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SCHOOL ON SYNCHROTRON RADIATION

6 November - 8 December 2000

Miramare - Trieste, Italy

Supported in part by the Italian Ministry of Foreign Affairs in connection with the SESEME project

Co-sponsors: Sincrotrone Trieste, Società Italiana di Luce di Sincrotrone (SILS) and the Arab Fund for Economic and Social Development

Synchrotron Radiation

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Synchrotron Radiation.

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Introduction.

A full and exact treatment of the properties of synchrotron radiation is a pretty demanding task for the teacher as well as for the student. Trying to give such a treatment in the limited time for this course should imply that we were to rush through the material with little time for discussions and interpretations.

I have thus rather chosen to transmit a somewhat more intuitive picture of synchrotron radiation which mainly is based on the Lorentz transformations and diffraction, both effects often taught at the high-school level. The hope is then that you will remember this picture if you in the future will rather be using the synchrotron radiation than calculating its characteristics. For the future expert in synchrotron light generation, it might serve as a tool when going to more elaborate studies.

1. Classical Dipole Radiation.

A charge is situated as shown below at $t=0$. The electric field lines are distributed homogeneously in space. The charge is then suddenly accelerated downwards. If we now freeze the picture at time $t=\tau$, the field lines will move with the charge. At a distance $L=c\,\tau$, we will still have the old field lines, since the new signal is propagating with the speed of light. We will thus see the vertical field component moving outwards from the charge with the speed of light.

This moving vertical field will then induce a circular magnet field around the charge and we have an electromagnetic wave moving outwards from the accelerated charge. The original vertical electric field lines will remain unchanged and we can thus expect that no radiated power is emitted in the vertical direction.

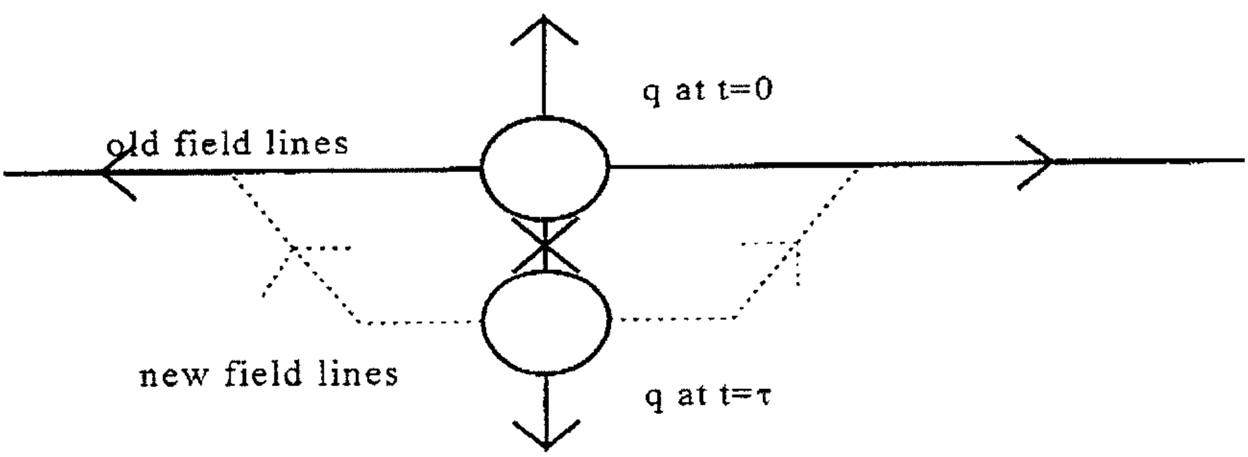

Fig. 1. Electric field from an accelerated charge.

From this, we can suspect that the outgoing electric field component can be written

$E \propto qa\cos\theta$ where a is the acceleration of the charge and θ is the angle to the horizontal plane.

The total power emitted is proportional to the square of the electric field or more exactly

$$P = \frac{2q^2}{3c}\left(\frac{a}{c}\right)^2 \qquad (1)$$

From this we learn:

1. The outgoing electric field vector is parallel to the acceleration.
2. The radiated power is proportional to the charge squared.
3. The intensity distribution looks like:

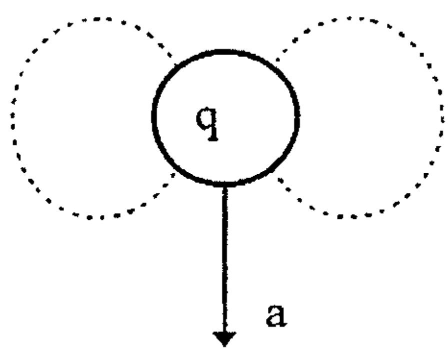

Fig. 2. Dipole emission power distribution.

2. Relativity and 4-vectors.

Synchrotron radiation (SR) is a relativistic version of the classical dipole radiation. To get the power, spectrum and angular distribution of SR we need a convenient relativistic formulation.

2.a. Classical transformations (v<<c).

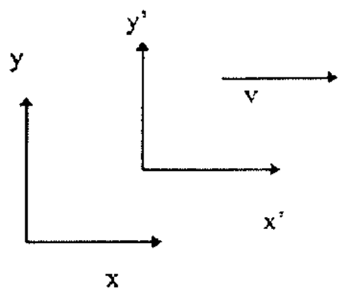

Fig. 3. The two co-ordinate systems.

Let's assume we are in the (x,y) system. Another co-ordinate system is moving along our x-axis with a constant speed v. If someone in that moving system were to measure a yardstick placed at rest in our system and also need some time for the measurement, he (she) will measure a shorter distance than we in our system.
We write this in the following form:

x'=x-vt
y'=y
t'=t

We put this in the matrix formulation

$$\begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & 0 & -v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} \qquad (2)$$

2.b The Lorentz transformation in the matrix formulation.

The Lorentz transformation is given by

$$x' = \frac{1}{\sqrt{1-\beta^2}}(x - \beta t)$$

$$y' = y \qquad (3)$$

$$z' = z$$

$$t' = \frac{1}{\sqrt{1-\beta^2}}\left(t - \frac{x\beta}{c}\right) \quad \text{where } \beta = \frac{v}{c}$$

We will now introduce the 4-vector (**r**)=(x,y,z,ict) and write the relation above in matrix form

$$\begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad (4)$$

When working with synchrotron radiation, $\gamma \approx 10^3 \to 10^4$. We can then often

approximate $\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}$

A 4-vector (**r**)=(x,y,z,ict) is invariant under Lorentz transformations, that is its length

$|r| = \sqrt{x^2 + y^2 + z^2 - c^2 t^2}$ is constant in all systems. (To prove this, just apply the Lorentz transformation to the vector (**r**).)

