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Special Relativity

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SPECIAL RELATIVITY

by

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Contents

1 Introduction

The theory of special relativity - from now on SR - has changed in a drastic and profound manner our way of thinking and has been considered together with quantum mechanics the most important physical theories, if not at all times, certainly of the 20^{th} century. The founder of the theory is considered to be A. Einstein (born 14/3/1879 in Ulm Germany and died 18/4/1955 in Princeton New Jersey USA) but as we shall see many people before him made important contributions.

The SR is not, like other scientific theories, a statement about matter that forms the physical world, but has the form of a condition that the explicit physical theories must satisfy. It is thus a form of description a kind of grammar of physics, prescribing which combinations of theoretical statements are admissible descriptions of the physical world. The theory came officially into existence in 1905 as a result of the union of two previously unrelated ideas, the notion that motion has a relative character and the notion that optics and mechanics are not two independent disciplines but must be rendered consistent one with the other. SR deals mainly about spacetime and its properties and admittedly changed our way of looking at space and time. However it was not mechanics from which all started but optics.

The impact of SR is great also out of the realm of physics but unfortunately it produced also a great confusion (as usually grammar does) to the non experts (and not only) including philosophers too! The statement of Poincare:
"According to the principle of relativity, the laws of physical phenomena must be the same for a fixed observer as for an observer who has a uniform motion of translation relative to him,

so that we have not, nor can we possibly have, any means of discerning whether or not we are carried along in such motion" when descended upon the world, it caused a great stir to many, dividing to those that simplified to: Einsteins theory says all is relative, and that has a profound influence on our ideas and to others that felt very uncomfortable about SR that asserts that one cannot determine and detect absolute velocities without looking outside, claiming that this was self evident. As we shall see both statements expressed as above are either obvious or trivial. It is obvious that things depend upon your frame of reference as can be asserted by many observations (after all a person looks different from the front than from the back) and its simply not true that one cannot detect any motion except by looking outside (uniform rotation about a fixed axis can be).

However people where impressed not because of the principle of relativity quoted above but rather from the fact that while for a moment electrodynamics suggested that absolute motion could be detected, soon to be found experimentally that it could not.

To better understand all that it is now necessary to make a small tour through the physical theories of the past.

2 History

Who Invented Special Relativity?

2.1 Before 1905

The fact that motion is not an absolute property of a body but a relation between a body and an observer was known at least as early as Galileo. From his Dialogue Concerning the Two Chief World Systems, in his most important work, which he began writing in 1626 he asks the reader to imagine the following though experiment (excerpts):

...have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.

This is the first formulation of "The Principle of Relativity". In other words, "the mechanical laws of physics are the same for every observer moving uniformly with constant speed in a straight line". Such an observer who "moves uniformly with constant speed in a straight line" (i.e., "moves with constant velocity") is a special type of observer: called an "Inertial" Observer. (From now on, we use "inertial" instead of "moves with constant velocity") So, we can restate:

Galileo's Principle of Relativity

The mechanical laws of physics are the same for every inertial observer. By observing the outcome of mechanical experiments, one cannot distinguish a state of rest from a state of constant velocity.

In this experiment Galileo contradicts Aristotle (384-322 b.C.) who considered that uniform motion was a quite different state from rest, needing some continual intervention to produce it. However he was the first to set laws and descriptions about space, time and motion:

- Every sensible body is by its nature somewhere. (Physics,Book 3, 205a:10)
- Time is the numeration of continuous movement. (Physics, Book 4, 223b:1)

Aristotle was interested in motion. He realised that motion can be understood by seeing how the location of an object changed. And that one could talk about "one object moving faster than another" by comparing how much the location of each changed in some interval of time. Aristotle believed that "being at rest" was the natural state of motion of any object. If an object were in motion, then there must be some agent that is responsible for that motion. And when that agent stops, the motion stops. According to Aristotle, there is a privileged being: The Prime Mover. He is the first agent, responsible for moving
objects, which, in turn, move other objects. The Prime Mover, he argued, must be at Absolute Rest. By "absolute" rest, we mean that all observers will universally agree on that state of rest.
Aristotle considered that rest is an absolute state (in which Earth was) and everything else rotated about. Do not think "oh how wrong or ignorant he was", of course he was wrong but it took people more than 2000 years to really understand it.
The considerations of Galileo were taken account of by I. Newton (1642-1727) who in his Prinzipia adheres to the idea of absolute space and time but introduces also the concepts of relative space and relative time. He expresses Galileo's relativity as: The motion of bodies included in a given space are the same amongst

themselves, whether the space is at rest or moves uniformly in a straight line. Observe the word rest, that shows his adherence to the Aristotelian spacetime.

Little progress can be made further than this without considering the interaction between mechanics and optics. The earliest detailed discussion of this relation was given by Euler in a paper he wrote in 1739. At that time there was the controversy between the wave and particle theories of light and Euler noticed a critical difference between the theories in the measurement of aberration (change in the measured position of the stars because of the motion of the Earth) by Bradley (1728). He calculates the aberration angle considering a moving source or a moving observer for both cases (wave and particle theory). For the wave assumption he derives a different answer and realises that he has to include the "medium" in which the wave motion took place. Euler was the first of a long list of investigators (though by far the earliest) to realise the possibility of an experimental check on the question of whether a ballistic or a wave theory of light was correct; and so of a question concerning the relationship between mechanics and optics.

In 1810 Arago studied the refraction of light through a prism when the light had come from different stars and was received in a telescope. If the light consists of a wave motion in a medium through which the earth is moving, the amount of refraction in the prism ought to depend on the direction in which the telescope is pointed (light from different directions will have different speeds relative to the telescope), however he found no difference and wrote to Fresnel about this. Fresnel replied by drawing heavily on analogy of acoustics and deducing from this analogy the way in which a solid material could be expected to drag along the wave.

Fizeau (1859) in order to test the Fresnels formula measured the speed of light when it passes through a long pipe containing water. The value of speed is compared in two cases when the water is at rest, and when it is moving with speed v. If c is the speed of light in the air, the speed when the water is at rest is given by c/n where n is the refractive index of water (about 4/3). When the water moves with speed v the Fresnel formula predicts:

$$velocity = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right)$$

showing that the light is dragged on by the water at a slower velocity than the water has itself.

The experiment which was the most subtle attempt to detect the motion of the Earth with respect to the medium of transmission of light (and so with respect to the preferred inertial frame at rest!) was that of Michelson (1881) and by Michelson and Morley (1887) and as we know with negative results.

Even before the astonishing and "paradoxical" result of Michelson, the wave theory of light had taken a definitive form, which served to pin-point the difficulties very clearly. Riemann had noticed that the expression

$$1/\sqrt{\mu\kappa}$$

(where μ is the magnetic permeability and k the dielectric constant) which occurs in transforming from one system of electrical units to another, has for vacuum a value near to the velocity of light (and has the dimensions of velocity). That this was no coincidence was strongly emphasised by the measurements of Weber and Kohlrausch (1856).

James C. Maxwell (1831-1879), greatly influenced by Riemann's coincidence. He completed and unified the theories of electricity and of magnetism set down by Carl F. Gauss (1777-1855) Andre M. Ampere (1775-1836) and Michael Faraday (1791-1867) Although the laws of electricity and of magnetism according to Gauss, Ampere, and Faraday worked remarkably well, there was a glaring problem: taken together, these laws did not "conserve charge". In other words, for these laws (as written) to work, one had to allow charge to be created or destroyed. And this is not a good thing. (Additionally, from the form of the equations of these theories, he noticed an interesting symmetry (a similarity) in the way the electric field and the magnetic field appeared. It wasn't a perfect symmetry, however.)
Maxwell modified Ampere's Law by adding a single term to it. This was what was needed to make the laws consistent with the conservation of charge. (It also made the above symmetry closer to being a perfect symmetry.)

However, the addition of this term led to a remarkable prediction: the existence of electromagnetic waves. With the full set of equations, Maxwell was able to calculate the speed of these waves. He found that their speed was a constant, independent of the nature of the electric and magnetic fields. What Maxwell found was that electromagnetic waves travelled at the speed of light. Maxwell had just discovered a fundamental constant of nature: **the speed of light**. It just "popped out" of the full set of equations.

The knowledge that the electromagnetic field was spread with a velocity essentially the same as the speed of light caused Maxwell to postulate that light itself was an electromagnetic phenomenon. Maxwell wrote an article on Ether for the 1878 edition of Encyclopaedia Britannica. He proposed the existence of a single ether and the article tells of a failed attempt by Maxwell to measure the effect of the ether drag on the earth's motion. He also proposed an astronomical determination of the ether drag by measuring the velocity of light using Jupiter's moons at different positions relative to the earth.

Thus, the Maxwell equations not only unify the theories of electricity and of magnetism, but of optics as well. In other words, electricity, magnetism, and light could all be understood as aspects of a single object: the electromagnetic field. Quite a remarkable achievement!

As a consequence, the Maxwell equations made the physical prediction that "light travels with the same speed, in all directions". In other words, "a spherical pulse of light will appear spherical".

But there's a problem....

Galileo's Relativity and Newton's Mechanics
(GALILEO'S SPACETIME)
......vs.
Maxwell's Electrodynamics

Recall that Galileo's Principle of Relativity says that the mechanical laws of physics are the same for every inertial observer. And this led to the understanding that there is no public notion of speed---no universal agreement on what the speed of an object is---the speed of an object is a private "relative" concept. This is what led us to abandon Aristotle's notion of absolute rest and replace his spacetime with Galileo's Spacetime.

However, there is a serious problem: Galileo's Spacetime is incompatible with Maxwell's Laws of Electrodynamics and Optics. The source of the problem is the appearance of a "constant speed"--a fundamental constant of nature: the speed of light--

automatically built into the Laws of Electromagnetism and Optics. (It turns out that if this speed were infinite, then there would be no conflict between Galileo's Spacetime and Maxwell's Laws of Electromagnetism and Optics. But, this speed is not infinite. So, there is a conflict.)

Thus, Maxwell's Electrodynamics is incompatible with Galileo's Space time.

At this point one had several possibilities:

A) Maxwell equations were incorrect

B) Galilean relativity applied to classical mechanics but optics had a preferred reference system, the frame of which the luminiferous ether was at rest

C) There existed a relativity principle for both classical mechanics and optics but it was not Galilean relativity i,e, mechanics (Newton's law) had to change

Since Maxwell equations were only 20 years old at the time, it seemed almost obvious that these equations must be wrong so the thing to do was to change them in such a way that under Galilean transformation did not change form (covariance) and thus Galilean relativity was satisfied. When this was tried, the new terms that had to be included into the equations led to predictions of new electrical phenomena that did not exist at all when tested experimentally , so this attempt had to be abandoned.

In the mean time, prompted by Maxwell's ideas, Michelson began his own terrestrial experiments and in 1881 he reported "The result of the hypothesis of a stationary ether is shown to be incorrect, and the necessary conclusion follows that the hypothesis is erroneous". So it seemed that also the second hypothesis was wrong: No special reference system for the light.

Lorentz wrote a paper in 1886 where he criticised Michelson's experiment and really was not worried by the experimental result which he dismissed being doubtful of its accuracy. Michelson was persuaded by Thomson and others to repeat the experiment and he did so with Morley, again reporting that no effect had been found in 1887. It appeared that the velocity of light was independent of the velocity of the observer. [Michelson and Morley were to refine their experiment and repeat it many times up to 1929.]

Thus only the third choice remained but at the time was not realised. Already as early as 1887 Voigt first wrote down the transformations

$$x' = x - vt,\ y' = y/g,\ z' = z/g,\ t' = t - vx/c^2$$

and showed that certain equations were invariant under these transformations. These transformations, with a different scale factor, are now known as the Lorentz equations and the group of Lorentz transformations gives the geometry of special relativity. All this was unknown to Voigt who was writing on the Doppler shift when he wrote down the transformations. Voigt corresponded with Lorentz about the Michelson-Morley experiment in 1887 and 1888 but Lorentz does not seem to have learnt of the transformations at that stage. Lorentz however was now greatly worried by the new Michelson-Morley experiment of 1887.

In 1889 a short paper was published by the Irish physicist George FitzGerald in Science. The paper "The ether and the earth's atmosphere" takes up less than half a page and is non-

technical. FitzGerald pointed out that the results of the Michelson-Morley experiment could be explained only if:
... the length of material bodies changes, according as they are moving through the ether or across it, by an amount depending on the square of the ratio of their velocities to that of light". Lorentz was unaware of FitzGerald's paper and in 1892 he proposed an almost identical contraction in a paper which now took the Michelson-Morley experiment very seriously. When it was pointed out to Lorentz in 1894 that FitzGerald had published a similar theory he wrote to FitzGerald who replied that he had sent an article to Science but I do not know if they ever published it . He was glad to know that Lorentz agreed with him for I have been rather laughed at for my view over here . Lorentz took every opportunity after this to acknowledge that FitzGerald had proposed the idea first. Only FitzGerald, who didn't know if his paper had been published, believed that Lorentz had published first!

Larmor wrote an article in 1898 "Ether and matter" in which he wrote down the Lorentz transformations (still not written down by Lorentz) and showed that the FitzGerald-Lorentz contraction was a consequence. Lorentz wrote down the transformations, now named after him, in a paper of 1899, being the third person to write them down. He, like Larmor, showed that the FitzGerald-Lorentz contraction was a consequence of the Lorentz transformations.

The most amazing article relating to special relativity to be published before 1900 was a paper of Poincaré "La mesure du temps" which appeared in 1898. In this paper Poincaré says
... we have no direct intuition about the equality of two time intervals.

The simultaneity of two events or the order of their succession, as well as the equality of two time intervals, must be defined in such a way that the statements of the natural laws be as simple as possible.
By 1900 the concept of the ether as a material substance was being questioned. Poincaré, in his opening address to the Paris Congress in 1900, asked "Does the ether really exist?" In 1904 Poincaré came very close to the theory of special relativity in an address to the International Congress of Arts and Science in St Louis. He pointed out that observers in different frames will have clocks which will
... mark what on may call the local time. ... as demanded by the relativity principle the observer cannot know whether he is at rest or in absolute motion.

The year that special relativity finally came into existence was 1905. June of 1905 was a good month for papers on relativity, on the 5th June Poincaré communicated an important work "Sur la dynamique de l'electron" while Einstein's first paper on relativity was received on 30th June. Poincaré stated that It seems that this impossibility of demonstrating absolute motion is a general law of nature. After naming the Lorentz transformations after Lorentz, Poincaré shows that these transformations, together with the rotations, form a group.

