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WORKSHOP ON MATHEMATICS IN INDUSTRY

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I. SOME CLASSICAL PROBLEMS IN DYNAMIC OPTIMIZATION AND THE GENERAL PROBLEM OF OPTIMAL CONTROL

II. THE CALCULUS OF VARIATIONS

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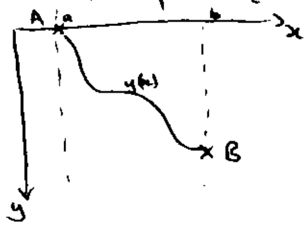
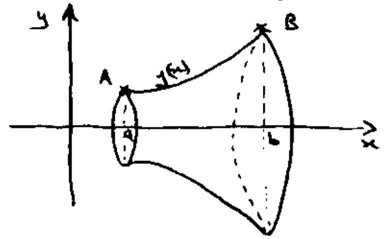
I : Some Classical Problems in Dynamic Optimization and the General Problem of Optimal Control.

Consider the following 3 classical problems

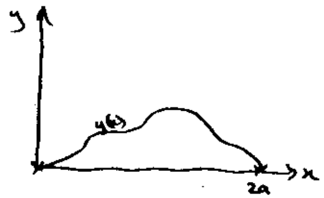
Problem (1): Given two points A, B not on the same level, find the curve joining A, B such that the time of descent for a particle constrained to move under gravity and without friction along the curve, is a minimum



Problem (2): Given two points A, B find the curve joining them which when rotated about the x-axis results in a surface of minimum area.



Problem (3): Given a piece of string of fixed length $2l$ and attached to two points $(0,0)$, $(2a,0)$ in the x, y plane, find the shape of the string that together with the x-axis encloses the maximum area.



Concerning problems (1) and (2) we note the following features

F(i) We are given a certain set of curves Σ to consider. Σ is called the SET OF COMPETING FUNCTIONS

In (1), (2) Σ is likely to be the set of smooth functions on $[a,b]$ (i.e. $y(x) \in C'[a,b]$) satisfying

$$y(a) = y_a \quad y(b) = y_b \qquad (1)$$

F(ii) To each $y(x) \in \Sigma$ there is assigned a real number (descent time in (1), surface area in (2)). Such an assignment is called a FUNCTIONAL.

Thus a functional is a mapping from a set of curves (Σ) into the real numbers. It is thus more general than a f^n and a special notation $I[y]$ is adopted to denote the real number assigned to $y(x)$.

$$\therefore \quad I : \Sigma \to \mathbb{R} \qquad (2)$$

F(iii) From all the numbers $I[y]$, find the smallest one. The associated function $y(x)$ is then the solution to the problem.

The above represents the general format of a DYNAMIC OPTIMISATION PROBLEM, the word 'dynamic' being used to signify that the solution is a set of points (a curve) rather than a single point.

In order to see what functionals are likely to be of interest it is useful to generate those for problems (1), (2).

In the case of problem (1) if s denotes distance along the curve measured from A and v is the speed of the particle then

$$v = \frac{ds}{dt}$$

If T is the sliding time then clearly

$$T = \int_0^T 1\,dt = \int_0^{s_B} \frac{dt}{ds}\,ds = \int_0^{s_B} \frac{ds}{v}$$

$$\text{i.e. } T = \int_a^b \frac{(1+y'^2)^{1/2}}{v}\,dx \qquad (3)$$

We thus require an expression for v in terms of $y(x)$. This is obtained by invoking the Conservation of Energy Principle. Since the particle starts from rest, take the zero P.E. level to be such that

Total Energy at A = 0

Total Energy at any other point (x,y) $= \frac{1}{2}mv^2 - mgy$ (since y is measured downwards)

$\therefore$ Conservation of Energy states that

$$\frac{1}{2}mv^2 - mgy = 0$$

$$\text{i.e. } v = \sqrt{2gy}$$

Hence from (3)

$$T[y] = \frac{1}{\sqrt{2g}}\int_a^b \frac{(1+y'^2)^{1/2}}{y^{1/2}}\,dx \qquad (4)$$

is the functional for problem (1).