A special 4-vector is the "proper time"

$$\tau = \sqrt{t^2 - \frac{x^2 + y^2 + z^2}{c^2}} = t\sqrt{1 - \frac{v^2}{c^2}} = \frac{t}{\gamma}$$

The proper time, that is the time measured in the rest frame system, is thus conserved in all systems. In the rest frame system τ=t.

Now we can define the 4-velocity

$$u \equiv \frac{dr}{d\tau} = \frac{dr}{dt}\left(\frac{dt}{d\tau}\right) = (\dot{r}\gamma, ic\gamma)$$

The 4-velocity must also be conserved under the Lorentz transformations since it consists of the ratio of two 4-vectors which don't vary under the Lorentz transformations.
Then, the 4-momentum

$$p \equiv m_0 u = (\gamma m_0 v_x, \gamma m_0 v_y, \gamma m_0 v_z, ic\gamma m_0) \qquad (5)$$

must also be 4-vector since we only have multiplied the 4-velocity with a constant m_0. $\gamma m_0 c^2$ is the total energy E.

We multiply a 4-momentum with c and get

$$\sqrt{\left((\gamma m_0 c^2)^2 - \bar{p}^2 c^2\right)} = \sqrt{E^2 - \bar{p}^2 c^2} = E_0 = m_0 c^2 \qquad (6)$$

We have thus the relation between total energy, momentum and rest energy.

Finally, we can now define the 4-force:

$$F \equiv \frac{dp}{d\tau} = \frac{dp}{dt}\frac{dt}{d\tau} = (\gamma\frac{dp}{dt}, \gamma\frac{i}{c}\frac{dE}{dt}) \qquad (7)$$

which also is an invariant.

3. Radiated power at relativistic velocities.

Let us rewrite (1) to

$$P = \frac{2q^2}{3c^3}\frac{1}{m_0^2}\left(\frac{d\bar{p}}{dt}\right)^2 \qquad (8)$$

This is the classical formulation. We now suspect that this classical expression is only an approximation and replace $\left(\frac{d\bar{p}}{dt}\right)$ force with the four-force $\left(\frac{dp}{d\tau}\right)$. These expressions are of course identical in the rest frame system. Assuming that the energy loss by the particle is replaced ($\frac{dE}{dt} = 0$), we get the relativistic expression

$$P = \frac{2q^2}{3c^3 m_0^2}\gamma^2\left(\frac{d\bar{p}}{dt}\right)^2 \quad (9)$$

For a charged particle in a magnet field

$$\frac{dp}{dt} = \frac{\gamma m_0 v^2}{\rho} = \frac{\gamma m_0 \beta^2 c^2}{\rho}$$

so we end up with

$$P = \frac{2q^2 c}{3\rho^2}\beta^4\gamma^4 \quad (10)$$

In more comfortable "engineering" units we get the energy loss/turn of an electron in a storage ring

$$\Delta E(keV) = 88.5\frac{E^4(GeV)}{\rho(m)} \quad (11)$$

4. Angle distribution.

We showed earlier that for the classical case

$$dP \propto q^2 a^2 \cos^2\theta\delta\theta$$

The power is then peaked within +- 1 radians.
We can now use a Lorentz transformation to see how this angle transforms when moving from the rest frame (unprimed) system to the lab system (primed).

$$\begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ i\frac{E'}{c} \end{pmatrix} = \begin{pmatrix} h\upsilon'\frac{\cos\theta'}{c} \\ h\upsilon'\frac{\sin\theta'}{c} \\ 0 \\ ih\frac{\upsilon'}{c} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} * \begin{pmatrix} h\upsilon\frac{\cos\theta}{c} \\ h\upsilon\frac{\cos\theta}{c} \\ 0 \\ ih\frac{\upsilon}{c} \end{pmatrix} \quad (12)$$

Thus:

$$\upsilon'\cos\theta' = \gamma\upsilon(\cos\theta - \beta)$$
$$\upsilon'\sin\theta' = \upsilon\sin\upsilon \quad (13)$$

This gives us the Doppler shift

$$\upsilon' = 2\gamma\upsilon \quad \text{for } \theta=\pi,\ \beta=1 \quad (14)$$

$$\tan\theta' = \frac{\sin\theta}{2\gamma} \quad \text{or } \theta' = \frac{\theta}{2\gamma} \text{ for small angles.}$$

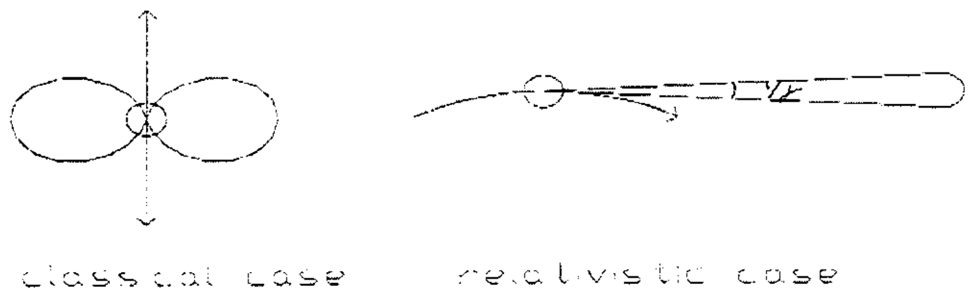

Fig. 3. Light intensity distribution in rest frame and lab frame systems.

5. Energy spectrum.

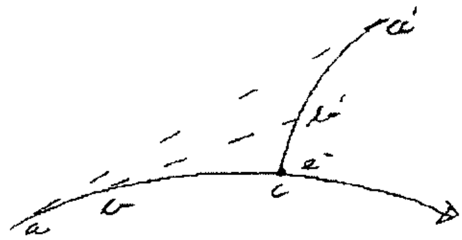

Fig. 4. Wave-front from a relativistic electron.

A relativistic electron is bent in a magnet field. Let's look on the light waves emitted at the points a, b and c. The electron, being relativistic is moving close below the speed of light. When the electron is situated at point c, the wave front emitted at point a is situated at the point a', the distance a-a' being somewhat larger than the distance a-c. The same applies to the light emitted at the point b, but the difference in path length between b-b' and b-c is somewhat smaller.
The snapshot shown above when the electron is situated at point c shows us the wavefront being the sum of all previously emitted waves c-b'-a' which is bent to the right.

Let us now look at another snapshot where we look at a storage ring with only one electron circulating from above.