2.2 The Big Moment

In 1905, Albert Einstein published "On the Electrodynamics of Moving Bodies".

3. *Zur Elektrodynamik bewegter Körper;*
von A. Einstein.

Daß die Elektrodynamik Maxwells — wie dieselbe gegenwärtig aufgefaßt zu werden pflegt — in ihrer Anwendung auf bewegte Körper zu Asymmetrien führt, welche den Phänomenen nicht anzuhaften scheinen, ist bekannt. Man denke z. B. an die elektrodynamische Wechselwirkung zwischen einem Magneten und einem Leiter. Das beobachtbare Phänomen hängt hier nur ab von der Relativbewegung von Leiter und Magnet, während nach der üblichen Auffassung die beiden Fälle, daß der eine oder der andere dieser Körper der bewegte sei, streng voneinander zu trennen sind. Bewegt sich nämlich der Magnet und ruht der Leiter, so entsteht in der Umgebung des Magneten ein elektrisches Feld von gewissem Energiewerte, welches an den Orten, wo sich Teile des Leiters befinden, einen Strom erzeugt. Ruht aber der Magnet und bewegt sich der Leiter, so entsteht in der Umgebung des Magneten kein elektrisches Feld, dagegen im Leiter eine elektromotorische Kraft, welcher an sich keine Energie entspricht, die aber — Gleichheit der Relativbewegung bei den beiden ins Auge gefaßten Fällen vorausgesetzt — zu elektrischen Strömen von derselben Größe und demselben Verlaufe Veranlassung gibt, wie im ersten Falle die elektrischen Kräfte.

Beispiele ähnlicher Art, sowie die mißlungenen Versuche, eine Bewegung der Erde relativ zum „Lichtmedium“ zu konstatieren, führen zu der Vermutung, daß dem Begriffe der absoluten Ruhe nicht nur in der Mechanik, sondern auch in der Elektrodynamik keine Eigenschaften der Erscheinungen entsprechen, sondern daß vielmehr für alle Koordinatensysteme, für welche die mechanischen Gleichungen gelten, auch die gleichen elektrodynamischen und optischen Gesetze gelten, wie dies für die Größen erster Ordnung bereits erwiesen ist. Wir wollen diese Vermutung (deren Inhalt im folgenden „Prinzip der Relativität“ genannt werden wird) zur Voraussetzung erheben und außerdem die mit ihm nur scheinbar unverträgliche

Einstein's paper is remarkable for the different approach it takes. It is not presented as an attempt to explain experimental results. In the discussion of the relationship between mechanics and optics he is directing the attention to the need of a proper operational definition of simultaneity for distant events. Instead of dealing with Maxwells equations at the beginning, he starts from kinematics where he defines simultaneity. In the introduction Einstein says

... the introduction of a light-ether will prove to be superfluous since, according to the view to be developed here, neither will a space in absolute rest endowed with special properties be introduced nor will a velocity vector be associated with a point of empty space in which electromagnetic processes take place. The theory to be developed is based -like all electrodynamics- on kinematics of the rigid body, since assertions of any such theory have to do with relationships between rigid bodies (systems of co-ordinates),clocks, and electromagnetic processes. **Insufficient consideration of this circumstance lies at the root of the difficulties which the electrodynamics of moving bodies at present encounters.**

Inertial frames are introduced which, by definition, are in uniform motion with respect to each other. The whole theory is based
on two postulates:

1. The laws of physics take the same form in all inertial frames.

2. In any inertial frame, the velocity of light c is the same whether the light is emitted by a body at rest or by a body in uniform motion.

[In 1983, the General Conference on Weights and Measures officially defined the speed of light to be

$$c = 299{,}792{,}458 \text{ meters/second},$$

and the meter, instead of being a primary measure, became a secondary quantity, defined in terms of the second and the speed of light. See also RADAR]

Einstein now deduces the Lorentz transformations from his two postulates and, like Poincaré proves the group property. Then the FitzGerald-Lorentz contraction is deduced. With the help of the mathematician Herman Minkowski (1849-1909) (who gave us the idea to think in terms of "Spacetime", not just space and time separately), Einstein proposed a new model for Spacetime to replace Galileo's Spacetime.

"Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

Also in the paper Einstein mentions the clock paradox. Einstein called it a theorem that if two synchronous clocks C1 and C2 start at a point A and C2 leaves A moving along a closed curve to return to A then C2 will run slow compared with C1. He notes that no paradox results since C2 experiences acceleration while C1 does not.

In September 1905 Einstein published a short but important paper in which he proved the famous formula

$$E = mc^2$$

2.3 Afterwards

The first paper on special relativity, other than by Einstein, was written in 1908 by Planck. It was largely due to the fact that relativity was taken up by someone as important as Planck that it became so rapidly accepted. At the time Einstein wrote the 1905 paper he was still a technical expert third class at the Bern patent office. Also in 1908 Minkowski published an important paper on relativity, presenting the Maxwell-Lorentz equations in tensor form. He also showed that the Newtonian theory of gravitation was not consistent with relativity.

The main contributors to special relativity were undoubtedly Lorentz, Poincaré and, of course, the founder of the theory Einstein. It is therefore interesting to see their respective reactions to the final formulation of the theory. Einstein, although he spent many years thinking about how to formulate the theory, once he had found the two postulates they were immediately natural to him. Einstein was always reluctant to acknowledge that the steps which others were taking due to the Michelson-Morley experiment had any influence on his thinking.

Poincaré's reaction to Einstein's 1905 paper was rather strange. When Poincaré lectured in Göttingen in 1909 on relativity he did not mention Einstein at all. He presented relativity with three postulates, the third being the FitzGerald-Lorentz contraction. It is impossible to believe that someone as brilliant as Poincaré had failed to understand Einstein's paper. In fact Poincaré never wrote a paper on relativity in which he mentioned Einstein. Einstein himself behaved in a similar fashion and Poincaré is only mentioned once in Einstein's papers. Lorentz, however, was praised by both Einstein and Poincaré and often cited in their work.

Lorentz himself poses a puzzle. Although he clearly understood Einstein's papers, he did not ever seem to accept their conclusions. He gave a lecture in 1913 when he remarked how rapidly relativity had been accepted. He for one was less sure.

As far as this lecturer is concerned he finds a certain satisfaction in the older interpretation according to which the ether possesses at least some substantiality, space and time can be sharply separated, and simultaneity without further specification can be spoken of. Finally it should be noted that the daring assertion that one can never observe velocities larger than the velocity of light contains a hypothetical restriction of what is accessible to us, a restriction which cannot be accepted without some reservation.

Despite Lorentz's caution the special theory of relativity was quickly accepted. In 1912 Lorentz and Einstein were jointly proposed for a Nobel prize for their work on special relativity. Indeed, Wilhelm Wein proposed that the Nobel prize of 1912 be awarded jointly to Lorentz and Einstein, saying

The principle of relativity has eliminated the difficulties which existed in electrodynamics and has made it possible to predict for a moving system all electrodynamic phenomena which are known for a system at rest... From a purely logical point of view the relativity principle must be considered as one of the most significant accomplishments ever achieved in theoretical physics... While Lorentz must be considered as the first to have found the mathematical content of relativity, Einstein succeeded in reducing it to a simple principle. One should therefore assess the merits of both investigators as being comparable.

The Nobel committee was at first cautious and waited for experimental confirmation, and thus the prize for 1912 was awarded to Dalen, and neither Einstein nor Lorentz nor anyone else was ever awarded a Nobel prize for either the special or general theories of relativity. By the time such confirmation was available Einstein had moved on to further momentous work . This is sometimes considered to have been an injustice to Einstein, although in retrospect it's conceivable that a joint prize for Lorentz and Einstein in 1912, as Wein proposed, assessing "the merits of both investigators as being comparable", might actually have diminished Einstein's subsequent popular image as the sole originator of both special and general relativity. Einstein never received a Nobel prize for relativity.

On the other hand, despite the fact that special relativity can, in a sense, be regarded as "just" an interpretation of Lorentz's theory, it is clearly an extraordinarily profound interpretation, with consequences extending far beyond Lorentz's electrodynamics. As Einstein later recalled, the new feature was the realization that the bearing of the Lorentz transformation transcended its connection with Maxwell's equations and was concerned with the nature of space and time in general. In any case Einstein was only one of several individuals (including Maxwell, Poincare, Fitzgerald, Lorentz, Planck, Mach, Milne, and Minkowski) responsible for the "relativity revolution".

3.1 Speed of light

Aristotle thought it was infinite, Galileo tried unsuccessfully to measure it with lanterns on hilltops, a Danish astronomer found it first by observing Jupiter's moons. Rival Frenchmen found it quite accurately about 1850, but a far more precise experiment was carried out in 1879 in Annapolis, Maryland by Albert Abraham Michelson.

The first recorded discussion of the speed of light (I think) is in Aristotle, where he quotes Empedocles as saying the light from the sun must take some time to reach the earth, but Aristotle himself apparently disagrees, and even Descartes thought that light travelled instantaneously. Galileo, unfairly as usual, in Two New Sciences (page 42) has Simplicio stating the Aristotelian position,

SIMP. Everyday experience shows that the propagation of light is instantaneous; for when we see a piece of artillery fired at great distance, the flash reaches our eyes without lapse of time; but the sound reaches the ear only after a noticeable interval.

Of course, Galileo points out that in fact nothing about the speed of light can be deduced from this observation, except that light moves faster than sound. He then goes on to suggest a possible way to measure the speed of light. The idea is to have two people far away from each other, with covered lanterns. One uncovers his lantern, then the other immediately uncovers his on seeing the light from the first. This routine is to be practised with the two close together, so they will get used to the reaction times involved, then they are to do it 3 or 4 km apart, or even further using telescopes, to see if the time interval is perceptibly lengthened. Galileo claims he actually tried the experiment at distances of about one km, and couldn't detect a time lag. **From this one can certainly deduce that light travels at least ten times faster than sound.**

The first real measurement of the speed of light came about half a century later, in 1676, by a Danish astronomer, Ole Römer, working at the Paris Observatory. He had made a systematic study of Io, one of the moons of Jupiter, which was eclipsed by Jupiter at regular intervals, as Io went around Jupiter in a circular orbit at a steady rate. Actually, Römer found, for several months the eclipses lagged more and more behind the expected time, until they were running about eight minutes late, then they began to pick up again, and in fact after about six months were running eight minutes early. The cycle then repeated itself. Römer realised the significance of the time involved-just over one year. This time period had nothing to do with Io, but was the time between successive closest approaches of earth in its orbit to Jupiter. The eclipses were furthest behind the predicted times when the earth was furthest from Jupiter.

The natural explanation was that the light from Io (actually reflected sunlight, of course) took time to reach the earth, and took the longest time when the earth was furthest away. From his observations, Römer concluded that light took about twenty-two minutes to cross the earth's orbit. This was something of an overestimate, and a few years later Newton wrote in the Principia (Book I, section XIV): "For it is now certain from the phenomena of Jupiter's satellites, confirmed by the observations of different astronomers, that light is propagated in succession (NOTE: I think this means at finite speed) and **requires about seven or eight minutes to travel from the sun to the earth**." This is essentially the correct value. Of course, to find the speed of light it was also necessary to know the distance from the earth to the sun. During the 1670's, attempts were made to measure the parallax of Mars, that is, how far it shifted against the background of distant stars when viewed simultaneously from two different places on earth at the same time. This (very slight) shift could be used to find the distance of Mars from earth, and hence the distance to the sun, since all relative distances in the solar system had been established by observation and geometrical analysis. According to Crowe (Modern Theories of the Universe, Dover, 1994, page 30), they concluded that the distance to the sun was between 64 and 145 million km. Measurements presumably converged on the correct value of about 149.668 million km soon after that, because it appears Römer (or perhaps Huygens, using Römer's data a short time later) used the correct value for the distance, since the speed of light was calculated to be **201,168 km per second, about three-quarters of the correct value of 299,792 km/s**. This error is fully accounted for by taking the time light needs to cross the earth's orbit to be twenty-two minutes (as Römer did) instead of the correct value of sixteen minutes.

The next substantial improvement in measuring the speed of light took place in 1728, in England. An astronomer James Bradley, sailing on the Thames with some friends, noticed that the little pennant on top of the mast changed position each time the boat put about, even though the wind was steady. He thought of the boat as the earth in orbit, the wind as starlight coming from some distant star, and reasoned that the apparent direction the starlight was "blowing" in would depend on the way the earth was moving. Another possible analogy is to imagine the starlight as a steady downpour of rain on a windless day, and to think of yourself as walking around a circular path at a steady pace. The apparent direction of the incoming rain will not be vertically downwards-more will hit your front than your back.

Bradley reasoned that the apparent direction of incoming starlight must vary in just this way, but the angular change would be a lot less dramatic. The earth's speed in orbit is about 32 km /s, he knew from Römer's work that light went at about 10,000 times that speed. That meant that the angular variation in apparent incoming direction of starlight was about the magnitude of the small angle in a right-angled triangle with one side 10,000 times longer than the other, about one two-hundredth of a degree. Notice this would have been just at the limits of Tycho's measurements, but the advent of the telescope, and general improvements in engineering, meant this small angle was quite accurately measurable by Bradley's time, and he found the velocity of light to **be 297,728 km/sec,** with an accuracy of about one percent.

The problem is, all these astronomical techniques do not have the appeal of Galileo's idea of two guys with lanterns. It would be reassuring to measure the speed of a beam of light between two points on the ground, rather than making somewhat indirect deductions based on apparent slight variations in the positions of stars. We can see, though, that if the two lanterns are 15 km apart (i.e in total 30 km) , the time lag is of order one-ten thousandth of a second (0.1 msec), and it is difficult to see how to arrange that. This technical problem was solved in France about 1850 by two rivals, Fizeau and Foucault, using slightly different techniques.

In Fizeau's apparatus, a beam of light shone between the teeth of a rapidly rotating toothed wheel, so the "lantern" was constantly being covered and uncovered. Instead of a second lantern far away, Fizeau simply had a mirror, reflecting the beam back, where it passed a second time between the teeth of the wheel. The idea was, the blip of light that went out through one gap between teeth would only make it back through the same gap if the teeth had not had time to move over significantly during the round trip time to the far away mirror. It was not difficult to make a wheel with a hundred teeth, and to rotate it hundreds of times a second, so the time for a tooth to move over could be arranged to be a fraction of one ten thousandth of a second. The method worked.

