $r = 1$. This either implies $s = 1$ or $q = \infty$. In the first case, (7.5) easily follows from the embedding of BV into L^2. In the second case (7.5) is weaker than (7.2). Indeed the L^∞ norm which is used in (7.5) is replaced in (7.2) by a much weaker one.

Theorem 5 generalizes to any dimension $n > 2$. It then reads

$$\|f\|_{n/n-1} \leq C(\|f\|_{BV})^{(1-1/n)}(\|f\|_\alpha)^{1/n} \tag{7.6}$$

where $\|f\|_\alpha$ is now defined as the optimal constant C for which one has $| < f, g_{a,b} > | \leq C$ with $g_{a,b} = ag(a(x-b)), a > 0, b \in \mathbb{R}^n$, and $g(x) = exp(-|x|^2)$. In other words $\|f\|_\alpha$ is the norm of f in the homogeneous Besov space $B_\infty^{-(n-1),\infty}$.

Every function with a bounded variation belongs to $L^{\frac{n}{n-1}}$ and it is natural to ask the following question: what would happen if a function f both belongs to BV and to L^q for some $q > n^*$ where $n^* = \frac{n}{n-1}$?

One guesses that this size estimate on a BV function should imply improved regularity. That is what the following theorem is telling.

The Lebesgue space L^q is contained inside the larger space $C^{-\beta}$ when $\beta = -n/q$. Then the theorem we have in mind reads the following

Theorem 6 *If* $0 \le s < 1/p$, $1 < p \le 2$ *and* $\beta = \frac{1-sp}{p-1}$, *then*

$$\|f\|_{L^{p,s}} \le C\|f\|_{BV}^{1/p}\|f\|_{-\beta}^{1-1/p} \tag{7.7}$$

where $\|.\|_{-\beta}$ *stands for the norm of* f *in the homogeneous Besov space* $B_{\infty}^{-\beta,\infty}$.

The proof of Theorem 6 still relies on Theorem 4 and wavelet techniques.

Returning to L^2 norms, we unsuccessfully tried to prove the following theorem

Theorem 7 *In any dimension* $n \ge 1$, *let us assume that a function* f *both belongs to* BV *and to the homogeneous space* $B_{\infty}^{-1,\infty}$. *Then we have*

$$\|f\|_2 \le C(\|f\|_{BV}\|f\|_{\epsilon})^{1/2} \tag{7.8}$$

where $\|f\|_{\epsilon}$ *is norm of* f *in the Besov space* $B_{\infty}^{-1,\infty}$.

The Besov norm of f can be defined as the optimal constant C for which one has $| < f, g_{a,b} > | \le Ca^{2-n}, a > 0,\ b \in \mathbb{R}^n$. Let us observe that BV is contained in L^2 if and only if $n = 2$. In other words when $n = 1$ or $n > 2$, the assumption $f \in B_{\infty}^{-1,\infty}$ complements $f \in BV$ and both are needed to get an L^2 estimate.

The proof of this theorem requires new estimates on wavelet coefficients of BV functions which are sharpening Theorem 4. Indeed Albert Cohen, Wolfgang Dahmen, Ingrid Daubechies and Ron DeVore proved the following theorem

Theorem 8 *In any dimension* $n \ge 1$, *let us assume* $\gamma < n - 1$ *where* γ *is a real exponent. Then for* $f \in BV(\mathbb{R}^n)$ *and* $\lambda > 0$, *one has*

$$\sum_{\{|c(j,k)| > \lambda 2^{-j\gamma}\}} 2^{-j\gamma} \le C\lambda^{-1}\|f\|_{BV} \tag{7.9}$$

where $c(j,k) = \int_{\mathbb{R}^n} f(x)2^j\psi(2^jx - k)\,dx$

It is easily seen that this estimate is false when $\gamma = n - 1$. If one returns to the two-dimensional case, it is clear that Theorem 8 implies Theorem 4. Indeed (7.9) is Theorem 4 when $\gamma = 0$. But Theorem 8 is saying more and it is not difficult to construct sequences $c(j,k)$ belonging to weak-l^1 for which (7.9) is not fulfilled. For instance pick $\alpha \in (0,1)$ and let F_j be a sequence of finite sets of integers of cardinality $N_j \approx 2^{\alpha j}$. Then the sequence defined by $c(j,k) = 2^{-\alpha j}$ for $k \in F_j$ will belong to weak$-l^1$ but does not fulfil (7.9) when $\gamma = \alpha$.

Knowing Theorem 8, the proof of Theorem 7 is an excercise.

8 Improved Poincaré's inequality

The standard Poincaré's inequality reads as follows: If Ω is a connected bounded open set in the plane with a Lipschitz boundary $\partial\Omega$, then there exists a constant $C = C_\Omega$ such that for every f in $BV(\Omega)$ we have

$$\int_\Omega |f(x) - m_\Omega(f)|^2 \, dx \le C \|f\|_{BV}^2 \tag{8.1}$$

Here $m_\Omega(f)$ denotes the mean value of the function f over Ω.

