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REGIONAL STUDIES OF CRUSTAL ANELASTICITY USING BODY AND SURFACE WAVES

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Regional studies of crustal anelasticity using body and surface waves

Some Theoretical Aspects of Seismic Wave Propagation in an Absorptive Medium

A. Selected References

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Anderson, D.L., A. Ben-Menahem, and C.B. Archambeau, Attenuation of seismic energy in the upper mantle, J. Geophys. Res., 70, 1441-1448, 1965.

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VII. (cont)

B. Introduction

1. Mechanical models for time-dependent stress-strain relationships

The concepts of viscoelasticity (and some of the mechanical models) can be introduced simply in terms of two basic components,

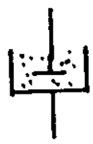
a spring

in which stress (force) is linearly related to strain (extension)

$$F = E\,a$$

where F is the force, E is the elastic constant of the spring, and a is extension. (The nomenclature follows Bland).

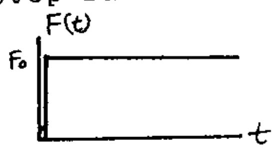
a dashpot



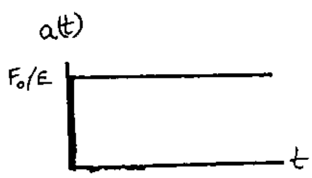
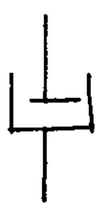
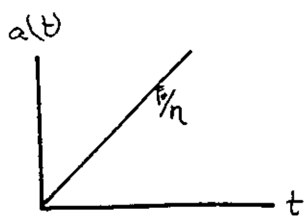
in which stress is linearly related to time rate of change of extension

$$F = \eta \frac{da}{dt}$$

To illustrate these models and combinations of them, let us suppose that a step-function of applied force is used

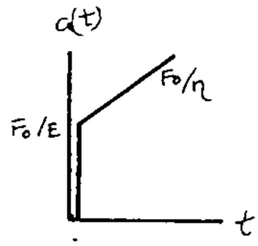


and inquire as to the resulting extension or strain.

Model	Extension	Extension computed for specified force	Force computed for specified extension
Spring 		$a = F/E$	$F = E\,a$
		This is the basic model used in elementary elasticity theory.	
		$a = \frac{1}{\eta}\int F\,dt$	$F = \eta \frac{da}{dt}$
		This is the basic model used for viscous fluid theory.	

Model	Extension	Extension, for specified force	Force, for specified extension

Maxwell Solid



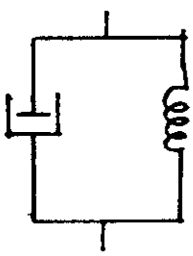
$$a = \frac{F}{E} + \frac{1}{\eta}\int F\,dt \qquad \frac{1}{\eta}F + \frac{1}{E}\frac{dF}{dt} = \frac{da}{dt}$$

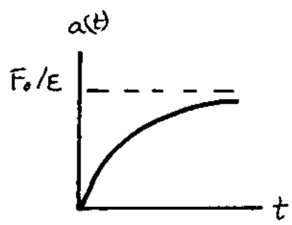
Note the implication that extension increases without limit.

This model has been used for long-time stresses such as post-glacial uplift by Gutenberg and others. See Walcott, 1970, Jour. Geoph. Res. v. 75, p. 3941.

Application to earthquake recurrence times: Bonafede, 1982.

Voigt Solid

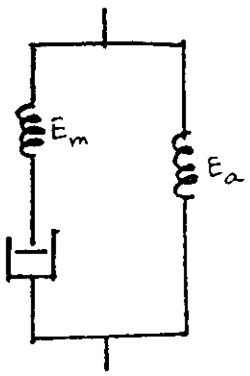


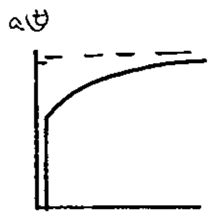


$$\eta a + E\int a\,dt = \int F\,dt \qquad F = Ea + \eta\frac{da}{dt}$$

This model has found favor with short-period phenomena such as seismic wave propagation (for example, Collins, 1960, Geophysics, v. 25, p. 483).

"Standard Linear Solid"





$$F + \left(\frac{\eta}{E_m}\right)\frac{dF}{dt} = E_a a + \frac{\eta}{E_m}(E_m+E_a)\frac{da}{dt}$$

This more versatile (but more complex) model includes all of the preceding as special cases.

Comment here concerning the terms viscous, viscoelastic, etc.

VIII.B (cont)

2. Linear vs non-linear viscoelasticity

In all theories of viscoelasticity, we consider stress-strain relationships of the form

$$F(\sigma, \epsilon, t) = 0$$

where σ and ϵ are stress and strain, respectively. Elasticity theory as used in earlier sections is the special case,

$$F(\sigma, \epsilon) = 0$$

<u>Linear</u> viscoelasticity (or elasticity) theory is the special case in which we require that F involve only first powers of stress and strain, excluding terms like

$$\sigma^2 \qquad \epsilon^2 \qquad \epsilon\sigma$$

The differential equations <u>do</u> remain linear, however, if terms appear of the form

$$\int \sigma \, dt \qquad \frac{d^2\epsilon}{dt^2}$$

The advantages of retaining a linear theory, so long as it can accommodate the observations, include

1) The mathematics is usually simpler.
2) Linear transforms such as Fourier and Laplace*can be used.
3) Effects are additive and commutative. We can apply forces in any time sequence and obtain the same final result, i.e. the process is commutative. The additive statement may be put: If stress σ_1 produces strain ϵ_1 and stress σ_2 produces strain ϵ_2, then stress $\sigma_1 + \sigma_2$ will produces strain $\epsilon_1 + \epsilon_2$.

The property of superposition is often taken as the defining criterion for linearity; see for example Savage and Hasegawa (1967).

A practical implication of linearity (Aki and Richards, p. 168) is that a wave may be analysed or synthesized into its Fourier components, each of which can be studied separately.

3. Elementary seismic wave propagation in an attenuating medium

Preceding sections have assumed the simplest possible stress-strain relationship, namely Hooke's Law in which stress components are linearly related to strain components.

The stringency of this assumption can be relaxed in various ways, as touched upon in section III.A.5. The theory of finite strain, for example, permits the stress-strain relationship to include higher powers than the first, or cross products (which means that the corresponding theory has become non-linear elasticity).

In the present section, we extend Hooke's Law to include time-dependent stress-strain relationship. This does <u>not</u> mean that we must give up linear elasticity, with all its advantages of superposition, Fourier analysis, etc.

The consequences for wave propagation may be summarized as follows:

1) The most significant effect is a <u>frequency-dependent</u> attenuation of the wave as it travels, usually represented by an amplitude multiplying factor

$$e^{-\alpha D}$$

where D is the path distance and α is the attenuation coefficient.

Experimental data suggest that α is frequency dependent, and that linear dependence upon frequency is a good approximation in many earth materials over a wide frequency range,

$$\alpha = \alpha_0 \omega .$$

Since a transient waveform contains many frequencies, it follows that <u>the transient waveform will change shape as it propagates</u>.

2) The propagation velocity is also frequency-dependent.

The effect is second order so the measured travel times will not usually be affected significantly. However, the cumulative effect on pulse shape for a transient waveform can be very significant.

If velocity depends upon frequency, then we have dispersion. We will call this "material dispersion" to distinguish it from "geometrical dispersion" such as we encountered in section V.

A practical consequence of this effect for earthquake seismology and the composition of the deep interior arises when we attempt to compare velocities computed at short periods (body waves) from velocities computed at long periods (surface waves and normal modes of oscillation). As noted by Dziewonski (1979), the relative difference between velocities at a period of 1 hour and 1 second is of the order of 1% and must be considered very significant.

In the present section, we limit our attention to (1) plane waves propagating in the x-direction (to avoid unnecessary complications in arriving at the underlying principles), and (2) a single frequency, w.

We will demonstrate that both the attenuation factor and the velocity must depend upon the attenuation constants as well as the elastic constants.

In subsequent sections, we will consider transient signals which include many frequencies. Since the attenuation constants and the elastic constants will, in general, be frequency-dependent, it will follow that the attenuation factor and the velocity must also be frequency-dependent. We will find further that their dependences upon frequency must be related to each other (so intimately that one can in principle be computed from the other), in order to satisfy the condition of causality.

We start by reviewing one-dimensional single-frequency wave propagation in a purely elastic medium.

For either elastic or anelastic medium, we have a one-dimensional equation of motion

$$d \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}$$

and a one-dimensional definition of strain $\epsilon = \frac{\partial u}{\partial x}$

The elastic case diverges from the anelastic case at the stress-strain relationship. For the elastic case, we have

$$\sigma = E \epsilon$$

leading to

$$d \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2}$$

This has a general solution

$$u = f\left(t \pm \frac{x}{V_0}\right) \quad \text{with } V_0 = \sqrt{\frac{E}{d}}$$

of which a particular example might be

$$u = u_0 \cos \omega \left(t - \frac{x}{V_0}\right)$$

We observe that the peak amplitude is independent of distance and that the velocity does not depend explicitly upon frequency (although it is possible that E or d might do so, in which case velocity would also).

The anelastic case uses a different stress-strain relationship; for example

$$\sigma = [E_1 + i E_2] \epsilon$$

$$\text{OR} = |E| e^{i\delta} \epsilon \qquad \left\{ \begin{array}{l} |E| = [E_1^2 + E_2^2]^{1/2} \\ \tan \delta = \frac{E_2}{E_1} \end{array} \right.$$

which upon substitution gives

$$d \frac{\partial^2 u}{\partial t^2} = (E_1 + i E_2) \frac{\partial^2 u}{\partial x^2}$$

Let us seek a trial solution of the form

$$u = u_0 \epsilon^{-\alpha x} \epsilon^{i\omega(t - \frac{x}{V})}$$

which suitably represents an attenuating wave. We substitute and find, after some tedious algebra,

$$\alpha = \frac{\omega}{V} \tan \frac{\delta}{2}$$

$$V = \sqrt{\frac{E_1}{d}} \frac{1}{\cos \frac{\delta}{2} \sqrt{\cos \delta}}$$

corresponding to Kolsky's equations 18 and 19. We note that for $E_2 = 0$ these reduce to the elastic-medium results above.

For small attenuation, we can use the standard series expansions,

$$\sin x = x - \cdots$$
$$\cos x = 1 - \frac{x^2}{2} + \cdots$$
$$(1+x)^{\pm\frac{1}{2}} = 1 \pm \frac{x}{2} + \cdots$$

and find that, accurate to terms in δ^2,

$$\alpha = \frac{\omega}{2V} \delta \left[1 - \frac{\delta^2}{4} + \cdots\right]$$

$$V = \sqrt{\frac{E_1}{d}} \left[1 + \frac{3}{8} \delta^2 + \cdots\right]$$

Discussion:

Recalling that δ is a measure of the degree of anelasticity ($\delta = 0$ in the elastic case), we observe that attenuation depends upon it to the first order but velocity only to the second order.

If δ is independent of frequency, then it follows that V is independent of frequency and α is proportional to w^{+1}.

If we wished to express the attenuation in terms of Q rather than δ, then using

$$\frac{1}{Q} = \tan \delta \xrightarrow{\sim} \delta$$

we conclude that the attenuation term has the form

$$\epsilon^{-\frac{\omega x}{2VQ}}$$

which agrees with Aki and Richards, p. 169.

Alternative derivation: Minster, p.168, reference in VII.A.1.

VII. (cont)

C. Alternative (but equivalent) parameters which may be used to represent the attenuating properties of a medium

1. Complex values for velocities or elastic constants

Basically, we require a mathematical model which describes a time lag or phase shift between stress and strain. This can be accomplished by making into complex quantities one or more of the following:

elastic constants
velocities
wave numbers
(even frequencies)

The problem is analogous to an electrical circuit in which the direct-current resistance, V = RI, is generalized to an alternating-current impedance, V = ZI with Z = R + iX.