In problem (2) the required functional is

$$S[y] = 2\pi \int_a^b y(1+y'^2)^{1/2}\,dx \qquad (5)$$

Hence in problems (1), (2) the functional to be considered is of the basic form

$$I[y] = \int_a^b f(x, y(x), y'(x))\,dx \qquad (6)$$

Although dynamic optimisation problems generally are likely to lead to a variation of the strict form (6) (i.e. more unknown functions $y(x)$ in the integrand or multiple integrals if y is a function of many variables) it is a general observation that dynamic optimisation problems are concerned with functionals which contain integrals.

Problem (3) represents another dynamic optimisation problem

Whose functional adopts the form (6). The functional here is

$$A[y] = \int_a^b y \, dx \qquad (7)$$

Note however that in this case there is an additional feature in the problem, namely the curves must all have fixed length. Mathematically this condition is formulated as

$$L[y] \triangleq \int_a^b (1+y'^2)^{1/2} \, dx = 2l \qquad (8)$$

Hence in problem (3) there are additional restrictions on the curves $y(x)$ that are to be considered, over and above the usual restriction imposed in F(i). Restrictions of this type are called CONSTRAINTS and any problem containing such a feature is called a DYNAMIC CONSTRAINED OPTIMIZATION PROBLEM.

There are various forms a constraint may take. It could be integral, algebraic or differential in nature. The constraint arising from an optimal control problem is usually a set of differential equations representing the equations of motion of the system.

An optimal control problem however has several interesting features of its own. Some typical problems are

PROBLEM 4: Transfer a rocket from the earth to the moon in such a way that the minimum amount of fuel is used and the rocket makes a "soft landing".

PROBLEM 5: Minimise the time of interception of an aircraft and a missile.

PROBLEM 6: Find the temperature profile along a tubular chemical reactor which maximises the yield of the product.

Note the following features

FE(i): a dynamical system (describing the physical system) to be controlled.

In the above problems the dynamical system is described by ordinary differential equations. However these may equally well be partial differential equations or even not deterministic but stochastic in nature.

It will be assumed in this presentation that the describing differential equations are of the form

$$\dot{x}(t) = f(x(t), u(t), t) \qquad (9)$$

where $x(t) \in \mathbb{R}^n$ $u(t) \in \mathbb{R}^l \ \forall t$.

$x(t)$ is the STATE VECTOR

$u(t)$ is the CONTROL or INPUT VECTOR.

We denote the solution of (9) starting from $x^{(0)}$ at $t = t_0$ under the effect of the control $u(t)$ by

$$x(t; t_0, x^{(0)}, u) \qquad (10)$$

FE(ii): a set of allowable controls

Consideration of the above problems shows that a physically meaningful set of controls is the set U of sectionally continuous functions whose values $u(t)$ lie in $\Omega \ \forall t$, where $\Omega \subseteq \mathbb{R}^l$. Thus

$$u(t) \in U \Rightarrow \left\{ \begin{array}{l} u(t) \in C_s[t_0, t_1]^l \\ u(t) \in \Omega \quad \forall t \in [t_0, t_1] \end{array} \right\} \qquad (11)$$

The set Ω is called the CONTROL CONSTRAINT SET and if $\Omega = \mathbb{R}^l$ then the control is said to be UNCONSTRAINED, otherwise (i.e. $\Omega \subset \mathbb{R}^l$) the control $u(t)$ is said to be CONSTRAINED.

Note that in the Chemical Reactor Problem (problem (6))

$$\Omega = \{T \ ; \ T \in \mathbb{R}, \ 0 \le T \le T^{(u)}\}$$

FE(iii): an objective for the system.

In the above problems we are trying to reach a certain state at a specific time. Thus we are trying to end up in some set S where

$$S \subseteq \mathbb{R}^n \times [t_0, t_1]$$

S is called the TARGET SET and its elements are ordered pairs (x,t) which represent the objectives of the problem.