Such an estimate cannot be true in $\mathbb{R}^n$ for $n \ge 3$ since BV is not locally embedded in L^2. However there exists an improvement on Poincaré's inequality which (a) is valid in any dimension and (b) is sharpening the standard one in the plane.

We need a Banach space $C^{-1}(\Omega)$ for measuring the oscillations of a function f on Ω. Once more the Besov space $B_\infty^{-1,\infty}$ will be used. The first definition reads: $f \in C^{-1}(\Omega)$ iff f is the restriction to Ω of some distribution belonging to $B_\infty^{-1,\infty}$.

Here is an equivalent definition. We write $f \in C^{-1}(\Omega)$ if $f = \Delta F$ where F is the restriction to Ω of a function G defined on $\mathbb{R}^n$ and belonging to the Zygmund class. The Zygmund class is defined by the classical condition that a constant C should exist such that

$$|G(x+y) + G(x-y) - 2G(x)| \le C|y| \quad x, y \in \mathbb{R}^n \tag{8.2}$$

The norm of f in $C^{-1}(\Omega)$ is denoted by $\|f\|_*$ and is defined as the infimum of these constants C. This infimum is computed over all extensions G of F such that $f = \Delta F$ on Ω.

The improvement we have in mind is valid in any dimension and reads as follows:

Theorem 9 *Let Ω be a connected bounded open set in $\mathbb{R}^n$. Let us assume that Ω has a smooth boundary $\partial\Omega$. With the preceding notations, there exists a constant $C = C_\Omega$ such that, for every function f both belonging to $BV(\Omega)$ and to $C^{-1}(\Omega)$, we have*

$$\int_\Omega |f(x) - m_\Omega(f)|^2 \, dx \leq C\|f\|_{BV}\|f\|_* \tag{8.3}$$

Let us insist on the fact that $BV(\Omega)$ is not contained in $C^{-1}(\Omega)$. A counter-example is given by $f(x) = |x|^{-n+1}(\log|x|)^{-2}$ when Ω is the ball $|x| < 1/2$.

For proving Theorem 9, one introduces local coordinates on some annular neighbourhood R of $\partial\Omega$ defined by $x = y + t\nu$, $x \in R$, $y \in \partial\Omega$, $t \in (-\eta, \eta)$. We have denoted by ν the interior unit vector at to $\partial\Omega$ at y. Let us extend f into F as follows: $F(y + t\nu) = \pm f(y + t\nu)$ where $\pm$ is the sign of t.
Finally Theorem 7 is applied to this new function F, once it has been cut by a convenient cut-off function.
The key fact which enters in the proof of Theorem 9 is that this odd extension operator is both continuous with respect to the Besov norm and the BV norm. An even extension operator would also be fine for the BV norm but certainly not for our Besov norm.

9 Wavelet coefficients of integrable functions

Albert Cohen et al. proved that wavelet coefficients of functions in $L^1(\mathbb{R}^n)$ have some interesting and important properties.

This work was motivated by a striking discovery by P.L. Lions and R. DiPerna [15]. Lions and DiPerna observed that some velocity averages arising in the context of Boltzmann equation are more regular than expected. More precisely they proved the following theorem

Theorem 10 *Let Ω be a bounded open subset of $\mathbb{R}^n$ and $f(x,v)$, $x \in \mathbb{R}^n$, $v \in \Omega$ be any function satisfying the following two properties* *(a)* $f \in L^2(\mathbb{R}^n \times \Omega)$

(b) $v.\nabla_x f \in L^2(\mathbb{R}^n \times \Omega)$

Then the "velocity average"

$$\bar{f}(x) = \int_\Omega f(x, v)dv \tag{9.1}$$

belongs to the Sobolev space $H^{1/2}(\mathbb{R}^n)$

This was later sharpened by M. Bézard [1]. Finally Ronald DeVore and Guerguana Petrova [14] proved the following

Theorem 11 *If* $1 < p < \infty$ *and if*

(a) $f \in L^p(\mathbb{R}^n \times \Omega)$

(b) $v.\nabla_x f \in L^p(\mathbb{R}^n \times \Omega)$

then the velocity average $\bar{f}(x)$ *belongs to the homogeneous Besov space* $\dot{B}_p^{s,p}(\mathbb{R}^n)$ *where* $s = \inf(1/p, 1 - 1/p)$.

When $p = 2$ this is exactly the Lions-DiPerna theorem.

The proof of Theorem 11 has been simplified by A. Cohen and we will follow his presentation.

A. Cohen writes

$$g(x, v) = f(x, v) + v.\nabla_x f(x, v) \tag{9.2}$$

and denotes by $\bar{f}(x)$ the velocity average defined by (9.1). Finally A.Cohen studies the linear operator T which maps $g(x, v)$ on $\bar{f}(x)$.