For example, suppose that the stress is sinusoidal and that the strain lags behind it by a phase angle δ

$$\sigma = \sigma_0 \, e^{i\omega t}$$

$$\epsilon = \epsilon_0 \, e^{i(\omega t - \delta)}$$

Thus we can write

$$\sigma = Y \epsilon$$

with
$$Y(\omega) = E_1(\omega) + i\,E_2(\omega)$$

$$\tan \delta = \frac{E_2(\omega)}{E_1(\omega)}$$

We note that E_1 is the constant applicable to the purely elastic situation where $E_2 = 0$. Thus it is called the dynamic modulus, and E_2 is called the dynamic loss.

In section VII.E.3, we will show that E_1 and E_2 are not independent quantities, and that knowledge of one over the entire frequency range will define the other.

As another example, suppose we wished to describe a one-dimensional plane wave with an attenuation factor,

$$u(x,t) = u_0 \, e^{-\alpha x} \, e^{i\omega(t - x/c)}$$

We observe that this can be written as

$$u(x,t) = u_0 \, e^{i(\omega t - kx)}$$

if we simply define a complex wavenumber

$$k(\omega) = \frac{\omega}{c(\omega)} - i \, \alpha(\omega)$$

The same result can be achieved in many other ways. For example, White (1966, p. 95) replaced elastic constants with

$$\lambda(\omega) \rightarrow \lambda(\omega) + i \, \mathrm{sgn}(\omega) \, \lambda^*(\omega)$$
$$\mu(\omega) \rightarrow \mu(\omega) + i \, \mathrm{sgn}(\omega) \, \mu^*(\omega)$$

leading to stress-strain relationships like

$$p_{xy} = [\mu + i \, \mathrm{sgn}(\omega) \, \mu^*(\omega)] \, \epsilon_{xy}$$

(Note: $\mathrm{sgn}(x) = +1$ if $x > 0$ and $= -1$ if $x < 0$.
When we look at the Fourier transforms, we see that this term is required to keep strain and stress as real functions of time)

This approach leads to complex velocities (see for example Yamakawa and Sato, 1964). If we consider shear velocity for illustration (compressional velocity proceeds in the same way), then

$$V_s^*(\omega) = \left[\frac{\mu(\omega) + i\,\mu^*(\omega)}{D}\right]^{1/2}$$

$$= \underbrace{\left[\frac{\mu(\omega)}{D}\right]^{1/2}}_{\substack{\text{Elastic} \\ \text{velocity} \\ V_S}} \left[1 + i\,\underbrace{\frac{\mu^*(\omega)}{\mu(\omega)}}_{\frac{1}{Q_s(\omega)}}\right]^{1/2}$$

(Q will be introduced in next section)

$$V_s^*(\omega) = V_{s'}\left[1 + \frac{i}{Q_s(\omega)}\right]^{1/2}$$

$$\cong V_s\left[1 + \frac{i}{2\,Q_{s'}(\omega)}\right] \quad \text{for large Q}$$

This formulation is widely used in attenuated surface wave studies. Schwab and Knopoff (1972, p.146) show that the Haskell matrix method for layered media can still be used for attenuating media with minor modifications if we introduce complex velocities,

$$\frac{1}{\alpha} = \frac{1}{A_1} - i A_2 \qquad \frac{1}{Q_\alpha} = 2A_1A_2$$

$$\frac{1}{\beta} = \frac{1}{B_1} - i B_2 \qquad \frac{1}{Q_\beta} = 2B_1B_2$$

$$\frac{1}{C} = \frac{1}{C_1} - i C_2 \qquad \frac{1}{Q_c} = 2C_1C_2$$

2. Differential operator representation

A quite general formulation of the stress strain relationship for linear functions is

$$a_0 \sigma + a_1 \frac{\partial \sigma}{\partial t} + a_2 \frac{\partial^2 \sigma}{\partial t^2} + \cdots = b_0 \epsilon + b_1 \frac{\partial \epsilon}{\partial t} + b_2 \frac{\partial^2 \epsilon}{\partial t^2} + \cdot$$

(Kolsky, 1960, eq. 11.
Hunter, 1960, eq. 38.
Bland, 1960, eq. 86;
Alfrey and Gurney, 1958, eq. 36).

This may be abbreviated

$$P\sigma = Q\epsilon$$

$$P = a_0 + a_1 \frac{\partial}{\partial t} + \cdots$$

$$Q = b_0 + b_1 \frac{\partial}{\partial t} + \cdot$$

Comparing it with the complex modulus representation, we see the equivalence,

$$E_1(\omega) + i\, E_2(\omega) = \frac{b_0 + (i\omega) b_1 + (i\omega)^2 b_2 + \cdots}{a_0 + (i\omega) a_1 + (i\omega)^2 a_2 + \cdots}$$

We note that all of the mechanical models in Section VII.B are described by differential equations of this form:

Maxwell solid: $a_0 \sigma + a_1 \frac{\partial \sigma}{\partial t} = b_1 \frac{\partial \epsilon}{\partial t}$

Voigt solid: $a_0 \sigma = b_0 \epsilon + b_1 \frac{\partial \epsilon}{\partial t}$

Standard linear solid: $a_0 \sigma + a_1 \frac{\partial \sigma}{\partial t} = b_0 \epsilon + b_1 \frac{\partial \epsilon}{\partial t}$

Jeffreys solid (Gutenberg, equation 9.1): $a_0 \sigma + a_1 \frac{\partial \sigma}{\partial t} = b_1 \frac{\partial \epsilon}{\partial t} + b_2 \frac{\partial^2 \epsilon}{\partial t^2}$

Simplified equations such as these may be useful in matching experimental results over a limited frequency range, but none of them is universally useful. Kolsky, page 68, notes that the standard linear solid is much more satisfactory for wave propagation than either the Maxwell or Voigt solid. The basic question is, how wide is the frequency spectrum associated with the source; if it is not too wide, then it may be possible to adjust the constants in one of the simpler models to give satisfactory agreement with observations.

Experimental results can of course be matched as closely as desired by taking enough terms in P and Q, but the mathematics quickly becomes impossible. What is equivalent, the model may be generalized by taking a large number of Voigt or Maxwell elements in series or parallel, each with its own characteristic constant (Kolsky, p. 69; Bland, p. 7). The viscoelastic behaviour is then defined in terms of a distribution function of retardation times (using Voigt models in series) or relaxation times (using Maxwell models in parallel).

VII.C (cont)

3. Q; the Specific Dissipation Function

References:

Aki and Richards, p. 168

The most common parameter used in geophysics to describe attenuation is Q. By analogy with the "Q" for an electrical circuit, often used to specify the sharpness of a resonant circuit, we define Q for anelasticity as

$$\frac{2\pi}{Q} = \frac{\text{Energy dissipated in taking a specimen through one complete stress cycle}}{\text{Maximum stored elastic energy per unit volume during one complete stress cycle.}}$$

The reciprocal of Q is sometimes called Specific Dissipation Function.

Q is related to other modes of representing attenuation as:

complex elastic modulus: $\frac{1}{Q} = \tan\delta = \frac{E_2}{E_1} = \frac{\Delta}{\pi}$

attenuation coefficient: $\alpha = \frac{\omega}{2cQ}$

logarithmic decrement: $\Delta = \frac{2\pi\alpha c}{\omega}$

Comment: Exploration seismologists prefer to express attenuation in db/cycle, as for example 0.2 db/cycle. Numerically, attenuation in db/cycle is about 27/Q.

Comment: Persons who wish to see attenuation discussed in an exploration context should see p. 117 et seq in Anstey, N., 1977, Seismic interpretation, the physical aspects: IHRDC, 625 pp.

Let us derive the last two relationships for the case of a damped harmonic wave with displacement u:

$$u = C\ e^{-\alpha x} \cos (kx - \omega t + \beta).$$

At a <u>fixed</u> time, the displacement may be graphed as:

Unless the damping is excessively heavy (so that the peaks do not occur exactly at the peaks of the cosine function), the peaks occur at

kx_1 = some constant, say K, and

$kx_2 = K + 2\pi$,

so that the logarithmic decrement becomes

$$\Delta = \ln \frac{u_1}{u_2}$$

$$= \ln \left[e^{-\alpha(x_1 - x_2)} \right]$$

$$= \frac{2\pi\alpha}{k}$$

$$= \frac{2\pi\alpha c}{\omega}$$ where c is the propagation velocity.

But since the peak stored elastic energy is proportional to $(\text{displacement})^2$, we have

$$2\pi/Q = (u_1^2 - u_2^2)/u_1^2 \quad = 1 - (u_2/u_1)^2 \quad = 1 - e^{-2\Delta}$$

which, for small Δ , may be expanded in series to yield

$$\frac{1}{Q} = \frac{\Delta}{\pi}$$

VII.C.3 (cont)

It should be noted that there are several definitions for Q and that they are not all equivalent. Aki and Richards p. 168 discuss it; see also O'Connell and Budiansky, 1979.

Matters are further complicated because a particular earth material will have different values of Q for

- compressional waves, Q_α
- shear waves Q_β
- Rayleigh waves Q_R
- losses in pure compression Q_κ
- losses in pure shear Q_μ

and in a layered or otherwise inhomogeneous medium there will be an effective Q; I will give expressions for this in VII.G.4.

The following relationships are given by Jordan (p. 8, in Dziewonski and Boschi, 1980):

$$Q_\beta^{-1} = Q_\mu^{-1}$$

$$Q_\alpha^{-1} = \left[1 + \frac{4}{3}\frac{\mu}{\kappa}\right] Q_\kappa^{-1} + \left[1 + \frac{3}{4}\frac{\kappa}{\mu}\right] Q_\mu^{-1}$$

(Note the introduction of $Q^{-1} = 1/Q$ as a more fundamental quantity than Q itself; this usage is increasingly common).

In many earth materials, losses in pure compression are often much smaller than losses in pures shear,

$$Q_\kappa^{-1} \ll Q_\mu^{-1}$$

In that case, if we take for illustration $\sigma = 0.25$ which implies $\kappa = \frac{5}{3}\mu$, we see from the above that

$$Q_\alpha^{-1} \doteq \frac{4}{9} Q_\beta^{-1}$$

4. Creep function and relaxation function of Boltzmann

This approach was first set forth by Boltzmann (1876) on the basis of linear superposition, and it remains a satisfactory starting point from a physical point of view.

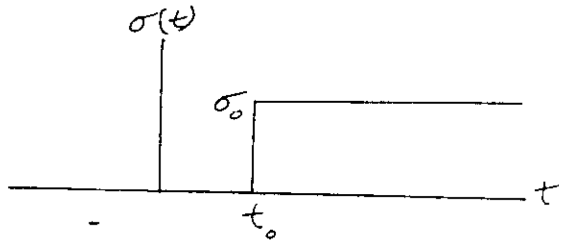
We assert that the mechanical behaviour of a material is a function of its entire previous loading history, and that if a material is subjected to a number of separate deformations, the subsequent behaviour of the specimen can be calculated by a simple addition of the effects which would be produced when the deformations took place singly.

In the limit, we can break an arbitrary force function into small increments and express the result as a summation or integral.

If the deformation is three-dimensional, then both shear and dilatational strains will occur; the effects of these must be treated separately.