FE(iv): a performance index

It is clear ~~that~~ from problems 4 and 5 that again the performance index should contain an integral. Problem 6 further suggests that some function of the end state might also be present. We take as a performance index incorporating both these requirements

$$J[x^{(0)}, t_0, u(t), x, t] = \phi(x,t) + \int_{t_0}^{x} F(x(\tau), u(\tau), \tau)\, d\tau \qquad (12)$$

Let $u(t)$ be a control transferring $(x^{(0)}, t_0)$ to S and let $t_f\ (>t_0)$ be the first instant when this is achieved

$$\text{i.e. } (x(t_f; t_0, x^{(0)}, u), t_f) \in S \qquad (13)$$

Write

$$x(t_f; t_0, x^{(0)}, u) = x_f \qquad (14)$$

and

$$J[x^{(0)}, t_0, u] \triangleq J[x^{(0)}, t_0, u, x_f, t_f] \qquad (15)$$

Thus $J[x^{(0)}, t_0, u]$ is the value of the performance index for the control $u(t)$ w.r.t. the target set S.

The general optimal control problem may then be stated as

– "Find that control $u(t) \in U$ which transfers $(x^{(0)}, t_0)$ to S and which minimises $J[x^{(0)}, t_0, u]$.

THE CALCULUS OF VARIATIONS:

Suppose we wish to minimise a functional $I: \Sigma \to \mathbb{R}$ over a given set of competing functions $\Sigma \subset S$ where S is some underlying vector space of functions.

Let $y^* \in \Sigma$ be a minimising curve. We wish to consider perturbations of this curve of the form

$$y(x) = y^*(x) + h(x) \qquad (1)$$

Now not all functions $h(x) \in S$ will be acceptable as perturbations of $y^*(x)$, since it is possible that $y(x)$ as constructed in (1) is not a competing function. ∴ we say that $h(x) \in S$ is an ADMISSIBLE VARIATION OF $y^*(x)$ if

$$y^*(x) + h(x) \in \Sigma \qquad (2)$$

and we will denote the SET OF ADMISSIBLE VARIATIONS by H.

Under fairly mild assumptions it may be established that if $h(x) \in H$ then so also is $\theta h(x) \in H$ for $\theta \in \mathbb{R}$.

$$\text{i.e. } y^* + \theta h(x) \in \Sigma \quad \text{if } h \in H \qquad (3)$$

Since (3) is a competing function, I is defined for this function and

$$I[y^* + \theta h] \triangleq I(\theta) \qquad (4)$$

i.e. for a fixed $y^*(x)$ and a fixed $h(x)$, the functional $I[y^* + \theta h]$ may be thought of purely as a function of the parameter θ.

Now the function $I(\theta)$ has a minimum at $\theta = 0$ since $y^*(x)$ is minimising for $I[y]$. Thus invoking the well-known necessary condition for a minimum of a function of a single variable gives

$$\left.\frac{dI(\theta)}{d\theta}\right|_{\theta=0} = 0 \qquad (5)$$

Definition 1: The quantity

$$\left.\frac{dI(\theta)}{d\theta}\right|_{\theta=0} \triangleq \left.\frac{d}{d\theta} I[y^* + \theta h]\right|_{\theta=0} \qquad (6)$$

is called the FIRST VARIATION OF $I[y]$ AT $y^*(x)$. It is usually denoted by $\delta I[h]$ since it depends on the function $h(x)$ used to ~~construct~~ perturb $y^*(x)$ and hence is itself a functional.

In view of (5) & (6) we obtain the central theorem in variational calculus

THEOREM 1: A necessary condition for the functional $I[y]$ to attain a minimum over some set of competing functions Σ_1 for some $y^* \in \Sigma_1$ is that the first variation of $I[y]$ at y^* should vanish

$$\text{i.e.}\quad \delta I[h] = 0 \tag{7}$$

for all admissible variations $h(x)$.