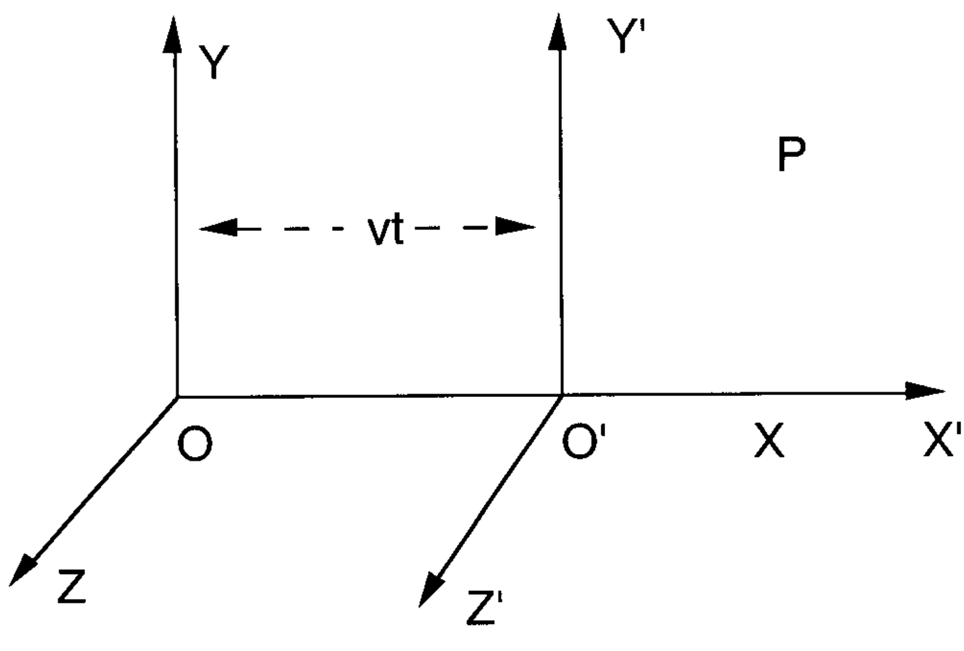
Foucault's method was based on the same general idea, but instead of a toothed wheel, he shone the beam on to a rotating mirror. At one point in the mirror's rotation, the reflected beam fell on a distant mirror, which reflected it right back to the rotating mirror, which meanwhile had turned through a small angle. After this second reflection from the rotating mirror, the position of the beam was carefully measured. This made it possible to figure out how far the mirror had turned during the time it took the light to make the round trip to the distant mirror, and since the rate of rotation of the mirror was known, the speed of light could be figured out. These techniques gave the speed of light **with an accuracy of about 1609 km per second.**

Albert Michelson was born in 1852 in Strzelno, Poland and his family soon after migrated to Annapolis USA. Michelson in 1875 became an instructor in physics and chemistry at the Naval Academy, under Lieutenant Commander William Sampson. Michelson met Mrs. Sampson's niece, Margaret Heminway, daughter of a very successful Wall Street tycoon, who had built himself a granite castle in New Rochelle, NY. Michelson married Margaret in New Rochelle in 1877.
At work, lecture demonstrations had just been introduced at Annapolis. Sampson suggested that it would be a good demonstration to measure the speed of light by Foucault's method. Michelson soon realised, on putting together the apparatus, that he could redesign it for much greater accuracy, but that would need money well beyond that available in the teaching demonstration budget. He went and talked with his father in law, who agreed to put up $2,000. Instead of Foucault's 21 meter to the far mirror, Michelson had about 700 m along the bank of the river Severn, a distance he measured to one tenth of an inch. He invested in very high quality lenses and mirrors to focus and reflect the beam. His final result was **186,355 miles per second, with**

possible error of 30 miles per second (299,909.30 km/s with an error of 48.3 km/s). This was twenty times more accurate than Foucault, made news in the New York Times, and Michelson was famous while still in his twenties. In fact, this was accepted as the most accurate measurement of the speed of light for the next forty years, at which point Michelson measured it again.

3.2 Galilean Transformations

Let O'X'Y'Z' a co-ordinate system moving with respect of the OXYZ system with uniform velocity v along the X axis:



The position of a point P can be expressed using a set of co-ordinates (x,y,z) or (x',y',z'). The relationship of the co-ordinates in the two systems is clear from the above diagram and can be expressed as:

$$x' = x - vt,$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Are Newton's laws the same for the moving system? (i.e. proving Galileo's relativity)
Take Newton's second law: F= d(m dx /dt)/dt (tacitly assuming that m is a constant -which as we now know it is not exactly true- we have F= m d^2x/dt^2) and substitute the above transformation. From the first equation of the transformations we have dx'/dt'= dx/dt-v and $d^2x'/dt'^2=d^2x/dt^2$ - dv/dt. Since dv/dt=0 (uniform velocity) $d^2x'/dt'^2=d^2x/dt^2$
We have thus proven that Newton's law keeps its form i.e. is the same in any inertial system.
This property is called Covariance (=form invariance) and it is very important since it indicates whether a theory respects the transformation under question.

3.3 Covariance of the wave equation

The preservation of the form of the equations of classical mechanics under the Galilean transformations is in contrast to the change in form of the equations governing wave phenomena. Suppose that a field $\psi(x',t')$ satisfies the wave equation

$$\left(\sum_i \frac{\partial^2}{\partial x_i'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\psi = 0$$

in the reference frame O'. Here we have partial derivatives that they transform from x',t' to x, t according to the following rule:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x} = \frac{\partial}{\partial x'}$$

for the space partial derivative and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x}$$

for the time partial derivative. Then the wave equation in the O frame becomes:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}v\nabla\frac{\partial}{\partial t} - \frac{1}{c^2}v\nabla v\nabla\right)\psi = 0$$

Thus the form of the wave equation (including Maxwell's) is not invariant under Galilean transformations. Furthermore no kinematic transformation of ψ can restore the form.

The lack of invariance is not always a sign of weakness of the theory. For example for sound waves the lack of invariance under

Galilean transformations is quite acceptable. Sound waves are compressions and rarefactions in a transmitting medium and the preferred system is obviously the one where the medium is at rest. So it was thought also for electromagnetism. However there the medium seemed truly ethereal with no manifestation or purpose other than to support the propagation.

NB: The Schrödinger equation is invariant under Galilean transformations since there is a wave function that can restore the lack of covariance of the partial derivatives.

3.4 Michelson and the Ether

By the late 1800's, it had been established that light was wavelike, and in fact consisted of waving electric and magnetic fields. These fields were thought somehow to be oscillations in a material ether, a transparent, light yet hard substance that filled the universe (since we see light from far away). Michelson devised an experiment to detect the earth's motion through this ether, and the result contributed to the development of special relativity.

Detecting the ether wind was the next challenge Michelson set himself after his triumph in measuring the speed of light so accurately. Naturally, something that allows solid bodies to pass through it freely is a little hard to get a grip on. But Michelson realised that, just as the speed of sound is relative to the air, so the speed of light must be relative to the ether. This must mean, if you could measure the speed of light accurately enough, you could measure the speed of light travelling upwind, and compare it with the speed of light travelling downwind, and the difference of the two measurements should be twice the wind speed. Unfortunately, it wasn't that easy. All the recent accurate measurements had used light travelling to a distant mirror and coming back, so if there was an ether wind along the direction between the mirrors, it would have opposite effects on the two parts of the measurement, leaving a very small overall effect. There was no technically feasible way to do a one-way determination of the speed of light.

At this point, Michelson had a very clever idea for detecting the ether wind. As he explained to his children (according to his daughter), it was based on the following puzzle:

Suppose we have a river of width w (say, 100 meters), and two swimmers who both swim at the same speed v (say, 5 m /s). The river is flowing at a steady rate, say 3 m/s. The swimmers race in the following way: they both start at the same point on one bank. One swims directly across the river to the closest point on the opposite bank, then turns around and swims back. The other stays on one side of the river, swimming upstream a distance (measured along the bank) exactly equal to the width of the river, then swims back to the start. Who wins?

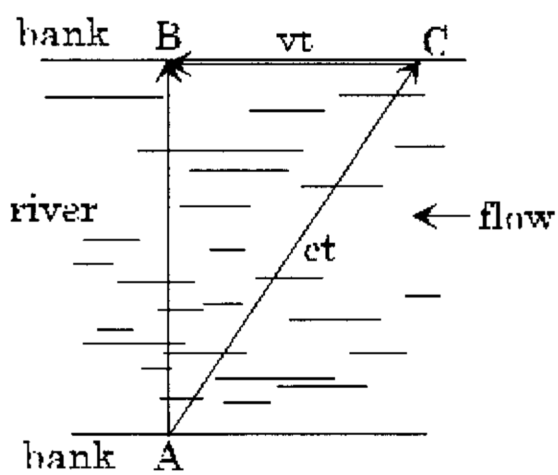


Figure 1 In time t, the swimmer has moved ct relative to the water, and been carried downstream a distance vt.

The swimmer going upstream and back will need 62.5 seconds. The swimmer going across the flow is trickier but choosing correctly the upstream angle (from a 3,4,5 m/s triangle) so that the net movement is directly across, the swimmer gets across in 25 seconds, and back in the same time, for a total time of 50 seconds. The cross-stream swimmer wins. This turns out to true whatever their swimming speed is provided they can swim faster than the current.

Michelson's great idea was to construct an exactly similar race for pulses of light, with the aether wind playing the part of the river. The scheme of the experiment is as follows: a pulse of light is directed at an angle of 45 degrees at a half-silvered, half transparent mirror, so that half the pulse goes on through the glass, half is reflected.

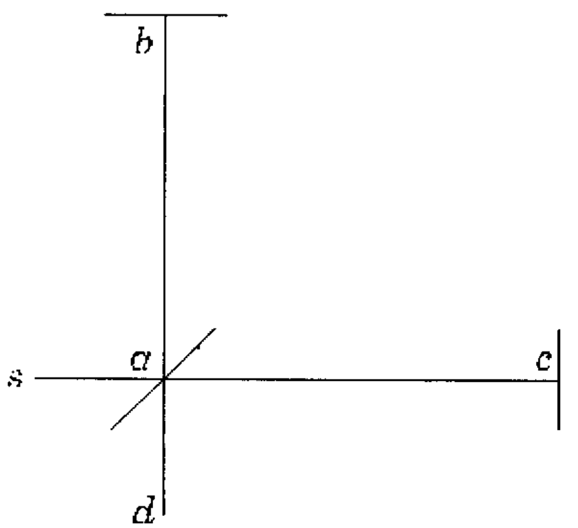


Figure 2 This diagram is from the original paper. The source of light is at s, the 45 degree line is the half-silvered mirror, b and c are mirrors and d the observer.

These two half-pulses are the two swimmers. They both go on to distant mirrors which reflect them back to the half-silvered mirror. At this point, they are again half reflected and half transmitted, but a telescope is placed behind the half-silvered mirror as shown in the figure so that half of each half-pulse will arrive in this telescope. Now, if there is an ether wind blowing, someone looking through the telescope should see the halves of the two half-pulses to arrive at slightly different times, since one would have gone more upstream and back, one more across stream in general. To maximise the effect, the whole apparatus, including the distant mirrors, was placed on a large turntable so it could be swung around.

The time delay one expects to find between the arrival of the two half-pulses of light can be calculated as follows. Taking the speed of light to be c km/s relative to the ether, and the ether to be flowing at v km/s through the laboratory, to go a distance D km upstream will take $D/(c-v)$ seconds, then to come back will take $D/(c+v)$ seconds. The total roundtrip time upstream and downstream is the sum of these, which works out to be $2Dc/(c^2-v^2)$, which can also be written as $(2D/c)/(1-v^2/c^2)$. Now, we can safely assume the speed of the ether is much less than the speed of light, otherwise it would have been noticed long ago, for example in timing of eclipses of Jupiter's satellites, this means v^2/c^2 is a very small number, and we can use some handy mathematical facts to make the algebra a bit easier.
First, if x is very small compared to 1, $1/(1-x)$ is very close to $1+x$. (You can check it with your calculator.) Another fact we shall need in a minute is that for small x, the square root of $1+x$ is very close to $1+x/2$. Anyway, the roundtrip upstream-downstream time can be taken, to an excellent approximation, to be $(2D/c)(1+v^2/c^2)$.

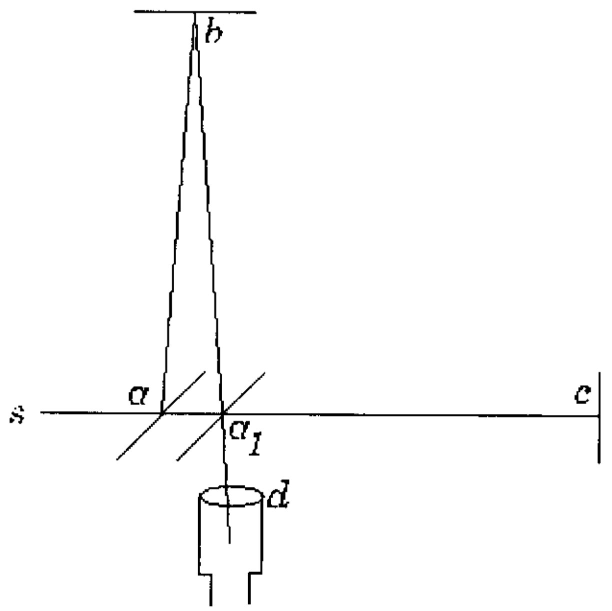


Figure 3 This is also from the original paper, and shows the expected path of light relative to the ether with an ether wind blowing.