An elementary creep function, $\psi(t)$:

Suppose that we apply a step function of stress to a body,



(For example, suppose we pull on one end of a bar and measure the elongation:

$$\sigma = \sigma_0 \; H(t - t_0)$$

VII.C.4 (cont)

The resulting strain will arise from three causes:

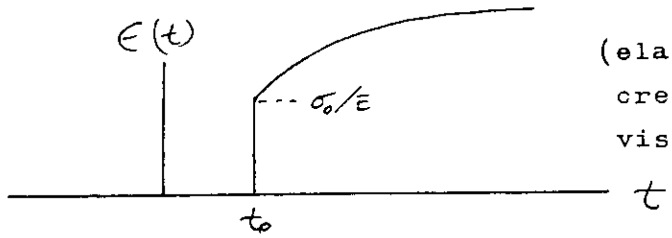
(1) elastic, where E is an elastic constant

$$\frac{\sigma_0}{E} H(t-t_0)$$

(2) long term viscous flow, if any

$$\frac{1}{\eta}(t-t_0) \; H(t-t_0)$$

(3) creep, approaching a finite strain at infinite time



(elastic plus creep, without viscous flow)

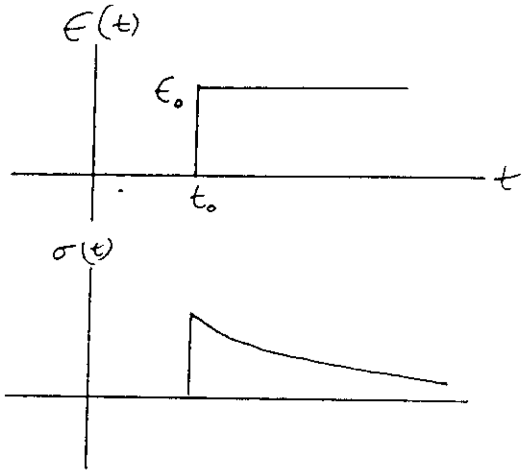
$$\epsilon(t) = \frac{\sigma_0}{E}\left[1 + \psi(t-t_0)\right] H(t-t_0)$$

where $\Psi(t)$ is a positive, monotonically increasing function with

$\Psi(\infty)$ finite.

An elementary relaxation function, $\phi(t)$.

Suppose that we apply whatever stress is necessary to produce and maintain a step function in strain



$$\epsilon(t) = \epsilon_0 \; H(t-t_0)$$

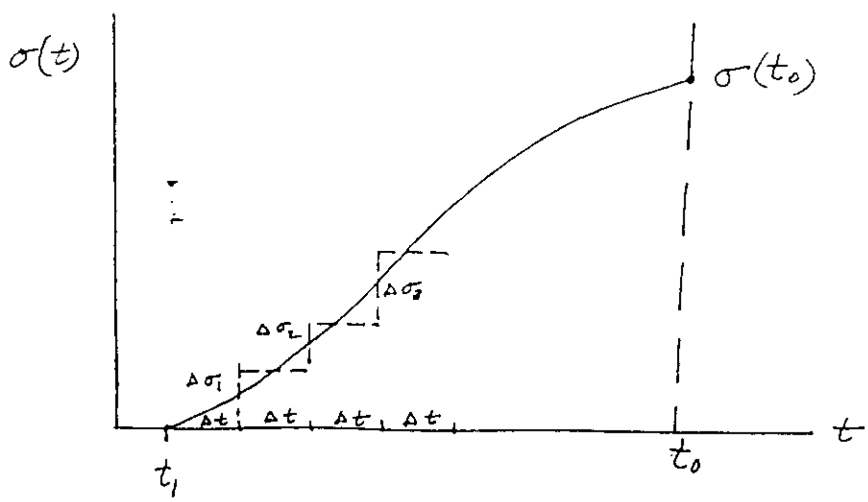
where $\phi(t)$ is a positive, monotonically increasing function,

$$\sigma(t) = E\,\epsilon_1 \left[1 - \phi(t-t_0)\right] H(t-t_0)$$

$$0 \leq \phi(t) \leq 1$$

Generalization to an arbitrary input time function.

Consider, for example, the strain produced by the following stress input:



Measured at t_o, the contributions from the stress increments are

$$
\begin{aligned}
E\,\epsilon(t_o) = {} & \Delta\sigma_1 + \Delta\sigma_1\,\psi(t_o - t_1) \\
& + \Delta\sigma_2 + \Delta\sigma_2\,\psi(t_o - t_1 - \Delta t) \\
& + \Delta\sigma_3 + \Delta\sigma_3\,\psi(t_o - t_1 - 2\Delta t) \\
& + \cdots
\end{aligned}
$$

$\downarrow$ $\qquad$ $\downarrow$ multiply by $dt(1/dt)$ and integrat

$\sum \Delta\sigma$ $\qquad$ $\downarrow$

$\downarrow$

$\sigma(t_o)$ $\qquad$ $\displaystyle\int_{-\infty}^{t_o} \dot\sigma(t)\,\psi(t_o - t)\,dt$

with $\dot\sigma(t) \equiv \dfrac{d\sigma}{dt}$

or

$$\int_{-\infty}^{t_o} \sigma(t)\,\dot\psi(t_o - t)\,dt$$

We can use similar arguments for the relaxation function, arriving at the "Boltzmann formulation of the law of superposition":

$$E\,\epsilon(t_o) = \sigma(t_o) + \int_{-\infty}^{\infty} \dot\sigma(t)\,\psi(t_o - t)\,dt$$

$$\frac{1}{E}\,\sigma(t_o) = \epsilon(t_o) - \int_{-\infty}^{\infty} \dot\epsilon(t)\,\phi(t_o - t)\,dt$$

where we have changed the upper limit from t_o to ∞ since both of the functions are 0 for negative values of their arguments.

The creep and relaxation functions are interrelated, as may be seen by taking Laplace transforms (Minster, p. 162)

$$\left[1 + s\,\bar{\Psi}(s)\right]\left[1 - s\,\bar{\Phi}(s)\right] = 1$$

We note that the integrals are of the form of convolution integrals:

$$\int_{-\infty}^{\infty} \dot{\sigma}(t)\,\psi(t_o - t)\,dt = \dot{\sigma}(t) * \psi(t)$$

VII (cont)

D. Seismic pulse propagation, and an introduction to the causality and linearity conditions

1. An example to illustrate the consequences of failing to satisfy causality

Let us consider a one-dimensional plane wave propagating as

$$e^{-\alpha x} \quad e^{i\omega(t - x/v)}$$

and let us take V=constant and α a linear function of frequency,

$$\alpha = \frac{\omega}{2QV}$$

Assuming the signal is a pulse, we will need an integral over frequency.

At distance x=0, we can call the waveform f(0,t) which will have a Fourier transform F(0,w).

At distance x, the waveform will have attenuated by an amount

$$e^{\frac{-|\omega|}{2QV}x}$$

and will have been phase shifted by an amount

$$e^{\frac{-i\omega}{V}x}$$

so that the waveform at x will be the inverse Fourier transform

$$f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{|\omega| x}{2QV}} \; e^{-\frac{i\omega x}{V}} \; F(0,\omega)\, e^{i\omega t}\, d\omega$$

As a specific example, let us take $f(0,t) = \delta(t)$ with transform $F(0,w) = 1$. Using symmetry properties,

$$f(x,t) = \frac{1}{\pi}\int_0^\infty e^{-\frac{\omega x}{2QV}} \cos\omega\left(t-\frac{x}{V}\right) d\omega$$

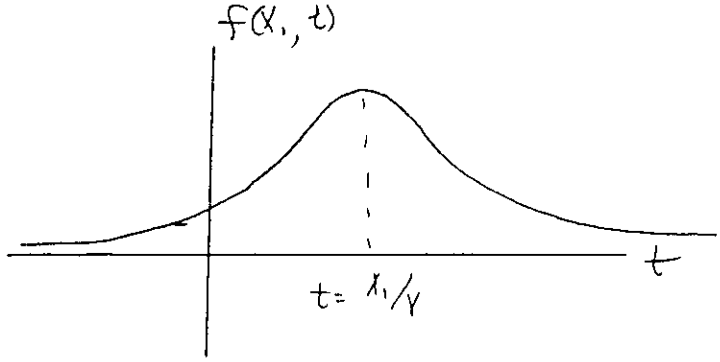
This is a standard integral of the form

$$\int_0^\infty e^{-ax} \cos bx \; dx = \frac{a}{a^2+b^2}$$

hence

$$f(x,t) = \frac{1}{2\pi QV}\,\frac{x}{\left(\frac{1}{2QV}\right)^2 x^2 + \left(t-\frac{x}{V}\right)^2}$$

At a fixed $x=x_1$, this looks like



We observe that this has non-zero value even for $t = -\infty$, which is obviously impossible for a source impulse which occurs at $t = 0$.

This illustrates a "non-causal" solution, one in which the effect appears <u>before</u> the cause happens. The requirement that no effect can appear before the cause happens is called the requirement of "causality".

VII.D.1 (cont)

The introduction of causality leads to the requirement that velocity be frequency-dependent, i.e. dispersion is required.

The following two examples show computations with and without causality. The first example is a model study on plexiglass; circles are observational data points. (i) shows the computed results without causality condition, whereas (ii) shows the same results with it, producing much better agreement with observations as well a causal time signal (Hunter, 1960).

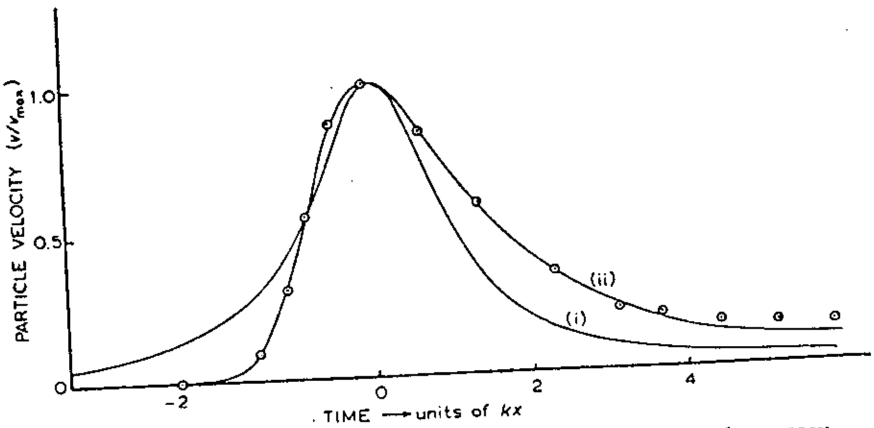


Fig. 10. Comparison of theoretical and experimental pulse shapes for perspex.
(i) Symmetric theoretical pulse neglecting variation of phase velocity with frequency.
(ii) Theoretical pulse obtained numerically by KOLSKY [1956] including variation of phase velocity with frequency.
○ Experimental results for perspex at $x = 600$ cm, with $kx = 50\ \mu s$ (KOLSKY, [1956]).

Related to this is the following figure which displays cumulative degradation of the waveform following multiple transits of a highly-attenuating rod:

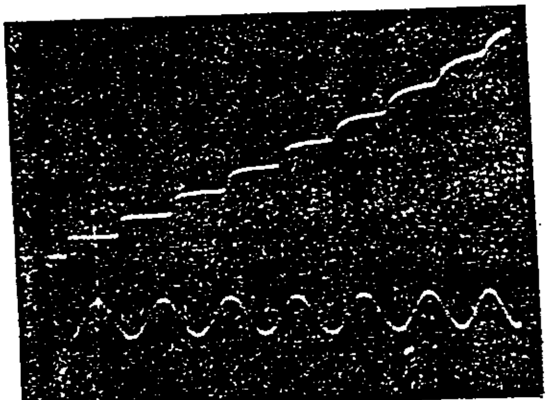


Fig. 8. Oscillograph record of displacement of end of polymethyl methacrylate rod 46 cm long and 1.25 cm diameter when 5 mg charge of lead azide has been detonated at opposite end. (Period of timing wave 500 microseconds.)

The second example is from Wuenschel (1965) and shows waveforms taken at various depths in a drill hole through the Pierre Shale in Colorado. "A"shows the signal observed at a depth of 91 feet. This signal is then used to predict the signal to be expected at a depth of 491 feet. This is shown in "B" as computed without causality and in "C" as computed with causality; the actual signal observed at 491 feet is shown as a solid line in both cases.

It can be seen that the use of a frequency-dependent velocity is required to produce satisfactory agreement.

