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Basic Electromagnetism

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R.P. Walker, Sincrotrone Trieste, Nov. 2000

Contents

1. Introduction

Both electrostatic and magnetic effects were known since ancient times:

- amber, when rubbed, will attract small pieces of matter
- certain mineral ores (e.g. "loadstone") can attract small pieces of iron

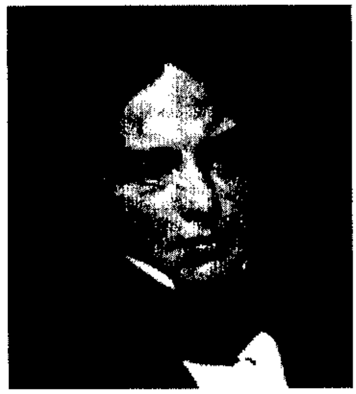
The study of electrostatics and magnetism proceeded more or less independently until the 19th Century when a number of important discoveries were made:

1819: **Oersted** discovers that an electric current can produce magnetic forces the same as those produced by permanent magnets

Oersted's compass

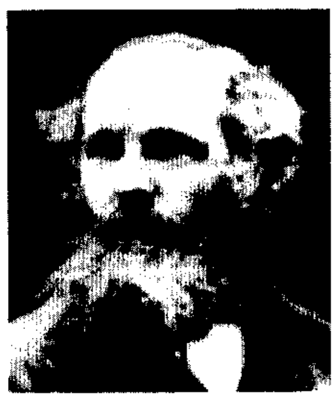
1820-1830 : These effects were studied systematically principally by **Ampere**, Biot and Savart.

1831: **Faraday** discovers that a transient flow of current can be produced by a change in the magnetic flux threading the circuit



Electricity and magnetism are now linked.

1865: **Maxwell**'s Equations predict the existence of electromagnetic waves



Not only electricity and magnetism are linked, but now light, and all electromagnetic radiation, are described by a single Unified theory – Electrodynamics

"*the most significant event of the 19th Century*" (Feynman)

Electrodynamics deals mainly with macroscopic phenomena – bulk effects in which large numbers of atoms and molecules are involved. Forces between individual atoms, as well as quantum mechanical effects are therefore not included. Nevertheless, using only simple models for how electric and magnetic fields affect materials a wide range of electromagnetic phenomena can be studied.

2. Reminder on Fields, Vectors and their Derivatives

Fields

A 'field' is simply a quantity that is distributed over a region of space, i.e. any function f(x,y,z)

i/ scalar field: e.g. the temperature of the air $T(x, y, z)$

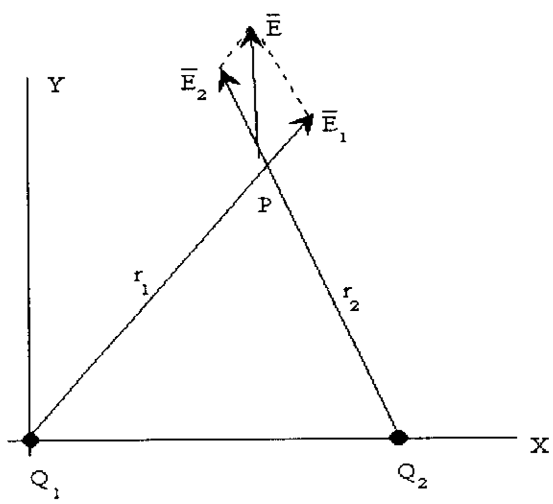
ii/ vector field: e.g. the velocity of the air $\overline{v}(x, y, z)$,

Electric and Magnetic fields are vector fields:

$$\overline{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

Vector algebra

i/ addition of vectors:



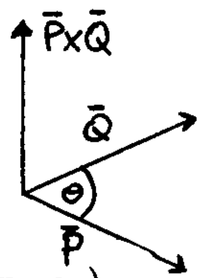
i.e. equivalent to adding separately the 3 components:

$$E_x = E_{1,x} + E_{2,x}, \quad E_y = E_{1,y} + E_{2,y}, \quad E_z = E_{1,z} + E_{2,z}$$

ii/ scalar product:

$$\overline{P} \bullet \overline{Q} = PQ\cos\theta = P_x Q_x + P_y Q_y + P_z Q_z$$

e.g. The magnitude of the Electric Field along a given direction $\hat{s}$ (vector of unit length) is given by:



$$E = \overline{E}\cos\theta = E_x s_x + E_y s_y + E_z s_z = \overline{E} \bullet \hat{s}$$

iii/ vector product :

$$\overline{P} \times \overline{Q} = PQ\sin\theta = \left(P_x Q_y - P_y Q_x, P_x Q_z - P_z Q_x, P_y Q_x - P_x Q_y\right)$$

<u>Derivatives of fields</u>

We use the differential operator ∇: (which is itself a vector)

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

The three possible first derivatives are:

$$\nabla T = \text{grad T} = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) \qquad \text{a vector}$$

$$\nabla \bullet \overline{v} = \text{div } \overline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \qquad \text{a scalar}$$

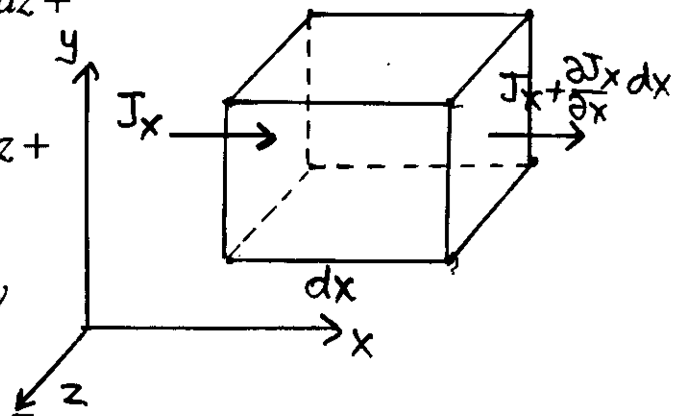
$$\nabla \times \overline{v} = \text{curl } \overline{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z},\ \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z},\ \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) \quad \text{a vector}$$

The divergence of a vector field

If a vector, e.g. $\bar{J}$, corresponds to the flux of some quantity in a particular direction, per unit time, per unit surface area, then the flux crossing a surface $d\bar{S}$ (a vector with magnitude equal to the area of the surface and direction normal to the surface) is $\bar{J} \bullet d\bar{S}$.

For an elementary cube of size *dx, dy, dz*

$$\Sigma(\bar{J} \bullet d\bar{S}) = \left[\left(J_x + \frac{\partial J_x}{\partial x} dx\right) - J_x\right] dy\, dz + \left[\left(J_y + \frac{\partial J_y}{\partial y} dy\right) - J_y\right] dx\, dz + \left[\left(J_z + \frac{\partial J_z}{\partial z} dz\right) - J_z\right] dx\, dy$$



$$= \left[\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right] dx\, dy\, dz \quad = (\nabla \bullet \bar{J})\, dV$$

The divergence of a vector is the outward flux per unit volume.

Integrating over a finite volume, the contribution from adjacent surfaces cancels everwhere except on the outer surface:

$$\int_S \bar{J} \bullet d\bar{S} = \int_V \nabla \bullet \bar{J}\; dV \qquad \text{(Divergence Theorem)}$$

We use the same concepts for electric and magnetic fields, even though there is no real flow.

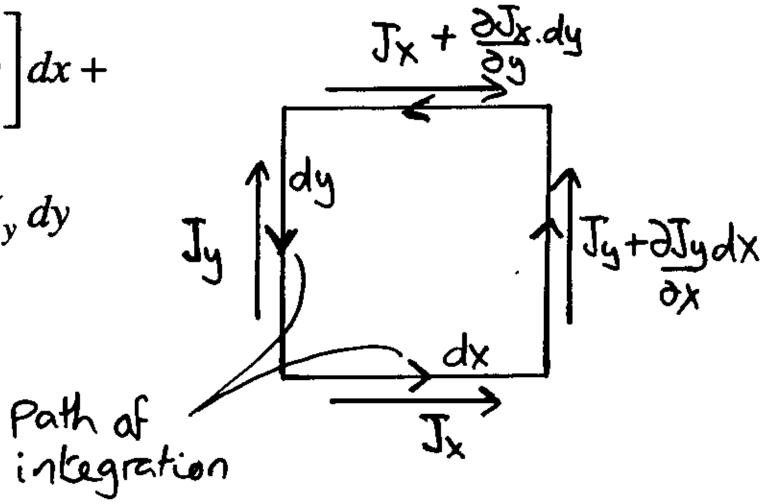
The curl of a vector field

Curl $\bar{J}$ is related to the "circulation" of a vector around a closed loop, defined as $\oint \bar{J} \bullet d\bar{l}$, the integral of the tangential component around the loop.