Now what about the cross-stream time? The actual cross-stream speed must be figured out as in the example above using a right-angled triangle, with the hypotenuse equal to the speed c, the shortest side the ether flow speed v, and the other side the cross-stream speed we need to find the time to get across. From Pythagoras' theorem, then, the cross-stream speed is the square root of (c^2-v^2). Since this will be the same both ways, the roundtrip cross-stream time will be $2D/\mathrm{sqrt}(c^2-v^2)$. This can be written in the form $(2D/c)/\mathrm{sqrt}(1-v^2/c^2)$, which can be approximated as $(2D/c)\mathrm{sqrt}(1+v^2/c^2)$ or $(2D/c)(1+v^2/2c^2)$. The two roundtrip times thus will differ by an amount of:

$$(2D/c)\ (1+v^2/c^2)-(2D/c)(1+v^2/2c^2)=(2D/c)v^2/2c^2$$

Now, 2D/c is just the time the light would take if there were no ether wind at all, say, a few millionths of a second. If we take the ether windspeed to be equal to the earth's speed in orbit, for example, v/c is about 1/10,000, so v^2/c^2 is about 1/100,000,000. This means the time delay between the pulses reflected from the different mirrors reaching the telescope is about one-hundred-millionth of a few millionths of a second. It seems completely hopeless that such a short time delay could be detected. However, this turns out not to be the case, and Michelson was the first to figure out how to do it. The trick is to use the interference properties of the light waves. Instead of sending pulses of light, as we discussed above , Michelson sent in a steady beam of light of a single colour. This can be visualised as a sequence of in going waves, with a wavelength one hundred-thousandth of a cm or so. Now this sequence of waves is split into two, and reflected as previously described. One set of waves goes upstream and downstream, the other goes across stream and back. Finally, they come together into the telescope and the eye. If the one that took longer is half a wavelength behind, its troughs will be on top of the crests of the first wave, they will cancel, and nothing will be seen. If the delay is less than that, there will still be some dimming. However, slight errors in the placement of the mirrors would have the same effect. This is one reason why the apparatus is built to be rotated. On turning it through 90 degrees, the upstream-downstream and the cross-stream waves change places. Now the other one should be behind. Thus, if there is an ether wind, if you watch through the telescope while you rotate the turntable, you should expect to see variations in the brightness of the incoming light.

To magnify the time difference between the two paths, in the actual experiment the light was reflected backwards and forwards several times, like a several lap race. Michelson calculated that an ether wind speed of only one or two km a second would have observable effects in this experiment, so if the ether windspeed was comparable to the earth's speed in orbit around the sun, it would be easy to see. **In fact, nothing was observed**. The light intensity did not vary at all. Some time later, the experiment was redesigned so that an ether wind caused by the earth's daily rotation could be detected. Again, nothing was seen. Finally, Michelson wondered if the ether was somehow getting stuck to the earth, like the air in a below-decks cabin on a ship, so he redid the experiment on top of a high mountain in California. Again, no ether wind was observed. It was difficult to believe that the ether in the immediate vicinity of the earth was stuck to it and moving with it, because light rays from stars would deflect as they went from the moving faraway ether to the local stuck ether.

The only possible conclusion from this series of very difficult experiments was that the whole concept of an all-pervading ether was wrong from the start. Michelson was very reluctant to think along these lines. In fact, new theoretical insight into the nature of light had arisen in the 1860s from the brilliant theoretical work of Maxwell, who had written down a set of equations describing how electric and magnetic fields can give rise to each other. He had discovered that his equations predicted there could be waves made up of electric and magnetic fields, and the speed of these waves, deduced from experiments on how these fields link together, would be 186,300 miles per second. This is, of course, the speed of light, so it is natural to assume that light is made up of fast-varying electric and magnetic fields. But this leads to a big problem: Maxwell's equations predict a definite speed for light, and it is the speed found by measurements. But what is the speed to be measured relative to? The whole point of bringing in the ether was to give a picture for light resembling the one we understand for sound, compressional waves in a medium. The speed of sound through air is measured relative to air. If the wind is blowing towards you from the source of sound, you will hear the sound sooner. If there isn't an ether, though, this analogy doesn't hold up. So what does light travel at 299,792 km per second relative to?

There is another obvious possibility, which is called the emitter theory---the light travels at 299,792 km/s relative to the source of the light. The analogy here is between light emitted by a source and bullets emitted by a machine gun. The bullets come out at a definite speed (called the muzzle velocity) relative to the barrel of the gun. If the gun is mounted on the front of a tank, which is moving forward, and the gun is pointing forward, then relative to the ground the bullets are moving faster than they would if shot from a tank at rest. The simplest way to test the emitter theory of light, then, is to measure the speed of light emitted in the forward direction by a flashlight moving in the forward direction, and see if it exceeds the known speed of light by an amount equal to the speed of the flashlight. Actually, this kind of direct test of the emitter theory only became experimentally feasible in the nineteen-sixties. It is now possible to produce particles, called neutral pions, which decay each one in a little explosion, emitting a flash of light. It is also possible to have these pions moving forward at 296,740 km/s when they self destruct, and to catch the light emitted in the forward direction, and clock its speed. It is found that, despite the expected boost from being emitted by a very fast source, the light from the little explosions is going forward at the usual speed of 299,792 km/s . In the last century, the emitter theory was rejected because it was thought the appearance of certain astronomical phenomena, such as double stars, where two stars rotate around each other, would be affected. Those arguments have since been criticised, but the pion test is unambiguous. The definitive experiment was carried out by Alvager et al., Physics Letters 12, 260 (1964).

3.5 Lorenz Transformations

Already as early as 1887 Voigt working on Doppler shifts wrote down transformations that leave invariant the wave and Maxwell's equations. These transformations with a different scale factor are now known as Lorentz transformations although he wrote them down in 1899 :

Let us define:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Then the transformations between O and O′ read:

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - \frac{v}{c^2}x)$$

With these transformations the partial derivatives become:

$$\frac{\partial}{\partial x} = \gamma(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'})$$

$$\frac{\partial}{\partial t} = \gamma(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'})$$

Squaring them and substituting into the wave equation we arrive at the following expression after some simple rearranging of terms:

$$\gamma^2\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c}\frac{\partial^2}{\partial t^2} - \frac{v^2}{c^2}(\frac{\partial^2}{\partial x^2} - \frac{1}{c}\frac{\partial^2}{\partial t^2})\right]\psi = 0$$

Which readily becomes:

$$\gamma^2(1-\frac{v^2}{c^2})(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})\psi = 0$$

Remember that

$$\gamma^2(1-\frac{v^2}{c^2}) = 1$$

and thus the equation remains invariant.

Now how about Newton's laws? Are covariant under Lorentz transformations?

First let us see how velocities are transformed. Taking the differentials of x' and t' transformations we obtain:

$$dx' = \gamma(dx - vdt)$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

And dividing left and right parts one has:

$$\frac{dx'}{dt'} \equiv u'_x = \frac{u_x - v}{1-\frac{vu_x}{c^2}}$$

This is a funny way to "add" velocities but as we shall see it is the correct way! Remember that with the Galilean transformations the denominator was missing.

Differentiating the above expression in dt' and using the total derivative property that:

$$\frac{d}{dt'} = \frac{dt}{dt'}\frac{d}{dt}$$

One obtains the acceleration that has an equally complicated form:

$$\frac{du'_x}{dt'} = \frac{1}{\gamma^3(1-\frac{u_x v}{c^2})^3}\frac{du_x}{dt}$$

And when u=v :

$$\frac{du'_x}{dt'} = \gamma^3 \frac{du_x}{dt}$$

At first sight this transformation seems quite in conflict with Newton's laws of motion. If in the O' system the second Newton's law was F=d(mu')/dt' in the system O assuming m =constant, it will have a new form i.e. Newton's Law **As expressed above**, IS NOT INVARIANT.

However this turns out not to be the case for the following reason, that provides also the ingredients for the modification. Firstly we have from the definition of γ the following relation:

$$\frac{1}{\gamma^2} + \frac{u^2}{c^2} = 1$$

And differentiating this gives us:

$$-\frac{2}{\gamma^3}\frac{d\gamma}{dt} + \frac{2u}{c^2}\frac{du}{dt} = 0$$

As a result is follows that:

$$\frac{d}{dt}(\gamma u) = \gamma\frac{du}{dt} + \gamma^3\frac{u^2}{c^2}\frac{du}{dt} = \gamma^3\frac{du}{dt}$$

Since the final result here is exactly the quantity to be defined as the acceleration we can, by inserting the mass, rewrite Newton's second law in the form:

$$\frac{d}{dt}(\gamma mu) = F$$

A very interesting result as we shall see, but not understood at the time the Lorentz equations were written

3.6 Relativistic Effects

3.6.1 Fitzgerald Contraction using Formulae

Consider a rigid rod stationary in O' and lying along the X' axis. Let x'_1 and x'_2 the two ends so its length as measured in the O' system is $L'=x'_2-x'_1$.

At the instant t in O suppose that these ends occupy the positions x_1 and x_2 so that $\mathbf{L=x_2-x_1}$

Under Galilean transformations one can immediately see that the rod will appear as having the same length in both systems:

$$x_2' = x_2 - vt$$

$$x_1' = x_1 - vt$$

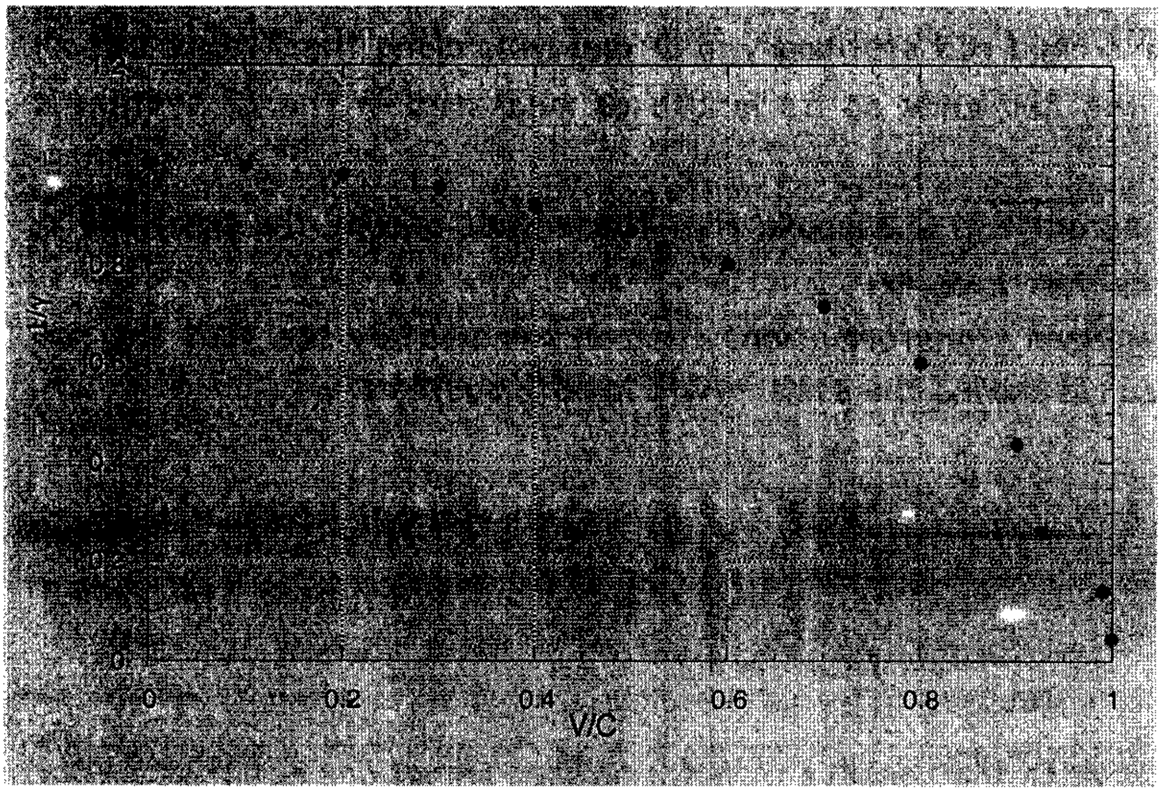
$$L' = L$$

However under the Lorentz transformations things are not that simple, applying the LT one finds that:

$$L = L' / \gamma$$

This seems to be funny since it shows that the length L' seen from O appears different from the length as seen at O'.

Now lets us see how $1/\gamma$ is behaving with V:

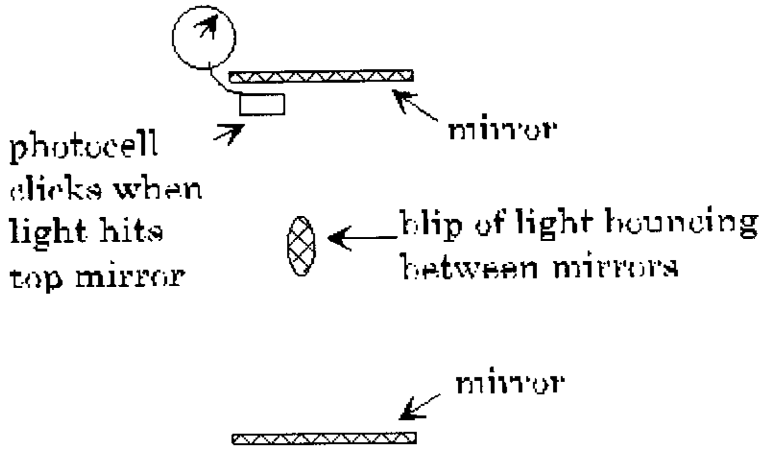


Therefore the length of the bar suffers contraction when it is moved! This is called the Fitzgerald contraction.

This contraction is not to be thought of as a physical reaction of the rod to its motion and as belonging to the same category of physical effects as the contraction of a metal rod when it is cooled. It is due to a changed relationship between the rod and the instruments measuring its length as we shall shortly see.