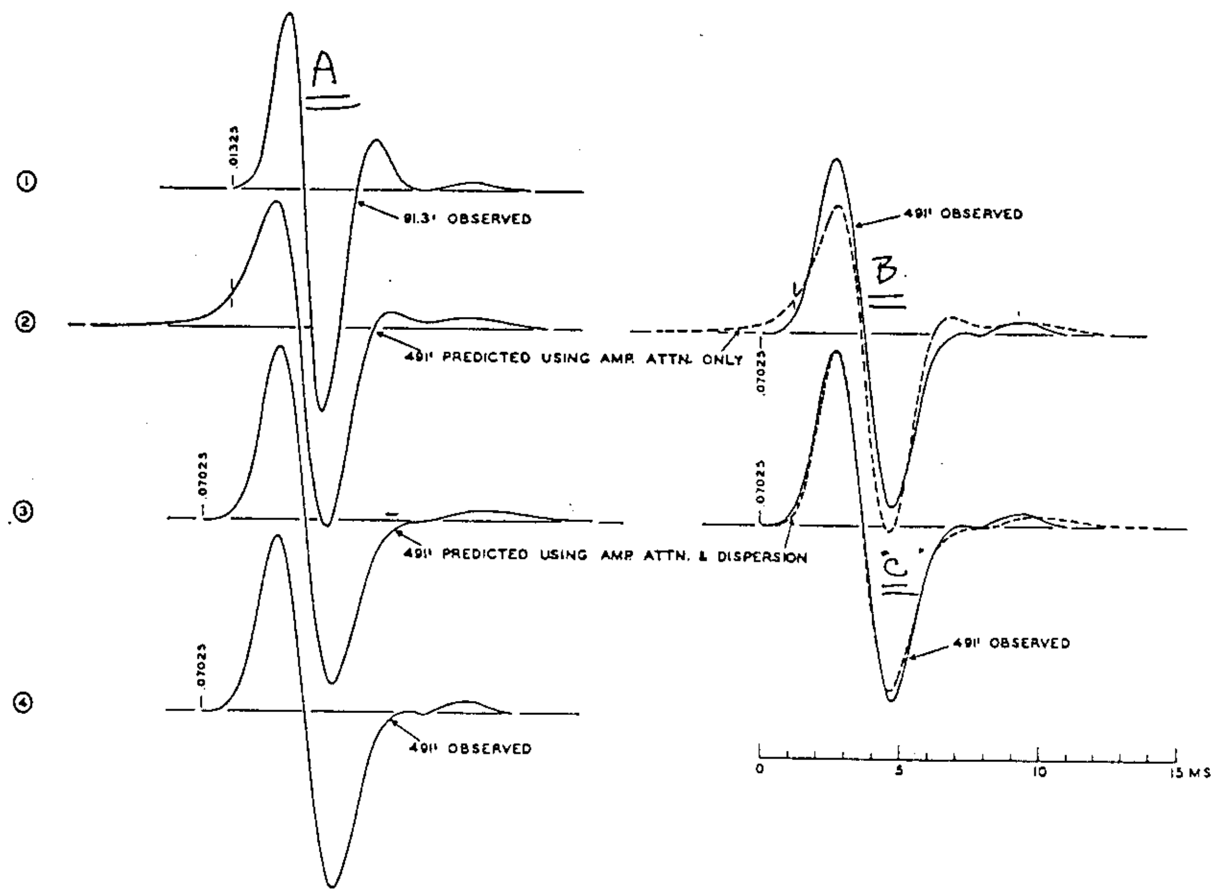


FIG. 3. Comparison between predicted and observed pulse waveforms. Record T.

VII.D (cont)

2. Summary of causality and linearity condition requirements.

Discussions of seismic wave attenuation often lead into discussions of three common assumptions:

(1) The attenuation process is linear.

I discussed this and cited references in VII.B.2. Aki and Richards put it (p. 171): The indirect evidence for retaining linearity is simply that it has led to self-consistent results in so many careful analyses of seismic data.

It should be noted, however, that some physical models for attenuation are non-linear, as for example solid friction model.

(2) Q is independent of frequency.

Since the attenuation constant is given by

$$\alpha = \frac{1}{2VQ}\,\omega$$

this is equivalent to the assumption that α is proportional to frequency to the first power.

We will return to this in connection with observational data, section VII.H. Current consensus is that, in the seismic frequency band, Q is either constant or only weakly dependent on frequency.

It should be noted that a constant-Q attenuation factor introduces unexpected difficulties into mathematical models of attenuation. We shall see some of these in section VII.F.

(3) Causality must be satisfied.

"Causality" describes the requirement that an effect cannot happen before its cause has taken place. In the context of seismic wave propagation, we can say that if the signal from a source at r=0 propagates with a velocity V and if the signal is 0 prior to t=0, then the received signal must be 0 prior to t - r/V.

We shall see that imposition of causality leads to two results:

1) Velocity is dependent upon frequency, i.e. dispersion exists.
2) The attenuation coefficient and the velocity are interrelated, and in fact knowledge of either one over all frequencies suffices to compute the other.

VII. (cont)

E. Interrelationships between attenuation and velocity imposed by the causality condition

1. A digression on the Hilbert Transform and its geophysical applications**

a) Basic relationships

The Hilbert transform of f(t) is

$$f'(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - t} \, d\tau$$

which is unusual among transforms in that one time function is transformed into another time function. We note also that the integrand has a singularity at $\tau = t$, so we may have to take the "principal part" of the integral, $P_r\int_{-\infty}^{\infty}$

The transform operation can be expressed as a convolution

$$f'(t) = f(t) * \left(\frac{-1}{\pi t}\right)$$

as can be seen by substitution into the general formula for convolution. Taner et al, eq 15, give a sampled-data version of this formula.

In the Fourier transform domain, if F(w) is the transform of f(t), then the transform of f'(t) is

$$F'(\omega) = i \quad \operatorname{sgn}(\omega) \quad F(\omega)$$

This reveals a simple physical interpretation for the Hilbert transform, namely a 90° phase shift of all frequency components in the original signal. (The "sgn(w)" is required to provide convergence of the integral for negative w).

** References cited in this section are given in section VII.A.3.

VII.E.1a(cont)

Some sample Hilbert transform pairs (Bracewell, p.272; Ansell, App e)

$f(t)$	$f'(t)$
$\sin t$	$\cos t$
$\sin at$	$\sin\left(at + \frac{\pi}{2}\right)$
$\cos t$	$-\sin t$
$\cos at$	$\cos\left(at + \frac{\pi}{2}\right)$
$\delta(t)$	$-\frac{1}{\pi t}$
$\frac{1}{t}\sin t$	$\frac{1}{t}(\cos t - 1)$
$e^{\pm iat}$	$e^{\pm i\left(at + \frac{\pi}{2}\right)}$

We surmise that a seismic signal which is converted to its Hilbert transform may undergo some dramatic changes. Show examples from Pilant, p. 91.

We can generalize from a constant phase shift of $+\pi/2$ to a constant phase shift of ϕ. The result (Pilant, p. 92) is

$$f(t)\cos\phi - f'(t)\sin\phi$$

if ϕ is a phase delay, or + if ϕ is a phase advance (Aki and Richards, p. 158 bottom). We encounter phase <u>delay</u> in attenuation, and phase <u>advance</u> when a body wave touches a caustic.

Example of:

1) A convenient source waveform, with three adjustable parameters.

   (For this example, a 20hz pulse, the parameters used were

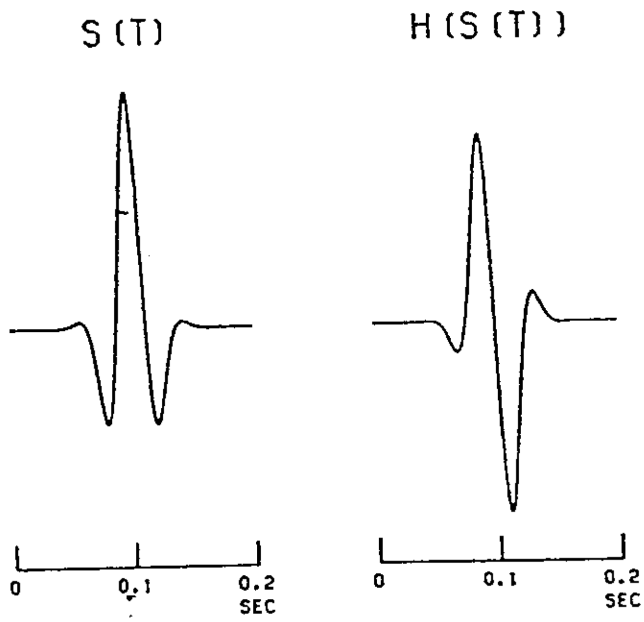
   f_m = 20 hz
   γ = 3
   ν = 0°)

2) Hilbert transform of a function, right side.

(Source: Langston and Lee, BSSA v. 73, 1983, p. 1851)

$$S(t) = e^{-(2\pi f_m/\gamma)^2\tau^2} \cos(2\pi f_m \tau + \nu)$$

$$H[S(t)] = -e^{-(2\pi f_m/\gamma)^2\tau^2} \sin(2\pi f_m \tau + \nu)$$



We can proceed further by carrying out the integration in w in previous equation, Pilant 9-36.

$$f'(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t')\,dt' \int_0^{\infty} e^{-\alpha\omega} \sin\omega(t'-t)\,d\omega$$

where we handle the divergence of the integrand by multiplying by the convergence factor, $e^{-\alpha w}$, and then letting $\alpha \to 0$

The appropriate standard integral needed here (Dwight 863.1)

$$\int_0^{\infty} e^{-ax} \sin mx \, dx = \frac{m}{a^2+m^2} \qquad a>0$$

hence

$$\int_0^{\infty} e^{-\alpha\omega} \sin[\omega(t'-t)] = \frac{t'-t}{\alpha^2+(t'-t)^2} \underset{\alpha\to 0}{\Longrightarrow} \frac{1}{t'-t}$$

so

$$\boxed{f'(t) = \frac{1}{\pi}\,\mathrm{Pr}\int_{-\infty}^{\infty} \frac{f(t')\,dt'}{t'-t}} \tag{5}$$

where P means the principal part. This is the final result for the Hilbert transform pair,

$$f(t) \longleftrightarrow f'(t)$$

Note that the Hilbert transform is still a function of <u>t</u>.

Comment on the "principal part", P of the integration.

The intent here is to integrate close to but not quite up to the singularity at t'=t from both sides. The argument is that the integrand is infinite-positive on one side and infinite-negative on the other, so that the integral cancels out over this tiny range of integration (for which we may assume f(t') to be a constant).

Several methods are available for computing the Hilbert transform. Cerveny (1976) and Choy and Richards (1975) discuss these. Lin and Kosloff give an FFT computational scheme.

The simplest approach is via equation (2). We start from given f(t) and compute (perhaps using Fast Fourier Transform) $F(w)=F_R(w)+iF_I(w)$. The usual requirements exist to keep f(t) real, namely F_R is even in w and F_I is odd.

We next multiply by i to give 90° phase shift and by sgn(w) to achieve a real function for f'(t):

$$F'(\omega) = i \, \mathrm{sgn}(\omega) \, F(\omega)$$
$$= \left[-F_I \, \mathrm{sgn}(\omega)\right] + i \left[F_R \, \mathrm{sgn}(\omega)\right]$$

and take the inverse transform to obtain f'(t).

Comment: Choy and Richards p. 59 and Aki and Richards p. 158 state that the above steps may be stated: Compute F(w), interchange the real and imaginary parts (with a sign change for the resulting real part using my sign convention or for the resulting imaginary part using opposite), and inverse transform.

It doesn't look quite so simple to me. It looks like there has to be an additional sign change for negative frequencies due to the sgn(w) above. If we are using the FFT with sample points 1 to N , then the procedure would seem to be:

1) Compute F_R and F_I for sample points 1 to N+1.
2) Compute F_R' and F_I' for the transformed function as

$$\left.\begin{aligned} F_R' &= -F_I \\ F_I' &= F_R \end{aligned}\right\} \times \begin{cases} +1 & 1 \le i \le \frac{N}{2} \\ -1 & \frac{N}{2}+1 \le i \le N \end{cases}$$

3) Take the inverse FFT to get f'(t).