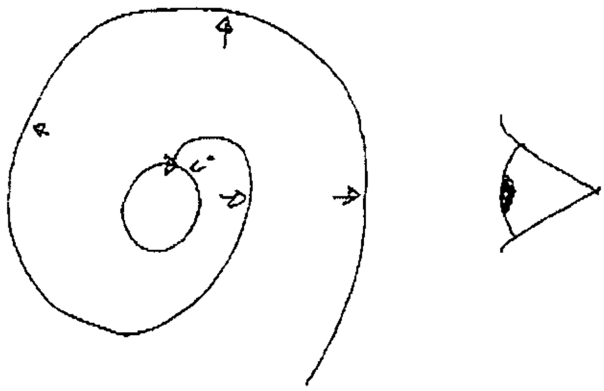

Fig. 5. Snapshot from above of the light emitted from a circulating electron.

As the particle rotates in the ring of circumference O, the light will be emitted in a spiral D with the distance O between the turns. An observer looking into the storage ring from the side will thus see light flashes of a frequency c/O.
To find the wavelength of the emitted light, we now calculate the width of the wave front.

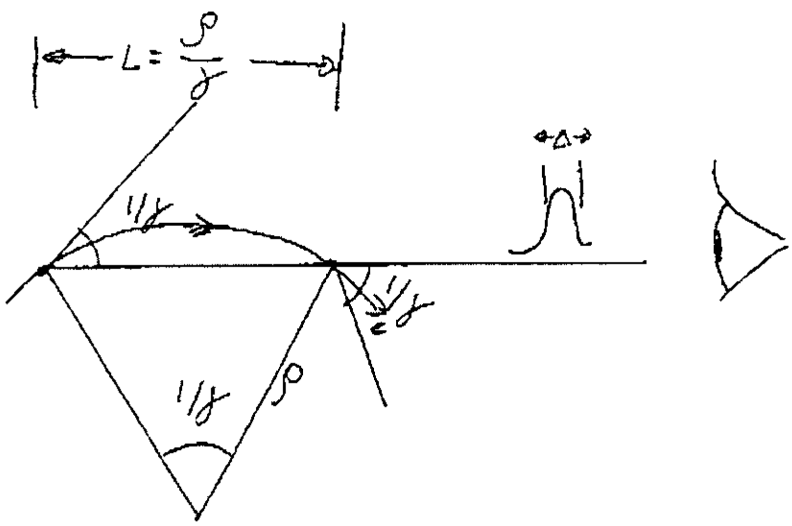

Fig. 6. The observers view of a radiating electron.

We remember that the angle of emission, which is typically 1 radian in the rest frame of the electron, is reduced with a factor $\frac{1}{2\gamma}$ in the lab system. The light from the electron can thus only hit the observer when the electron is situated between the points a-b, that is over a distance $\frac{\rho}{\gamma}$. (This distance is in most cases just a mm or so). Now we look at how much the electron is lagging behind the light emitted at point a. This path length difference must be

$$\Delta = \frac{\rho}{\gamma}(1-\beta) \approx \frac{\rho}{2\gamma^3} \text{ since } \gamma >> 1$$

The observer will see light pulses of width Δ separated by the distance O.

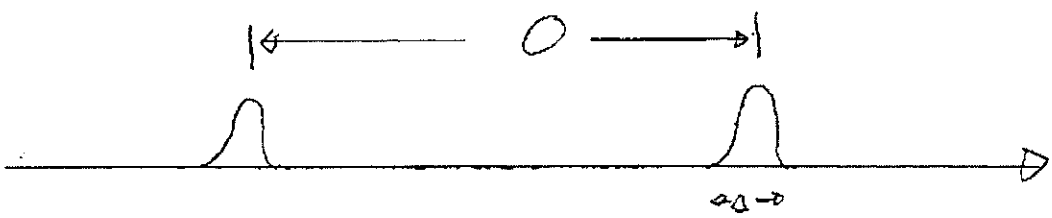

Fig. 7. Pulse train from the electron in fig 5.

Looking now for the harmonics to the rotational frequency we notice that we have harmonic coefficients of equal magnitude until we approach wavelengths comparable with the pulse length $\Delta = \frac{\rho}{\gamma^3}$, the cut-off wavelength. A more strict treatment defines the critical wavelength

$$\lambda_c = \frac{4\pi}{3}\frac{\rho}{\gamma^3} \tag{16}$$

Half of the total power radiated at wavelengths shorter than the critical wavelength, half at longer.
The intensity spectrum is shown below.

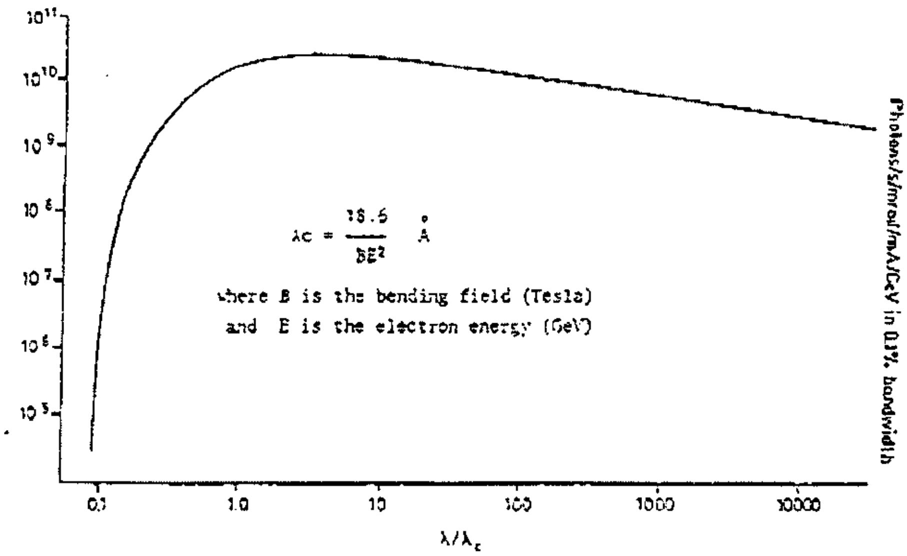

Fig. 9. Synchrotron radiation spectral distribution .

6. Angle of emission.

We have shown earlier, that the angle of emission at the critical wavelength is $1/\gamma$ by using the Lorentz transformation. We will now look into the angle distribution somewhat more in general.
Lt's assume we have a long light source, it could be an undulator or the part of a dipole source visible to an observer.