A partial Fourier transformation in x gives

$$\hat{g}(\xi, v) = (1 + iv.\xi)\hat{f}(\xi, v) \tag{9.3}$$

Then Plancherel identity and standard calculations yield

$$\|\bar{f}(x)\|_{H^{1/2}} \leq C\|g\|_{L^2(R^n \times \Omega)} \tag{9.3}$$

and T maps $L^2(\mathbb{R}^n \times \Omega)$ into $H^{1/2}(\mathbb{R}^n)$. This is the Lions-DiPerna theorem.

Moreover the uniform boundedness in v of $(1 + iv.\xi)^{-1}$ as a multiplier of $\mathcal{F}L^1(R^n)$ implies that T maps $L^1(\mathbb{R}^n \times \Omega)$ to $L^1(\mathbb{R}^n)$.

In order to prove Theorem 11, it then suffices to use the real interpolation method of Lions and Peetre and show that $B_p^{s,p}$ is the interpolation space between $H^{1/2} = B_2^{1/2,2}$ and L^1. Since all Besov spaces admit trivial characterizations by size properties of wavelet coefficients, it remains to study the wavelet coefficients of functions in L^1. We now concentrate on that task.

The normalization which will be used is the following. We write ψ_λ for $\psi(2^j x - k)$ and the wavelet coefficients of f are now $c(\lambda) = < f, \psi_\lambda >$. They are indexed by $\Lambda = \mathbb{Z} \times \mathbb{Z}^n \times F$ where F is a finite set with cardinality $2^n - 1$. Next we denote by Q_λ the corresponding dyadic cube defined by $2^j x - k \in [0, 1)^n$. The theorem on wavelet coefficients of $L^1(\mathbb{R}^n)$ functions says the following:

Theorem 12 *For any real exponent γ larger than 1, there exists a constant $C = C_{\gamma,n}$ such that for f in $L^1(\mathbb{R}^n)$ and for $\tau > 0$, one has*

$$\sum_{\{|c(\lambda)| > \tau |Q_\lambda|^\gamma\}} |Q_\lambda|^\gamma \leq C\tau^{-1} \|f\|_1 \tag{9.4}$$

In other words the wavelet coefficients $c(\lambda), \lambda \in \Lambda$, belong to to a weighted weak-l^1 space where the weighting factor is $|Q_\lambda|^\gamma$. Theorem 12 yields the required interpolation theorem we were looking for. Moreover Theorem 12 complements Theorem 8. Indeed Theorem 12 can be applied to the gradient of a BV function and the normalizations are adjusted in such a way that Theorem 12 corresponds to $\gamma > n$ in Theorem 8.

10 The role of oscillations in some nonlinear PDE's

For a number of nonlinear evolution equations, blowup may happen even if the initial condition is smooth and compactly supported. It is clear that such a blowup needs to be defined with respect to some functional norm and that some norms might become infinite when $t \to t_0$ while other would remain bounded. Proving that some strong norm does not blowup as long as a weaker norm remains under control is quite interesting but often rather hard. Such 0-1 laws will be at the heart of our discussion and we will construct weak norms which do control stronger ones. Our favourite example is the nonlinear heat equation. For Navier-Stokes equations the occurence of an eventual blowup is still an open problem. The nonlinear Schrödinger equation will also be treated. In these first two examples the weaker norm which will be used is denoted by $\|.\|_*$ and $\|f\|_*$ is small when f is oscillating. If the nonlinear Schrödinger equation is being excluded, the main message of this chapter is the following slogan: *blowup does not happen when the initial condition is oscillating.*

This assertion is easily proved for the nonlinear heat equation and was already known in the case of the Navier-Stokes equations, as both Peter Constantin and Roger Temam told me. We will later on return to their sharp comments.

What is completely new is that Besov spaces are manifesting again (Theorems 16 and 19). Exactly as it happened when we were modeling textures, an oscillating pattern is defined as a function which has a small norm in a Besov space with negative regularity index. If the initial condition is such an oscillating pattern, the corresponding solution does not blow up.

As often in mathematics, a discovery raises new problems. Here we want to find the sharpest theorem in the direction given by these heuristic considerations. It means measuring oscillations with the weakest norm. In the case of Navier-Stokes equations, the best result was obtained by Herbert Koch and Daniel Tataru [24].

The space they used is no longer a Besov space and is defined as the collection of functions or vectors u_0 that can be written as $u_0 = \partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3$ where f_j, $j = 1, 2, 3$ belong to the John and Nirenberg space BMO. This Koch and Tataru space is one of the spaces which were used for modeling textures.

11 A first model case: the nonlinear heat equation

Our first model case will be the following heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u^3, & (x,t) \in \mathbb{R}^3 \times (0,\infty) \\ u(x,0) = u_0(x) \end{cases} \tag{11.1}$$

where $u = u(x,t)$ is a real valued function of (x,t), $x \in \mathbb{R}^3$ and $t \geq 0$.