(Omit this page from lecture, but retain for reference):

Proof of the transform relationship: Let us accomplish this by taking the inverse transform of

$$F'(w) = i\ \mathrm{sgn}(w)\ F(w) = F(\omega)\ e^{i\frac{\pi}{2}\mathrm{sgn}(\omega)}$$

and show that it equals $f'(t)$ as given above.

$$f'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\frac{\pi}{2}\mathrm{sgn}(\omega)}\ F(\omega)\ e^{i\omega t}\,d\omega \qquad (1)$$

OR

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} i\ \mathrm{sgn}(\omega)\ F(\omega)\ e^{i\omega t}\,d\omega \qquad (2)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} i\ \mathrm{sgn}(\omega)\left[\int_{-\infty}^{\infty} f(t')\,e^{-i\omega t'}\,dt'\right] e^{i\omega t}\,d\omega \qquad (3)$$

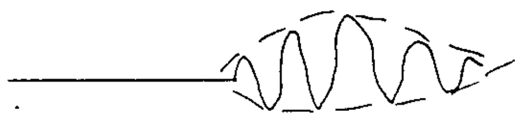
$$= \frac{1}{\pi}\int_{0}^{\infty} d\omega \int_{-\infty}^{\infty} f(t')\ \sin\omega(t'-t)\,dt' \qquad (4)$$

VII.E.1(cont)

b) Motivation (see Liu and Kosloff for other geophysical exampl

Application 1: Determine the envelope of a waveform.

We can subjectively construct the envelope of a seismic or other waveform. For example,



The technique here considered provides an objective way to do this, where the desired result may otherwise be difficult to arrive at. Farnbach treats this.

Application 2: Determine instantaneous amplitude, phase, and frequency of a signal.

Taner et al focus on this application.

Application 3: Compute the effect on a signal of shifting all frequency components by a <u>constant</u> phase, say $\pi/2$

This situation occurs in several seismic applications: seismic pulses reflected at angles beyond the critical, waves propagating through a caustic (focussing region), surface waves on a spherical earth passing through the antipodal (180°) points and 360° points. See Choy and Richards.

Application 4: To compute elementary seismograms, it is necessary to know the Hilbert transform of the source-time function (Cerveny, Jour. Geoph., v. 46, 1979, p. 137).

Application 5: The Hilbert transform appears in the Kramers-Kronig ("dispersion") relations which we develop in the following section, in connection with our attempt to create causal functions.

Application 6: The WKBJ method of Chapman (1978) for computing synthetic seismograms includes application for Hilbert transforms.

Application 7: Interpretation of magnetic anomalies (Mohan et al, 1982).

c) The Hilbert transform in "complex envelope" analysis

For further discussion, see

Farnbach, 1975 Kirlin et al, 1984

Taner and others, 1979. Hailey and Kirlin, 1984.

Robertson and Nogami, 1984.

They cite references to other literature such as electrical engineering.

Bracewell, p. 268

As noted under (a) above, our goal is to come up with an envelope to a given waveform. This envelope will have an instantaneous amplitude, instantaneous phase, and instantaneous frequency.

Taner et al call the instantaneous amplitude the "reflection strength". He makes the obvious point that its peak does not necessarily coincide with the peak of the waveform.

The instantaneous frequency should be more precise as a measurement of frequency than the usual process of measuring times between peaks and converting to frequency.

Definition: The "complex trace" (Taner et al), or the "complex envelope" (Farnbach), or the "analytic signal" (Bracewell) is defined as

$$F(t) = f(t) \pm i f'(t)$$

where - is appropriate using my choice of signs for the Fourier transform (Farnbach; Bracewell) and + is appropriate for the opposite choice (Taner et al) (Pilant; Aki and Richards).

From this, we can compute (I will use - sign in above) the

- instantaneous amplitude

$$|F(t)| \doteq \left[f(t)^2 + f'(t)^2 \right]^{1/2}$$

Per Julian (oral, 1979), if our original signal had been narrow-band filtered, the instantaneous amplitude peak gives arrival time of the peak hence group velocity vs frequency.

- instantaneous phase

$$\phi(t) = \tan^{-1} \frac{-f'(t)}{f(t)}$$

- instantaneous frequency

$$\omega(t) = \frac{d\phi}{dt}$$

$$= \frac{f \dfrac{df'}{dt} - f' \dfrac{df}{dt}}{f^2 + f'^2}$$

Direct computation of the complex trace is particularly simple. Since

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$f'(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [+i\, \mathrm{sgn}(\omega)]\, F(\omega)\, e^{i\omega t}\, d\omega$$

$$F(t) = f(t) - i\, f'(t)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 + \mathrm{sgn}(\omega)]\, F(\omega)\, e^{i\omega t}\, d\omega$$

Thus to compute F(t) from f(t),

1) Obtain the transform F(w) of f(t).
2) Multiply by 2 for positive frequencies and 0 for negative frequencies, or
   If using DFT, multiply by 2 for indices 1 to N/2
   and multiply by 0 for indices N/2 + 1 to N
3) Take inverse Fourier transform.

The following figure from Farnbach shows a seismic signal and its complex envelope. A following Figure 4 (not reproduced here) shows the improvement in defining arrival time of a signal using the complex envelope.

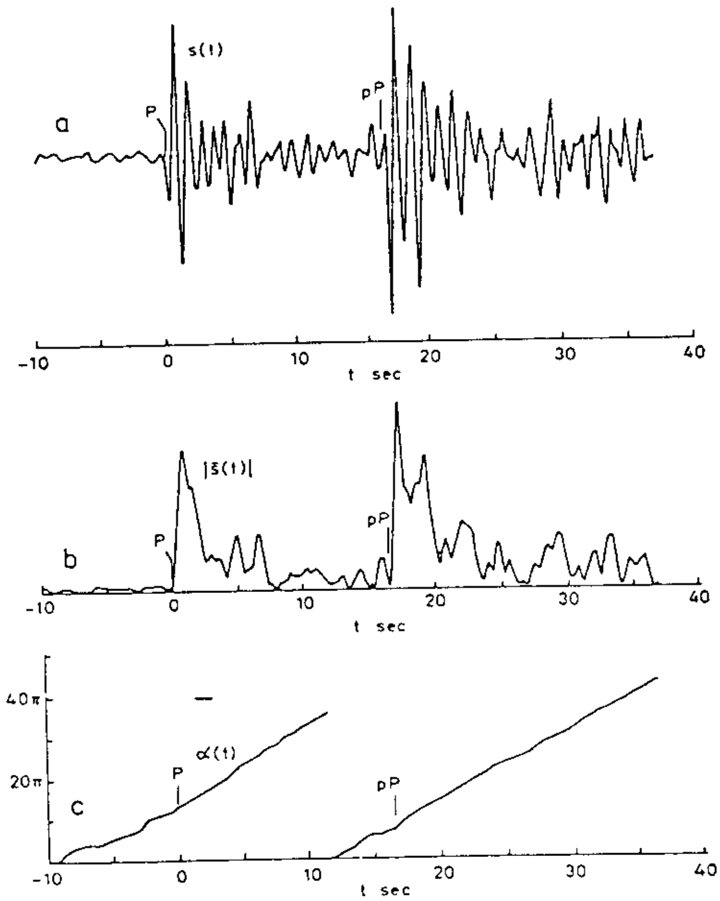


Fig. 3. A seismic signal and its complex envelope: (a) Real signal, (b) envelope, (c) angle function.

VII.E.1(a) (cont)

A use for the analytic signal

Reference: Arditty et al, p. 322 in 1982 SEG Annual Meeting Abstrac

They are concerned with sorting out various wave types in a seismic logging tool.measurement: I quote:

In order to study these different waves we have computed for each trace its analytic signal and decomposed it into is module (Modulus?), instantaneous phase,and instantaneous frequency. Each serves a different purpose.

The module-section allows us to study the amplitude variatior in terms of wave types, lithology variation, and travel time.

The instantaneous phase section is useful to follow coherent signal in zones where it is difficult to point an arrival by its amplitude. It is also a check to avoid phase shift in automatic picking.

The instantaneous frequency section gives information on the frequency variation in terms of lithology and travel time. If the energy of a specific arrival is not a good criterion to pick the arrival, the frequency display may be used to distinguish between different modes of propagation.

These are interesting comments. I should try the methods on some synthetic waveforms.

d) Hilbert transform in synthetic seismograms

Cerveny (1976) states that in asymptotic ray theory we are called on to deal with

$$F(t,\psi) = f(t)\cos\psi + f'(t)\sin\psi$$

This formulation appeared above. I haven't yet worked out how it emerges in ray theory.

In my lecture notes on Chapman's WKBJ method for synthetic seismograms (section X.N), I make extensive use of Hilbert transforms.

While the point is discussed in more detail in the next subsection, it should be noted that passage of a ray through a turning point involves a 90° phase shift which equates to a Hilbert transformation.

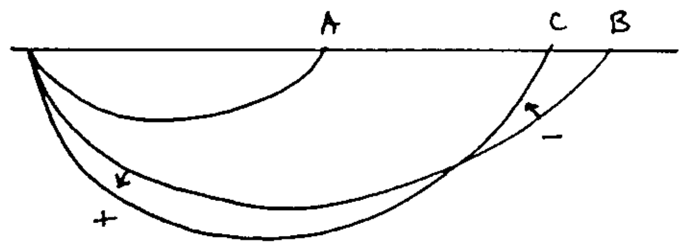
The following reference on page 97 shows the formulas by which to achieve this. Basically one computes the signal just above the turning point on the way down, multiplies it by the phase factor which they give in equation 34, and state that this equals the signal on the way up.

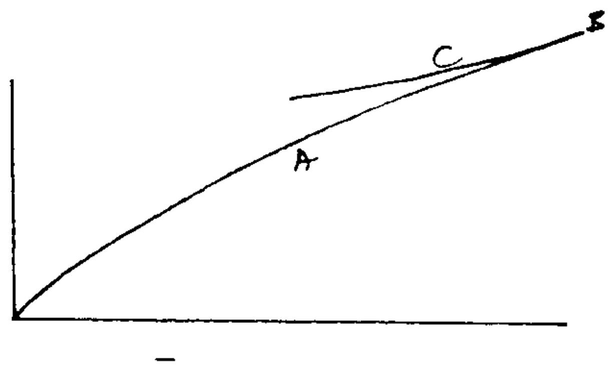
Hill, N., and P. Wuenschel, 1985, Numerical modeling of refracation arrivals in complex areas: Geophysics, v. 50, p. 90-98.

VII. E. (cont)

e) Hilbert transform at a caustic

Choy and Richards (1975) give a physical explanation of what is happening. I don't fully grasp their argument, but it seems to rely on the proposition that over a receding portion of a travel time branch created by a caustic, the cross-sectional area δA change sign from + to − upon crossing the caustic. Since the amplitude is proportional to $\delta A^{\frac{1}{2}}$, we are left with a phase shift of $(-1)^{\frac{1}{2}} = i$.





This apparently occurs also for PP, SS, where the travel time is mini-max (maximum for small changes along the ray path, and minimum for small changes perpendicular to it). The above examples appear to apply to the receding branches of P, S, and PKP.

Choi and Hron (1981) show that the phase shift due to caustics is intimately related to the occurrence of turning points along the ray. They derive an expression relating this phase shift to the number of turning points.