For an elementary surface of size *dx, dy* the integral becomes:

$$\sum \bar{J} \bullet d\bar{l} = J_x dx - \left[J_x + \frac{\partial J_x}{\partial y} dy \right] dx + \left[J_y + \frac{\partial J_y}{\partial x} dx \right] dy - J_y\, dy$$

$$= \left[\frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y} \right] dx\, dy$$



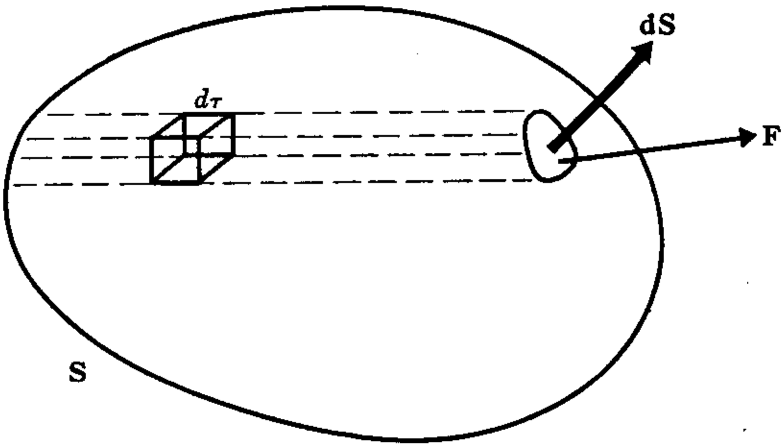
i.e. in general

$$\oint \bar{J} \bullet d\bar{l} = (\nabla \times \bar{J}) \bullet d\bar{S}$$

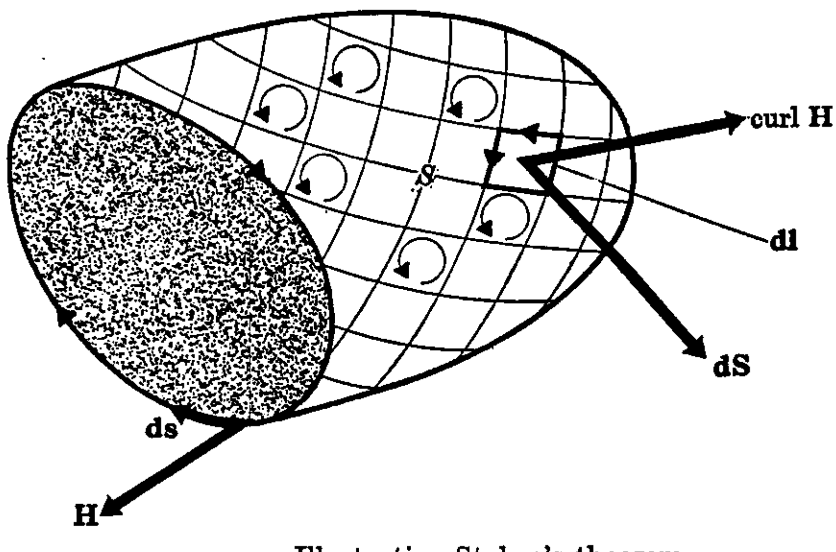
The curl of a vector in a particular direction, per unit area, is the "circulation" around a loop perpendicular to that direction.

Integrating over a finite surface, the contribution from adjacent loops cancels everywhere except on the outer loop :

$$\oint_L \bar{J} \bullet d\bar{l} = \int_S (\nabla \times \bar{J}) \bullet d\bar{S} \qquad \text{(Stokes's Theorem)}$$



Illustrating the divergence theorem.



Illustrating Stokes's theorem.

Second derivatives

In all there are 5 combinations of second derivatives involving grad, div, curl:

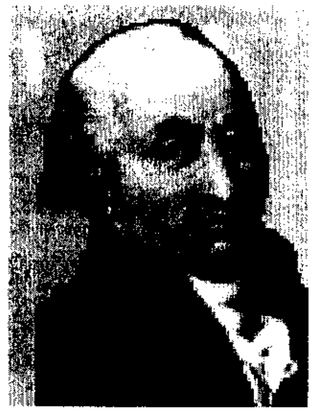
$$\text{curl}(\text{grad}T) = \nabla \times (\nabla T) = 0$$
$$\text{div}(\text{grad}T) = \nabla \bullet (\nabla T) = \nabla^2 T$$
$$\text{grad}(\text{div}\,\bar{v}) = \nabla(\nabla \bullet \bar{v})$$
$$\text{div}(\text{curl}\,\bar{v}) = \nabla \bullet (\nabla \times \bar{v}) = 0$$
$$\text{curl}(\text{curl}\,\bar{v}) = \nabla \times (\nabla \times \bar{v}) = \nabla(\nabla \bullet \bar{v}) - \nabla^2 \bar{v}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is an operator that can act on a scalar, or on all components of a vector

NB] knowing the divergence and curl of a vector at every point specifies completely the vector field.

3. **Electrostatics** - *the study of static electric charges*

Coulomb's Law (1785):
the force between two charges q_1, q_2 is inversely proportional to the square of the distance between them, r, and is directed along the line between them:



$$F = C\frac{q_1 q_2}{r^2}$$

or in vector notation: $$\overline{F} = C\frac{q_1 q_2}{r^3}\,\overline{r}$$

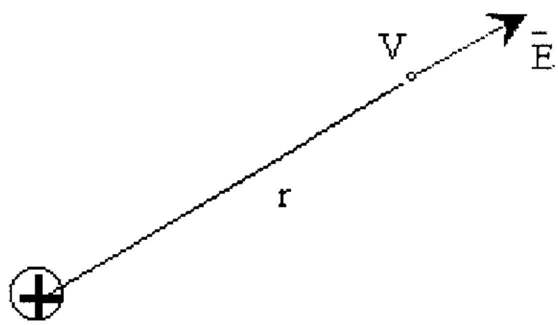
In MKS units, with F in Newtons, q in Coulombs, r in metres:

$$\overline{F} = \frac{1}{4\pi\varepsilon_o}\frac{q_1 q_2}{r^3}\,\overline{r}$$

where ε_o is an experimental constant (the 'permittivity of free space') = 8.85 10^{-12} coulomb^2 newton^{-1} metre^{-2}

The force that charge q_2 experiences can be described as being due to the presence of an **Electric Field** produced by the charge q_1. Thus:

$$\overline{F} = q_2\,\overline{E} \quad \text{and} \quad \overline{E} = \frac{q_1}{4\pi\varepsilon_o r^3}\,\overline{r}$$



<u>Electric Potential</u>

The work done by a force $\overline{F}$ acting on an object as it moves through a small distance $d\bar{l}$ is :

$$W = \overline{F} \bullet d\bar{l} = F_x\,dx + F_y\,dy + F_z\,dz$$

Applying this to the case of a unit charge in an Electric Field, the work done <u>by</u> the field is $\overline{E} \bullet d\bar{l}$ and the work done <u>against</u> the field is $-\overline{E} \bullet d\bar{l}$

The work done against an Electric Field in moving a unit charge from point A to point B defines a difference in **Electric Potential**:

$$\phi = -\int_A^B \overline{E} \bullet d\bar{l}$$

- the potential is a scalar quantity
- the difference in potential between two points depends only on the position of the points, not on the path between them
- single valued at any point
- for a distribution of charges, the total potential is the sum of the potentials for the individual charges

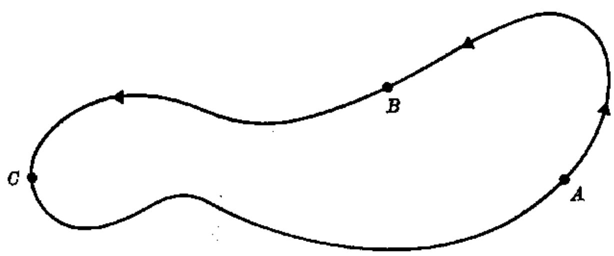
By convention the potential at a distance r from a charge q is the work done in bringing a unit charge from infinity, where by definition $\phi = 0$

$$\phi = -\frac{q}{4\pi\varepsilon_o}\int_{\infty}^{r}\frac{1}{r^2}\,dr = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{r}\right]_{\infty}^{r} = \frac{q}{4\pi\varepsilon_o r}$$

Since $\quad d\phi = -\overline{E}\bullet d\overline{l} = -(E_x\,dx + E_y\,dy + E_z\,dz)$

it follows that $\quad \dfrac{d\phi}{dx} = -E_x \quad \dfrac{d\phi}{dy} = -E_y \quad \dfrac{d\phi}{dz} = -E_z$

i.e.: $\quad \overline{E} = -\text{grad}\,\phi, \quad \text{or} \quad \overline{E} = -\nabla\phi$



For any complete loop,

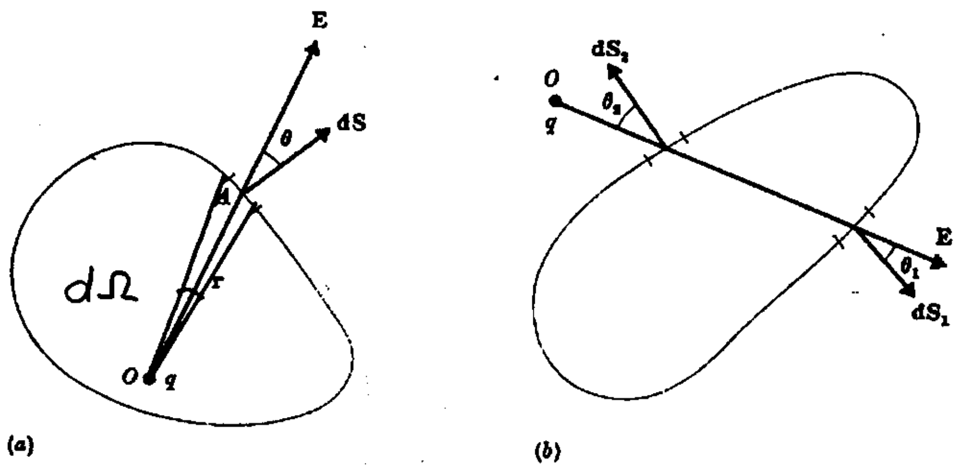
$$-\oint \overline{E}\bullet d\overline{l} = (\phi_B - \phi_A) + (\phi_C - \phi_B) + (\phi_A - \phi_C) = 0$$

Hence, by Stokes's Theorem,

$$\nabla \times \overline{E} = 0$$

This also follows directly from the fact that $\overline{E} = -\nabla\phi$, since for any vector, $\text{curl}(\text{grad}\,T) = \nabla(\nabla T) = 0$

Gauss's Theorem



S is a closed surface surrounding a charge q. The integral of the normal component of the Electric Field over the surface is:

$$\int \overline{E} \bullet d\overline{S} = \int E\cos\theta \, dS = \frac{q}{4\pi\varepsilon_o}\int \frac{\cos\theta}{r^2}dS$$

but the solid angle subtended by the element dS is given by:

$d\Omega = dS\cos\theta / r^2$ and $\int d\Omega = 4\pi$ for a closed surface, therefore

$$\int \overline{E} \bullet d\overline{S} = \frac{q}{\varepsilon_o}$$

By the same argument the contribution from external charges is zero.