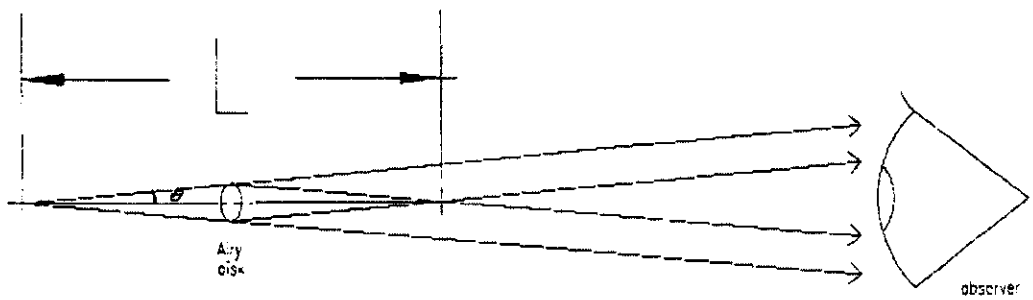

Fig. 10. Light emitted from a long, thin light source.

Let us assume that the light is emitted within an angle θ. The observer will then see the as it came from a shining disk situated in the middle of the line source. This disk has then a diameter d=Lθ. We will then use the Frauenhofer diffraction relation which tells us

$$d \sin\theta \approx d\theta = \lambda$$
$$d = L\theta \qquad (18)$$

which yields

$$\theta = \sqrt{\frac{\lambda}{L}}$$
$$(19)$$
$$d = \sqrt{L\lambda}$$

Let us now take the case of bending magnet radiation:

$$L = \rho\theta$$

yields $\theta^3 = \dfrac{\lambda}{\rho}$ or $\theta \propto \lambda^{\frac{1}{3}}$

Since θ=1/γ at the critical wavelength we get

$$\theta = \frac{1}{\gamma}\left(\frac{\lambda}{\lambda_c}\right)^{\frac{1}{3}} \qquad (20)$$

7. Matching.

Some figures of merit for a synchrotron radiation ring are flux and brilliance. The flux Φ is defined as

$$\Phi = \frac{n_f}{s, 0/00\frac{\Delta E}{E}, \Psi}$$ expressed in photons/(s, 0/00 energy spread, mrad horizontally)

The brilliance (brightness) B is the flux density in phase space defined as

$$B = \frac{n_f}{s, 0/00\frac{\Delta E}{E}, \Sigma_x, \Sigma'_x, \Sigma_y, \Sigma'_y}$$ expressed in photons/(s, permille energy spread, mm², mrad²)

Σ_i is the RMS sum of the electron and photon beam sizes (i=x,y)

$$\Sigma_i = \sqrt{\sigma_{i,e}^2 + \sigma_{i,f}^2}$$

and Σ'_i is defined likewise for the angular spread.

The flux is mainly defined by the circulating current, the number of emitting poles and the electron energy.
The brilliance case is somewhat more complicated. Let us first consider the electron and photon beam emittances.
In the electron case we have

$\varepsilon_{e,i} = \sigma_{e,i}\sigma'_{e,i}$, the emittance being defined by the lattice parameters. The photon emittance is given by

$\varepsilon_{f,i} = \sigma_{f,i}\sigma'_{f,i} = \frac{\lambda}{4\pi}$ (for gaussian distributions).
The photon emittance is defined by the diffraction as discussed earlier and is given by the radiation wavelength only. If we are to maximise the brilliance from a storage ring we need to decrease the electron emittance down to the value given by diffraction. If we are operating in the x-ray domain, say around 1 Å, we need an electron emittance of 10^{-11} rad m, which really is a challenge. Third generation light sources have an horizontal emittance of 10^{-9} - 10^{-8} rad m and a vertical emittance some two orders of magnitude smaller.
The situation is somewhat different for VUV and soft x-ray rings, which have similar electron beam emittances but the photon beam emittance is naturally orders of magnitudes larger than in the x-ray case.
In this context we should also point out the potential of linac-based electron sources. The electron beam emittance is reduced during the acceleration as

$\varepsilon = \frac{\varepsilon_n}{\gamma}$ where an optimised electron gun can achieve ε_n around 10^{-6} rad m. An x-ray diffraction limited linac source then needs an electron energy of at least 10 GeV while some GeV is needed for a VUV soft x-ray linac. The emittance demands on a IR linac are more relaxed.
A small electron beam emittance is however not sufficient to optimise the brilliance. We also need to match the diffraction defined beam size and angular spread to those of the electron beam. A striking example is to compare the brilliance from a dipole magnet to that from an undulator.

For this purpose, we can use the parameter numbers for the MAX II ring:

$\varepsilon_{x,e} = 10^{-8}$ rad m	Bending magnet	Straight section
$\sigma_{x,e} =$	0.1 mm	0.3 mm
$\sigma'_{x,e} =$	0.1 mrad	0.03 mrad
εy,e = 10^{-10} rad m		
$\sigma_{y,e} =$	0.03 mm	0.014 mm
$\sigma'_{y,e} =$	0.003 mrad	0.007 mrad
L =	3 mm	2.5 m
λ=1 nm		
$\sigma_f =$	0.0002 mm	0.004 mm

$\sigma'_r =$ 0.4 mrad 0.02 mrad

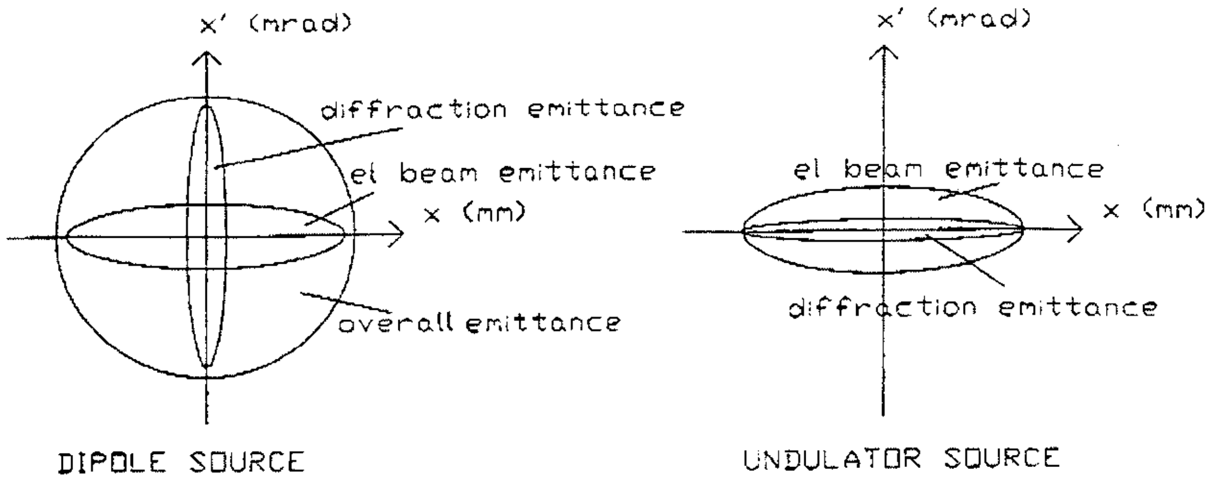

Fig. 11. Electron and photon beam emittances (horisontal).