In the following calculation, u is assumed to be a classical solution to (11.1) with enough regularity and with appropriate size estimates. All L^p-norms will be finite by assumption and all integrations by parts will be legitimate.
Multiplying (11.1) by u and integrating over $\mathbb{R}^3$ yields $\frac{d}{dt}\|u\|_2^2 = -2\|\nabla u\|_2^2 + 2\|u\|_4^4$ which means that the evolution will depend on the competition between $\|u\|_4^4$ and $\|\nabla u\|_2^2$.

This remark paves the way to the following theorem.

Theorem 13 *(J. Ball, H.A. Levine and L. Payne). If u_0 is a smooth compactly supported function which does not vanish identically and if*

$$\|\nabla u_0\|_2 \leq \frac{1}{\sqrt{2}}\|u_0\|_4^2 \tag{11.2}$$

then the corresponding solution of (11.1) *blows up in finite time: there exists a finite T_0 such that $\|u(.,t)\|_2$ is unbounded as t reaches T_0.*

Theorem 13 raises the following problem: does there exist a function space norm $\|.\|_*$ and a positive η with the following properties

(a) $\|f\|_4^2 \leq C\|\nabla f\|_2\|f\|_*$
(then (11.2) would say that $\|f\|_*$ is large)

(b) if u_0 is smooth and compactly supported and if $\|u_0\|_* \leq \eta$, then there exists a global (in time) smooth solution $u(x,t)$ to (11.1) (no blowup)

(c) $\|f_\lambda\|_* = \|f\|_*$ if $f_\lambda(x) = \lambda f(\lambda x)$

(d) $\|f\|_*$ is small if f is oscillating?

Let us comment (c). If $u(x,t)$ is a solution to (11.1), so are $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ for every positive λ. If $p \neq 3$, the L^p-norm is not invariant under these rescalings and a condition like $\|u_0\|_p \leq \eta$ for some small η is not relevant for a global existence of the corresponding solution to (11.1). Indeed by a convenient rescaling of the initial condition u_0, this smallness requirement can be reached. This remark explains condition (c).

The simplest norm fulfilling (c) is the L^3 norm for which F.Weissler [40] proved the following.

Theorem 14 *For a positive constant η, the condition $\|u_0\|_3 < \eta$ implies the existence of a global solution $u(x,t) \in C([0,\infty), L^3(\mathbb{R}^3))$ to (11.1).*

Uniqueness was proved by F. Weissler inside a Banach space Y which is smaller than the "natural space" $C([0,\infty), L^3(\mathbb{R}^3))$. This smaller space is defined by imposing the following condition on $u(x,t)$

$$t^{1/2}\|u(.,t)\|_\infty \leq \eta \tag{11.3}$$

Let us denote the heat semigroup by $S(t)$. Then imposing this growth condition on the linear evolution $S(t)u_0$ is equivalent to saying that u_0 belongs to our old friend $B_\infty^{-1,\infty}$. This is not a restriction since this Besov space contains L^3.

In her thesis, Elide Terraneo constructed a striking counter-example showing that uniqueness of solutions $u(x,t)$ to (11.1) in $C([0,\infty), L^3(\mathbb{R}^3))$ could not be expected in general [39]. This explains the role of (11.3).

This situation sharply contrasts with what happens for Navier-Stokes equations. T. Kato proved the analogue of Theorem 14. Kato's proof is close to Weissler's approach and (11.3) is playing a very important role in the construction. For quite a long time, uniqueness of Kato's solutions $v(x,t) \in C([0,\infty), L^3(\mathbb{R}^3))$ was an open problem. Finally uniqueness was proved by Giulia Furioli, Pierre-Gilles Lemarié-Rieusset and Elide Terraneo without assuming (11.3). The interested reader is referred to [20] or [28].

Theorem 14 says that $\|.\|_3$ fulfils (b). Moreover one has $\|f\|_4^2 \leq \|\nabla f\|_2 \|f\|_3$ but these two answers to our program are far form being optimal ones. Indeed the L^3 norm can be replaced by a much weaker one for which (d) holds.

This weaker norm is a Besov norm. The relevance of Besov norms in (a) is explained by the following Gagliardo-Nirenberg inequality.

Lemma 4 *For any function f belonging to the homogeneous Sobolev space H^1, we have*

$$\|f\|_4^2 \le C\|\nabla f\|_2 \|f\|_B \tag{11.4}$$

where B is the homogeneous Besov space $B_\infty^{-1,\infty}$.

Lemma 4 suggests that the weak norm $\|f\|_*$ fulfilling (a) to (d) might be the Besov norm $\|f\|_B$. This Besov norm is the weakest one since $B_\infty^{-1,\infty}$ is the largest function space whose norm is translation invariant and fulfils (c). We do not know if it is the case but the following theorem gives an example of a norm fulfilling (a) to (d).