Since this is true for one charge, it must be true also for a distribution of charges:

$$\int \overline{E} \bullet d\overline{S} = \frac{\Sigma q}{\varepsilon_o} \qquad \text{(Gauss's theorem)}$$

The flux of the electric field out of a closed surface is equal to the sum of the charges within it.

Applying now the divergence theorem:

$$\int \overline{E} \bullet d\overline{S} = \int \nabla \bullet \overline{E}\, dV$$

and since for a distribution of charge density $\rho(x,y,z)$

$$\Sigma q = \int \rho dV$$

we have:

$$\int (\nabla \bullet \overline{E})\, dV = \frac{\int \rho dV}{\varepsilon_o}$$

Since this is true for any arbitrary dV it follows that:

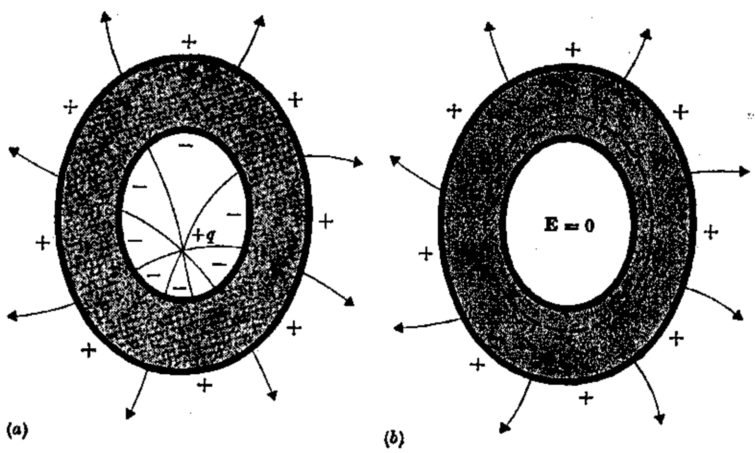
$$\nabla \bullet \overline{E} = \frac{\rho}{\varepsilon_o}$$

- the differential form of Gauss's Theorem

Application of Gauss's Theorem to a hollow conductor

Since charges are free to move in a conductor, it follows directly that there can be no static charges, or electrostatic field, within a conductor. Any static charge must therefore reside on the surface.

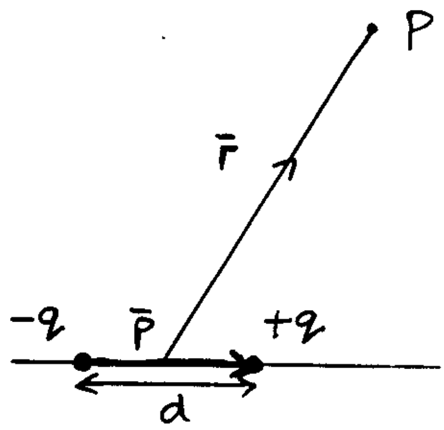
Applying Gauss's Theorem to a surface entirely within a hollow conductor, since there is no field it follows that the total charge within the surface is zero.



Thus either there is no charge (b), or no net charge (a) i.e. the charge induces equal and opposite charges on the inner surface. even if the outside surface is charged.

Note that this provides a sensitive way of testing the inverse-square law of the electrostatic force on which Gauss's Theorem relies.

The Electric Dipole



An electric dipole consists of two charges of equal magnitude, q, separated by a small distance, d. ("Small" means we are only interested in the fields at large distances, r, with respect to d)

The potential of such a dipole is:

$$\phi = \frac{\overline{p} \bullet \overline{r}}{4\pi \varepsilon_o r^3}$$

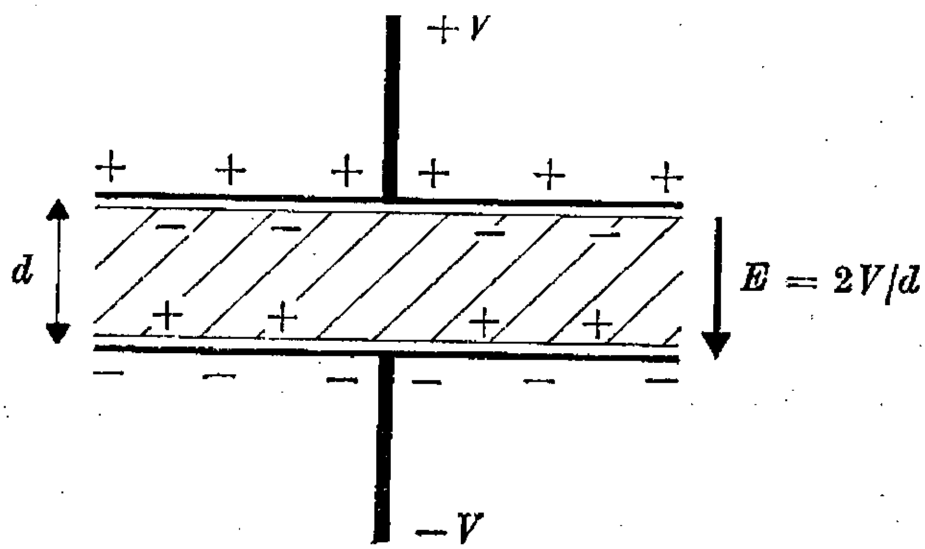
where $\overline{p} = q\overline{d}$, the dipole moment

Note that the potential decreases as $1/r^2$, and the field as $1/r^3$.

The importance of the electric dipole lies in the fact that atoms can behave as a tiny electric dipole under the influence of an electric field. Some molecules also have a net dipole moment, even in the absence of an external field.

Dielectrics

Faraday noticed that when insulating material is placed between the plates of a capacitor, held at constant voltage, the charge on the plates increased.



Dielectrics are insulators, i.e. with no free electrons. In the presence of an Electric Field the charge distribution of each atom shifts slightly creating a small electric dipole. The net effect is that the surfaces of the dielectric become charged, inducing opposite charges on the plates of the capacitor, and hence increasing the capacitance.

Defining the dipole moment (qd) per unit volume as $\overline{P}$, the positive charge passed across a surface $d\overline{S}$ is given by $\overline{P} \bullet d\overline{S}$.

The total polarization charge accumulated in a volume V is then:

$$Q_{\text{pol}} = -\int \overline{P} \bullet d\overline{S}$$

Using the Divergence Theorem, and the fact that $Q_{\text{pol}} = \int \rho_{\text{pol}}\, dV$ we obtain:

$$\rho_{\text{pol}} = -\nabla \bullet \overline{P}$$

The Electric Field then derives from the sum of free charges and the induced polarization charges:

$$\nabla \bullet \overline{E} = \frac{\rho_{\text{free}} + \rho_{\text{pol}}}{\varepsilon_o}$$

<u>If</u> we define a new vector $\overline{D}$ such that $\overline{D} = \varepsilon_o \overline{E} + \overline{P}$, then

$$\nabla \bullet \overline{D} = \rho_{\text{free}} \qquad \text{(Gauss's theorem in a dielectric)}$$

NB] The earlier equation is still true, even in dielectrics, provided both free charges and induced charges are included; separating out the equation for the free charges only is merely a convenience.

For a linear, isotropic dielectric

$$\overline{P} = \chi \varepsilon_o \overline{E}$$

χ = 'electric polarizability', or 'susceptibility'

$$\overline{D} = \varepsilon_o \overline{E} + \overline{P} = \varepsilon_o (1 + \chi) \overline{E}$$

or,

$$\overline{D} = \varepsilon \varepsilon_o \overline{E}$$

where $\varepsilon = 1 + \chi$ is the 'dielectric constant'

In this case:

$$\nabla \bullet \overline{E} = \frac{\rho_{\text{free}}}{\varepsilon \varepsilon_o} \qquad \text{(linear, isotropic dielectric)}$$

However, the relationship between $\overline{P}$ and $\overline{E}$ can be more complicated: $\overline{P}$ (and hence $\overline{D}$) may not be in the same direction as $\overline{E}$, and may not be linearly proportional to it.

4. Steady Currents

The current density $\bar{J}$ is defined as the rate at which charge passes per unit surface area per unit time (Coulomb s^{-1} m^{-2}, or A m^{-2}). The total current in a circuit $\hat{I}$ is then

$$I = \int \bar{J} \bullet d\bar{S}$$

Since electric charges cannot be created or destroyed, the rate of increase of total charge inside a volume must equal the net flow of charge into the volume

i.e. $$\int \frac{\partial \rho}{\partial t} dV = -\int \bar{J} \bullet d\bar{S}$$

Transforming the surface integral to a volume integral (Divergence theorem) gives:

$$\int \frac{\partial \rho}{\partial t} dV = -\int (\nabla \bullet \bar{J}) dV$$

and hence: $$\nabla \bullet \bar{J} = -\frac{d\rho}{dt}$$

known as the "Equation of Continuity"

5. Magnetic effects of steady currents (Magnetostatics)

The magnetic field $\overline{B}$ at a position $\overline{r}$ due to an element of conductor $d\overline{s}_1$ in which a current I_1 is flowing is given by:

$$d\overline{B} = \frac{\mu_o}{4\pi r^3} I_1 \, (d\overline{s}_1 \times \overline{r}) \qquad \text{(Biot-Savart Law)}$$

where μ_o is defined to be equal to $4\pi 10^{-7}$.