3.6.2 Relativistic clocks and lengths

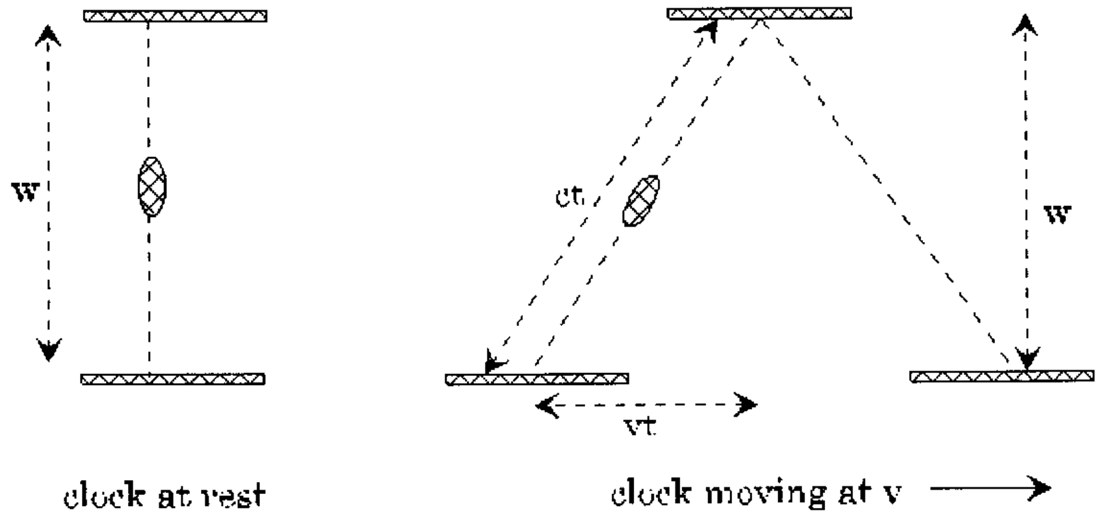
In order to define time and simultaneity one has to define a simple but reliable clock. Imagine that our (inertial) frames of reference is calibrated (had marks at regular intervals along the walls) to measure distances, and has a clock to measure time, that is easy to understand in any frame of reference. Instead of a pendulum swinging back and forth, which wouldn't work away from the earth's surface anyway, we have a blip of light bouncing back and forth between two mirrors facing each other. We call this device a light clock. To really use it as a timing device we need some way to count the bounces, so we position a photocell at the upper mirror, so that it catches the edge of the blip of light. The photocell clicks when the light hits it, and this regular series of clicks drives the clock hand around, just as for an ordinary clock. We ignore the technical difficulties such as: the driving of the photocell will eventually use up the blip of light , the need of some provision to reinforce the blip occasionally, such as a strobe light set to flash just as it passes and thus add to the intensity of the light etc. Admittedly, this may not be an easy way to build a clock, but the basic idea is simple.



It's easy to figure out how frequently our light clock clicks. If the two mirrors are a distance D apart, the round trip distance for the blip from the photocell mirror to the other mirror and back is 2D. Since we know the blip always travels at c, we find the round trip time to be 2D/c, so this is the time between clicks. This isn't a very long time for a reasonable sized clock! The crystal in a quartz watch "clicks " of the order of 10,000 times a second. That would correspond to mirrors about 15 km apart, so we need our clock to click about 1,000 times faster than that to get to a reasonable size. Anyway, let us assume that such purely technical problems have been solved.

3.6.3 Looking at Somebody Else's Clock

Let us now consider two observers, A and B, each equipped with a calibrated inertial frame of reference, and a light clock. To be specific, imagine A standing on the ground with his light clock next to a straight railroad line, while B and his clock are on a large flatbed railroad wagon which is moving down the track at a constant speed v. A now decides to check B's light clock against his own. He knows the time for his clock is 2D/c between clicks. How long does he think that B's blip takes to make a round trip? The one thing he's sure of is that it must be moving at c, relative to him according to the second postulate of relativity. So to find the round trip time, all he needs is the round trip distance. This will not be 2D, because the mirrors are on the flatbed wagon moving down the track, so, relative to A on the ground, when the blip gets back to the top mirror, that mirror has moved down the track some since the blip left, so the blip actually follows a zigzag path as seen from the ground.



Suppose now the blip in B's clock on the moving flatbed wagon takes time t to get from the bottom mirror to the top mirror as measured by A standing by the track. Then the length of the "zig" from the bottom mirror to the top mirror is necessarily ct, since that is the distance covered by any blip of light in time t. Meanwhile, the wagon has moved down the track a distance vt, where v is the speed of the wagon. This should begin to look familiar---it is precisely the same as the problem of the swimmer who swims at speed c relative to the water crossing a river flowing at v! We have again a right-angled triangle with hypotenuse ct, and shorter sides vt and D.

From Pythagoras theorem , then,

$$c^2t^2 = v^2t^2 + D^2$$

so

$$t^2(1-\frac{v^2}{c^2}) = D^2 / c^2$$

and, taking the square root of each side, then doubling to get the round trip time, we conclude that A sees the time between clicks for B's clock to be:

$$t = \frac{2D}{c}\gamma$$

This means that A sees B's light clock to be going slow---a longer time between clicks---compared to his own identical clock. Obviously, the effect is not dramatic at everyday speeds. Nevertheless, the effect is real and can be measured, as we shall discuss later.

It is important to realise that the only reason we chose a light clock, as opposed to some other kind of clock, is that its motion is very easy to analyse from a different frame. The observer B could have a collection of clocks on the wagon, and would synchronise them all. For example, he could hang his wristwatch right next to the face of the light clock, and observe them together to be sure they always showed the same time. Remember, in his frame his light clock clicks every 2D/c seconds, as it is designed to do. Observing this scene from his position beside the track, the observer A will see the synchronised light clock and wristwatch next to each other, and, of course, note that the wristwatch is also running slow by the factor γ. In fact, all clocks are slowed down by this factor according to the observer A. The observer B is ageing more slowly because he's moving!

But this isn't the whole story -- we must now turn everything around and look at it from B's point of view. His inertial frame of reference is just as good as A's. He sees the A's light clock to be moving at speed v (backwards) so from her point of view his light blip takes the longer zigzag path, which means his clock runs slower than B's. That is to say, each of them will see the other to have slower clocks, and be ageing more slowly. This phenomenon is called time dilation. It has been verified in recent years by flying very accurate clocks around the world on jetliners and finding they register less time, by the predicted amount, than identical clocks left on the ground. Time dilation is very easy to observe in elementary particle physics, as we shall discuss in the next section.

3.6.4 Fitzgerald Contraction using reasoning

Consider now the following puzzle: suppose B's clock is equipped with a device that stamps a notch on the track once a second. How far apart are the notches? From B's point of view, this is pretty easy to answer. He sees the track passing under the wagon at v meters per second, so the notches will of course be v meters apart. But A sees things differently. He sees B's clocks to be running slow, so he will see the notches to be stamped on the track at intervals of γ x seconds (so for a relativistic train going at $v = 0.8c$, the notches are stamped at intervals of $5/3 = 1.67$ seconds). Since A agrees with B that the relative speed of the wagon and the track is v, he will assert the notches are not v meters apart, but $v \times \gamma$ meters apart, a greater distance. Who is right? It turns out that A is right, because the notches are in his frame of reference, so he can go over to them with a tape measure and check the distance. This implies that as a result of the motion of the B observer, B observes the notches to be closer together by a factor $1/\gamma$ than they would be at rest. This is called the Fitzgerald contraction, and applies not just to the notches, everything looks somewhat squashed in the direction of motion!

3.6.5 Experimental Evidence for Time Dilation: Dying Muons

The first clear example of time dilation was provided over fifty years ago by an experiment detecting muons. These particles are produced at the outer edge of our atmosphere by incoming cosmic rays hitting the first traces of air. They are unstable particles, with a "half-life" of 1.5 microseconds, which means that if at a given time you have 100 of them, 1.5 microseconds later you will have about 50, 1.5 microseconds after that 25, and so on. Anyway, they are constantly being produced many km up, and there is a constant rain of them towards the surface of the earth, moving at very close to the speed of light. In 1941, a detector placed near the top of Mount Washington (at 2000 meters above sea level) measured about 570 muons per hour coming in. Now these muons are raining down from above, but dying as they fall, so if we move the detector to a lower altitude we expect it to detect fewer muons because a fraction of those that came down past the 2 km level will die before they get to a lower altitude detector. Approximating their speed by that of light, they are raining down at c, which turns out to be, conveniently, about 300 meters per microsecond. Thus they should reach the 1550 meter level 1.5 microseconds after passing the 2 km level, so, if half of them die off in 1.5 microseconds, as claimed above, we should only expect to register about 570/2 = 285 per hour with the same detector at this level. At the 1km level, about 280/2 = 140 per hour, are expected, at 550 meters about 70 per hour, and at ground level about 35 per hour (approximately).

To summarise: given the known rate at which these raining-down unstable muons decay, and given that 570 per hour hit a detector near the top of Mount Washington, we only expect about 35 per hour to survive down to sea level. In fact, when the detector was brought down to sea level, it detected about 400 per hour! How did they survive? The reason they didn't decay is that as observed from us in their frame of reference, much less time had passed. Their actual speed is about 0.994c, corresponding to a time dilation factor of about 9, so in the 6 microsecond trip from the top of Mount Washington to sea level, their clocks register only 6/9 = 0.67 microseconds. In this period of time, only about one-quarter of them decay.

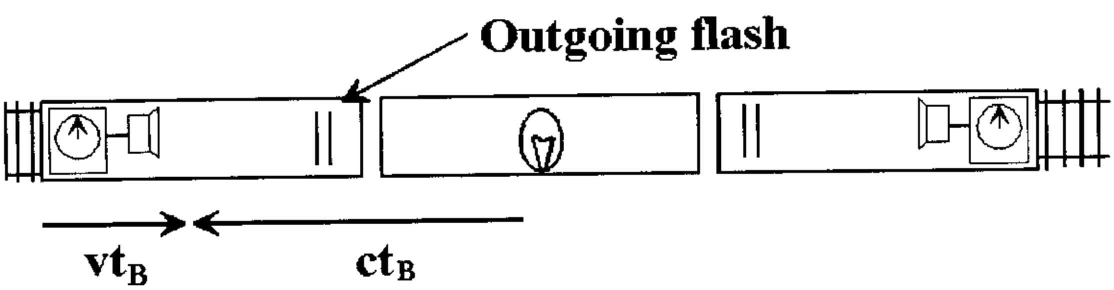
What does this look like from the muon's point of view? How do they manage to get so far in so little time? To them, Mount Washington and the earth's surface are approaching at 0.994c, or about 298 meters per microsecond. But in the 0.67 seconds it takes them to get to sea level, it would seem that to them sea level could only get 204 meters closer, so how could they travel the whole 2 km from the top of Mount Washington? The answer is the Fitzgerald contraction---to them Mount Washington is squashed in a vertical direction (the direction of motion) by a factor of γ, the same as the time dilation factor, which for the muons is 9. So, to the muons, Mount Washington is only 204 meter high---this is why they can get down it so fast!

3.6.6 Simultaneity

Suppose we want to synchronise two clocks that are some distance apart. We could stand beside one of them and look at the other through a telescope, but we'd have to remember in that case that we are seeing the clock as it was when the light left it, and correct accordingly. Another way to be sure the clocks are synchronised, assuming they are both accurate, is to start them together. How can we do that? We could, for example, attach a photocell to each clock, so when a flash of light reaches the clock, it begins running. If, then, we place a flashbulb at the midpoint of the line joining the two clocks, and flash it, the light flash will take the same time to reach the two clocks, so they will start at the same time, and therefore be synchronised.

Let us now put this whole arrangement - the two clocks and the midpoint flashbulb - on a train, and we suppose the train is moving at some speed v to the right. Let's examine carefully at the clock-synchronising operation as seen from the ground. In fact, an observer on the ground would say the clocks are not synchronised by this operation! The basic reason is that he would see the flash of light from the middle of the train travelling at c relative to the ground in each direction, but he would also observe the back of the train coming at v to meet the flash, whereas the front is moving at v away from the bulb, so the light flash must go further to catch up. In fact, it is not difficult to figure out how much later the flash reaches the front of the train compared with the back of the train, as viewed from the ground. First recall that as viewed from the ground the train has length,

$$L/\gamma$$



Letting t_B be the time it takes the flash to reach the back of the train, it is clear from the figure that

$$vt_B + ct_B = \frac{L}{2\ \gamma}$$

In a similar way, the time for the flash of light to reach the front of the train is (as measured by a ground observer)

$$vt_F - ct_F = \frac{L}{2\ \gamma}$$

Therefore the time difference between the starting of the two clocks, as seen from the ground, is:

$$t_F - t_B = (\frac{1}{c-v} - \frac{1}{c+v})\frac{L}{2\ \gamma} = \gamma\frac{v}{c^2}L$$

Remember, this is the time difference between the starting of the train's back clock and its front clock as measured by an observer on the ground with clocks on the ground. However, to this observer the clocks on the train appear to tick more slowly, by the factor $1/\gamma$, so that although the ground observer measures the time interval between the starting of the clock at the back of the train and the clock at the front as shown above, he also sees the slow running clock at the back actually reading vL/c^2 seconds at the instant he sees the front clock to start.

To summarise: as seen from the ground, the two clocks on the train (which is moving at v in the x-direction) are running slowly, registering only $1/\gamma$, seconds for each second that passes. Equally important, the clocks-which are synchronised by an observer on the train-appear unsynchronised when viewed from the ground, the one at the back of the train reading vL/c^2 seconds ahead of the clock at the front of the train, where L is the rest length of the train (the length as
measured by an observer on the train).

Note that if L = 0, that is, if the clocks are together, both the observers on the train and those on the ground will agree that they are synchronised. We need a distance between the clocks, as well as relative motion, to get a disagreement about synchronisation.

SPACE AND TIME INFLUENCE
EACH OTHER AND ARE REALLY LOCAL !

3.7 Einstein's derivation

As we have seen Einstein found all that with good reasoning and needed not to know about the Lorentz transformations. Got it by understanding that each system has its own time. Now how one can get from first principles those transformations? How one can reformulate mechanics? This is a vast subject but I will try and give you some flair. The only assumptions made are the two postulates of relativity and the homogeneity and locality of space and time.

Let $O(t,x,y,z)$ a co-ordinate system and $O'(\tau,\xi,\eta,\zeta)$ moving with velocity v relatively to O. If we place $x'=x-vt$, it is clear that a point at rest in O' must depend from x',y,z and must be independent of time τ. Thus as a first step we determine τ as a function of x',y,z, and t.