Theorem 15 *Let $\|.\|_*$ denote the norm in the Besov space $B_6^{-1/2,\infty}$. Then there exists a positive constant η such that if the initial condition u_0 satisfies $u_0 \in L^3$ and $\|u_0\|_* \le \eta$ then the corresponding solution of the nonlinear heat equation will be global in time and belongs to $C([0,\infty), L^3(\mathbb{R}^3))$. Moreover there exists a constant C such that*

$$\|u(t)\|_* \le C\|u_0\|_*, \quad t \ge 0 \tag{11.5}$$

The homogeneous Besov space $B_6^{-1/2,\infty}$ needs to be defined. We start with $\varphi(x) = (2\pi)^{-n/2} exp(-|x|^2/2)$ and let S_j be the convolution operator with $2^{3j}\varphi(2^j x), j \in \mathbb{Z}$. Then we have

Definition 3 *f belongs to $B_6^{-1/2,\infty}$ if and only if a constant C exists such that $\|S_j(f)\|_6 \le C2^{j/2}, \quad j \in \mathbb{Z}$. The optimal C being the norm of f in $B_6^{-1/2,\infty}$.*

Theorem 15 is not optimal but it improves on Theorem 14. On one hand,

$$L^3 \subset B_6^{-1/2,\infty} \subset B_\infty^{-1,\infty} \tag{11.6}$$

and these embeddings are provided by Bernstein's inequalities. On the other hand, Theorem 15 is consistent with the guess that the oscillating character of the initial condition implies the global (in time) existence of the corresponding solution. Indeed one can easily check that

$$\|\cos(\omega x)\varphi(x)\|_* \le C|\omega|^{-1/2}\|\varphi\|_6 \tag{11.7}$$

as $|\omega|$ tends to infinity. When $|\omega|$ is large enough, then the smallness requirement is met and the corresponding solution is global in time. A last observation concerns scale invariance. The norm in $B_6^{-1/2,\infty}$ has the same invariance as the L^3 norm does and this invariance is consistent with the one we found in the nonlinear heat equation.

The experience we gained on this specific example will now be used to attack the much more difficult Navier-Stokes equations.

12 The Navier-Stokes equations

We now consider the Navier-Stokes equations decribing the motion of some incompressible fluid. The fluid is assumed to be filling the space and there are no exterior forces. Then the Navier-Stokes equations read as follows

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v - (v.\nabla)v - \nabla p \\ \nabla.v = 0 \\ v(x,0) = v_0 \end{cases} \tag{12.1}$$

Here $v = (v_1, v_2, v_3)$, $v_j = v_j(x,t)$, $x \in \mathbb{R}^3, t \geq 0$, the pressure is a scalar and the Navier-Stokes equations are a system of four equations with four unknown functions v_1, v_2, v_3 and p.

The notation ∇p means the gradient of the pressure, $\nabla.v$ means $\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$. Moreover $(v.\nabla)v = v_1\partial_1 v + v_2\partial_2 v + v_3\partial_3 v$ which is a vector.

If the velocity $v(x,t)$ is not a smooth function of x, then multiplying some components of v with derivatives of some other components might be impossible. That is why $(v.\nabla)v$ should be rewritten as $\partial_1(v_1 v) + \partial_2(v_2 v) + \partial_3(v_3 v)$ which makes sense whenever v is locally square integrable.

Navier-Stokes equations have some remarquable scale invariance properties. First they commute with translations in x and $t \geq 0$. Moreover if the pair $v(x,t), p(x,t)$ is a solution of (12.1) and if for every $\lambda > 0$ we dilate this solution into

$$\begin{cases} v_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \\ p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t) \end{cases} \tag{12.2}$$

then (v_λ, p_λ) is also a solution to the Navier-Stokes equations. The initial condition is replaced by

$$v_\lambda(x,0) = \lambda v_0(\lambda x) \tag{12.3}$$

We observe that this scale invariance is exactly the same as the one we met in the non-linear heat equation.

The problem we want to address is the possible blowup of solutions in finite time. We are aiming to attack this problem by following the same heuristic approach as in the nonlinear heat equation setting. Our guess is the following: if the initial condition is (everywhere) oscillating, then the corresponding solution to Navier-Stokes equations should be global in time. Moreover this global solution will keep for ever some additionnal qualitative regularity of the initial condition. For instance an initial condition which is $\mathcal{C}^\infty$ and is sufficiently oscillating will lead to a global solution which will also be $\mathcal{C}^\infty$ in the time-space variables. Such a result seems inconsistent. Indeed a function cannot at the same time be smooth and oscillating. However this objection disappears if the smoothness assumption is not given a quantitative form while the oscillations are defined by a specific threshold η and by imposing $\|v_0\|_* < \eta$. Here $\|f\|_*$ is a norm which is small whenever f is oscillating. This norm might be one of the norms which has been used in image processing in order to model textures. In other words, we are now assuming that our initial condition v_0 is a function which has a small norm in a function space containing generalized functions. The norm of a function f in such a function space takes advantage of the oscillating character of f. At the same time our v_0 may be extremely large in function spaces like the Hölder or Sobolev spaces. We will denote by B a Banach space of smooth functions. For instance B can be the Sobolev space H^m or the usual $\mathcal{C}^m$, $m \geq 1$.

