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SMR/161 - 02

COLLEGE ON

REPRESENTATION THEORY OF LIE GROUPS

(4 November - 6 December 1985)

TOPOLOGICAL GROUPS - I

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These are preliminary lecture notes, intended only for distribution to participants.

CHAPTER I

TOPOLOGICAL GROUPS

§1. GENERALITIES

Let G be a group and consider on G a topology such that

1) the map $G \times G \to G$ defined by $(x,y) \mapsto xy^{-1}$ is continuous

REMARKS. 1. Condition 1) is equivalent to the fact that the two maps $(x,y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.
2. The discrete topology in G satisfies condition 1).

For any given element $g \in G$, the left and right translations $x \mapsto gx$, $x \mapsto xg$ and the inner automorphism $x \mapsto gxg^{-1}$ are homeomorphisms of the group G onto itself.
Since by varying $g \in G$ the left or right translations act transitively on G, the topology of G is completely determined by a fundamental system of neighborhoods of the identity element e. If U is a neighborhood of e, also $U^{-1} = \{g \in G : g^{-1} \in U\}$ is a neighborhood of e.

LEMMA 1.1 *The closure $\bar{A}$ of a subset $A \subset G$ is the intersection of all the subsets UA (or, equivalently, AU) when U varies among the neighborhoods of e.*

PROOF. It turns out that $x \in \bar{A}$ if, and only if, for every neighborhood U of e, $Ux \cap A \neq \emptyset$ (or, equivalently, $xU \cap A \neq \emptyset$), i.e., if, and only if, there exist $y \in A$, $z \in U$ such that $zx = y$ ($xz = y$). Therefore $x \in \bar{A}$ if, and only if, $x \in U^{-1}A$ ($x \in AU^{-1}$). Since U^{-1} is a generic neighborhood of e, the assertion follows. □

A neighborhood W of e such that $W^{-1} = W$ is called a symmetric neighborhood of e. Clearly $W = U \cap U^{-1}$ is a symmetric neighborhood of e. This fact implies that the symmetric neighborhoods of e are a fundamental system of neighborhoods of e.

For a given neighborhood B of e, let U_1 and U_2 be neighborhoods of e such that $U_1 U_2 \subset B$. The set $W = U_1 \cap U_1^{-1} \cap U_2 \cap U_2^{-1}$ clearly satisfies the conditions of the following

LEMMA 1.2. *For every neighborhood B of e there exists a symmetric neighborhood W of e such that $W^2 \subset B$.*

DEFINITION 1.3 *A topological group is a group G endowed with a T_0 topology satisfying condition 1).*

By remark 2, any group G with the discrete topology is a topological group; it will be called a discrete group.

Other examples of topological groups are the additive subgroups of the field of real numbers $\mathbb{R}$, with the Euclidean topology of the real line, or the additive subgroups of the complex field $\mathbb{C}$, with the Euclidean topology of the plane, or the multiplicative subgroups of $\mathbb{R}\setminus\{0\}$ and $\mathbb{C}\setminus\{0\}$, with the induced topology.

A <u>homomorphism</u> of a topological group G into a topological group G' is a continuous homomorphism of G into G'.

A <u>topological subgroup</u> of a topological group G is a subgroup of G, which is a topological group with respect to the induced topology.

The reason for condition T_0 on the topology lies in the following

<u>PROPOSITION 1.4.</u> <u>Let G be a group with a topology satisfying condition 1), and let F be the closure of e. F is a closed subgroup of G and is minimal with respect to set inclusion, among the closed subgroups of G. Moreover F is a normal subgroup of G.</u>

<u>PROOF</u> For every $x \in F$ we have $e \in Fx^{-1}$. Since Fx^{-1} is a closed subset, it turns out that $F \subset Fx^{-1}$, i.e. $Fx \subset F$, for all $x \in F$ and therefore $FF = F$. On the other hand, since F^{-1} is closed and since $e \in F^{-1}$, we have $F \subset F^{-1}$ and hence $F = F^{-1}$

Therefore F is a subgroup of G, and, by definition, is contained in every closed subgroup of G.

For all $g \in G$, gFg^{-1} is closed and contains the identity element e of G. Hence $F \subset gFg^{-1}$, yielding $g^{-1}Fg \subset F$ for all $g \in G$, and therefore $g^{-1}Fg = F$ for all $g \in G$. This means that F is a normal subgroup of G. □

Since F is closed, all the F-cosets are closed subsets, and then the quotient topology on G/F is T_1.

Let H be a closed normal subgroup of G, and let G/H be the quotient group, endowed with the quotient topology. G/H is a topological group. By Proposition 1.4, $F \subset H$, and hence there exists a unique surjective homomorphism $\psi : G/F \to G/H$ such that the following diagram commutes

$$\begin{tikzcd} & G \arrow[dl, "\pi_F"'] \arrow[dr, "\pi_H"] & \\ G/F \arrow[rr, "\psi"'] & & G/H \end{tikzcd}$$

<u>Exercise</u> Prove that the homomorphism ψ is continuous, i.e. that ψ is a homomorphism of topological groups

<u>LEMMA 1.5</u> <u>If G is a topological group, then for all $g \neq e$ there is a symmetric neighborhood W of e such that $W \cap gW = \emptyset$.</u>