VII.E.1 e (cont)

Butler (1979) notes that the waveforms for SS do indeed correspond favorably to Hilbert-transformed S waveforms. The following figure shows

top - S

center - Hilbert-transformed S

bottom - SS as observed at twice the S-distance

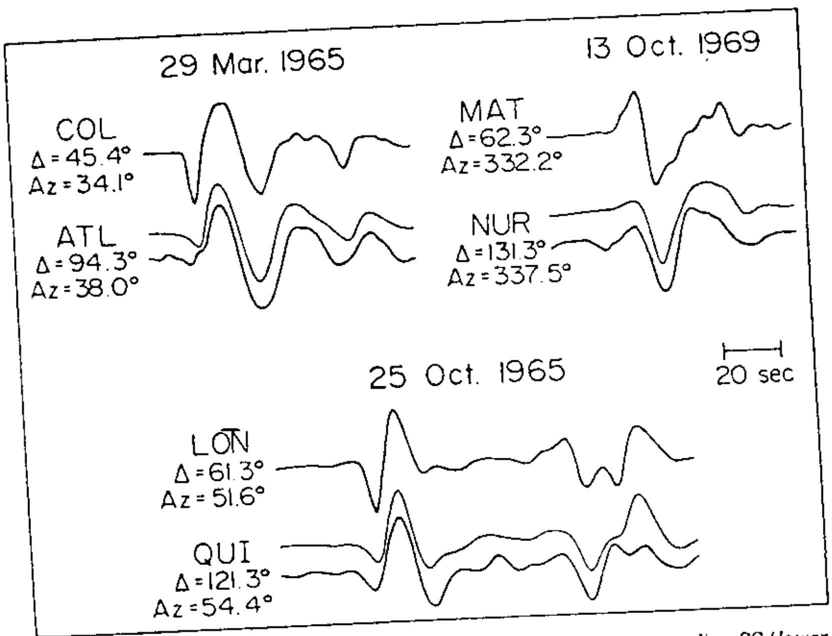


FIG. 2. Recordings of *S* and *SS* for three earthquakes where the station recording *SS* (*lower trace*) is roughly twice the distance from the source and at nearly the same azimuth as the station recording direct *S* (*upper trace*). The *center trace* for each event illustrates the effect of applying a Hilbert transform to the *S* wave (*upper trace*) to imitate the effect of the *SS* caustic.

f) Wide-angle seismic reflections

In section IV..B.3, we noted that wide-angle reflections may produce total reflection with a frequency-<u>in</u>dependent phase shift. The discussion above in section VII.F.2(b) was along these lines. We noted also that this effect can be viewed as a complex reflection coefficient.

Two papers which consider this problem for seismic exploration are Levy and Oldenburg (1982) and Ulrych and Walker (1982). They point out that if ϵ symbolizes the phase of the complex reflection coefficient, then an input waveform w(t) emerges following reflection as

$$\cos\epsilon \; w(t) \; + \; \sin\epsilon \; \mathcal{H}\{w(t)\}$$

They interpret this to mean that upon reflection, "the positive frequencies in w(t) are advanced by ϵ and the negative frequencies are retarded by ϵ .

We showed in the earlier section, for the special case $\epsilon = \pi/2$, that the altered waveform will differ considerably from the input waveform. If w(t) is delta-like, the output will have an emergent portion of significant amplitude at times preceding those expected from geometrical ray computations.

(The gist of their papers is to seek a method for deconvolving waveforms which have undergone this kind of modification).

VII.E (cont)

2. The Kramers-Kronig dispersion relationships

Sources of alternative derivations:

My derivation follows Collins, Mathematical Methods for Physicists and Engineers, Reinhold Publ. Co. p. 222.

Aki and Richards, p. 173, have a derivation which I find hard to follow.

Van Kampen (Jour. Physique, v. 22, 1961, p. 179) has a good general derivation.

Mathews and Walker, Mathematical Methods of Physics, W. A. Benjamin Co., p. 129, have a derivation.

Elmore and Heald, Physics of Waves, McGraw Hill, p 443 have a derivation based on a limiting process which I think is questionable.

Pilant p. 329 and Aki and Richards p. 173 have totally different derivations. Aki and Richards couch the argument in the special context of wave propagation with complex wave number,

$$k(\omega) = \frac{\omega}{c(\omega)} + i\,\alpha(\omega)$$

from which they derive the result that velocity and attenution must be related as

$$\frac{\omega}{c(\omega)} = \frac{\omega}{c(\infty)} + \mathcal{H}\left[\alpha(\omega)\right]$$

VII. E.2 (continued).

The topic we consider here in connection with the interdependence of velocity and attenuation for waves propagating in an anelastic medium actually has wide applicability and generality. The subject is sometimes referred to simply as "the dispersion relations"*. It occurs in exactly identical mathematical form (with different physical interpretation for the parameters) in

- wave propagation in anelastic media
- electromagnetic waves in conducting dielectrics
- scattering of nuclear particles
- time and frequency domain response of electrical networks
- magnetic materials

We can illustrate intuitively the nature of the problem in the following way. Suppose we have frequency-dependent attenuation of a propagating seismic wave. At distance r_1, we have a pulse

$$f_1\left(t - \frac{r_1}{V}\right) \qquad f_1 \to 0 \quad t < \frac{r_1}{V}$$

with transform

$$F_1(\omega) = F_{1R}(\omega) + i\, F_{1I}(\omega)$$

After propagating to a distance r_2, each frequency component has been attenuated by a factor

$$e^{-\alpha_0 \omega (r_2 - r_1)}$$

so that the transform at r_2 will be

$$F_2(\omega) = F_1(\omega)\, e^{-\alpha_0 \omega (r_2 - r_1)}$$

Taking the inverse transform of this will give us the waveform at r_2 <u>but</u> in general this inverse transform will be non-zero for $-\infty < t < \infty$. This violates causality ("no output before the input") because a signal appears at r_2 before it appears at r_1.

To satisfy causality, we must shift the phases of the Fourier components at r_2 in such a way as to achieve perfect cancellation for all times prior to $t - r_2/V$. It turns out that this requires that V depend upon frequency also, and in a very specific way.

*A term applied to any integral relationship between the real and imaginary parts of a function of a complex variable.

The question we consider is, if $f(t)$ has the property that it is 0 for negative t, what does this imply about the properties of the transform $F(w) = F_R(w) + iF_I(w)$?

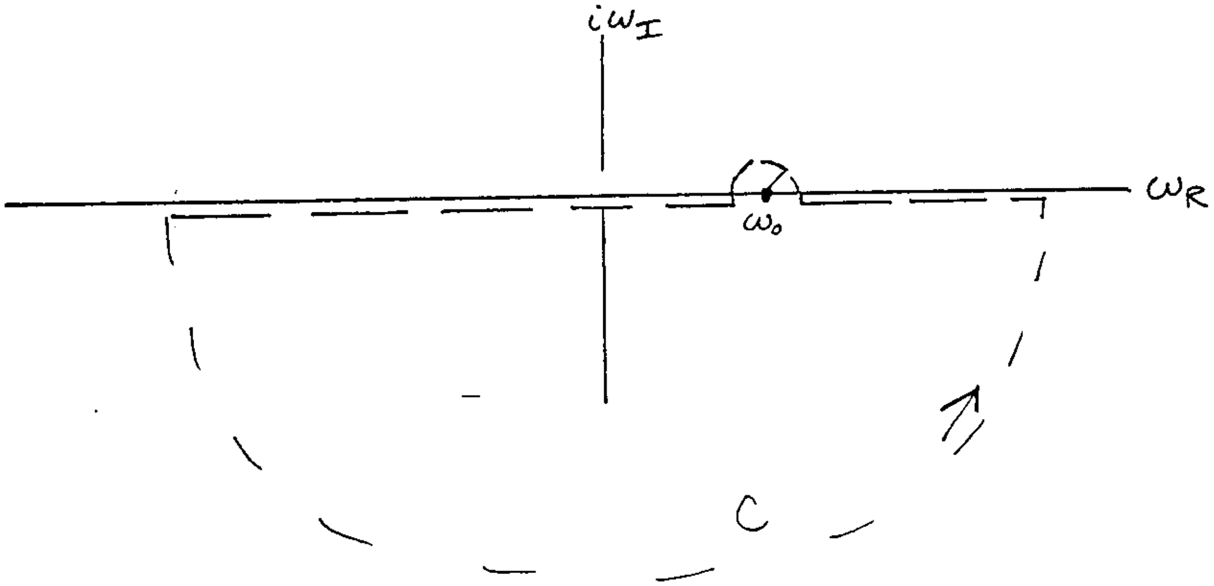
We assert first that if $f(t)$ has the above property, then $F(w)$ is analytic in the lower half of the w-plane (i.e. with $w = w_R + iw_I$, that $w_I \leq 0$). (Comment: This applies with the normal sign convention for the Fourier transform pair. Aki and Richards use the reverse convention, in which case the statement applies to the upper half of the w-plane).

Writing

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i[\omega_R + i\omega_I]t}\, dt$$

we see that the lower limit can be replaced by 0, and that the assertion is true.

We now consider the contour,



We can apply Cauchy's Theorem (Churchill, p. 90) which says that for any point interior to the curve C, within and on which $f(w)$ is analytic, we can write

$$F(\omega_0) = \frac{1}{2\pi i} \int_C \frac{F(\omega)}{\omega - \omega_0}\, d\omega$$

where the contour is taken in the counterclockwise direction.

For the present problem, we find

$$2\pi i\, F(\omega_0) = \int\limits_{\substack{\text{infinite}\\ \text{semi-circle}}} + \Pr \int_{\infty}^{-\infty} + \int\limits_{\substack{\text{small}\\ \text{semi-circle}}}$$

For the small semi-circle,

$$F(\omega) \simeq F(\omega_0)$$

$$\omega - \omega_0 = re^{i\theta}$$

$$d\omega = re^{i\theta}\, i\, d\theta$$

$$\int \Rightarrow \int_{\theta=0}^{\pi} \qquad \Rightarrow \pi i F(\omega_0)$$

hence finally

$$F(\omega_0) = -\frac{1}{\pi i} \Pr \int_{-\infty}^{\infty} \frac{F(\omega)\, d\omega}{\omega - \omega_0}$$

Note: The - sign arises because I used the conventional definition of Fourier transform, with the - sign in the exponent of the Forward Fourier Transform.

Aki and Richards, who use the opposite definition, correctly obtain a + sign here.

Two authors who use the conventional definition as I read them, yet who obtain a + sign here are Collins (book referenced above) and Lee and Solomon, 1978, Jour. Geoph. Res. v. 83, p. 3398. I don't see how these can be correct unless I have misread them.

Note: "Pr" mean principal value, i.e., we integrate <u>almost</u> up to w_o from both sides.

The remaining step is obtained by breaking F(w) into its real and imaginary parts:

$$F_R(\omega_0) + i\, F_I(\omega_0) = -\frac{1}{\pi i} \Pr \int_{-\infty}^{\infty} \frac{F_R(\omega') + i\, F_I(\omega')}{\omega' - \omega_0}\, d\omega'$$

Separating real and imaginary parts, we find finally that

$$F_R(\omega_o) = -\frac{1}{\pi} \Pr \int_{-\infty}^{\infty} \frac{F_I(\omega)}{\omega - \omega_o} d\omega$$

$$F_I(\omega_o) = \frac{1}{\pi} \Pr \int_{-\infty}^{\infty} \frac{F_R(\omega)}{\omega - \omega_o} d\omega$$

which agrees with Collins except that he gets opposite signs (which I think is incorrect).

If we recall from the previous section that f(t) has a Hilbert transform f'(t) given by,

$$f'(t) = \frac{1}{\pi} \Pr \int_{-\infty}^{\infty} \frac{f(t')}{t'-t} dt'$$

we see that $F_R(w_o)$ and $F_I(w_o)$ are a Hilbert transform pair.

An alternative form of the above can be written by taking note of the fact that F_R is an even function of w and F_I is an odd function. Upon substitution, this yields

$$F_I(\omega_o) = \frac{1}{\pi} \Pr \left[\int_{-\infty}^{0} + \int_{0}^{\infty} \right] \frac{F_R(\omega)}{\omega - \omega_o} d\omega$$

etc, which can be combined to give with $\omega' \triangleq -\omega$

$$F_R(\omega_o) = -\frac{2}{\pi} \Pr \int_0^{\infty} \frac{\omega F_I(\omega)}{\omega^2 - \omega_o^2} d\omega$$

$$F_I(\omega_o) = \frac{2\omega_o}{\pi} \Pr \int_0^{\infty} \frac{F_R(\omega)}{\omega^2 - \omega_o^2} d\omega$$

which again differs from Lee and Solomon eq. B3, for example by having opposite signs for both expressions.