Conjecture *Let $\|.\|_*$ be one of the norms which has been used in image processing in order to model textures. Then there exists a positive η such that if v_0 is smooth and satisfies the following two conditions $\nabla . v_0 = 0$, and $\|v_0\|_* < \eta$, then the corresponding solution of the Navier-Stokes equations belongs to $\mathcal{C}([0,\infty), B)$.*

Notice that we are not requiring that $\|v_0\|_B$ be small. This conjecture will be our guideline in this chapter and the best result will be Theorem 19 which combines a deep theorem by Herbert Koch and Daniel Tataru [24] and a nice observation by Pierre-Gilles Lemarié-Rieusset and his team [21].

As Roger Temam pointed out, the first result along these lines has been proved by H. Fujita and T. Kato in 1964 [19]. It reads the following

Theorem 16 *Let $H^{1/2}(\mathbb{R}^3)$ denote the usual homogeneous Sobolev space. Then there exists a positive constant η such that if v_0 belongs to $H^1(\mathbb{R}^3)$ and fulfils*

$$\nabla . v_0 = 0 \tag{12.4}$$

$$\|v_0\|_{H^{1/2}(\mathbb{R}^3)} \leq \eta \tag{12.5}$$

then there exists a unique global solution $v \in C([0,\infty), H^1(\mathbb{R}^3))$ to the Navier-Stokes equations.

As Roger Temam observed, this theorem is specially attractive if $\|v_0\|_{H^{1/2}}$ is much smaller than $\|v_0\|_{H^1}$. This is often the case since the first Sobolev norm is less demanding than the second one. Indeed in this situation, we do not pay too much for getting a global solution since the norm with which the initial condition is measured is weaker than the H^1 norm in which the global existence is proved. The following lemma tells us when $\|f\|_{H^{1/2}}$ is small while $\|f\|_{H^1}$ is large.

Lemma 5 *There exists a constant C such that*

$$\|f\|_{H^{1/2}} \leq C\sqrt{\|f\|_B \|f\|_{H^1}} \tag{12.6}$$

where B is the homogeneous Besov space $B_2^{-1/2,\infty}$.

The weak norm will be much smaller than the square root of the strong norm when our Besov norm is small. Does this Gagliardo-Nirenberg estimate mean that $B_2^{-1/2,\infty}$ is the space which needs to be used in our heuristic approach? It cannot be so since the Besov space B which is used does not enjoy the right scaling property. Indeed $f(x)$ and $f_\lambda(x) = \lambda f(\lambda x)$ do not have the same norm in B. However the homogeneous space $\|v_0\|_{H^{1/2}}$ is enjoying this scale invariance. But an oscillating initial condition has a large norm in this Sobolev space. We can conclude in saying that the Fujita-Kato theorem does not meet our expectations.

The Fujita-Kato theorem should be compared to a second theorem proved by Y. Giga and T. Miyakawa [22]. These mathematicians are focusing on the vorticity field $\omega = \text{curl}(v)$. Since u is divergence free, the mapping $v \mapsto \omega$ is an isomorphism for most of the function spaces which are being used in analysis. The inverse mapping is provided by The Biot-Savart law which reads

$$-4\pi v(x,t) = \int_{\mathbb{R}^3} |x-y|^{-3}(x-y) \times \omega(x,t)dy \qquad (12.7)$$

In other words, the mapping $\omega \mapsto v$ is smoothing of order 1 and any functional estimate on the vorticity field implies a corresponding estimate on the velocity field.

The motivation of Giga and Miyakawa was twofold. They wanted to model vorticity filaments in order to understand the evolution of such filaments. These vorticity filaments appear in numerical simulations of Navier-Stokes equations. At the same time Giga and Miyakawa wanted to construct some self-similar solutions to the Navier-Stokes equations. In order to achieve these goals, they modeled these vorticity filaments with the Guy David space of Radon measures μ which was already met in Section 5.

Definition 4 *A Radon measure satisfies the Guy David condition if and only if a constant C exists such that, for every ball B with radius R, we have*

$$|\mu|(B) \leq CR \qquad (12.8).$$

This Guy David space will be denoted by M. It is a dual space and will always be equipped with its weak* topology. In other words, $C([0,\infty), M)$ will always refer to the weak* topology.