The force exerted on another conductor element $d\overline{s}_2$ carrying current I_2 is then:

$$d\overline{F}_2 = I_2 \, (d\overline{s}_2 \times d\overline{B})$$

Units: with $\overline{F}$ in Newton, I in Amps, $d\overline{s}$ in metre, the unit of $\overline{B}$ is $NA^{-1}m^{-1}$, or equivalently Weber m^{-2}, now Tesla (T)

Notice that the field (and force) varies as $1/r^2$ as for electrostatic fields, but has a more complex spatial variation.

i/ Divergence of $\overline{B}$

It follows from the Biot-Savart Law that

$$\nabla \bullet \overline{B} = 0$$

By comparison with the result for electrostatics, it also means that there is no equivalent to electric charge,
i.e. **there are no free magnetic poles.**

ii/ Curl of $\overline{B}$

The Biot-Savart Law can also be used to deduce the following:

$$\nabla \times \overline{B} = \mu_o \overline{J}$$

expressing the fact that **magnetic field circulates around a current distribution.**

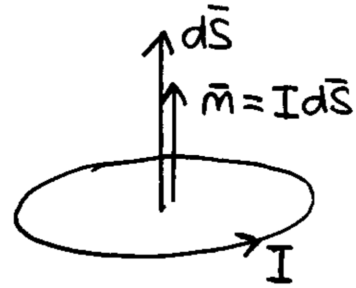
Applying Stokes's Theorem:

$$\oint \overline{B} \bullet d\overline{l} = \int (\nabla \times \overline{B}) \bullet d\overline{S} = \mu_o \int \overline{J} \bullet d\overline{S} = \mu_o I$$

which is Ampere's Law

The fact that $\nabla \times \overline{B} = \mu_o \overline{J}$ means that in general the magnetic field cannot be related to a scalar potential as in the case of electrostatic fields. However in current free regions $\nabla \times \overline{B} = 0$ and so a scalar potential can be used:

$$\overline{B} = -\nabla \psi$$



The Magnetic Dipole

The equivalent to an electric dipole is a current loop, for which it can be shown that (at a sufficient distance away):

$$\psi = \frac{\overline{m} \bullet \overline{r}}{4\pi r^3}$$

where $\overline{m} = I\, d\overline{S}$ is the magnetic dipole moment. The field distribution is therefore identical to that for an electrical dipole.

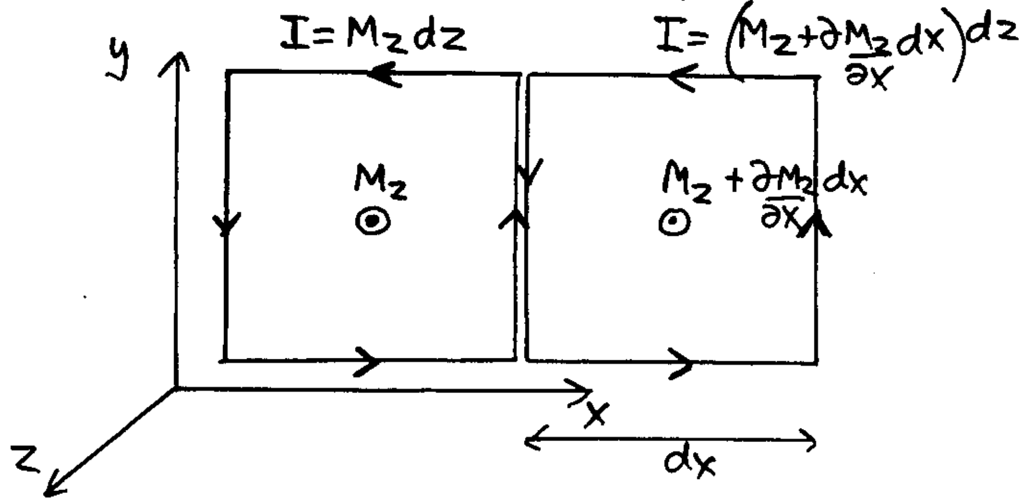
The significance is that, as already suggested at the time by Ampere, we can model – at least macroscopically - the effects of natural magnetism in terms of circulating atomic currents.

Magnetic Materials

Magnetic materials are those that exhibit a magnetic"polarization", i.e. a magnetic dipole moment.
In the case of permanent magnets, this can occur even in the absence of an external field.

Let $\overline{M}$ be the magnetic dipole moment per unit volume of material, called simply the Magnetization. For an elementary loop in the x-y plane, with magnetization in the z-direction:

$$I = M_z\, dx\, dy\, dz / dx\, dy = M_z\, dz$$



If $\overline{M}$ is constant, the current from neighbouring loops cancel, but in general:

$$I_y = \left[M_z - \left(M_z + \frac{\partial M_z}{\partial x} dx\right)\right] dz = -\frac{\partial M_z}{\partial x} dx\, dz$$

i.e. $$J_y = -\frac{\partial M_z}{\partial x}$$

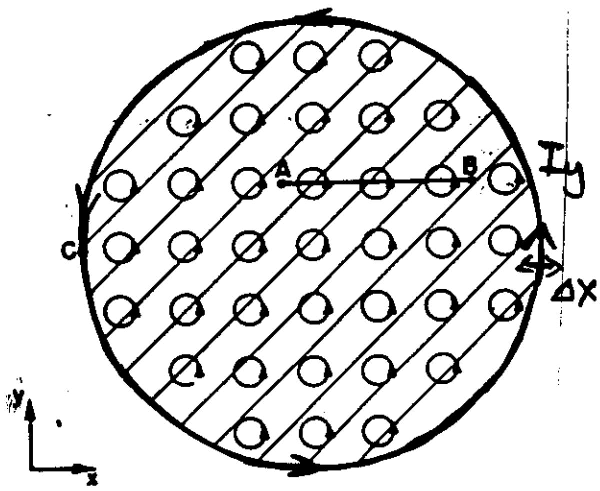
Similarly a current loop in the y-z plane, with an associated M_x can lead to a current in the y-direction. In total therefore:

$$J_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}$$

The right-hand-side of the above is the z-component of $\nabla \times \overline{M}$. Similar expressions can be derived for the other directions, so that finally:

$$J_{\text{mag.}} = \nabla \times \overline{M}$$

In a uniformly magnetized material $\nabla \times \overline{M}$ is zero everywhere except at the boundary i.e. the equivalent currents circulate around the surface of the material.



How big is this current ?
Suppose the magnetization changes from M_z to zero in a distance Δx, then $J_y = \frac{\partial M_z}{\partial x} = \frac{M_z}{\Delta x}$ and hence $I_y = M_z \Delta_z$.
If $\mu_o \overline{M} = 1$ Tesla, then for a 1 cm length of material, the equivalent current flowing on the outside is 8 kA !

The previous expression for curl B must therefore be modified to include also the magnetization currents as well as conduction currents:

$$\nabla \times \overline{B} = \mu_o (\overline{J}_{\text{cond.}} + \overline{J}_{\text{mag.}}) = \mu_o \overline{J}_{\text{cond.}} + \mu_o (\nabla \times \overline{M})$$

As in the electrostatic case, we can if we wish separate out the conduction currents from the magnetization currents by defining a new vector, $\overline{H}$

$$\overline{H} = \frac{\overline{B}}{\mu_o} - \overline{M}$$

so that: $$\nabla \times \overline{H} = \overline{J}_{\text{cond.}}$$

In many materials, the Magnetization is linearly proportional to the field:

$$\overline{M} = \chi \overline{H}$$

where χ is the "magnetic susceptibility". Then

$$\overline{B} = \mu_o (\overline{H} + \overline{M}) = \mu_o (1 + \chi) \overline{H} = \mu \mu_o \overline{H}$$

where $\mu = 1 + \chi$ is the "magnetic permeability", a similar quantity to the dielectric constant in electrostatics.

In this case,

$$\nabla \times \overline{B} = \mu \mu_o \overline{J}_{\text{cond.}}$$

The Magnetic Vector Potential

We saw above that in general we cannot define a scalar potential function ($\overline{B} = -\nabla \psi$) since $\nabla \times \overline{B} \neq 0$.

We can however make use of the fact that $\nabla \bullet \overline{B} = 0$.

Since for any vector, $\text{div}(\text{curl}\,\overline{v}) = \nabla \bullet (\nabla \times \overline{v}) = 0$
we can write

$$\overline{B} = \nabla \times \overline{A}$$

where $\overline{A}$ is called the magnetic vector potential.

One difficulty however is that this definition is incomplete since if $\overline{A}$ is one solution, so also is $\overline{A} + \nabla\varphi$:

$$\nabla \times (\overline{A} + \nabla\varphi) = (\nabla \times \overline{A}) + (\nabla \times \nabla\varphi) = (\nabla \times \overline{A})$$

In magnetostatics it is convenient to solve this problem by defining: $\nabla \bullet \overline{A} = 0$

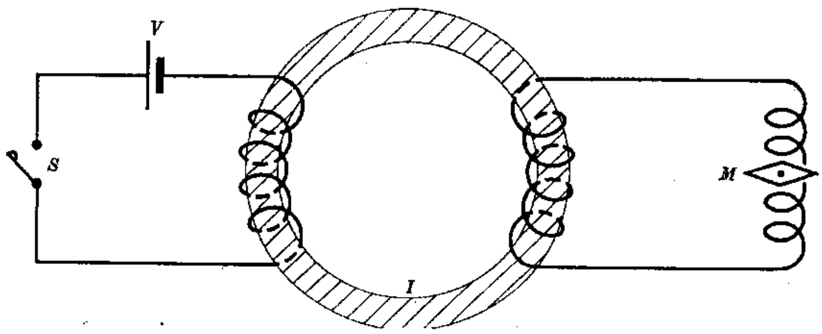
The potential of a magnetic dipole is:

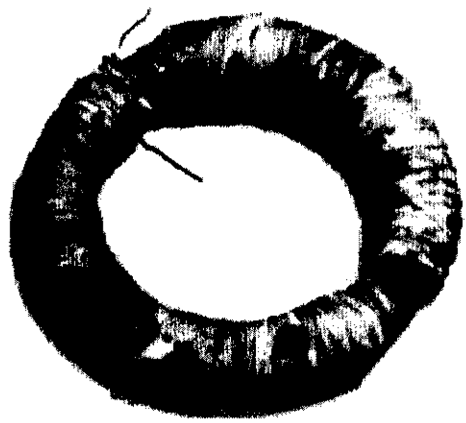
$$\overline{A} = \frac{\mu_o}{4\pi r^3}(\overline{m} \times \overline{r})$$

6. Magnetic Induction

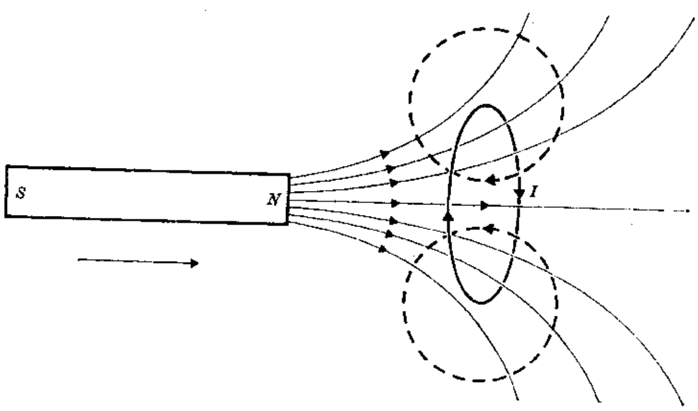
Faraday's experiments:

• When the first coil is connected to a battery, a current flows in the second coil causing a deflection of the magnetic needle





• If a permanent magnet is moved near a coil (or a coil is moved in the field of a magnet) a current is induced



Faraday's results can be summed up as follows:

When the magnetic flux through a circuit is changing an electromotive force (e.m.f.), V, is induced in the circuit, which is proportional to the rate of change of flux.

The direction of the e.m.f. is given by Lenz's Law:

The current induced is in such a direction to oppose the flux change causing the e.m.f.

Thus:

$$V = -\frac{d\left(\int \overline{B} \bullet d\overline{S}\right)}{dt}$$

But since $$V = \int \overline{E} \bullet d\overline{l}$$

We have $$\int \overline{E} \bullet d\overline{l} = -\frac{d\left(\int \overline{B} \bullet d\overline{S}\right)}{dt}$$

Using Stokes' Theorem $\int \overline{E} \bullet d\overline{l} = \int (\nabla \times \overline{E}) \bullet d\overline{S}$, ans since this is true for any surface, it must be that:

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt}$$

7. Maxwell's Equations

Maxwell realised that Ampere's Law :

$$\nabla \times \bar{B} = \mu_o \bar{J}$$

was not entirely correct.

Since for any vector $\nabla \bullet (\nabla \times \bar{P}) = 0$, when applied to Ampere's Law gives –

$$\nabla \bullet (\nabla \times \bar{B}) = \mu_o \nabla \bullet \bar{J} = 0$$

which contradicts the equation of continuity: $\nabla \bullet \bar{J} = -\dfrac{d\rho}{dt}$

How to resolve the problem ?

Combining Gauss's Law and the equation of continuity gives the following :

$$\nabla \bullet \bar{J} = -\frac{d\rho}{dt} \quad = -\varepsilon_o \frac{d}{dt}(\nabla \bullet \bar{E}) \quad = -\varepsilon_o \nabla \bullet \frac{d\bar{E}}{dt}$$

i.e.

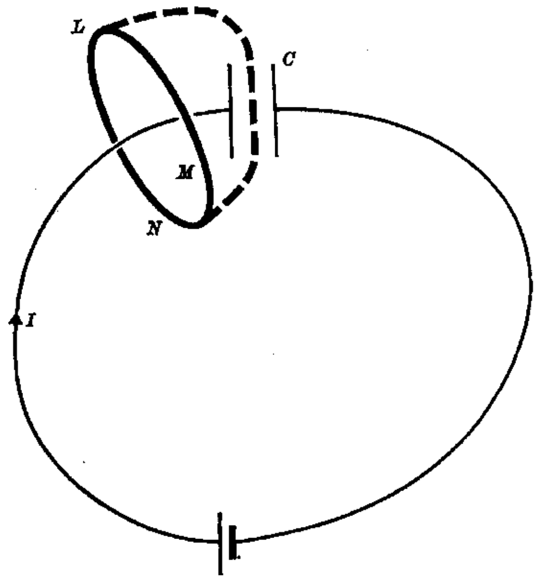
$$\mu_o \nabla \bullet \bar{J} + \mu_o \varepsilon_o \nabla \bullet \frac{d\bar{E}}{dt} = 0$$

which is therefore consistent with:

$$\nabla \times \bar{B} = \mu_o \bar{J} + \mu_o \varepsilon_o \frac{d\bar{E}}{dt}$$

Illustration of the new term

Consider a capacitor being charged by a battery, and two surfaces bounded by the same loop: Ampere's Law must be true for both surfaces



Surface S_1 is threaded by the current, I:

$$\oint \overline{B} \bullet d\overline{l} = \mu_o I$$

Surface S_2 is has no current, but a changing Electric Field:

$$\oint \overline{B} \bullet d\overline{l} = \mu_o \varepsilon_o \int \frac{d\overline{E}}{dt} \bullet d\overline{S} = \mu_o \varepsilon_o \frac{d}{dt} \int \overline{E} \bullet d\overline{S}$$

Applying Gauss's Law: $\int \overline{E} \bullet d\overline{S} = \frac{\int \rho \, dV}{\varepsilon_o} = \frac{q}{\varepsilon_o}$

We get: $$\oint \overline{B} \bullet d\overline{l} = \mu_o \frac{dq}{dt} = \mu_o I$$

i.e. results are therefore consistent (as they must be !).

$$\nabla \times \overline{B} = \mu_o \overline{J} + \mu_o \varepsilon_o \frac{d\overline{E}}{dt}$$

Note that in both the "proof" and in the illustration above, there are currents and free charges present. However the result is true in all cases.

In particular, when $\overline{J} = 0$ we see that :

a changing electric field creates a

circulation of magnetic field,

just as $$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt}$$

implies

a changing magnetic field creates a

circulation of electric field

Adding the new term has not simply patched up a hole in the theory, but, as realised by Maxwell, leads to a whole new range of electromagnetic phenomena.

Qu. Why was it never discovered experimentally ?

The full set of Maxwell's Equations are therefore:

$\nabla \bullet \overline{E} = \dfrac{\rho}{\varepsilon_o}$	Gauss's Theorem
$\nabla \bullet \overline{B} = 0$	No free magnetic poles
$\nabla \times \overline{B} = \mu_o \overline{J} + \mu_o \varepsilon_o \dfrac{d\overline{E}}{dt}$	Ampere's Law, modified by Maxwell
$\nabla \times \overline{E} = -\dfrac{d\overline{B}}{dt}$	Faraday's and Lenz's Law of magnetic induction

Remembering that :

$$\rho = \rho_{\text{cond.}} + \rho_{\text{pol.}}$$

with
$$\rho_{\text{pol.}} = -\nabla \bullet \overline{P}$$

and
$$\overline{J} = J_{\text{cond.}} + J_{\text{mag.}} + J_{\text{pol.}}$$

with
$$J_{\text{mag.}} = \nabla \times \overline{M} \quad \text{and} \quad \overline{J}_{pol.} = \frac{d\overline{P}}{dt}$$

8. Electromagnetic Waves

In a vacuum, with no conductivity ($\overline{J} = 0$) or charges ($\rho = 0$) Maxwell's Equations become:

$$\nabla \bullet \overline{E} = 0 \qquad\qquad \nabla \bullet \overline{B} = 0$$

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt} \qquad\qquad \nabla \times \overline{B} = \mu_o \, \varepsilon_o \, \frac{d\overline{E}}{dt}$$

A set of partial differential equations, that can be solved by eliminating one of the variables, using the relationship

$$\nabla \times (\nabla \times \overline{E}) = \nabla(\nabla \bullet \overline{E}) - \nabla^2 \, \overline{E}$$

Substituting in the above, we get:

$$\nabla \times \frac{d\overline{B}}{dt} = \frac{d}{dt}(\nabla \times \overline{B}) = \nabla^2 \, \overline{E}$$

hence:

$$\nabla^2 \, \overline{E} = \mu_o \, \varepsilon_o \frac{d^2 \, \overline{E}}{dt^2}$$

similarly we can obtain:

$$\nabla^2 \, \overline{B} = \mu_o \, \varepsilon_o \frac{d^2 \, \overline{B}}{dt^2}$$

Both of which are of the general form of a wave-equation !