VII. (cont)

F. Proposed attenuation-dispersion pairs to satisfy causality

1. Introduction

Two separate but inter-related topics enter into the following discussion:

a) If we impose causality, as seems physically plausible, then the preceding considerations require that velocity and attenuation constant be inter-dependent, that is, computable one from the other.

b) If the attenuation constant is frequency-dependent, then velocity must also be frequency-dependent.

If indeed the velocity is dependent upon frequency, then we are in the presence of "material dispersion", to be distinguished from "geometric dispersion" as discussed in section V.

As a specific example of the practical problems arising, we note that the frequencies differ by several orders of magnitude between earthquake short-period body waves and long-period surface waves and free oscillations. Thus, the velocity structure of the earth computed by these methods can be expected to display discrepancies.

We note also that attenuation tends to be significantly greater for shear type seismic waves (S waves; Love waves) than for compressional waves. Thus the frequency-dependence of velocity will be greater for shear waves.

A central experimental consideration lies in the frequency dependence of the attenuation constant, since this governs the resulting frequency dependence of velocity.

As we noted in an earlier section, the argument is usually expressed in the question, does Q depend upon frequency (and if so, how?) or is Q independent of frequency.

In the previous section, we demonstrated the following:

If a time function f(t) with Fourier transform F(w) is 0 for negative t, then the real and imaginary parts of the transform are related by being the Hilbert transforms (in frequency) of each other.

Expressed in terms of wave propagation, the absorption coefficient and the velocity must both depend upon frequency. If the frequency dependence of the absorption coefficient is specified, then the frequency dependence of the velocity can be computed.

For the specific case that α is a linear function of frequency hence Q is independent of frequency, the following relationship emerges:

$$\frac{C(\omega_1)}{C(\omega_2)} = 1 + \frac{1}{\pi Q} \ln\left(\frac{\omega_1}{\omega_2}\right)$$

This is usually attributed to Futterman (1962) but the same result emerges from other types of analysis.

In application, one usually takes w_2 as some reference frequency (1 hz is common for earthquake seismology) and computes other velocities with reference to it.

Some problems arise in this formula for very large or very small frequencies. The usual way out of this is to assert, following Futteman, that it applies only over the seismic band of frequencies.

VII.F. (cont)

2. Futterman attenuation dispersion pair

I will present here a Futterman-type derivation of the dispersion relations between attenuation and velocity. This follows Aki and Richards, p. 173.

The final result of the analysis as Futterman gave it was (Savage, 1976):

For any frequency in the seismic band, we take Q to be <u>almost</u> constant. A low frequency cutoff w_0 is defined to be much lower than any frequencies in the seismic band:

$$V_{PHASE} = C \frac{1}{1 - \frac{1}{\pi Q_0} \ln\left(\gamma \frac{\omega}{\omega_0}\right)} \quad {**}$$

$$V_{GROUP} = C \frac{1}{1 - \frac{1}{\pi Q_0} \left[1 + \ln\left(\gamma \frac{\omega}{\omega_0}\right)\right]}$$

$$Q = Q_0 \left[1 - \frac{1}{\pi Q_0} \ln\left(\gamma \frac{\omega}{\omega_0}\right)\right]$$

$\gamma = .577\ldots$ (Euler's constant,* emerging from one of the integrals)

"c" is the velocity at a very high frequency, w_1; we cannot extend it to infinite frequency because Q would beacome negative, but we will take it above the pass band of the instrumentation.

**To reconcile this with the expression on preceding page, note that for small x,

$$\frac{1}{1-x} \sim 1 + x$$

$$^{*}\ \gamma = \lim\ 1 + \frac{1}{2} + \cdots + \frac{1}{s} - \ln s$$

VII. F.2 (cont)

The argument as presented by Aki and Richards seems a little ad hoc to me, but essentially they claim that for one-dimensional plane waves we can represent the effect of propagating through a distance x as

$$f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x,\omega)\, e^{-i\omega t}\, d\omega$$

with $$F(x,\omega) = e^{-\alpha(\omega)x}\, e^{ikx}$$

and $$k = \omega\left[\frac{1}{c(\omega)} - \frac{1}{c_\infty}\right]$$

Note that I have accepted their sign convention for the Fourier transform pair, in order to follow their derivation.

The ad hoc portion appears to be the introduction of c_∞, the limiting value of velocity at infinite frequency. Lee and Solomon do something equivalent to it.

If we grant both of these points, then we can write

$$\log F(\omega) = -\alpha(\omega)x + i\omega\left[\frac{1}{c(\omega)} - \frac{1}{c_\infty}\right]x$$

They establish as an extension that the real and imaginary parts of log F(w) are also Hilbert transform pairs (p. 174; I shall not repeat it), leading finally to

$$\alpha(\omega) = \mathcal{H}\left[\omega\left(\frac{1}{c(\omega)} - \frac{1}{c_\infty}\right)\right]$$

$$\omega\left[\frac{1}{c(\omega)} - \frac{1}{c_\infty}\right] = \mathcal{H}\left[\alpha(\omega)\right]$$

The second of these is the more important. It can be written out:

$$\omega\left[\frac{1}{c(\omega)} - \frac{1}{c_\infty}\right] = \frac{1}{\pi}\, \mathrm{Pr}\int_{-\infty}^{\infty} \frac{\alpha(\omega')}{\omega'-\omega}\, d\omega'$$

At this point, some subtle arguments enter. They may be followed best in Aki and Richards, p. 172. They center around the proposition that the relationship we used in section 2 above, that attenuation with distance can be represented by

$$e^{-\frac{\omega x}{2\,C(\omega)\,Q}}$$

which implies that

$$\frac{\omega}{C_\infty} + \mathcal{H}\left[\alpha(\omega)\right] = 2Q\,\alpha(\omega)$$

cannot be satisfied by any Q which is <u>completely</u> independent of frequency.

On the other hand, experimental data suggest that Q is essentially independent of frequency <u>over the seismic band</u>. The approach taken, therefore, as suggested by Azimi et al (1968) is to introduce a term, $1 + \alpha_1 w$, where α_1 is such that the second term is negligible over seismic frequencies, not becoming significant until w is very large.

The implications of this are that

$$\alpha(\omega) = \frac{\alpha_0\,\omega}{1+\alpha_1\omega}$$

$$\mathcal{H}\left[\alpha(\omega)\right] = \frac{2\,\alpha_0\,\omega}{\pi\,\underbrace{(1-\alpha_1^2\omega^2)}_{\to 1.0}} \ln\left(\frac{1}{\alpha_1\omega}\right)$$

$$\frac{1}{C(\omega)} = \frac{1}{C_\infty} + \frac{2\alpha_0}{\pi}\ln\left(\frac{1}{\alpha_1\omega}\right)$$

Thus the ratio of phase velocities at two different frequencies in the seismic band can be written

$$\frac{C(\omega_1)}{C(\omega_2)} = 1 + \frac{1}{\pi Q}\ln\left(\frac{\omega_1}{\omega_2}\right)$$

They discuss this in considerable detail, arriving later at the same result from another line of reasoning.

VII.F. (cont)

3. A guided tour through other attenuation-velocity dispersion pairs

(a) For overviews of the various approaches, try:

Aki and Richards, p. 167-185.
Kjartansson, 1979
Lee and Solomon, 1978 Appendix B; 1979 p. 72
Mavko, 1979
Liu, Anderson, and Kanamori, 1976

(b) Dispersion relations may be presented either in the frequency domain using Kramers-Kronig relations, or in the time domain using Boltzmann creep (or relaxation) functions.

For any given frequency dependence of Q, both lead to the same dispersion relations.

(c) Nearly all approaches are based upon "constant Q" models, i.e. models in which Q is independent of frequency.

Most constant-Q models run into mathematical difficulties, and are forced to either

(1) assume Q is constant only within the seismic frequency band, and has other behaviour outside of it (Futterman, 1962;

or

(2) assume Q is almost constant but not quite (Strick, 1967,1970;

(d) A few approaches use <u>exactly-constant</u> Q but they lead to some exotic dispersion relations (Kjartansson, 1979;

$$\frac{c}{c_0} = \left(\frac{\omega}{\omega_0}\right)^{\gamma} \qquad c_0 = c(\omega_0)$$

with

$$\gamma = \frac{1}{\pi}\tan^{-1}\left(\frac{1}{Q}\right)$$

He started the analysis from the creep function $\phi(t) \sim t^{2\gamma}$

(e) In all of these approaches, the reader must be on guard against concealed assumptions that Q is very large compared to 1; such approximations would invalidate the results for small-Q materials like soils.

(f) Among the Futterman-type theories, we may note the following starting points for dependence of attenuation on frequency

Futterman, 1962 $\quad \alpha(\omega) = C_1 \omega \qquad \dfrac{1}{Q(\omega)} = 2 C_1 V_\infty$

Azimi et al, 1968 $\quad \alpha(\omega) = \dfrac{C_1 \omega}{1 + C_2 \omega} \qquad \dfrac{1}{Q(\omega)} = \dfrac{2 C_1 V_\infty}{1 + C_2 \omega}$

Azimi et al choose C_2 so that it remains insignificant in the seismic frequency band; they simply make explicit some of the high frequency conditions which Futterman introduces empirically.

(g) Among creep type theories, the most widely used one is based upon an experimental creep function of Lomnitz (1957):

$$\phi(t) = q \ln(1 + at)$$

which leads to (Lee and Solomon, 1979)

$$\frac{V(\omega)}{V(\infty)} = 1 + \frac{q}{2}\left[\gamma + \ln \frac{\omega}{a}\right] \text{ with } \frac{1}{Q} = q \frac{\pi}{2}$$

γ = A CONSTANT

For large Q, this approximates to the earlier expression,

$$\frac{V(\omega_1)}{V(\omega_2)} = 1 + \frac{1}{\pi Q_0} \ln\left(\frac{\omega_1}{\omega_2}\right)$$

For a comparison between Lomnitz and Futterman theories, see Savage and O'Neill (1975). Jeffreys (1958) proposed a generalization of Lomnitz function which leads to frequency-dependent Q: $\quad \phi(t) = \dfrac{q}{\nu}\left[(1 + at)^{\nu} - 1\right]$

(h) For perspective on theories using frequency-dependent Q, see Lee and Solomon, 1979, p. 73.

(i) Savage (letter, 12/82) comments that he doesn't like formulations of Futterman or of Strick because they are essentially mathematical inventions. Lomnitz theory, on the other hand, is based on real physics and ties together two previously unrelated phenomena, creep and attenuation.

(j) A third approach uses a simple mechanical model, for example the standard linear solid (section VII.B.1):

$$\sigma + \tau_\sigma \dot{\sigma} = M_R(\epsilon + \tau_\epsilon \dot{\epsilon})$$

but invokes a statistical distribution of such effects with different time constants $\tau_\sigma, \tau_\epsilon$ to approximate to a constant-Q situation. For this approach, see Liu et al (1976); Wielandt (1975);

(k) Current research on attenuation emphasizes the frequency dependence of Q^{-1} (peaks in the kilohertz range; see) and the role of pore fluids

(m) Strick (several papers) has developed an elaborate three-parameter dispersion model, starting from

attenuation constant $\alpha(\omega) \sim \omega^s$

where s is less than 1.0 but "may be arbitrarily close to it".

So far as I know, Strick's work is sound and valid. On the other hand, it starts from an ad hoc assumption (above) rather than any physical model. It gets quite complicated and leads to kind of precursor which he calls a "pedestal" (Strick, 1970).

Pilant is a colleague of Strick; his book contains an alternative presesntation of some of the material.