Note: The result is the counterpart of the Classical Analysis conditi[on]

$$f'(x) = 0 \tag{8}$$

Again as in the Classical Analysis situation the condition (7) is a necessary condition and it is no way sufficient.

It is the exploitation of the general condition (7) in the relevant ~~particular~~ situations ~~of the Calculus of Variations and Optimal Control~~ that yields the central results of the Calculus of Variations and Optimal Control.

Consider then the condition (7) in the case of the Simplest Problem of the Calculus of Variations.

Simplest Problem: From all the smooth curves $y(x)$ that join the two fixed points (a, y_a), (b, y_b) find the one that minimises

$$I[y] = \int_a^b f(x, y, y')\,dx \tag{9}$$

It is noted in this problem that

$$S \triangleq C'[a,b] \tag{10}$$

competing functions: $\Sigma_1 \triangleq \{\, y(x) \;;\; y(x) \in C^1[a,b],\ y(a) = y_a,\ y(b) = y_b \,\}$ (11)

admissible variations: $H \triangleq \{\, h(x) \;;\; h(x) \in C'[a,b],\ h(a) = h(b) = 0 \,\}$ (12)

Now for (9),

$$I(\theta) \triangleq I[y^* + \theta h] = \int_a^b f(x, y^* + \theta h, y^{*\prime} + \theta h') \,dx \tag{13}$$

and so

$$\delta I[h] \triangleq \frac{d}{d\theta} I[y^* + \theta h]\Big|_{\theta=0}$$

$$= \int_a^b \left\{ f_y\big|_* h + f_{y'}\big|_* h' \right\} dx \tag{14}$$

Integrating the second term by parts gives in view of the fact that $h \in H$,

$$\delta I[h] = \int_a^b \left\{ f_y\big|_* - \frac{d}{dx}\left(f_{y'}\big|_*\right) \right\} h(x)\,dx \tag{15}$$

$$= 0 \quad \forall\, h \in H.$$

By the arbitrariness of h it then follows that a necessary condition for a minimum is

$$f_y(x, y^*(x), y^{*\prime}(x)) - \frac{d}{dx}\left\{ f_{y'}(x, y^*(x), y^{*\prime}(x)) \right\} = 0 \tag{16}$$

Thus we have

THEOREM 2: A necessary condition that $y^*(x) \in \Sigma_1$ of (11) be a minimising curve for the functional $I[y]$ of (9) is that $y^*(x)$ satisfy the EULER EQUATION

$$f_y\big|_* - \frac{d}{dx}\left(f_{y'}\big|_*\right) = 0 \tag{17}$$

It is noted that the Euler equation (17) is a 2^{nd} order nonlinear ordinary differential equation since f is a known function.

$$\text{viz.}\quad \left(\frac{\partial^2 f}{\partial y'^2}\right)\cdot\frac{d^2y}{dx^2} + \left(\frac{\partial^2 f}{\partial y\partial y'}\right)\frac{dy}{dx} + \frac{\partial^2 f}{\partial x\partial y'} - \frac{\partial f}{\partial y} = 0 \qquad (18)$$

The solutions of Euler's Equation are called EXTREMALS but it is to be emphasised that ~~these~~ extremals are only suspected of being minimising curves. They are not necessarily minimising curves. Sufficient conditions for a minimum of a functional can be found but in general they are very difficult ~~to~~ to implement.

In general the Euler equation cannot be solved analytically and its numerical solution is complicated by the nature of ~~the~~ the boundary conditions ($y(a)=y_b$, $y(b)=y_b$) which are two point in nature.

Two special cases when ~~one~~ integration of the Euler Equation may be performed analytically are

CASE (1): $f \equiv f(x,y')$ $\qquad (19)$

then a first integral ~~of~~ (17) is

$$f_{y'} = \text{constant} \qquad (20)$$

CASE (2): $f \equiv f(y,y')$ $\qquad (21)$

then a first integral of (17) may be generated as

$$f - y'\cdot f_{y'} = \text{const.} \qquad (22)$$

In fact all of the functionals arising from problems (1)–(3) fall into the category of CASE (2).