We can see from fig 11 above, that the matching in phase space between the diffraction induced photon emittance and the electron beam, emittance is poor. The overall emittance must be transported down the beamline and we have diluted phase-space a factor of four. The situation is even worse in the vertical phase space where we have a dilution of a factor of 100.

If we now turn to the undulator case, we notice that the angle of emission of the emitted light is decreased considerable, due to the longer light source. The situation is also shown above for the horizontal phase space and we have no dilution at all. The dilution in vertical phase space is a factor of two only.

The undulator source brilliance is increased compared to the dipole case of two reasons; first we can use some 100 undulator poles and second the matching is some factor 100 better. We can thus expect the undulator brilliance to be some four orders of magnitude higher than in the dipole case, provided we have a relatively small electron beam emittance.

Below, we see the brilliance curves for the MAX I and MAX II storage rings. The undulator brilliance is as pointed out above some four orders of magnitude higher than that from bending magnets. The abscissa is given in energy units ($\varepsilon_{ph} = \frac{12400}{\lambda(\AA)}$)

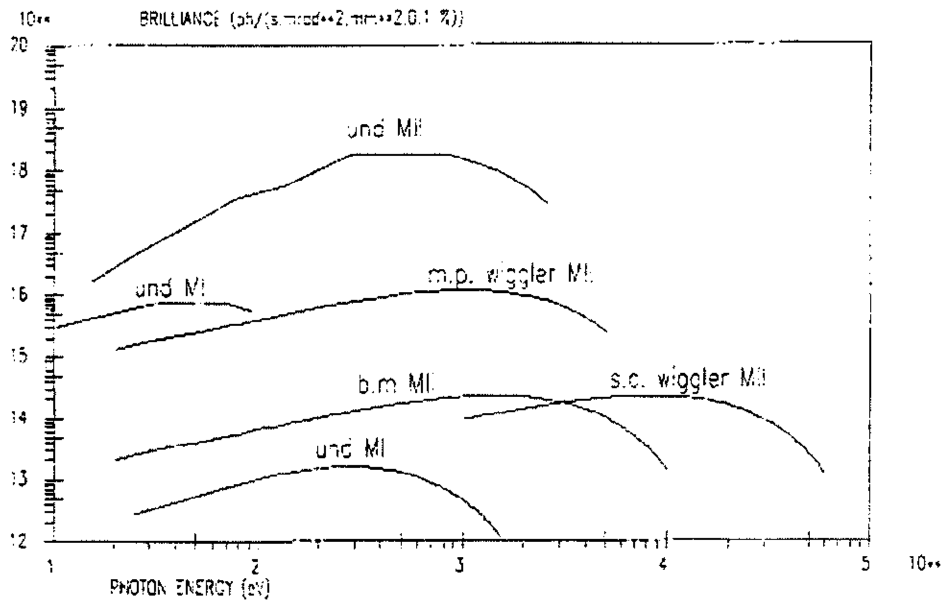

Fig. 13. Brilliance from dipoles and undulators in MAX II.

References:

For a more complete treatment of the properties of synchrotron radiation, the following litterature can be consulted:

J. Schwinger, Phys. Rev. 75, 1912 (1949)

J. D. Jackson, Classical Electrodynamics, Wiley & Sons, N. Y. (1975)

A. Hofmann, "Characteristics of Synchrotron radiation", in Proc. CERN Acc. School, Synchrotron radiation and Free Electron Lasers, 1989, CERN 90-03 (1990) p115

G. K. Green, Spectra and Optics of Synchrotron radiation, BNL Rep. No 50522 (1976)

S. Krinsky et al, Characteristics of Synchrotron radiation and its Sources, in Handbook on Synchrotron radiation, ed., E.-E. Koch, North-Holland, Amsterdam (1983) p65

R. Walker, Insertion Devices, This course.

Aug 7, 1992 LUNTDX/(NMT-7020)/1-8/(1992)

DIFFRACTION FROMSYNCHROTRON RADIATION SOURCES.

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Abstract. The Fraunhofer diffraction properties for a slit are recapitulated. The same technique is then used to deduct the diffraction relation for a gaussian transverse distribution of the light field. A synchrotron light source, could be a bending magnet or an undulator, is then treated in the same way as used for the other cases. Finally, the matching between the parameters of the electron beam and those for the light fields is discussed.

1. The single slit. A plane electromagnetic wave is passing a slit as shown in fig. 1. In the Frauenhof diffraction picture, each line segment at the slit is treated as a source point for spherical waves. Since the light intensity is constant over the slit, so is the electric field vector E. The difference in path length between different source points is

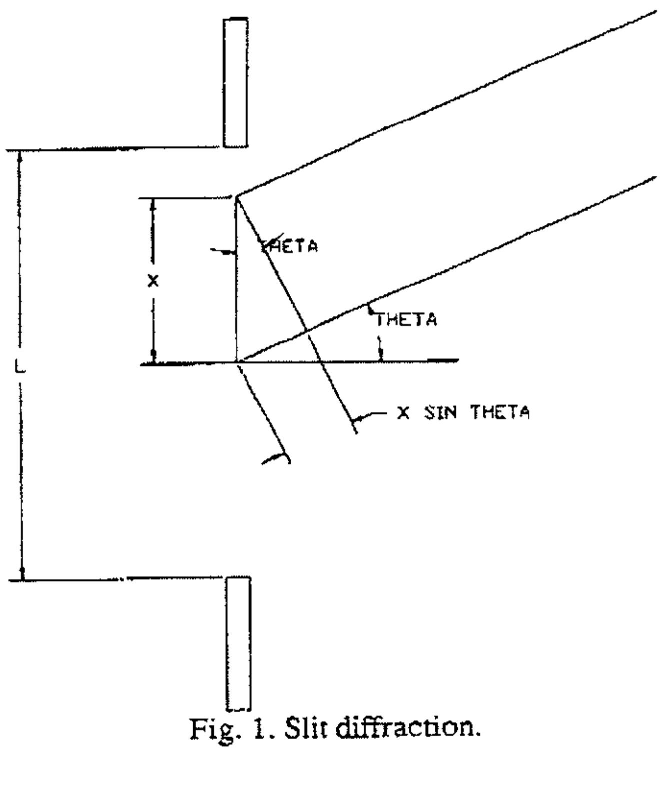

Fig. 1. Slit diffraction.