From the origin of O' let a ray be emitted at the time τ_0 along the X-axis to x' and at the time τ_1 be reflected thence to the origin O' arriving at τ_2 . Obviously

$$\tau_1=(\tau_2+\tau_0)/2$$

Since light has constant velocity also in O we must have $x'=ct-vt$ or $x'=ct+vt$ i.e. $t=x'/(c-v)$ or $t=x'/(c+v)$ depending from the direction of the ray relative to the origin O. Assuming τ to be function of the O co-ordinates then we have:

$$\frac{1}{2}\left[\tau(0,0,0,t)+\tau(0,0,0,t+\frac{x'}{c-v}+\frac{x'}{c+v})\right]=\tau(x',0,0,t+\frac{x'}{c-v})$$

Hence if x' is chosen infinitesimally small and expanding around $\tau(0,0,0,t)$ we obtain:

$$\frac{1}{2}(\frac{1}{c-v}+\frac{1}{c+v})\frac{\partial\tau}{\partial t}=\frac{\partial\tau}{\partial x'}+\frac{\partial\tau}{\partial t}\frac{1}{c-v}$$

or

$$\frac{\partial\tau}{\partial x'}+\frac{v}{c^2-v^2}\frac{\partial\tau}{\partial t}=0$$

Since τ should be a linear function (homogeneity) the solution of the above differential equation is:

$$\tau=a(t-\frac{v}{c^2-v^2}x')$$

Where a=a(v) a still unknown function. Substituting the x' value from x'=x-vt in the above equation and observing that $1+\gamma^2(v^2/c^2)=\gamma^2$ we have:

$$\tau=a(v)\gamma^2(t-\frac{v}{c^2}x)$$

defining $\phi(v)=a(v)\ \gamma$ the transformation is identical to Lorentz apart the function $\phi(v)$.

To obtain the spatial transformation assume a light ray emitted at $\tau=0$ to the direction of increasing ξ so we have $\xi = c\ \tau$ and from $x'=x-vt$ by substituting to the equation for τ expressed in t and x' :

$$\xi = a(v)\gamma^2 x' = a(v)\gamma^2 (x - vt)$$

Whereas the perpendicular co-ordinates simply are

$$\eta = \phi(v)y \ , \ \zeta = \phi(v)z$$

The function $\phi(v)=a(v)\ \gamma$ has still to be found. To found it consider a third inertial system O" that moves with velocity -v in respect of O'. Then by twofold application of the transformations one has:

$$x'' = \phi(-v)\gamma(-v)(\xi + vt) = \phi(v)\phi(-v)x$$

However since O and O" are at rest should $x=x''$ therefore $\phi(v)\ \phi(-v)=1$ i.e. identical transformation. On the other hand from symmetry the length of a perpendicular rod should not depend upon the direction of motion, thus

$$\frac{y}{\phi(v)} = \frac{y}{\phi(-v)}$$

or $\phi(v)= \phi(-v)$ and together with $\phi(v)\ \phi(-v)=1 \Rightarrow \phi(v)= 1$

Thus the Lorentz transformations have been obtained from the postulates of relativity using first principles.

3.8 Space-time

From all the above I hope it is clear now that what was needing reformulation was not electrodynamics but mechanics. Furthermore the very concept of time (and space) had to change since their relative measurement depend upon motion (e.g. simultaneity for one observer does not apply to another moving with velocity v). Thus the old way of mapping the position of a particle by using spatial co-ordinate systems and time as a universal parameter had to change. Since then lengths and times are not invariant what can be constructed that is invariant under the new theory? The constancy of the velocity of light will give the clue:

Assume that at time $t=\tau=0$, when the origin of O and O' co-inside, a light flash is emitted (spherical wave) propagated with velocity c . If (x,y,z) be a point in O just attained by the wave then:

$$x^2 + y^2 + y^2 = c^2 t^2$$

being the equation of a sphere. Performing the Lorentz transformations for the O' system it is easy to find that:

$$\xi^2 + \eta^2 + \zeta^2 = c^2 \tau^2$$

which is a spherical wave too with the same propagation velocity c when viewed from O'. Thus the above quantity is invariant!

A mathematical device due to H. Minkowski has now to be employed. A 4-vector is defined that has 3 space and 1 time components. Since all components should have the same units, the time component should be ct, in fact we replace the time co-ordinate t by a purely imaginary co-ordinate

x^0=ict. Thus a point in this system has co-ordinates x,y,z with respect to a rectangular Cartesian set of space axis and time t. In fact this has to be interpreted as a rectangular Cartesian four-dimensional Euclidean space (t is now a dimension not a parameter) called **Minskowski space-time**. A point P in this space-time is called an "**event**":

$$x^0 = ict \ , \ x^1 = x \ , \ x^2 = y \ , \ x^3 = z$$

The interval between 2 events P1 and P2 is defined to be:

$$\Delta s^2 = \left(\Delta x^0\right)^2 - \left(\Delta x^1\right)^2 - \left(\Delta x^2\right)^2 - \left(\Delta x^3\right)^2$$

and is easily shown to be invariant under Lorentz transformations. If a clock travels from P1 to P2 then the time interval it measures between these two events is called **proper time** $\Delta\tau$ with $c\Delta\tau=\Delta s$. As can be easily shown, for a light ray travelling from P1 to P2, $\Delta s^2=0$ and the interval is called **light-like** i.e this interval can be reached only by light. If $\Delta s^2> 0$ the interval is called **time-like** and can be reached by travelling from P1 to P2 with velocities less than that of light. Why we call it so? It has to do with the proper time. Suppose that a point P is in motion in O defining the events P1 and P2. Since $c\Delta\tau=\Delta s$ one can in general write:

$$d\tau = \sqrt{(dt^2 - \frac{1}{c^2} ds^2)} = \sqrt{(1 - \frac{v^2}{c^2})} \ dt$$

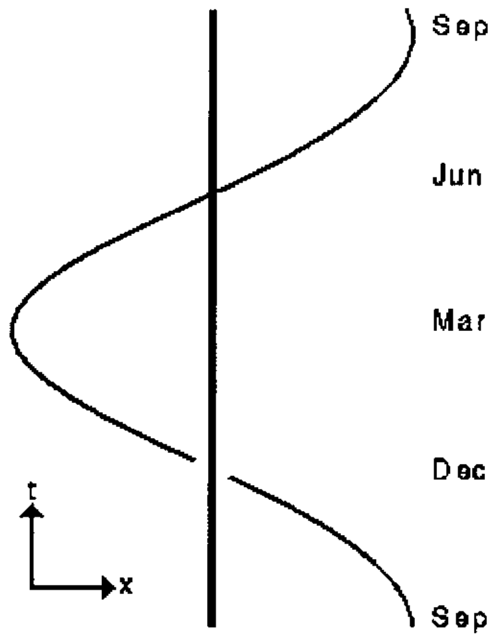
From the above relation it is obvious that if t is the time of clocks stationary at O, the proper time is the time shown by a clock moving along (as also defined) and its rate is slowed down by $1/\gamma$. Lets define a new inertial frame O' moving along the line connecting those two events with v=ds/dt <c. Relative to this frame the two events will occur at the same

point and therefore $d\tau^2=dt^2$. Clearly in this case $\tau^2>0$. That is why it is called time-like. Finally for $\Delta s^2< 0$ the interval is called **space-like** the proper time is negative $\tau^2<0$ and the events P1 and P2 are having no "causal" connection i.e. no information can pass from P1 to P2 because then it should travel faster than light, since $ds/dt > c$. In this case whatever happens to P1 can not have influence on P2 and vice versa.

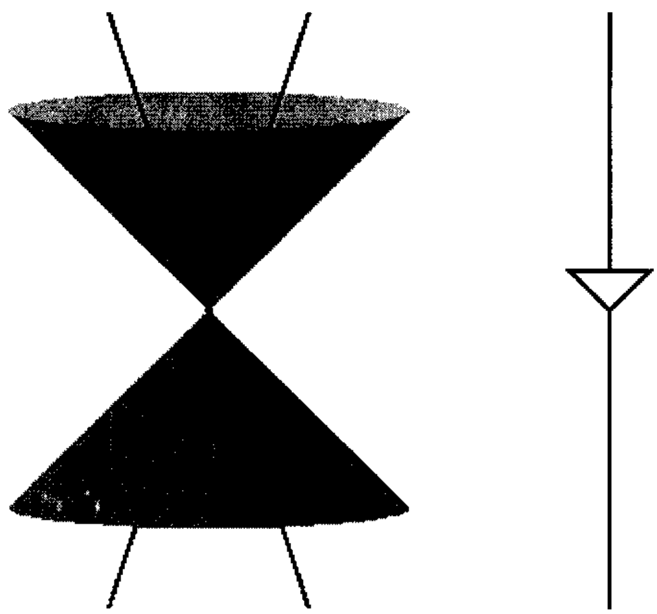
A good way to keep track of these concepts is the space-time diagram. A space-time diagram is nothing more than a graph showing the position of objects as a function of time. The usual convention is that time runs up the diagram, so the bottom is the past, or early times, and the top is the future, or late times. An event, on this graph describes both a position (the horizontal or x coordinate) and a time (the vertical or t coordinate).

The picture below shows a space-time diagram of the Earth going around the Sun. This figure uses perspective to try to show two spatial dimensions and the time axis on a two dimensional sheet of paper, but usually we will just show one spatial axis and avoid perspective.

The line representing the position of the Earth as a function of time is called a **worldline**. The slope of the worldline for a particle shows its velocity in the reference frame of the diagram. Because the speed of light is special in relativity, space-time diagrams are often drawn in units of seconds and light-seconds, or years and light-years, so a unit slope [45 degree angle] corresponds to the speed of light.



The set of all light speed world lines going through an event defines the light cones of that event: the past light cone and the future light cone. An example of light cones is shown above.



3.9 4-vectors

We have seen how the space-time 4-vector is defined. Without going into details one can re-formulate all mechanics by defining invariant (i.e. transform under Lorentz transformations) 4-vectors accordingly. Thus another 4-vector, the 4-velocity can be defined:

$$v^0 = ic\gamma \ , \ v^1 = \gamma v_x \ , \ v^2 = \gamma v_y \ , \ v^3 = \gamma v_z$$

Where γ here contains the amplitude of the velocities and the v_x etc. are the standard components of the velocity meaning that in time dt the distance travelled along the x direction will be v_x dt. Note that the Lorentz transformation takes the same form for all four vectors, therefore the square of each such 4-vector is invariant. The invariant of the velocity 4-vector is c^2. To see that assume that we have motion only in the direction of x axis and take the sum of the squares of the components:

$$V^2 = c^2\gamma^2 - \gamma^2 v^2{}_x = \gamma^2(c^2 - v^2{}_x) = c^2 = const$$

Next 4-vector to be defined is the 4-momentum. The key role here plays the momentum conservation. Demanding Σ**MV** to be conserved , from the definition of the 4-velocity we have Σ**MV** = ΣM γ **(v,ic)** . We know from the third law of Newton that Σmv =conserved so to have momentum conservation must **M** γ = **m** a very important relation indeed. This relation shows that in the new mechanics the mass is not a constant but depends on the relative velocity! With this assumption we have re-established the conservation of momentum. Then from the 0^{th} component of the relation we have that ΣM γ **ic** = Σm = constant i.e. conservation of mass.

Thus in the new mechanics the conservation of the 4-momentum gives both the conservation of momentum and mass of Newtonian mechanics. Now for v=0 => M=m therefore M is the of a particle when measured in an inertial frame in which it is stationary. M will be referred as the **rest mass or proper mass** and is usually denoted by **m_0** then m= γ **m_0** a very important relation that shows that the mass (inertia) increases with the relative velocity. Thus the definition of the 4-momentum **P** = **m_0 V** Since **m_0** is an invariant and V is a vector P is a vector too, with components:

$$P^0 = imc \;,\; P^1 = p_x \;,\; P^2 = p_y \;,\; P^3 = p_z$$

p is the classical momentum. The invariant for the 4-momentum is the rest mass:

$$\left(P^0\right)^2 - \left(\vec{P}\right)^2 = m^2c^2 - p^2 = m_0^{\ 2}\gamma^2(c^2 - v^2) = m_0^{\ 2}c^2$$

The Newton's second law can now be written as:

$$F = \frac{dP}{d\tau} = m_0 \frac{dV}{d\tau}$$

Where we note that the differentiation is with respect to the proper time which is a Lorentz scalar (scalar that transforms according to LT).
From the relation:

$$F = \frac{d}{d\tau}(imc, p) = \gamma(i\frac{dm}{dt}c, f)$$

One can define the 4-force components as:

$$F^0 = \gamma c \frac{dm}{dt} \;,\;\; F^i = \gamma f_i$$

where i=1,2,3 and fi (with i=x,y,z) the components of the classical three dimensional force.

The fact that the vectors V and F are orthogonal ($V^2=c^2$ and differentiating with respect to proper time one has $VdV/d\tau$ = VF=0) is having very important consequences: Substituting for V and F we obtain that:

$$vf - c^2\dot{m} = 0$$

But by definition vf is the rate at which f is doing work. It follows that the work done by the force acting on the particle during a time interval is:

$$\int_{t_1}^{t_2} vfdt = c^2 \int_{t_1}^{t_2} \dot{m}dt = m_2c^2 - m_1c^2$$

The classical equation of work is work done = increase in kinetic energy where $T=mv^2/2$ is the classical kinetic energy. However the above formula suggests that in SR we have to define the kinetic energy as:

$T=m\ c^2$+constant and for v=0 since T=0, the constant is $-m_0\ c^2$

Thus $T= m_0\,\gamma\, c^2 - m_0\, c^2$ and if v/c is small the factor γ (remember algebraic rules) approximates as $1 + v^2/2\ c^2$ and then $T=mv^2/2$, i.e. the classical mechanics expression. Thus the classical mechanics can be considered the low velocity limit of the relativistic mechanics.