Some extensions of the Simplest Problem are:

EXTENSION 1: Functionals of more than one unknown function:–

$$\text{i.e.}\quad I[y_1,\dots,y_n] = \int_a^b f(x,y_1,\dots,y_n,y_1',\dots,y_n')\,dx \qquad (23)$$

where

$$\begin{aligned} y_i(a) &= y_{ia} \qquad i=1,\dots,n \\ y_i(b) &= y_{ib} \end{aligned} \qquad (24)$$

are known.

In this case theorem 1 generates a system of Euler Equations

$$f_{y_i} - \frac{d}{dx}\left(f_{y_i'}\right) = 0 \qquad i=1,\dots,n \qquad (25)$$

for the determination of the unknown functions $y_1(x),\dots,y_n(x)$

EXTENSION 2: Variable End-points:

Suppose we wish to minimise

$$I[y] = \int_a^b f(x,y,y')\,dx \qquad (26)$$

where the final end-point (b,y_b) is no longer given ~~by~~ but constrained to lie somewhere on the curve

$$y = \psi(x) \qquad (27)$$

In this case ~~the~~ an extra term is present outside the integral in the condition (15). Some analysis then reveals the following conditions for the minimising curve to satisfy

$$f_y - \frac{d}{dx}(f_{y'}) = 0 \qquad \forall\, x\in[a,b] \qquad (28)$$

where b, y_b are determined from

$$f + (\psi' - y')f_{y'} = 0 \qquad \text{at } x=b \qquad (29)$$

and the relation (27).

(29) is called the TRANSVERSALITY CONDITION.

III: THE OPTIMAL CONTROL PROBLEM – THE VARIATIONAL APPROACH AND THE MINIMUM PRINCIPLE:

Consider the following problem

PROBLEM I: (Scalar, free end-point, fixed-time problem)

It is required to transfer the initial state $x^{(0)}$ at $t=t_0$ [redacted] of the scalar system

$$\dot{x}(t) = f(x(t), u(t), t) \tag{1}$$

to an unspecified destination at the terminal time L in such a way that the performance index

$$J[x,u] \triangleq \phi(x(L)) + \int_{t_0}^{L} F(x(t), u(t), t)\, dt \tag{2}$$

is minimised.

Note: there is a slight abuse of notation in that $J[x^{(0)}, t_0, u]$ rather than $J[x, u]$.

Recall that this control problem is said to be CONSTRAINED if the control constraint set Ω is such that

$$\Omega \subset \mathbb{R} \tag{3}$$

and UNCONSTRAINED if

$$\Omega = \mathbb{R} \tag{4}$$

In the unconstrained control situation a variational approach may be employed to generate the required necessary conditions. If however the control is constrained then a non-classical calculus of variations problem is obtained and the variational approach breaks down. The solution in this latter case is the subject of PONTRYAGIN'S MINIMUM (Maximum) PRINCIPLE

For simplicity we consider the case of an unconstrained control and indicate the modification to be made to the results to cover the constrained control case.

Adopting a variational approach the spaces of competing functions are

$$\left.\begin{aligned} u &: \quad \Sigma_1 = C_s[t_0, L] \\ x &: \quad \Sigma_2 = \{x(t)\,;\, x(t) \in C_s^1[t_0, L],\ x(t_0) = x^{(0)}\} \end{aligned}\right\} \tag{5}$$

The relevant sets of admissible variations are then

$$\left.\begin{aligned} u &: \quad H_1 = C_s[t_0, L] \\ x &: \quad H_2 = \{\xi(t)\,;\, \xi(t) \in C_s^1[t_0, L],\ \xi(t_0) = 0\} \end{aligned}\right\} \tag{6}$$

From (1) note that

$$\int_{t_0}^{L} \lambda(t)\{f(x,u,t) - \dot{x}\}\, dt = 0 \tag{7}$$

for an arbitrary function $\lambda(t) \in C_s^1[t_0, L]$, say.