Giga and Miyakawa proved the following theorem [22]

Theorem 17 *There exists a (small) positive number η such that whenever the initial condition $\omega_0(x)$ satisfies* div $\omega_0 = 0$ *together with the size condition (12.8) with $C < \eta$, there exists a solution $\omega(x,t) \in C([0,\infty), M)$ to the Navier-Stokes equation which agrees with this initial condition. Moreover there exists a constant C_1 such that the corresponding velocity v satisfies*

$$t^{1/2}\|v(.,t)\|_\infty \leq C_1 \qquad (12.9)$$

The Biot-Savart law enables us to lift this theorem from vorticities to velocities but we are not going to be more precise about the Banach space Γ describing these corresponding velocities. This space is a dual space, its norm is compatible with the scaling properties of the Navier-Stokes equations and it contains functions which are homogeneous of degree -1 which permitted to Giga and Miyakawa to build self similar solutions to the Navier Stokes equations. Seven years later, Marco Cannone and Fabrice Planchon proposed an other construction of self-similar solutions. We will later on explain their approach. However the Γ-norm of Giga and Miyakawa does not enjoy the crucial property that oscillating functions have small norms. That is why we still want to improve on their theorem.

Some progress was made by M. Cannone, F. Planchon. The norm in the Sobolev space $H^{1/2}$ which was used by Kato can be replaced by a weaker norm which is the Besov norm in the homogeneous space $B_q = B_q^{-(1-3/q),\infty}$ whenever $3 \leq q < \infty$. More precisely we have

Theorem 18 *There exists a positive constant η_q such that whenever the initial condition v_0 satisfies, for some $q \in [3, \infty)$,*

$$\nabla . v_0 = 0 \tag{12.10}$$

$$v_0 \in L^3(\mathbb{R}^3) \quad \text{and} \quad \|v_0\|_{B_q} < \eta_q \tag{12.11}$$

then the corresponding solution to Navier Stokes equations belongs to $C([0, \infty), L^3(\mathbb{R}^3))$ and is unique.

The homogeneous Besov space B_q is defined exactly the same way as in the special case $q = 2$ (see Theorem 15). We let φ be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} \varphi(x)dx = 1$. Then $\varphi_j(x) = 2^{3j}\varphi(2^j x)$ and S_j denotes the convolution operator with $\varphi_j(x)$. Finally a function or a distribution f belongs to the homogeneous Besov space $B_q^{-\alpha,\infty}$ if and only if $\|S_j(f)\|_q \leq C 2^{j\alpha}, j \in \mathbb{Z}$.

One should observe that this result does not contain the theorem obtained by Giga and Miyakawa. Indeed the function space used by these authors is not contained inside any space B_q, $3 \leq q < \infty$.

F. Planchon made the following remarks. The Banach spaces B_q are increasing with q in such a way that the conditions (12.9) seem to be less demanding as q grows. However the positive constant η_q which appears in (12.9) tends to 0 as q tends to infinity. Therefore comparing these distinct conditions is a delicate matter.

In this direction of large values of q, a main breakthrough was achieved by Herbert Koch and Daniel Tataru [24] who treated the limiting case $q = \infty$. As it is often the case, L^∞ should be replaced by the John and Nirenberg space BMO. Moreover the regularity exponent α which is $1 - 3/q$ tends to 1 as q tends to infinity and these remarks pave the way to the following definition

Definition 5 *We denote by $B = B_\infty$ the Banach space consisting of all generalized functions f which can be written as $f = \partial_1 g_1 + ... + \partial_3 g_3$ where g_j, $j = 1, 2$ and 3 belong to BMO.*
The norm in B_∞ is the infimum of the sum of the three BMO norms.

As usual H^m will denote the standard Sobolev space. Then a combination between the Koch-Tatar theorem [24] and a nice remark by Pierre-Gilles Lemarié-Rieusset [21] reads as follows

Theorem 19 *There exists a positive constant η such that the conditions (a) $\|v_0\|_B \leq \eta$ together with (b) $v_0 \in H^m$ and (c) $\nabla.u_0 = 0$ imply the existence of a global solution v of the Navier-Stokes equations. This solution belongs to $\mathcal{C}([0,\infty), H^m(\mathbb{R}^3))$.*

As it might be guessed, the Koch and Tataru space contains all the previous Besov spaces which were used in Theorem 18. One also should observe that the Koch and Tataru space is exactly the one which was used for modeling the two-dimensional textures. Moreover the Koch and Tataru theorem implies the Giga-Miyakawa result. It is indeed a simple exercise to check that $\Lambda^{-1}(\mu)$ belongs to BMO whenever μ satisfies the Guy David condition. Here $\Lambda = (-\Delta)^{1/2}$.