(n) Lamb (1962) tried a starting point that

attenution constant $\alpha(\omega) \sim \omega^{1/2}$

and examined the consequences. No one seems to have liked this.

4. A dispersion-attenuation pair derived from a creep functio

The classic creep function for this kind of analysis is given by Lomnitz (1956):

$$\psi(t) = \frac{1}{M_o}\left[1 + q \ln(1+at)\right]$$

where M_o, q, and a are constants, and a is a frequency much greater than the sample rate. Unlike almost all other analyses which are based on ad hoc assumptions about frequency-dependence of attenuation (Futterman, for example), Lomnitz' formulation was based upon actual experiments on transient creep in rock at low stress levels.

Despite this, I shall follow Kjartansson (1979) because

1) he presents an alternative creep function which includes Lomnitz' result as a special or limiting case (p. 4745),
2) he develops his theory in comprehensive fashion, and
3) his theory yields a Q which is exactly independent of frequency, and
4) he escapes from the artificial and arbitrary high and low frequency cutoffs which Futterman and others are forced to introduce.

We start with a creep function which plots as a straight line on a log-log plot,

$$\psi(t) \sim t^{b} \qquad t>0$$

arguing that a frequency-independent Q implies that the loss per cycle is independent of the time scale of oscillation. For later convenience in interpreting the results, we introduce some constants

$$\psi(t) = \frac{1}{M_o\,\Gamma(1+2\gamma)}\left(\frac{t}{t_o}\right)^{2\gamma} \qquad t>0$$

where Γ is the gamma function (essentially 1.0 for present purposes), t_o is an arbitrary reference time, and M_o has the dimensions of a modulus in the stress-strain relationship.

Note: This Kjartansson paper is a good one, but I rather prefer a more unifed approach of Muller (1983), section V.G.4.

Reviewing (and slightly rewriting) the results from VII.C.4 on creep and relaxation functions and converting to the symbols of Kjartansson, we have creep function Ψ (t) and relaxation function $\bar{\Psi}$ (t) in expressions like

$$E\ \epsilon(t_0) = \sigma(t_0) + \int_{-\infty}^{\infty} \dot{\Psi}(t)\ \sigma(t_0 - t)\, dt$$

$$\frac{1}{E}\ \sigma(t_0) = \epsilon(t_0) - \int_{-\infty}^{\infty} \dot{\bar{\Psi}}(t)\, \epsilon(t_0 - t)\, dt$$

elastic
terms

and if we omit the elastic terms and use the abbreviations

$$s(t) = \dot{\Psi}(t) = \frac{d\Psi}{dt}$$

$$m(t) = \dot{\bar{\Psi}}(t) = \frac{d\bar{\Psi}}{dt}$$

the relationships can be written as convolutions

$$\sigma(t) = m(t) * \epsilon(t)$$

$$\epsilon(t) = s(t) * \sigma(t)$$

$$\delta(t) = m(t) * s(t)$$

with transforms

$$\Sigma(\omega) = M(\omega)\ E(\omega)$$

$$E(\omega) = S(\omega)\, \Sigma(\omega)$$

$$1 = M(\omega)\ S(\omega)$$

VII.F.4 (cont)

Starting from asssumed creep function, we differentiate to get

$$s(t) = \frac{2\gamma}{M_0\,\Gamma(1+2\gamma)}\,\frac{1}{t}\left(\frac{t}{t_0}\right)^{2\gamma} \qquad t>0$$

take Fourier transform to get

$$S(\omega) = \frac{1}{M_0}\left(\frac{i\omega}{\omega_0}\right)^{-2\gamma} \qquad \omega_0 \equiv \frac{1}{t_0}$$

use last relationship on preceding page to get

$$M(\omega) = \frac{1}{S(\omega)} = M_0\left(\frac{i\omega}{\omega_0}\right)^{2\gamma} = M_0\left|\frac{\omega}{\omega_0}\right|^{2\gamma} e^{i\pi\gamma\,\mathrm{sgn}(}$$

and take the inverse transform and integrate to get

$$\bar{\Psi}(t) = \frac{M_0}{\Gamma(1-2\gamma)}\left(\frac{t}{t_0}\right)^{-2\gamma} \qquad t>0$$

Since Q is related to the phase angle between stress and strain as

$$1/Q = \tan\delta$$

Taking the phase angle as the exponent in M(w), we have

$$\frac{1}{Q} = \tan(\pi\gamma)$$

or $$\gamma = \frac{1}{\pi}\tan^{-1}\left(\frac{1}{Q}\right) \approx \frac{1}{\pi Q} \quad \text{for large Q}$$

and Q is exactly independent of frequency.

VII.F.4 (cont)

Next we can write* one-dimensional wave propagation in such a medium as

$$u(x,t) = e^{i(\omega t - kx)}$$

with $$k = \omega \left[\frac{\rho}{M(\omega)}\right]^{1/2}$$

or expressed in the form

$$u(x,t) = e^{-\alpha x} e^{i\omega\left(t - \frac{x}{c(\omega)}\right)}$$

we find following substitution $-ikx \Rightarrow -\left(\alpha + \frac{i\omega}{c}\right)x$

or $$i\omega\sqrt{\frac{\rho}{M(\omega)}} = \alpha + \frac{i\omega}{c}$$

that, separating real and imaginary parts, we require

$$c = c_0 \left|\frac{\omega}{\omega_0}\right|^{\gamma}$$

$$\alpha = \tan\left(\frac{\pi}{2}\gamma\right) \quad \mathrm{sgn}(\omega) \quad \frac{\omega}{c}$$

with $$c_0 = \sqrt{\frac{M_0}{\rho}} \Big/ \cos\left(\frac{\pi}{2}\gamma\right)$$

(I have verified these).

*If clarification of this step is required, see Aki and Richards p. 178.

VII.F.4 (cont)

If the source waveform is an impulse, then the impulse response is a function b(t) whose transform B(w) is obtained simply by omitting the e^{iwt} in the expression above, yielding

$$B(\omega) = e^{-x \frac{\omega_0}{c_0} \left|\frac{\omega}{\omega_0}\right|^{1-\gamma} \left[\tan \frac{\pi\gamma}{2} + i \operatorname{sgn}(\omega)\right]}$$

Kjartansson is unable to perform the inverse integration to get b(t) analytically (it can be done numerically), but he goes through some scaling arguments leading to the conclusion

$$\tau \sim T \sim \frac{1}{A} \sim \left(\frac{x}{c_0}\right)^{\beta}$$

where τ is pulse width, T is travel time, A is pulse amplitude and

$$\beta = \frac{1}{1-\gamma} \simeq 1 + \frac{1}{\pi Q}$$

The proportionality can be written that the pulse width is

$$\tau = C(Q) \frac{T}{Q}$$

where C(Q) depends only on Q and is fact nearly independent of Q for Q > 20.

He shows numerous examples of how pulse shape for the impulse response changes with distance and with time.

NOTE THAT WE ARE BACK, BY AN ENTIRELY DIFFERENT ROUTE, TO A CARPENTER-TYPE OPERATOR AND A $t^* = T/Q$ description.

Finally, we may tie this back into a Futterman type operator as follows, simultaneously demonstrating the approximations in Futterman theory and displaying the Kjartansson theory as more generally applicable.

For large Q, we can replace $\tan^{-1}$ with its argument so

$$\gamma = \frac{1}{\pi} \tan^{-1}\left(\frac{1}{Q}\right) \Rightarrow \frac{1}{\pi Q}$$

and the expression for velocity becomes

$$\frac{c}{c_0} = \left|\frac{\omega}{\omega_0}\right|^{\gamma} \Rightarrow \left|\frac{\omega}{\omega_0}\right|^{1/\pi Q}$$

(He also gives a simplified expression, eq 41, for the transform B(w).)

Making use of the general principles that

$$x = e^{\ln x}$$

and $$e^{\theta} = 1 + \theta + \frac{1}{2!}\theta^2 + \cdots$$

we see that

$$\frac{c}{c_0} = 1 + \frac{1}{\pi Q} \ln\left|\frac{\omega}{\omega_0}\right| + \frac{1}{2!}\left[\frac{1}{\pi Q} \ln\left|\frac{\omega}{\omega_0}\right|\right]^2 + \cdots$$

The first two terms are identically the Futterman result, and we see that the condition for its validity rests on the stipulation that the third term be negligible, or

$$\frac{1}{\pi Q} \ln\left|\frac{\omega}{\omega_0}\right| << 1$$

for all frequencies of interest.

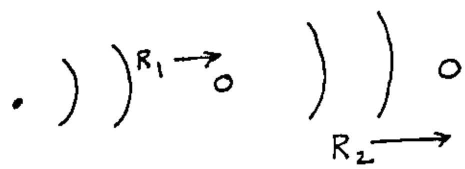
VII. (cont)

G. Practical computation techniques for introducing attenuation into synthetic seismogram computations.

1. Introduction

In this section, we present several versions of an "attenuation filter". The idea is that we solve the synthetic seismogram_problem for a purely <u>elastic</u> case, and then pass the resulting waveform through a filter which will simulate the effect of attenuation.

We will derive the filter for plane waves. For spherical or other wave types, we will handle the complications only for the elastic part of the problem and then use the plane wave attenuation filter. To illustrate this point suppose that we have waves spreading outward in a uniform medium from a point source:



We <u>observe</u> the signal at distance R_1 and wish to <u>predict</u> the signal at distance R_2. Then

$$f(R_2,t) = \frac{R_1}{R_2} f(R_1,t) * h(R_2-R_1,t)$$

where $h(x,t)$ is the impulse response of the attenuation filter, whose transform is the transfer function of the filter, $H(x,w)$.

For all of the following discussion, we will take as the starting point a transfer function of the form:

$$H(x,\omega) = e^{-\frac{|\omega| x}{2QV}} \; e^{-i\omega \frac{x}{V(\omega)}}$$

Nearly everyone seems to agree on this. The absolute value sign is needed to keep $h(x,t)$ real.

The important complication in the above lies in the dependence of velocity on frequency. <u>This is required to obtain a causal system</u> (i.e., $h(t)$ 0 for negative t).

Our starting point is Futterman's relationship given in earlier sections,

$$V(\omega) = V_o \left[1 - \frac{1}{\pi Q} \ln\left(\gamma \frac{\omega}{\omega_o}\right)\right]^{-1}$$

where γ = .577216 Euler's constant

and where

w_o = a "low frequency cutoff", namely some frequency well below the lowest frequency of interest, below which we assume that no absorption exists.

V_o = a reference or standard velocity at which no absorption exists

We now substitute this value for V(w) into the transfer function given in section 1 above. In the process, we impose the following conditions:

- the attenuation can be ignored from $0 < w < w_o$
- the transfer function must satisfy the complex conjugate condition,

  $$H(w) = H^*(-w)$$

  in order to keep the impulse response real.

The final result is the transfer function

$$H(\omega) = e^{-\frac{|\omega| x}{2 Q V_o}} \, e^{-i\omega \frac{x}{V_o}\left[1 - \frac{1}{\pi Q} \ln\left(\gamma \frac{|\omega|}{\omega_o}\right)\right]} \qquad |\omega| > \omega_o$$

$$= 1.0 \qquad -\omega_o < \omega < \omega_o$$

VII.G (cont)

2. The Carpenter attenuation operator based on the Futterman velocity-frequency relationship

a) Derivation of the frequency-domain version of the Carpenter operator

In a 1966 paper by E.W. Carpenter, the following steps were achieved:

- start from the transfer function from the previous page
- substitute for V(w) the velocity-frequency relationship given by Futterman (see section VII.F)
- convert the time scale from real time to time starting from the signal arrival time at the observing station
- take the inverse Fourier transform to get a time-domain attenuation operator, which can then be convolved with the observed signal to incorporate attenuation effects.

Carpenter's paper is difficult to read and in fact difficult to obtain since it was an internal British government document (copy in my file).

Further, I prefer to use the Carpenter operator in the frequency domain directly rather than using the time domain operator. This approach does introduce some complications, however, which I will treat in section (b) below.