We thus assume that the minimisation of the performance index (2) subject to (1) is equivalent to the unconstrained minimisation of

$$J[x,u] \triangleq \phi(x(L)) + \int_{t_0}^{L} [F(x(t), u(t), t) + \lambda\{f(x(t), u(t), t) - \dot{x}\}]\, dt \tag{8}$$

Defining the HAMILTONIAN

$$H(x,u,\lambda,t) \triangleq F(x,u,t) + \lambda f(x,u,t) \tag{9}$$

(8) becomes

$$J[x,u] = \phi(x(L)) + \int_{t_0}^{L} \{H(x,u,\lambda,t) - \dot{x}\}\, dt \tag{10}$$

Suppose $x_0(t)$, $u_0(t)$ denote the optimal pair of functions and consider the variations

$$\left.\begin{aligned} u(t,\theta) &= u_0(t) + \theta\, \eta(t) \\ x(t,\theta) &= x_0(t) + \theta\, \xi(t) \end{aligned}\right\} \qquad (11)$$

where $\eta \in H_1$, $\xi \in H_2$.

The performance index corresponding to this variation is

$$J[x_0+\theta\xi, u_0+\theta\eta] = \phi(x_0(L)+\theta\xi(L)) + \int_{t_0}^{L}\left\{H(x_0+\theta\xi, u_0+\theta\eta, \lambda, t) - \lambda(\dot{x}_0+\theta\dot{\xi})\right\}dt \qquad (12)$$

To invoke the ~~general~~ necessary condition of theorem 1 we require the first variation of $J[x,u]$. Now

$$\delta J[\xi,\eta] \hat{=} \frac{d}{d\theta} J[x_0+\theta\xi, u_0+\theta\eta]\Big|_{\theta=0}$$

$$= \frac{d}{dx}\phi(x_0(L))\cdot\xi(L) + \int_{t_0}^{L}\left\{\frac{\partial H}{\partial x}(x_0,u_0,\lambda,t)\,\xi + \frac{\partial H}{\partial u}(x_0,u_0,\lambda,t)\,\eta - \lambda\dot{\xi}\right\}dt \qquad (13)$$

Integrating the last term by parts and observing that $\xi \in H_2$ then yields

$$\delta J[\xi,\eta] = \left\{\frac{d}{dx}\phi(x_0(L)) - \lambda(L)\right\}\cdot\xi(L) + \int_{t_0}^{L}(H_{x_0}+\dot{\lambda})\,\xi\, dt + \int_{t_0}^{L} H_{u_0}\,\eta\, dt \qquad (14)$$

Note that (14) ~~already~~ indicates that we are already beginning to specify the function $\lambda(t)$. (We require $\lambda(t) \in C_s^1[t_0,L]$ for the integration by parts). We now complete the specification of $\lambda(t)$ by ~~also~~ insisting that

$$\left.\begin{aligned} \frac{d\lambda}{dt} &= -\frac{\partial H}{\partial x}(x_0,u_0,\lambda,t) \\ \lambda(L) &= \frac{d}{dx}\phi(x_0(L)) \end{aligned}\right\} \qquad (15)$$

Under appropriate differentiability conditions the existence and uniqueness theorem for solutions of (15) would hold. Then (15) determines $\lambda(t)$

In view of (15), the relation (14) reduces to

$$\delta J[\xi,\eta] = \int_{t_0}^{L} H_{u_0}\,\eta\, dt \qquad (16)$$

Invoking theorem 1 then shows that a necessary condition for a minimum is

$$\int_{t_0}^{L} H_{u_0}\,\eta\, dt = 0 \qquad (17)$$

for all $\eta(t) \in H_1$. In view of the arbitrariness of $\eta(t)$ we conclude that

$$H_{u_0} = 0 \qquad (18)$$

The condition (18) indicates that the optimal value of $u(t)$ at all times makes the value of $H(x,u,\lambda,t)$ stationary. In fact further analysis based on the SECOND VARIATION of $J[x,u]$ reveals that the optimal value of $u(t)$ at all times minimises H.