$$\nabla^2 X = \frac{1}{v^2}\frac{d^2 X}{dt^2}$$

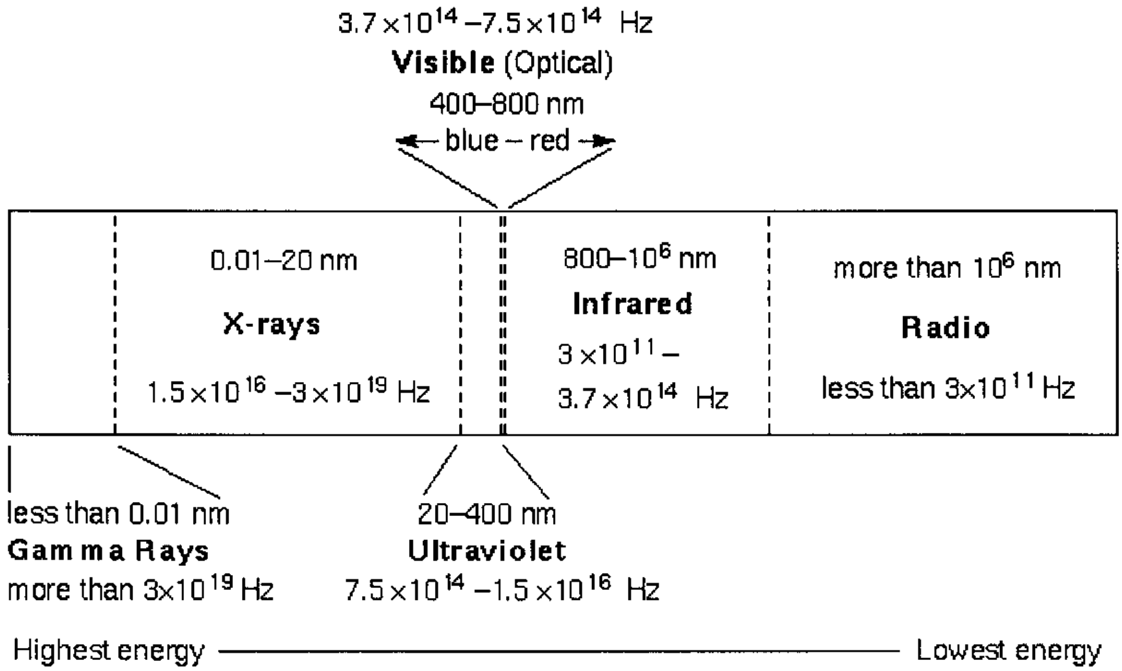
where X can be either a scalar or vector quantity.

Thus electric and magnetic fields can be propagated at a speed of

$$v = \frac{1}{(\mu_o\,\varepsilon_o)^{1/2}}$$

When Maxwell put in the numbers for μ_o, ε_o he obtained that the velocity agreed with the experimental value for the velocity of light, c.

"*We can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena*" (J. C. Maxwell)



The Electromagnetic Spectrum

Plane Waves

- waves in which there is no variation in a plane perpendicular to the direction of motion.

Assuming propagation along the z-axis, then $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$

which implies: $\frac{\partial E_z}{\partial z} = \frac{\partial E_z}{\partial t} = \frac{\partial B_z}{\partial z} = \frac{\partial B_z}{\partial t} = 0$

i.e. no z-components of electric or magnetic field;

the wave is purely transverse.

Examining the other components, they form two pairs:

$$\frac{\partial E_x}{\partial z} = \frac{\partial B_y}{\partial t}, \quad \frac{\partial B_y}{\partial z} = \mu_o \varepsilon_o \frac{\partial E_x}{\partial t}$$

and

$$\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}, \quad \frac{\partial B_x}{\partial z} = -\mu_o \varepsilon_o \frac{\partial E_y}{\partial t}$$

One pair involves (E_x, B_y) the other (E_y, B_x).

Thus we have two independent solutions, in each of which $\overline{B}$ is perpendicular to $\overline{E}$, differing only in the **plane of polarization**.

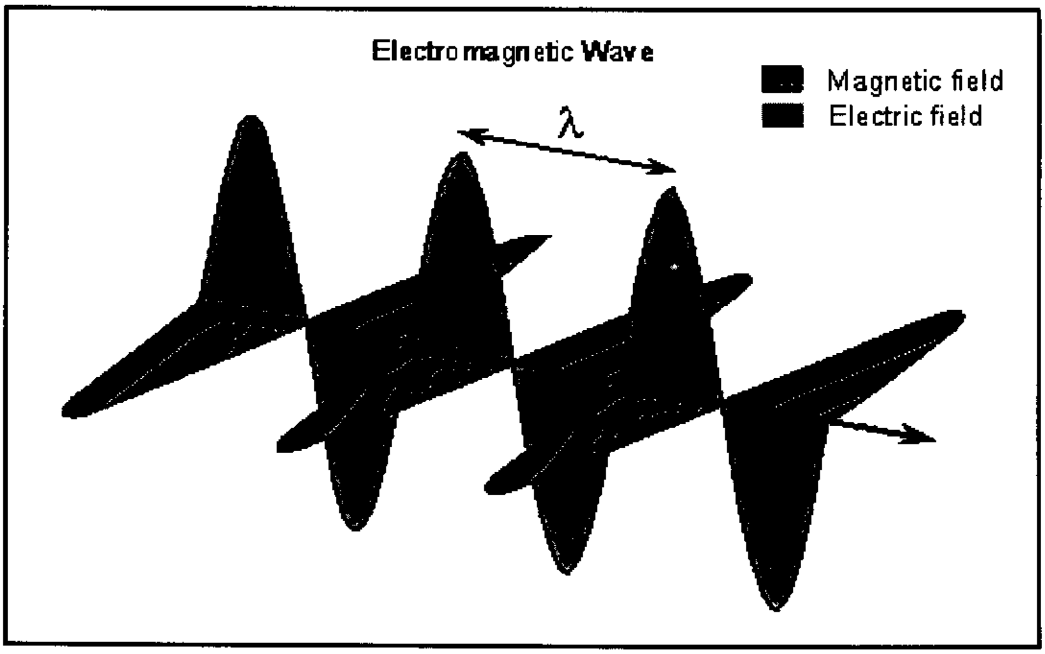
Since the Electric field is generally of most importance, we label the polarization according to this component.

The solution for forward propagating plane waves of a single frequency f, with corresponding wavelength λ, is :

$$E_x = A\,e^{\,i(kz-\omega t)}, \quad B_y = -\frac{A}{c}\,e^{\,i(kz-\omega t)}$$

where $k = \dfrac{2\pi}{\lambda}$, $\omega = 2\pi f$ and $c = f\lambda = \dfrac{\omega}{k}$

Thus, the oscillations of E_x and B_y are in phase :



Flow of Energy in Electromagnetic Waves

The flow of energy per unit area per second is given by the

Poynting vector: $$\overline{S} = \overline{E} \times \frac{\overline{B}}{\mu_o}$$

In the case of a plane wave $$S_z = \frac{E_x\,B_y}{\mu_o} = \varepsilon_o\,c\,E^2$$

Polarization

A general wave is composed of a superposition of plane waves. Consider a superposition of horizontally and vertically polarized radiation:

$$E_x = E_{xo} \cos(\omega t)$$
$$E_y = E_{yo} \cos(\omega t + \phi)$$

In the general case, there are 3 independent parameters E_{xo}, E_{yo}, ϕ. $\phi = 90^o$ with $E_{xo} = E_{yo}$ corresponds to circular polarization

A more useful description in terms of directly measurable quantities was introduced by G.G. Stokes (1852) :

Stokes Parameters — *difference in intensity between radiation polarized -*

$S_1 = I_x - I_y$ — *- linearly, in x, y directions*

$S_2 = I_{45^o} - I_{-45^o}$ — *- linearly, in +45o, -45o directions*

$S_3 = I_R - I_L$ — *- circularly, right and left-handed*

and

$S_0 = I_x + I_y = I_{45^o} + I_{-45^o} = I_R + I_L$ — *total intensity*

The relationship between them is :

$$S_o^2 = S_1^2 + S_2^2 + S_3^2$$

Normalizing to the total intensity (S_o) the polarization rates (or degrees) are defined as follows :

$$P_1 = \frac{S_1}{S_o} \quad P_2 = \frac{S_2}{S_o} \quad P_3 = \frac{S_3}{S_o}$$

where, $1 = P_1^2 + P_2^2 + P_3^2$

The above holds for a single wave only.
In general there is a summation of waves due of different sources, which leads to an un-polarized component :

$$S_o = \sqrt{S_1^2 + S_2^2 + S_3^2} + S_4$$
$$1 = \sqrt{P_1^2 + P_2^2 + P_3^2} + P_4$$

Electromagnetic Waves in Isotropic Dielectric/Magnetic Materials

In the most general case:

$$\nabla \bullet \overline{E} = 0 \qquad\qquad \nabla \bullet \overline{B} = 0$$

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt} \qquad\qquad \nabla \times \overline{B} = \mu_o(\nabla \times \overline{M}) + \mu_o\,\varepsilon_o\,\frac{d\overline{E}}{dt} + \mu_o\,\frac{d\overline{P}}{dt}$$

In the linear approximation:

$$\overline{P} + \varepsilon_o\,\overline{E} = \varepsilon\,\varepsilon_o\,\overline{E} \quad \text{and} \quad \overline{B} - \mu_o\overline{M} = \frac{\overline{B}}{\mu}$$

We get the same results as before with ε_o and μ_o replaced by $\varepsilon\,\varepsilon_o$ and $\mu\,\mu_o$ respectively.

The wave velocity therefore becomes

$$v = \frac{1}{(\mu\,\mu_o\,\varepsilon\,\varepsilon_o)^{1/2}} = \frac{c}{(\mu\,\varepsilon)^{1/2}}$$

Since $n = \frac{c}{v}$, defines the refractive index of the medium, we obtain the following approximation:

$$n = (\mu\,\varepsilon)^{1/2}$$

Electromagnetic Waves in a Conductive Medium

now we need to include also the conduction curent in the expression for curl $\overline{B}$,

$$\nabla \times \overline{B} = \mu \mu_o \varepsilon \varepsilon_o \frac{d\overline{E}}{dt} + \mu \mu_o \sigma \overline{E}$$

where $\overline{J}_{\text{cond.}} = \sigma \overline{E}$ (Ohm's Law)

σ is the conductivity.

which gives: $\nabla^2 \overline{E} = \mu \mu_o \varepsilon \varepsilon_o \frac{d^2 \overline{E}}{dt^2} + \mu \mu_o \sigma \frac{d\overline{E}}{dt}$

i.e. an extra damping term. Thus in a conductive medium there is dissipation of energy which gives a reduction of amplitude with distance travelled, $E \sim e^{-z/\delta}$

In a good conductor, an approximate expression for the 1/e distance (the "skin depth") is the following:

$$\delta \approx \left(\frac{2}{\sigma \omega \mu \mu_o} \right)^{1/2}$$

e.g. for copper, 3 μm at 500 MHz

9. Application of Electromagnetism to Accelerators

Motion of Charged Particles

From the force on an element of current carrying conductor

$$\overline{F} = I\,(d\overline{s} \times \overline{B})$$

if in length *ds* there are *N* charges the force per charge is:

$$\overline{F} = \frac{I}{N}\,(d\overline{s} \times \overline{B})$$

If the charges are of magnitude *q*, travelling with velocity *v*:

$$I = \frac{Q}{t} = \frac{Nq}{ds/v}$$

i.e. $$\overline{F} = q(\overline{v} \times \overline{B})$$

Adding the force due to an electric field, the total force in general is:

$$\overline{F} = q\,\overline{E} + q(\overline{v} \times \overline{B}) \qquad \text{(Lorentz force)}$$

For fast particles, the forces due to magnetic fields are much larger than those of electric fields – but are only transverse !