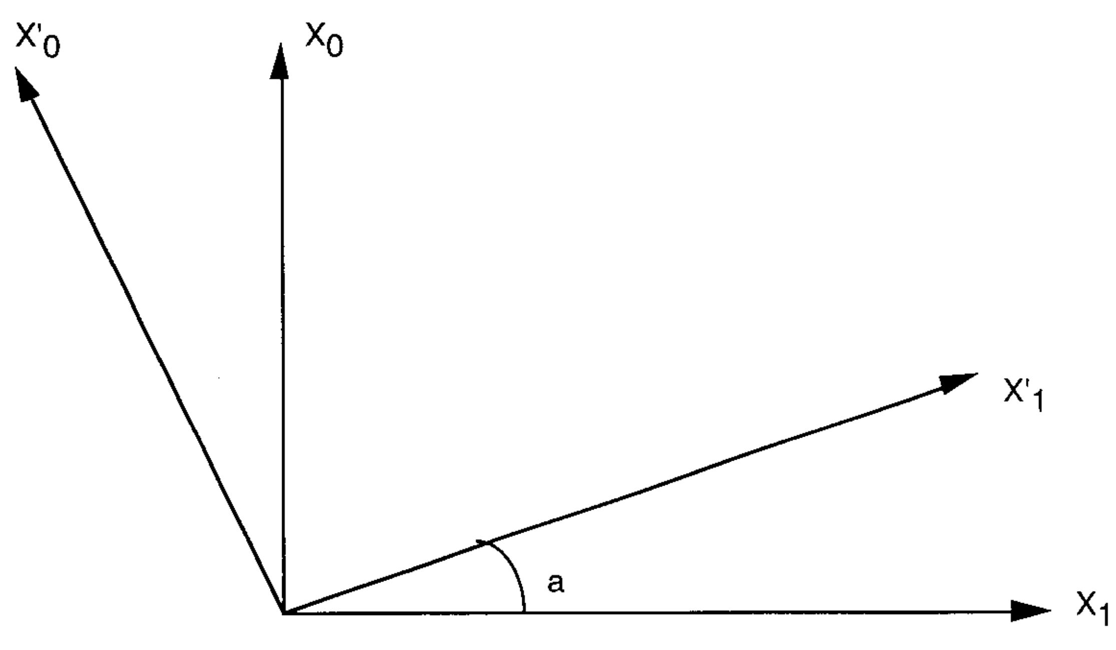
The equation $T = m_0 \gamma c^2 - m_0 c^2$ shows that any increase of kinetic energy increases the mass of the particle. However other forms of energy, like heat, can also be considered as an increase of the kinetic energy of the atoms. In general one can say that an increase of the energy increases the mass. The distinction between mass and energy that existed in classical mechanics does not exist in the new relativistic mechanics. All forms of energy E like mechanical ,thermal, electromagnetic are now taken to posses inertia of mass m according to Einstein's general equation:

$$\boxed{E = mc^2}$$

showing also that the only difference between mass and energy stays in conversion of units. This equation of Einstein is maybe the most known and famous, having the most dramatic confirmation of all times.

3.9.1 The special Lorentz transformations

Imagine now that the co-ordinate system O' rotates through an angle a parallel to the x_0x_1 plane,



The origin and axes x2, x3 unaffected by the rotation for simplicity. From the above figure clearly we have:

$$x_0' = -x_1 \sin a + x_0 \cos a$$
$$x_1' = \quad x_1 \cos a + x_0 \sin a$$
$$x_2' = \quad x_2$$
$$x_3' = \quad x_3$$

(remember that x_0=ict) Writing the above in 4-vector form we have that V'=AV where A is a 4X4 matrix:

$$A = \begin{bmatrix} \cos a & -\sin a & 0 & 0 \\ \sin a & \cos a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To interpret the above equation consider a stationary plane relative to O' with equation **bx'+cy'+dz'+e=0** for all t'. Its equation in the O system will then be: **(bcosa)x+cy+dz+e+ictbsina=0**. If c=d=e=0 the plain is the O'y'z' and we obtain: **x=-ict tana** i.e it is a plane parallel to Oyz displaced a distance -icttana along Ox. So if v is the speed of translation we have that

$$v = ic\tan a$$

This equation indicates that the angle a is imaginary and it is directly related to the speed of translation. We have tana=iu/c and hence:
(from the well known formulae cos=1/sqrt(1+tan^2) and sin=tan/sqrt(1+ tan^2))

$$\cos a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \;\; , \;\; \sin a = \frac{\frac{iv}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \frac{iv}{c}$$

Thus the matrix *A* becomes:

$$A = \begin{bmatrix} \gamma & -i\beta\gamma & 0 & 0 \\ i\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has been obtained just by rotating co-ordinate systems in the space-time and it is nothing else than the LT in a matrix form. Any 4-vector should be transformed according to V'=AV (in this case *A* is expressed in a simple form since the y,z transforms are not included)

3.9.2 The Relativistic Doppler Shift

This is a particular example of the Doppler Effect, first discussed in 1842 by the German physicist Christian Doppler for sound waves. Sound is generated by a vibrating object sending a succession of pressure pulses through the air. These pressure waves are analogous to the flashes of light. If you are approaching a sound source you will encounter the pressure waves more frequently than if you stand still. This means you will hear a higher frequency sound. If the distance between you and the source of sound is increasing, you will hear a lower frequency. This is why the note of a jet plane or a siren goes lower as it passes you. The details of the Doppler Effect for sound are a little different than those for light, because the speed of sound is not the same for all observers - it's 330 meters per second relative to the air.

The phase of a plane wave is an invariant quantity, the same in all coordinate frames. This is because the elapsed phase of a wave is proportional to the number of the wave crests that have passed the observer. Since this is a merely a counting operation it must be independent of coordinate frame. The phase thus can be written:

$$\phi = \omega t - kx = \omega' t' - k' x'$$

Where ω is the frequency and k the wave vector. The wave vector k can be generalised to a 4-vector with $\omega = ck_0$ and **k** the wave number, thus must transform as k'=Ak. (having established that k is a four vector $(ik_0, \mathbf{k})$ the invariance of phase comes naturally out since is the invariant scalar of 4-vector products) The transformation is:

$$k_0' = \gamma(k_0 - \vec{\beta}\vec{k})$$

$$k_{||}' = \gamma(k_{||} - \beta k_0)$$

$$k'_{\perp} = k_{\perp}$$

For light waves **k**=k_0 **n** with **n** the unit vector; thus from the first equation we have:

$$\omega' = \gamma\omega(1 - \beta\cos\theta)$$

where θ is the angle between the wave vector and the direction of velocity. This equation is the customary Doppler shift modified by the γ factor. However its presence means more than a "just" relativistic correction, it shows the existence of a transverse Doppler shift. When $\theta = \pi/2$ we have $\omega' = \gamma\,\omega$. This relativistic transverse Doppler shift has been observed spectroscopically with atoms in motion.(Ives-Stilwell experiment 1938)

An important astronomical application of the Doppler Effect is the red shift. The light from very distant galaxies is redder than the light from similar galaxies nearer to us. This is because the further away a galaxy is, the faster it is moving away from us, as the Universe expands. The light is redder because red light is low frequency light (blue is high) and we see low since galaxies are moving away from us.

4 Elements of covariant formulation in Electrodynamics

4.1 Current density

As we have said at the very beginning Maxwell's equations are invariant under LT. Since the subject of Electromagnetism is been covered by Dr. R. P. Walker I will only pin-point some formulation techniques.

Relative to an inertial frame O let ρ be the charge density and v its flow of velocity. Then, if j is the current density we have: $j = \rho\, v$ Since the charge can neither be destroyed or created, the equation of continuity

$$\nabla j + \frac{\partial \rho}{\partial t} = 0$$

will be valid for the charge flow in O. This equation must be valid in every inertial frame and hence must be expressible in a form which is covariant in space-time. Introducing the space-time co-ordinates x^i the continuity equation becomes:

$$\frac{\partial}{\partial x^0}(ic\rho) + \frac{\partial}{\partial x^i}(v_i \rho) = 0$$

where i=1,2,3 <-> x,y,z and repeated indexes are summed. This equation is covariant as required if $(ic\,\rho,\ \rho v_x,\ \rho v_y,\ \rho v_z)$ are the four components of a vector in space time. For, if **J** is this vector we have

$$\frac{\partial}{\partial x^i} J_i \equiv J_{i,i} = 0$$

with i=0,1,2,3 and this is covariant. Now since $\mathbf{J} = (ic\rho\ ,\ \rho\mathbf{v})$ from the definition of 4-velocity **V** we have that $\mathbf{J} = \rho\,\gamma\mathbf{V}$ and thus **J** is an invariant vector if $\rho\,\gamma$ is an invariant. Denoting the invariant by ρ_0 we have $\rho_0=\rho\,\gamma$. It follows that for v=0 $\rho_0=\rho$ and ρ_0 is the called **proper charge density** and J is called the **4-current density**, $J=\rho_0 V = (ic\rho\ ,\ j)$

4.2 Electric charge invariance

Let $d\omega_0$ be the volume of of a small element of charge as measured in the inertial frame O. The total charge $Q_0= \rho_0\ d\omega_0$. Due to the Fitzgerald contraction the volume measured in the system O' will be $d\omega = \gamma\ d\omega_0$ and therefore $Q= d\omega\ \rho = \gamma\ d\omega_0\rho = \rho_0\ d\omega_0 = Q_0$ q.e.d

One working further along these lines can define a 4-vector potential that has as components the scalar and vector potentials (ϕ,A) [see lecture by Dr. R. P. Walker] continue to define the electromagnetic field tensors etc. all respecting SR and LT.

4.3 Electric field of moving point charge

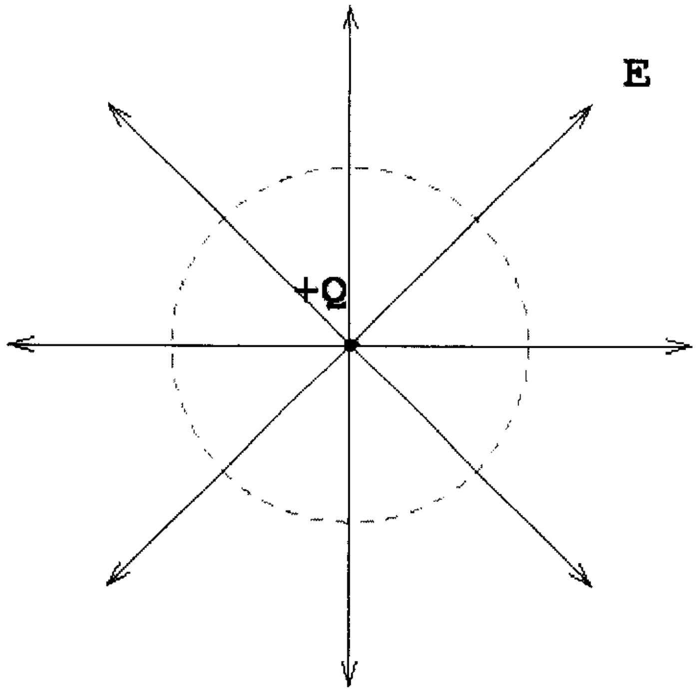
Consider the electric field about a point charge. The full vector field is

$$E = \kappa_c \frac{q}{r^2}\vec{e}_r \ ,\ \vec{e}_r = \frac{\vec{r}}{r}$$

($k_c = k_\varepsilon\ /2\pi$ and $k_\varepsilon = 4\ \pi$ in Gaussian (cgs) units or $1/\varepsilon$ in SI units, where ε is the electric permitivity). For simplicity consider just a field in the xy plane. Thus r=(x,y,0) and the filed components are:

$$E_{x(y)} = \kappa_c q\ \frac{x(y)}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

for the x (or y) field component.



Now consider a frame O' moving with velocity v in the positive x direction. The charge is now at rest at this system at the position O' and we would like to see how the field is seen by an observer located in the frame O with co-ordinates (0,b,0,t). In the frame O' the observer's point P (where the field is to be evaluated) has co-ordinates x'= -vt',y'=b,z'=0 and is at distance r'=SQRT(b^2+$(vt')^2$). The only co-ordinate needing transformation is t'=γ(t-(v/c^2) x)= γ t . Since O' is the rest frame for the charge the fields are:

$$E'_x = \kappa_c q \frac{\gamma v t}{\left(b^2 + (\gamma v t)^2\right)^{\frac{3}{2}}} \ , \ E'_y = \kappa_c q \frac{b}{\left(b^2 + (\gamma v t)^2\right)^{\frac{3}{2}}}$$

The LT transformations for the electric field components, are:

$$E_x = E'_x \ , \ E_y = \gamma E'_y$$

Thus:

$$E_x = \kappa_c q \ \frac{\gamma v t}{\left(b^2 + (\gamma v t)^2\right)^{\frac{3}{2}}} \ , \ E_y = \kappa_c q \ \frac{\gamma b}{\left(b^2 + (\gamma v t)^2\right)^{\frac{3}{2}}}$$

Taking the magnitude of the transformed field E= $(Ex^2 + Ey^2)^{1/2}$ and expressing it in terms of r = $(x^2 + y^2)^{1/2}$ =($(vt)^2 + b^2)^{1/2}$ after some tedious but straightforward manipulations we have:

$$E = \kappa_c q \ \frac{\vec{r}}{r^3 \gamma^2 \left(1 - \beta^2 \frac{b^2}{r^2}\right)^{\frac{3}{2}}}$$

since sinϕ=b/r where ϕ is the angle defined by the lines electric charge - observation point P and the x-axis. Hence,

$$\vec{E} = \kappa_c q \ \frac{\vec{r}}{r^3 \gamma^2 \left(1 - \beta^2 \sin^2 \phi\right)^{\frac{3}{2}}}$$

This field is radial but the lines of force are isotropically distributed only for β=0 while for $\beta \neq 0$ is weaker in the direction of motion than perpendicular to it. This is shown schematically in the next figure:

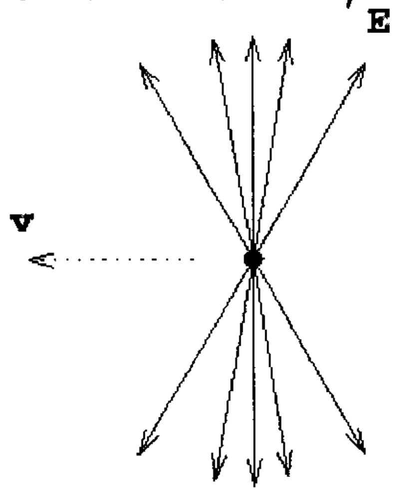


Figure: The electric field of a moving charge is concentrated in the directions perpendicular to the direction of motion. Thus for ϕ =0 we have

$$E_{\parallel} = \kappa_c q \ \frac{\vec{r}}{r^3 \gamma^2}$$

Whereas for $\phi = \pi/2$

$$E_{\perp} = \kappa_c q \ \frac{\vec{r}}{r^3 \gamma^2 (1-\beta^2)^{\frac{3}{2}}}$$

Therefore the ratio is:

$$\frac{E_{\parallel}}{E_{\perp}} = (1-\beta^2)^{\frac{3}{2}}$$

and for β=1 (speed of light) the longitudinal component is zero. The angle of the opening of field lines is $2\tan^{-1}(1/\gamma^3)$ thus for γ =1 (stationary) $2\tan^{-1}1 = \pi/2$ and the longitudinal and transverse fields have the some magnitude.