Before ending this section, we would like to say a few words about the proof of the Koch and Tataru theorem. It follows the general organization which was pioneered by Kato and Weissler. That is to say that the Navier-Stokes equations are rewritten as an integral equation. This is achieved by solving the linear heat equation. Let $S(t)$ denote the heat semigroup. We have $S(t)[f] = f \star \Phi(t)$ where $\Phi(t) = t^{3/2}\Phi(x/\sqrt{t})$ and $\Phi(x)$ is the usual gaussian fonction. Then we obtain

$$v(t) = S(t)v_0 + \Pi \int_0^t S(t-\tau) \sum_1^3 \partial_j v_j v(\tau) d\tau \tag{12.12}$$

Here Π denotes the Leray-Hopf projector on divergence free vector fields. In other words

$$\Pi(f_1, f_2, f_3) = (\sigma - R_1\sigma, \sigma - R_2\sigma, \sigma - R_3\sigma) \tag{12.13}$$

where $\sigma = R_1 f_1 + R_2 f_2 + R_3 f_3$. It implies that Π acts boundedly on all spaces which are preserved by the Riesz transforms R_1, R_2 and R_3.

Two points should be made. First the pressure $p(x,t)$ has disappeared from the Navier-Stokes equations and next the initial condition has been incorporated inside (12.12). Indeed the kernel of the Leray-Hopf operator is precisely the collection ∇p of curl-free vector fields.

We then rewrite (12.13) in a more condensed way as

$$v = g + B(v, v) \tag{12.14}$$

where v is viewed as a vector inside some function space X and g is a given vector in X. All functions are defined on $\mathbb{R}^3 \times (0, \infty)$. The difficult part of the proof is the construction of this Banach space X to which a Picard fixed point theorem will be applied.

In the Koch and Tataru proof X is defined as follows.

Definition 6 *The Banach space X consists of all functions $f(x,t)$ which are locally square integrable on $\mathbb{R}^3 \times (0,\infty)$ and which satisfy the following condition*

$$\|f\|_X = \sup \|t^{1/2} f(.,t)\|_\infty + \sup (|B(x,R)|^{-1} \int_{Q(x,R)} |f|^2 dy dt)^{1/2} < \infty$$

As usual $B(x,R)$ denotes the ball centered at x with radius R while $Q(x,R)$ is the Carleson box $B(x,R) \times [0, R^2]$. The supremum is computed over all such Carleson boxes and the right hand side is the norm of f in X.

Two facts need to be proved. First the function g in (12.14) should belong to X. Next the bilinear operator $B(v, v)$ should act boundedly from $X \times X$ to X.
The first fact is an easy consequence of the characterization of BMO by Carleson measures. This characterization can also be interpreted as a characterization of BMO by size conditions on wavelet coefficients. The second part of the proof is much deeper and the reader is referred to the beautiful paper by Koch and Tataru [24].

13 The nonlinear Schrödinger equation

We now consider the nonlinear Schrödinger equation which obeys the same scaling laws as the two preceding nonlinear PDE's. There are indeed two such equations depending on a $\pm$ sign. The Schrödinger equations with critical nonlinearity are the following evolution equations:

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta u = \epsilon |u|^2 u \\ u(x,0) = u_0, \quad x \in \mathbb{R}^3,\ t \in [0,\infty) \end{cases} \tag{13.1}$$

where ϵ is either -1 or 1 and $u = u(x,t)$ is a complex valued function defined on $\mathbb{R}^3 \times (0,\infty)$. If λ is any positive scale factor, then, for every solution $u(x,t)$ of (13.1), $\lambda u(\lambda x, \lambda^2 t)$ is also a solution of (13.1) for which the initial condition is $\lambda u_0(\lambda x)$. Therefore it is not unnatural to expect some similarities with both the nonlinear heat equation and the Navier-Stokes equations.

More precisely we might follow Kato and Fujita and expect (13.1) to be well posed for the critical Sobolev space $H^{1/2}(\mathbb{R}^3)$. Cazenave and Weissler [5] proved that it was the case under a smallness condition on the norm of the initial condition in $H^{1/2}(\mathbb{R}^3)$. Fabrice Planchon [38] extended this theorem and replaced the smallness condition $\|u_0\|_{H^{1/2}} \leq \eta$ by a weaker requirement which reads $\|u_0\|_* \leq \eta$ where the norm $\|.\|_*$ is the homogeneous Besov norm in $\dot{B}_2^{1/2,\infty}$.

Theorem 20 *With the preceding notations, there exists a positive constant η such that for every initial condition u_0 in $H^{1/2}(\mathbb{R}^3)$ satisfying $\|u_0\|_* \leq \eta$, there exists a solution $u(.,t)$ to the Scrödinger equation* (13.1) *which belongs to $C([0,\infty); H^{1/2}(\mathbb{R}^3))$.*

This theorem should be compared to Theorem 18. Indeed keeping the same notations as in Theorem 18, the Besov space which is used is B_2. Moreover this theorem implies the existence of many self-similar solutions to the nonlinear Schrödinger equation. Such solutions were previously proved to exist by Cazenave and Weissler [6] under much more restrictive regularity assumptions.

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