Circular Motion

requires a vertical magnetic field of strength:

$$q v B = \frac{\gamma m_o v^2}{r}$$

i.e. $$\frac{1}{r} = \frac{q B}{\gamma m_o v}$$

where r is the radius of curvature of the particle trajectory

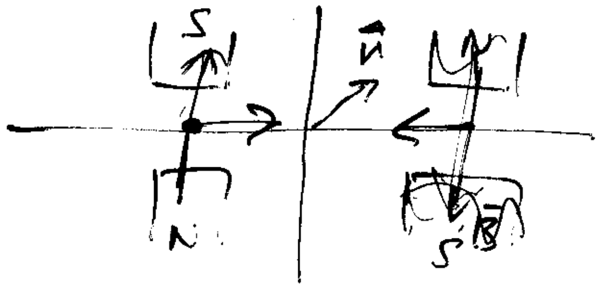
Example (ELETTRA):

For 2 GeV electrons, $\gamma = E/m_o c^2 = 3914$. With $B = 1.2$ T, we obtain $r = 5.5$ m.

To give a total angle of 360° we need a total magnet length of $2\pi r = 34.9$ m. Dividing into 24 units, the length of each "dipole magnet" is 1.45 m.

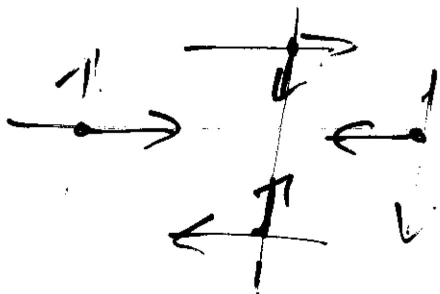
Focusing Fields

To achieve focusing we need a field that has a different direction on either side of the axis:



What about the other plane ?

We could try to rotate the previous picture:



Not a good idea. (Why ?)

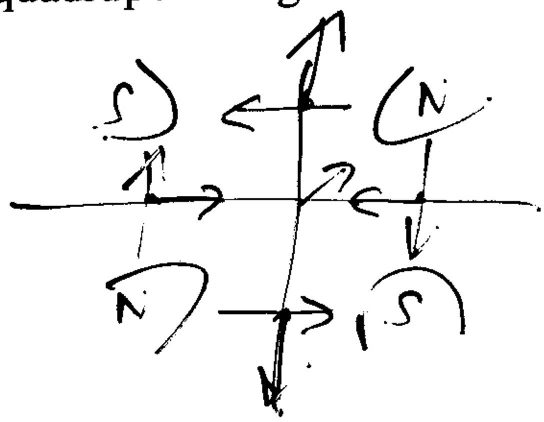
Go back to the previous picture and see what happens in the other plane:

Since $\nabla \times \overline{B} = 0$, we have that $\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$

A gradient in that plane also !

The simplest case is a linear variation: $B_y = Gx,\ B_x = Gy$,

produced by a "quadrupole magnet":



But it is defocusing in the other plane !

Solution:

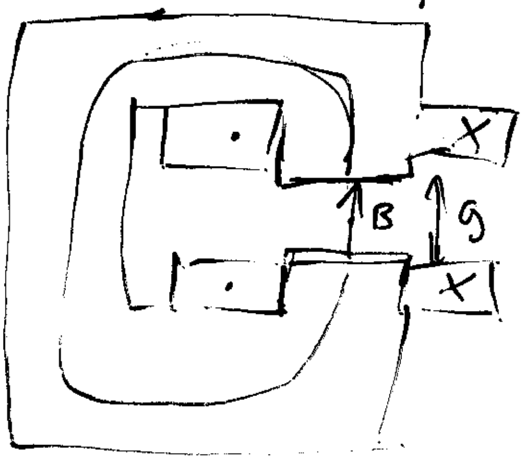
a sequence of focusing magnets of alternating sign, so that the net effect in both planes can be focusing.

(The principle of "Alternating Gradient Focusing")

Dipole and Quadrupole Magnets

Ampere's Law with magnetic materials:

$$\nabla \times \frac{\overline{B}}{\mu} = \mu_o \, \overline{J}_{\text{cond.}} \quad \text{, or} \quad \oint \frac{\overline{B}}{\mu} = \mu_o \, I$$



Consider a circuit in a dipole magnet, with air-gap g:

$$\oint \frac{\overline{B}}{\mu} = B_{\text{air}} \, g + \frac{B_{\text{iron}} \, l_{\text{iron}}}{\mu_{\text{iron}}} = \mu_o \, I$$

In general $B_{\text{iron}} \sim B_{\text{air}}$ (in fact at the pole tip $B_{\text{iron}} = B_{\text{air}}$, Why ?) and $\mu >> 1$, so that to a good approximation (a few %):

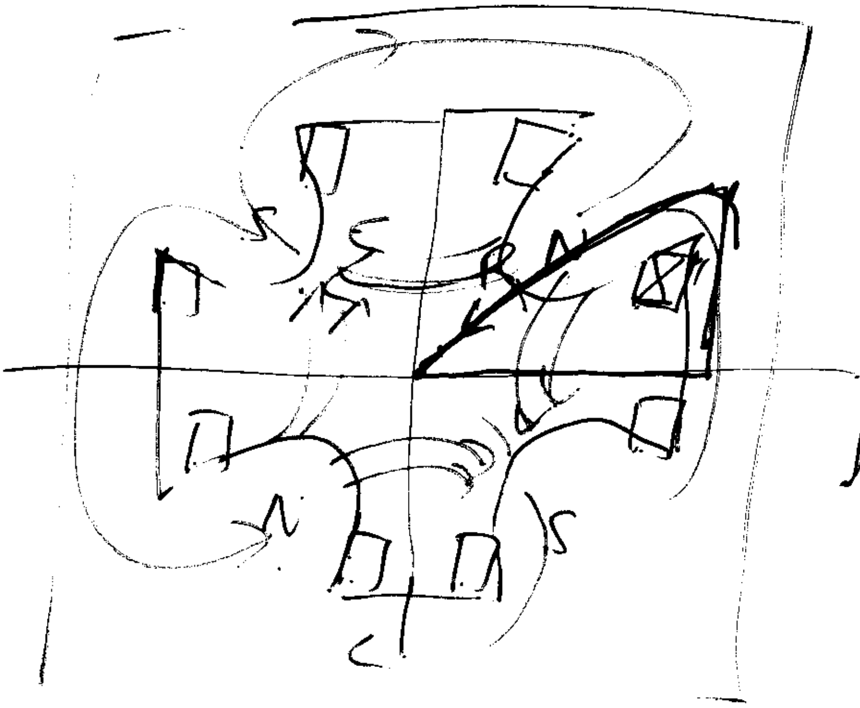
$$B_{\text{air}} = \frac{\mu_o (NI)}{g}$$

where we have introduced the fact that the total current threading the circuit is the product of the current, I, and the number of turns in the winding, N.

NI is known as the "Ampere-turns"

e.g. $$G = \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

for the ELETTRA dipole magnet with $B = 1.2$ T, and $g = 70$ mm, we need 68,000 Ampere-turns, which is achieved with a current of 1400 A and 48 turns.



$$B_y = Gx$$
$$B_x = Gy$$
$$B_r = Gr$$
$$\oint \underline{B} \cdot d\underline{l} = \int_0^R B_r \, dr = \int_0^R Gr \, dr = \frac{GR^2}{2} = \mu_0 (NI)$$

A similar calculation can be performed for the gradient, G (T/m), in a quadrupole magnet, giving:

$$G = \frac{2\mu_o (NI)}{R^2}$$

where R is the inscribed radius of the magnet.

<u>Acceleration</u>

We need an electric field in the direction of motion:

$$F_z = q\,E_z = \frac{d(\gamma m_o v)}{dt}$$

Why not use electrostatic fields ?

Can be used, but the maximum energy obtainable is limited by the maximum accelerating voltage that can be physically realized.

Also, cannot circulate the beam through a static E-field:

If $\partial/\partial t = 0$, $\nabla \times \overline{E} = 0$ and hence for any circuit in static fields

$\oint \overline{E} \bullet d\bar{l} = 0$ - no net acceleration !