The necessary conditions for a minimum of $J[x,u]$ are then summarised as follows (we give the vector version of the above conditions)

THEOREM 3: Let $u_0(t)$ be an admissible control ($\in U$) which operates for a period of time $[t_0, L]$ on the system

$$\dot{x}(t) = f(x(t), u(t), t) \qquad (x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^l) \tag{19}$$

initially in state $x^{(0)}$. In order that $u_0(t)$ be optimal it is necessary that there exists an n-vector function $\lambda(t)$ such that

(i) if $x_0(t)$ is the solution of (19) corresponding to $u_0(t)$, then $x_0(t)$, $\lambda(t)$ are solutions of the equations

$$\left.\begin{aligned} \frac{dx_0}{dt} &= \frac{\partial H}{\partial \lambda}(x_0(t), u_0(t), \lambda(t), t) \ \left(\equiv f(x_0(t), u_0(t), t)\right) \\ \frac{d\lambda}{dt} &= -\frac{\partial H}{\partial x}(x_0(t), u_0(t), \lambda(t), t) \end{aligned}\right\} \tag{20}$$

satisfying

$$x(t_0) = x^{(0)} \quad ; \quad \lambda(L) = \frac{\partial \phi}{\partial x(L)}(x_0(L)) \tag{21}$$

(ii) the function $H(x_0(t), u, \lambda(t), t)$ has an absolute minimum as a function of u at $u = u_0(t)$ for all $t \in [t_0, L]$.

$$\text{i.e.} \quad \min_{u \in \Omega} H(x_0(t), u, \lambda(t), t) = H(x_0(t), u_0(t), \lambda(t), t) \tag{22}$$

(iii) the function

$$M(t) \triangleq H(x_0(t), u_0(t), \lambda(t), t) \tag{23}$$

satisfies the relations

$$M(t) = -\int_t^L \frac{\partial H}{\partial t}(x_0(t), u_0(t), \lambda(t), t)\, dt \tag{24}$$

Stated in this form and when $\Omega \neq \mathbb{R}^l$ the above theorem is known as PONTRYAGIN'S MINIMUM PRINCIPLE.

Notes: (i) If L is fixed but $x(L)$ is constrained to satisfy

$$\psi(x(L)) = 0 \qquad (\psi \text{ a } q\text{-vector}, q \le n) \tag{25}$$

then

$$\Phi(x(L)) \triangleq \phi(x(L)) + \nu^T \psi(x(L))$$

replaces $\phi(x(L))$ in (21), where ν is an arbitrary constant q-vector.

(ii) If L and $x(L)$ are (not fixed) constrained to satisfy

$$\psi(x(L), L) = 0 \qquad (\psi \text{ a } q\text{-vector}, q \le n) \tag{26}$$

then

$$\Phi(x(L), L) \triangleq \phi(x(L), L) + \nu^T \psi(x(L), L) \tag{27}$$

replaces $\phi(x(L))$ in (21) where ν is again an arbitrary constant q-vector. There is then an additional condition appearing with (24), namely

$$\frac{\partial \Phi}{\partial L}(x_0(L), L) + M(L) = 0 \tag{28}$$

This last condition is the analogue of the transversality condition in the Calculus of Variations.

(iii) The solution to an optimal control problem involves the solution of the $2n$ ordinary differential equations (20) known as the STATE and COSTATE equations. Also at each time t in the integration process $u(t)$ is found by minimising the Hamiltonian H. The solution is complicated even numerically by the fact that the boundary conditions (21) are given at t_0 and L. Thus optimal control problems invariably give rise to "two-point boundary value problems".