$\Delta\lambda = x \sin\theta = x\,w$ *where* $w = \sin\theta$

The difference in phase is given by

$$\Delta\Phi = \frac{\Delta\lambda}{\lambda}\,2\pi = x\,w\,k \textit{ where } k = \frac{2\pi}{\lambda}$$

We can integrate over the slit to get the distant angle distribution of the light field

$$E = const \int_{-L/2}^{L/2} e^{i x k w} dx = \frac{2 \sin (k w L/2}{k w}$$

The light intensity is just the square of the field which gives us the the intensity distribution

$$I = const \frac{\sin^2 (k w L/2)}{(k w)^2}$$

The intensity distribution is seen in fig. 2 and we see that the first minimum is when

$k w L/2 = \pi$ *or* $L \sin \theta = \lambda$

which is the diffraction relation for a single slit.

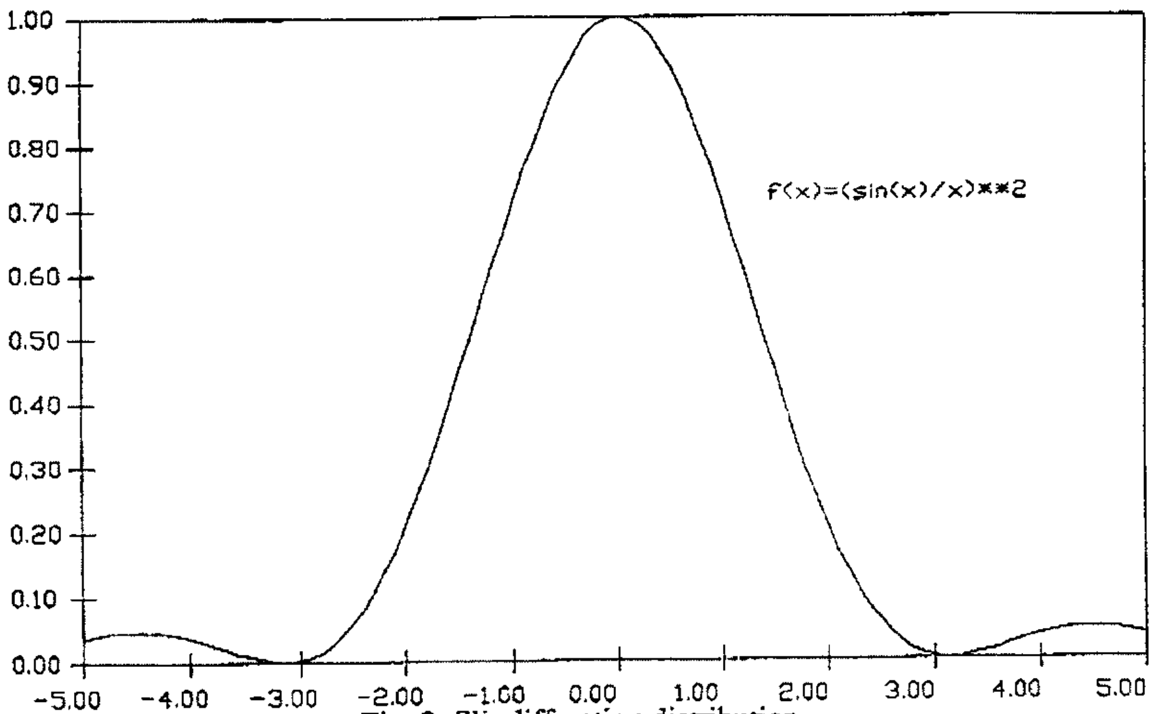

Fig. 2. Slit diffraction distribution.

2. Gaussian distributed beam.

We now proceed in the same way but replace the slit, which gives us a rectangular intensity distribution of the intensity with a light beam of gaussian distribution. The application is evident, the electron beam in a storage has a gaussian distributed intensity.

We treat here the one-dimensional case. The two-dimensional case gives easily the same result.

We assume a gaussian distributed intensity

$$I = const\ e^{-x^2/2\sigma^2}$$

We then get

$$E = const\ e^{-x^2/4\sigma^2}$$

Now, looking for the angular distribution we integrate the contributions from all parts of the field

$$E = const \int e^{-x^2/4\sigma^2} e^{ikxw}\, dx = const \int e^{-1/4\sigma^2 (x - 2i\sigma^2 kw)^2} e^{-(\sigma k w)^2}\, dx$$

This integral is easily sovled and we get

$$E = const\ e^{-(\sigma k w)^2}$$

The intensity angular distribution is then given by

$$I = const\ e^{-2(\sigma k w)^2} = const\ e^{-w^2/2\sigma_w^2}$$

This gives us the standard deviation for the angular distribution of the intensity

$$\sigma_w = \frac{1}{2\sigma k}$$

or the well-known dispersion relation for gaussian distributions

$$\sigma_w \sigma = \frac{\lambda}{4\pi}.$$

3. Long light sources.

A relativistic electron passing a magnet structure will emitt synchrotron radiation in the forward direction. Let us start with a very simple model.

The light is now assumed to be emitted at an angle θ towards the particle trajectory. An observer will see an shining disk placed at the middle of the magnet structure as seen in fig 3. For smal θ , we can approximat the diameter of this disk, the apparent source size to