One possibility is to use a time-varying magnetic field, i.e. magnetic induction, making direct use of the relationship

$\nabla \times \overline{E} = -\dfrac{d\overline{B}}{dt}$ and hence

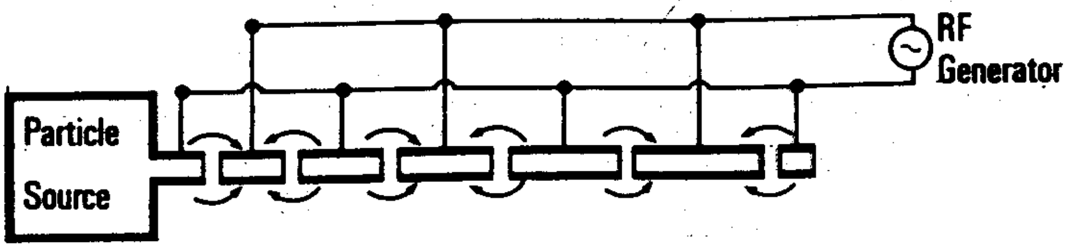
$$-\frac{d}{dt}\int \overline{B} \bullet d\overline{S} = \int (\nabla \times \overline{E}) \bullet d\overline{S} = \oint \overline{E} \bullet d\bar{l}$$

such as in the betatron accelerator, but this technique is limited to relatively low energies and is no longer used.

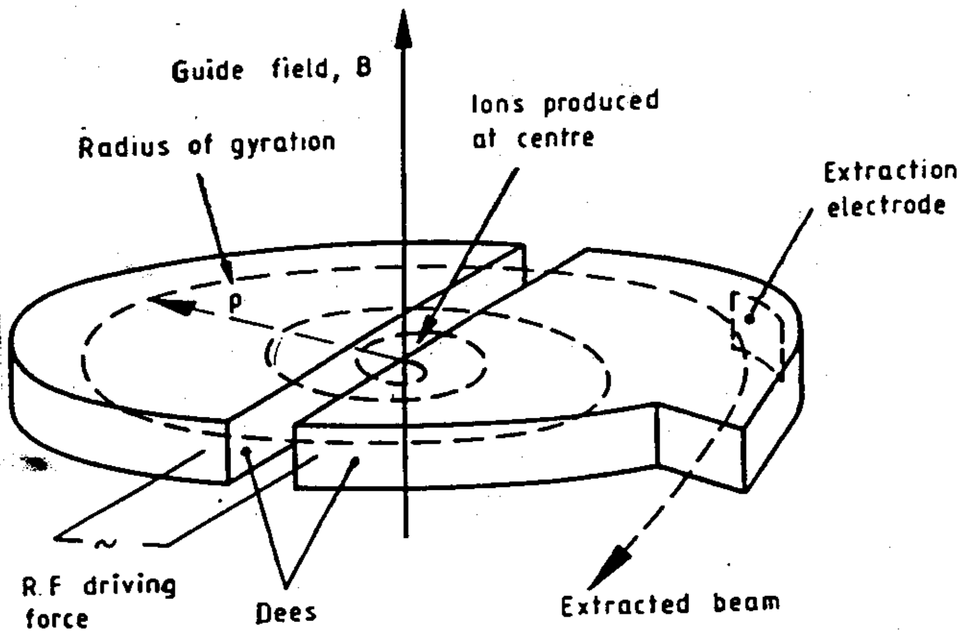
The main solution is to use time-varying electric fields, arranging that the particles arrive at the right moment to be accelerated.

Two examples of this type of acceleration, using relatively low frequency variation of the electric field-

"Drift-tube" linear accelerator:



Cyclotron:

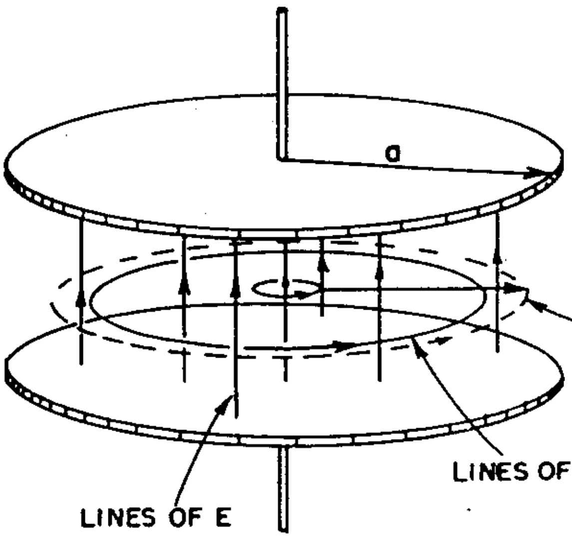


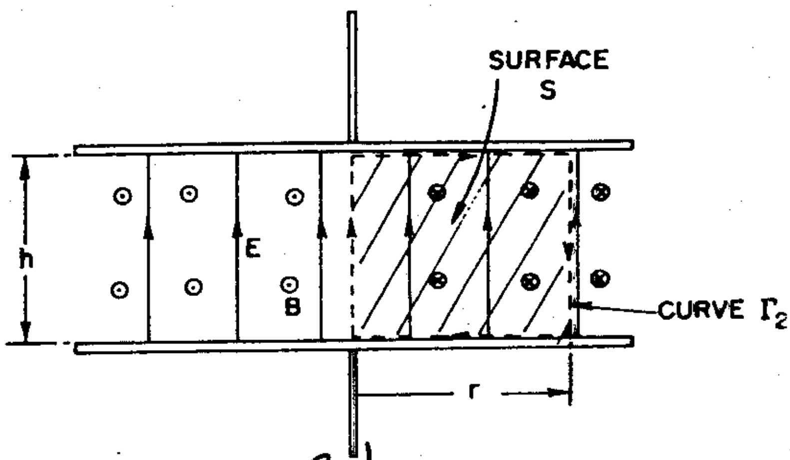
Both of these can be used for non-relativistic proton and heavier particle acceleration, not electrons.

Electrons (and higher energy protons) require using higher frequency fields.

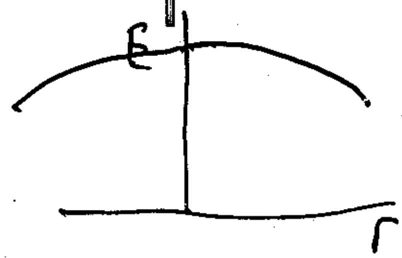
Consider the simplest single accelerating element, two plates of a capacitor, excited with a sinusoidal voltage:

$$E_z \approx E_o \cos(\omega t)$$





According to Maxwell's Equation:



$$\nabla \times \overline{B} = \frac{1}{c^2}\frac{d\overline{E}}{dt}$$

the varying electric field introduces an azimuthal magnetic field:

$$\int (\nabla \times \overline{B}) \bullet d\overline{S} = \oint \overline{B} \bullet d\overline{l} = \frac{1}{c^2}\frac{d}{dt}\int \overline{E} \bullet d\overline{S}$$

i.e. $$B_\theta \sim -\omega \sin(\omega t)$$

This magnetic field then in turn induces a change in the electric field according to:

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt}$$

i.e. $$\frac{\partial E_z}{\partial y} = \frac{\partial B_x}{\partial t} \sim \omega^2 \cos(\omega t)$$

The combined effect is to <u>reduce</u> the electric field towards the edges of the capacitor plates

A better approach is to consider the solution to Maxwell's Equations:

$$\nabla^2 \overline{E} = \frac{1}{c^2}\frac{d^2 \overline{E}}{dt^2}$$

i.e.

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = \frac{1}{c^2}\frac{d^2 E_z}{dt^2}$$

the solution of which is:

$$E_z \approx E_o \cos\left(\frac{\omega x}{\sqrt{2}c}\right)\cos\left(\frac{\omega y}{\sqrt{2}c}\right)\cos(\omega t)$$

We see that the variation in x and y, not present in the static case, depends on the frequency.

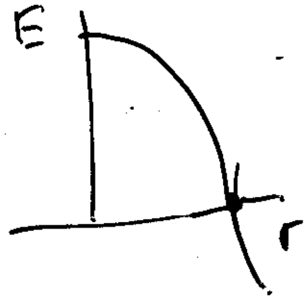
Let's examine the effect at x = 0.1 m

f (MHz)	$1-\cos\left(\frac{\omega x}{\sqrt{2}c}\right)$
1	10^{-6}
10	10^{-4}
100	0.01
500	0.26

Only at very high frequencies does the effect become significant

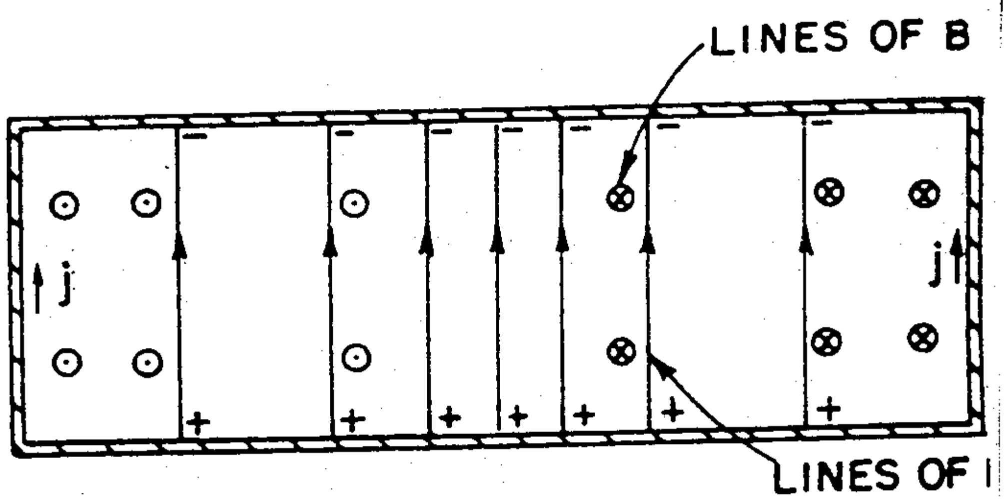
According to the above, at some point the field goes to zero :

$$x = \frac{\sqrt{2}\pi c}{2\omega}$$



e.g. x = 1 m at 100 MHz, 0.1 m at 1 GHz.

At this point the capacitor can be closed without affecting the field ! We have created a simple form of "resonant cavity".



Resonant cavities, of various forms, are the structures used to accelerate particles at high frequency. The dimensions of the structures are inversely proportional to the frequency.