Note that the electric field in the new frame is symmetric in the forwards and backwards directions. This follows immediately from the behaviour of the sine function and its appearance there in squared form. Physically though, the symmetry follows from the same behaviour of the tilted stick and the instantaneous nature of E.

4.4 Accelerators

As it has been mentioned before, Newtonian mechanics is the low velocity limit of the relativistic mechanics. In everyday life we do not observe clocks that delay or lengths that shorten or masses that grow larger. In some other cases relativistic effects like magnetism and electromagnets function without people realising that. However there are people that use relativity in their every day life like the high energy experimental physicists. Particle accelerators would not function if their builders did not know anything about SR.
A typical example is that of a Cyclotron.

- ***Cyclotrons- A brief detour***

Before the cyclotron was invented by Lawrence in 1929, the best method for accelerating nuclear particles to high energy was to put high voltage across a gap. Atoms of a gas were ionised near the gap, and an electric field across the gap accelerated the positively charged ions to an energy E given by the charge q times the voltage V: $E=qV$. The limitation on the evergy was the voltage that could be held across the gap before sparking occurred. Another method of acceleration was to use a linear accelerator containing a number of gaps whose voltage alternated in time to match the particle position, but this proved to be a very long and unwieldy machine at the time.

These difficulties suggested to Lawrence the idea of using a magnet to bend the ions around in cirles through high-voltage electrodes, which were called "dees" because of their shape.

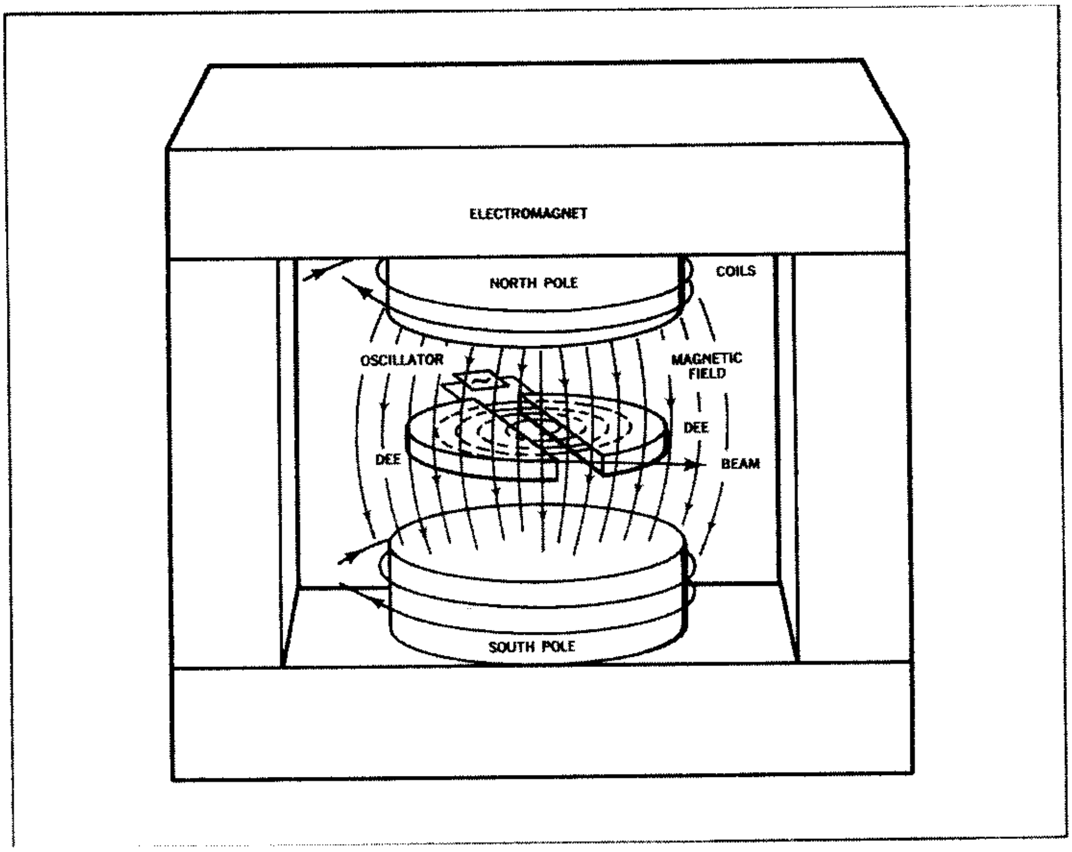


Figure 4.4-1: A cyclotron showing beam of particles being accelerated by dees in an electromagnet. Air is removed from the acceleration region by vacuum pumps. (Magnetic field lines are shown.)

An alternating accelerator dosage would be applied to the two dees. At a gap between the dees, the particles would be accelerated by the voltage, just as in a single gap accelerator. While a particle is completing a half-revolution inside the dee, the voltage would reverse and the particle would again be accelerated to the next gap.

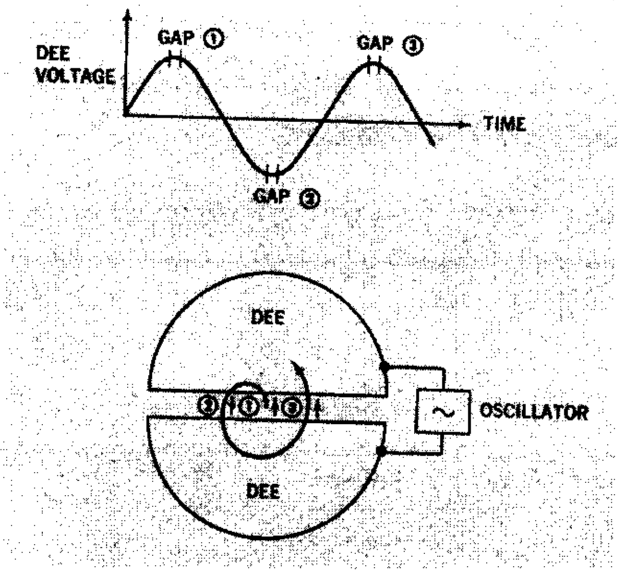


Figure 2: Particle orbit in cyclotron dees, showing gap crossings. Dee voltage reverses while particle is inside dee, giving acceleration at each gap.

This repetitive acceleration would continue until the particle would reach a high enough energy to get to the edge of the magnet. Much higher energies could now be obtained than in direct-current machines, because each gap was used many times instead of once.

The basic principle that made the cyclotron possible is the fact that an orbiting particle in a magnetic field takes the same time to make one revolution, regardless of radius or energy. Thus the alternating voltage on the dees can be set to match the revolution frequency of all the particles in the cyclotron. This is sometimes called the resonance condition, because the accelerating force varies so as to be always in the direction of particle motion, as it is when you push a swing. Mathematically the particle revolution frequency can be derived as follows. A particle of mass m and charge q moving with speed v through a magnetic field of strength B feels a force qvB bending it in a circle of radius r. Setting the mass times the acceleration (v^2/r) towards the center equal to the force toward the center (Newton's second law of motion), one gets:

$$\frac{mv^2}{r} = qvB$$

The angular velocity is v/r. So we find that:

$$\frac{v}{r} = \frac{qB}{m}$$

But the revolution period $T=2\pi r/v$ therefore the revolution frequency $f=1/T$ is :

$$f = \frac{1}{2\pi}\frac{qB}{m}$$

So the revolution frequency is thus independent of radius or energy (still velocity v is very low). The particle will remain in step, or in renosance, with the constant-frequency altering voltage on the dees. Using this principle, Lawrence built a series of cyclotrons at Berkeley in the 1930's, culminating in the 60-inch Croker Cyclotron in 1939. The 60-inch cyclotron accelerated protons to 12 MeV and alpha particles to 48 MeV.

At these energies an interesting effect began to appear. Einstein's theory of relativity predicts that when particles are accelerated to high speeds, they become heavier. Thus from the above equation the revolution frequency would decrease, and the particle would cross the dee gaps successively later. In the 60-inch cyclotron, the particles reached a speed 16% that of light. At this speed the relativistic effect makes them 1% heavier than when they were at rest. So they would circulate 1% too slowly, and in 25 revolutions they would slip 25%, or from the peak to zero voltage at the gap crossing.

Another condition that limits the maximum energy of a conventional cyclotron is the requirement for keeping the beam centered vertically inside the dees. The vertical beam space is only 2 or 3 inches. Since the beam travels about 1000 feet during it's spiral path outward, there must be a force to push the particles back towards the central , or median, plane between the magnet poles, when they deviate from in. This force is provided by the curved magnetic field lines in the magnet gap. The force on the particle is perpendicular to the velocity and magnetic field lines. It thus provides a restoring or focusing action back toward the median plane, for particles above or below it. To obtain this focusing , the magnetic field must decrease at larger radius. This causes the particle rotation frequency to decrease with radius. Unfortunately this change in frequency is in the same direction as that due to the relativistic mass increase. As a result, the number of particle revolutions in a conventional cyclotron is limited to about 40 before the particle slips to the decelerating part of the dee voltage. With the normal dee voltages of 175 kilovolts dee-to-dee, this gives an energy of 12-MeV protons. Since this energy is much higher than in the direct-current accelerators of the 1930's, many conventional cyclotrons were built throughout the world until the 1950's. One way to correct the phase slip factor is to decrease the frequency of the Dee voltage as the particle accelerates. The decrease in dee frequency would just much the decrease in particle frequency and acceleration would continue for thousands of turns to a very high energy (Synchro-cyclotrons, McMillan at Berkeley and Veksler in Russia)

- **Synchrotrons**

Lets take as our second example ELETTRA the 250 m of circumference, 2 GeV third generation synchrotron light source of Trieste. A synchrotron is a circular accelerator which has an (or more) electromagnetic resonant cavity to accelerate the particles as they pass through the cavity many times. As the particle's energy increases, the strength of the magnetic field that is used to steer them must be changed. This change must be carefully synchronised with the energy change, hence the name synchrotron. Due to synchrotron radiation - which is an electromagnetic radiation emitted by the charged particles when they are accelerated (i.e. when their velocity vector changes with time either in amplitude or in direction) -that comes out of the bending magnets and the insertion devices the electrons loose energy. This energy is given back to the electrons by the electromagnetic resonant cavity so that the energy does not change thus no further change in field is needed. However in ELETTRA electrons are injected at the energy of 1 GeV while they are used at 2 or 2.4 GeV for synchrotron light. During this process we have to change the magnetic field to keep the electrons at the same trajectory else they will get lost on the vacuum chamber wall. So to say from an initial mass of $m_1 = m_0 + 1\ GeV/c^2$ the electron finally arrives at $m_2 = m_0 + 2\ GeV/c^2$.

Firstly lets us see what is the electron's velocity at those energies. From $m = \gamma m_0$ therefore $\gamma = m/m_0 = E/E_0 = 1 + T/E_0$ Since $E_0 = 0.51099\ MeV/c^2$ (remember that mass and energy differ only by a

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

conversion factor) we get γ_1 = 1957 and γ_2 = 3913.9 Defining as $\beta = v/c$ we have β_1 =0.999999869 or v = 0.999999869 c and β_2 = 0.999999967 or v = 0.999999967 c Thus the electrons move practically at the speed of light and therefore no frequency slip occurs when increasing the energy. (For comparison the velocity of a proton with 2 GeV kinetic energy would only be Vp=0.947642 c while at 1 GeV Vp=0.875c)

Now in order to calculate the strength of the magnetic fields one needs to keep the electrons in a closed trajectory, the necessary field strength is given from the requirements of the radius of curvature and of course from the energy and circumference of the ring. For ELETTRA the radius of curvature or bending radius is 5.5 m. It can be proven that this bending radius is given from the following relation (using the Lorenzt force on a charge and the centrifugal force)

$$B(T) = \frac{P(GeV/c)}{0.2998 \ \ \rho(m)}$$

Where in the parenthesis are the units (Tesla, meters and GeV/c). P is the particle momentum P=mv or in our case P=mc. One can easily prove that

$$P = \frac{E_0\sqrt{(\gamma^2 - 1)}}{c}$$

with E_0 rest mass of the particle = 0.51099 MeV/c^2. Putting all this together we find that the field strength should be 1.2 T (or 12 kGauss). The field needed to bend a very slow moving electron at the same radius is T=0.0003 Tesla or 3 Gauss (only 5 times larger than earth's magnetic field). Accelerators appreciate the mass increase and so understand relativity!

Thus partially thanks to SR (and also to the Italian government) ELETTRA can function well.

I hope that by now it became evident to you that relativity is not that far away from our every day life as some of us already know.

5 Reminder Summary

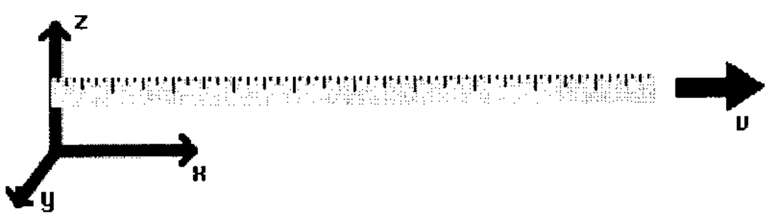
Just reminding you the SR important formulae. Let the rest frame be unprimed. Then:

$$\Delta t = \gamma \, \Delta t', \; l = \frac{l'}{\gamma}, \; m = \frac{m'}{\gamma}$$

With:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \; , \; \beta = \frac{v}{c}$$

To get a visual idea of the contraction consider a moving meter:



The Lorentz-Fitzgerald transformation formula for length contraction is given above. Here is a normal meter stick, with the contracted one below it.

The ruler at 0.7c is 0.8 meters long.

Also, one second has dilated to seconds.

Conceptual Framework of Relativity

Idea	Experiment
The measurement of absolute velocity is impossible	Michelson - Morley
The velocity of light is independent of source or detector velocity- a universal constant	Aberration of star positions Sharp double-star images
Lorentz transformations	
Galilean transformations and concept of universal time should be abandoned	
Length contraction	Muon decay in atmosphere
Time dilatation	Muon decay, relativistic Doppler effect, fine structure in atomic spectra
Relativistic mass	Accelerators, Cyclotron
Velocity of light c as speed limit	Cerenkov radiation
Energy mass relationship $E=mc^2$	Binding energy of nucleons, Nuclear fission and fusion

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