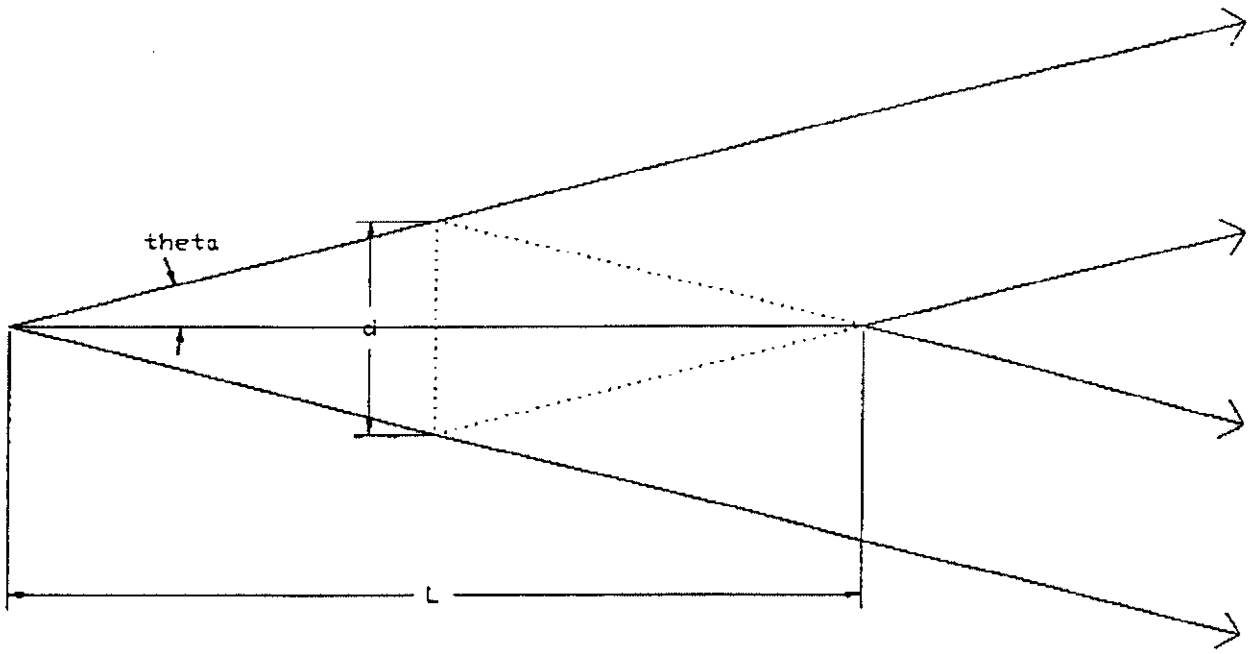

Fig. 3. Light from a long source.

$d = L\theta$

The dispersion relation for slits will then give

$d\,\theta = \lambda$

or

$L\,\theta^2 = \lambda$ *which gives us* $\theta = \sqrt{\lambda/L}$

Let us now look a bit deeper into the problem (Fig. 4.). At time t=0, the electron is at position x=0 and emitts a spherical light wave. A moment later, when the particle is at position x, it likewise emitts another spherical wave. These waves are out of phase

$$\Delta\Phi = x\,((1-\cos\theta)+(1-\beta))\,k$$

where βc is the effective speed in the x-direction of the particle.

The angle-dependent part of the phase shift can then be written for small angles

$$\Delta\Phi = x\frac{\theta^2}{2}k$$

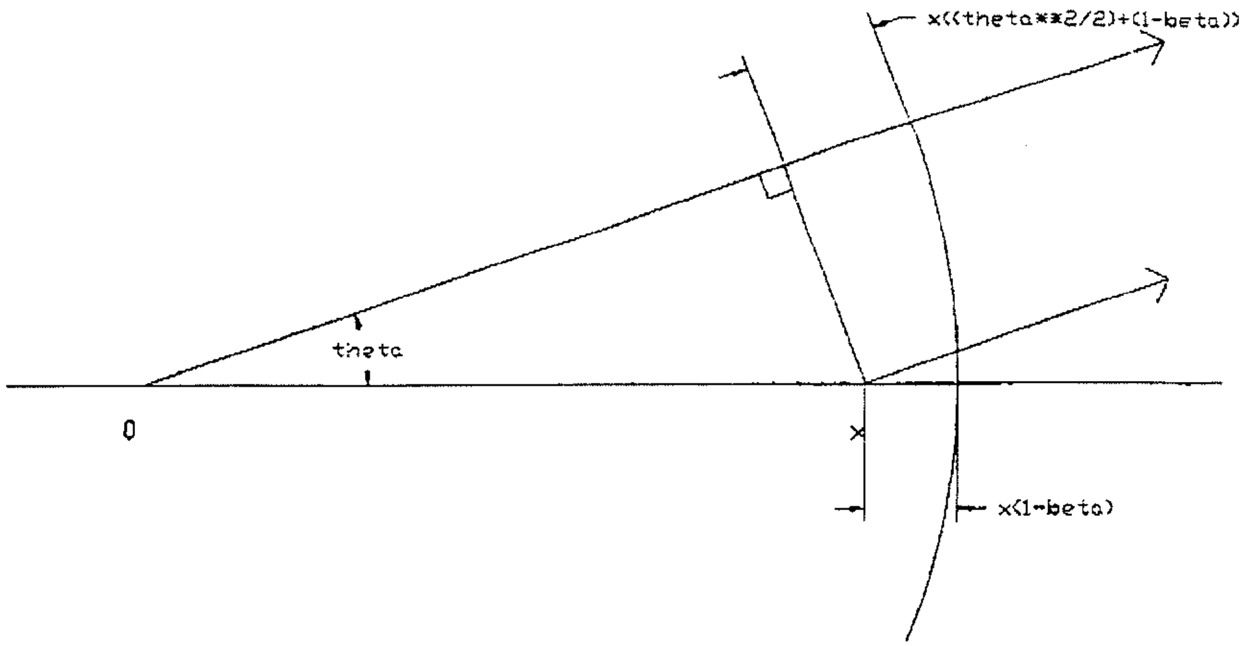

Fig. 4. Wavelength slip at long sources.

Integration over the total length of the magnet structure

$$E = const \int_{-L/2}^{L/2} e^{i k x \frac{\theta^2}{2}} dx$$

The light intensity is then given by

$$I = const \frac{\sin^2(k \frac{L}{2} \frac{\theta^2}{2})}{(k \frac{\theta^2}{2})^2}$$

The angular intensity distribution is seen in fig. 5. The first intensity zero takes place for

$$\theta = \sqrt{2 \lambda / L}$$

4. The gaussian approximation.

We are left with one problem: the electron beam is gaussian distributed and we must take this into account when calculating the total effective source size and angular spread of the light. We will now make a gaussian approximation of the angular spread σ'_r induced by diffraction. Then, we will use the diffraction relation for gaussian distributions to calculate the source size induced by diffraction, σ_r. We can then add the standard deviations for the electron beam and the diffraction related ones quadratically to get the peak light intensity in phase space, the brilliance.

The goal to get the peak light intensity in phase space leads us to choose a standard deviation for the gaussian approximation so that the peak of the two distributions coincide.

A rotational symmetric gaussian distribution has the following normalized form

$$f_{gauss}(\theta) = \frac{1}{2\pi\sigma^2} e^{-\theta^2/2\sigma_\theta^2}$$

and the corresponding one for the diffraction distribution

$$f_{diff}(\theta) = \frac{2a}{\pi^2}\frac{\sin^2(a\theta^2)}{(a\theta^2)^2} \quad where\ a = \frac{2\pi}{\lambda}\frac{L}{4}$$

Equalizing the peak values of the two distributions yields

$$\sigma_\theta = \sigma'_r = \sqrt{\frac{\lambda}{2L}}$$

The corresponding (not to equal-looking) distributions are seen in fig. 5.

The diffraction relation for gaussian distributions gives us now

$$\sigma_r = \frac{\sqrt{\lambda L}}{2\sqrt{2}\,\pi}$$

In this approximation we get the total intensity distribution in phase space

$$\frac{d^4 I}{d\theta_x\, d\theta_y\, dx\, dy} = \frac{I_o}{(2\pi)^2}\frac{1}{\Sigma'_x \Sigma'_y \Sigma_x \Sigma_y} e^{-\theta_x^2/2\Sigma'^2_x} e^{-\theta_y^2/2\Sigma'^2_y} e^{-x^2/2\Sigma_x^2} e^{-y^2/2\Sigma_y^2}$$

where I_o is the total number of photons per time unit and energy interval.

$\Sigma_i = \sqrt{\sigma_i^2 + \sigma_r^2}$

$\Sigma'_i = \sqrt{\sigma'^2_i + \sigma'^2_r}$

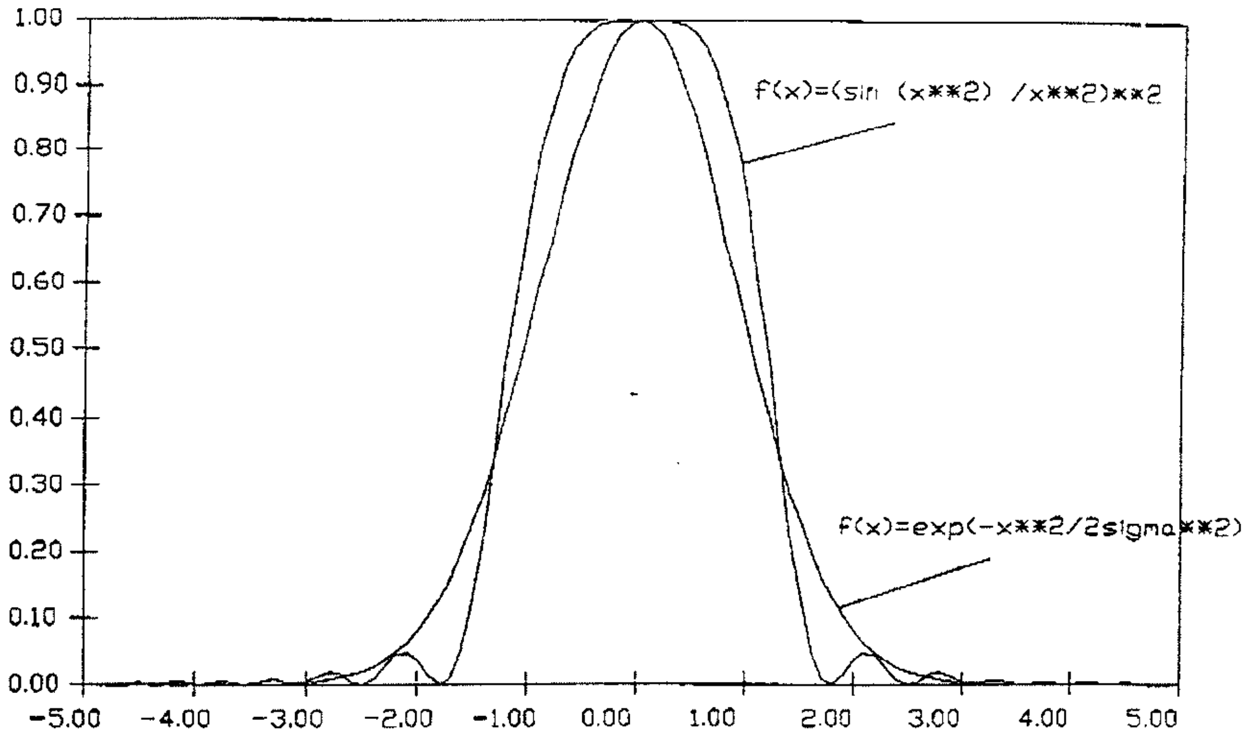

Fig. 5. Gaussian approximation.

It is now natural to define the brilliance as the maximum intensity density

$$B = \frac{I_0}{(2\pi)^2} \frac{1}{\Sigma_x \Sigma_y \Sigma'_x \Sigma'_y}$$

5. Matching.

From the brilliance definition, we see that it pays off to decrease the electron beam emittance

$$\varepsilon_i = \sigma_{i,e}\, \sigma'_{i,e}$$

until it reaches the diffraction-induced emittance $\sigma_r\, \sigma'_r$ or

$$\varepsilon_i = \frac{\lambda}{4\pi}$$

This is, however, not enough. The electron beam must also be matched to the diffraction conditions.The latter can be expressed as

$$\frac{\sigma_r}{\sigma'_r} = \frac{L}{2\pi}$$

The corresponding relation for the electron beam emittance can be written

$$\frac{\sigma_{i,e}}{\sigma'_{i,e}} = \beta_i$$ *where β is the Twiss function.*

The design aim must then be to get

$$\beta_i = \frac{\sigma_r}{\sigma'_r}$$

Let first concider a bending magnet source. In this case σ'_r approximately equals $1/\gamma$ which through the dispersion relation for gaussian distributions yields $\sigma_r = \frac{\lambda\gamma}{4\pi}$.

The optimum β value is then

$$\frac{\sigma_r}{\sigma'_r} = \frac{\lambda\gamma^2}{4\pi}$$

which for most typical cases is around 10^{-5} - 10^{-6} m while the minimum β function which can be attained is around 0.1 m. The electron beam is thus heavily mismatched to the light characteristics. The electron beam size is generally three orders of magnitude to large while the electron beam angulars spread is much smaller than the diffraction defined one.

A much better matching is achieved when long undulators can be used.The minimum mean β function value which can be attained in a straight section of length L is around $\beta = L/2$. This is only a factor of π from a perfect match.

Acknowledgements. Thanks are due to prof Helmut Wiedemann, SSRL, and dr Richard Walker, ELETTRA, for most interesting discussions.