Proof. Since G is a T_1 topological space, if every neighborhood of e contains g, then there exists a neighborhood V of g not containing e. In this case gV^{-1} would be a neighborhood of e not containing g. Therefore there exists a neighborhood B of e which does not contain g. Let now W be a neighborhood of e as in Lemma 1.2. If

$$W \cap gW \neq \emptyset$$

i.e. if there exist w_1 and w_2 in W such that

$$w_2 = g w_1$$

we obtain a contradiction since

$$g = w_2 w_1^{-1} \in W W^{-1} = W^2 \subset B. \qquad \square$$

Given two distinct points x and y in G, let W be a neighborhood of e such that

$$W \cap y^{-1} x W = \emptyset .$$

It turns out that $yW \cap xW = \emptyset$. It follows that

PROPOSITION 1.5 *A topological group is a Hausdorff space*

Since a Hausdorff space is a T_1-space, Proposition 1.4 yields the

LEMMA 1.7 *Every continuous homomorphism of the group G in a topological group is constant on the closure of e in G.*

We will now turn our attention to the subgroups of G.

PROPOSITION 1.8 *Every open subgroup of G is a closed subgroup.*

Proof: If $H \subset G$ is a subgroup, then its complement $\complement H$ can be obtained as

$$\complement H = \bigcup_{g \notin H} gH$$

which is an open set if H is open. $\square$

A criterion to decide if a subgroup of a topological group is open is given by the following

LEMMA 1.9 *A subgroup H of G is open, if, and only if, its interior $\mathring{H}$ is not empty.*

Proof. If H has an internal point, g, i.e. if there exists a neighborhood U of e in G such that $gU \subset H$, then for all $h \in H$ it turns out that

$$hU = hg^{-1} g U \subset hg^{-1} H \subset H .$$

Therefore every $h \in H$ is internal, i.e. H is open $\square$.

The following result extends Proposition 1.6 to the quotient groups.

PROPOSITION 1.10 *Let H be a closed subgroup of the topological group G, and let G/H be the quotient space (the set whose elements are the left-cosets) with the quotient topology. The canonical* projection $\pi : G \to G/H$ is an open map. Moreover G/H is a Hausdorff space.

Proof. If $A \subset G$ is open, AH is open too. Since for any $A \subset G$

$$\pi^{-1}(\pi(A)) = AH$$

$\pi(A)$ is an open set in G/H, and therefore π is an open map.

Let now x and y be two points of G such that $\pi(x) \neq \pi(y)$. Since yH is a closed set, there exists a neighborhood B of e such that $Bx \cap yH = \phi$. If W is a symmetric neighborhood of e such that $W^2 \subset B$, it turns out that

$$WxH \cap WyH = \phi .$$

In fact, if this is not the case, there exist two points w_1 and w_2 in W and two points h_1, h_2 in H such that

$$w_1 x h_1 = w_2 y h_2 ,$$

i.e.

$$Bx \ni w_2^{-1} w_1 x = y h_2 h_1^{-1} \in yH$$

which contradicts $Bx \cap yH = \phi$. Therefore the two neighborhoods $\pi(Wx)$ and $\pi(Wy)$ of $\pi(x)$ and $\pi(y)$ are disjointed.

COROLLARY 1.11. *If H is a closed normal subgroup of G, the quotient group G/H, endowed with the quotient topology, is a topological group. The canonical projection of G onto G/H is an open homomorphism.*

A subgroup of G on which G induces the discrete topology is called a discrete subgroup of G. A subgroup H of G is a discrete subgroup if, and only if, the identity e is an isolated point of H.

PROPOSITION 1.12 *A discrete subgroup of G is closed.*

Proof.

Let H be a non-closed subgroup of G. There exists a point $g \in \overline{H} \setminus H$ for which every neighborhood contains some element of H different from g. Let U be any neighborhood of e in G. There exists a symmetric neighborhood V of e in G such that $V^2 \subset U$. Let h' and h'' be points in H, $h' \neq h''$, both different from g, and contained in gV. It turns out that $g^{-1}h' \in V$, $g^{-1}h'' \in V$ and hence

$$h'^{-1} h'' = (g^{-1}h')^{-1}(g^{-1}h'') \in V^2 \subset U$$

Therefore every neighborhood U of e in G contains some element of H different from e, i.e. e is not an isolated point of H, and thus H is not discrete $\square$

Let $\{G\}_{\lambda \in \Lambda}$ a family of topological groups and let $G = \prod_{\lambda \in \Lambda} G_\lambda$. Let us define in G a product by setting, for $g = (g_\lambda)$ and $g' = (g'_\lambda)$, $(g_\lambda, g'_\lambda \in G_\lambda)$

$$gg' = (g_\lambda g'_\lambda)$$

In this way G has a group structure with identity element $e = (e_\lambda)$. If G is endowed with the product topology, it follows that

THEOREM 1.13 G is a topological group (which we will call the product of the topological groups G_λ)

Proof

Let U be a neighborhood of gg'^{-1} in G, for the product topology. There exists a finite set $\lambda_1, \dots, \lambda_n$ of indexes $\lambda \in \Lambda$ and - in correspondence - a finite number of neighborhoods, $U_{\lambda_1}, \dots, U_{\lambda_n}$ of $g_{\lambda_1} g'^{-1}_{\lambda_1}, \dots, g_{\lambda_n} g'^{-1}_{\lambda_n}$ (respectively) in $G_{\lambda_1}, \dots, G_{\lambda_n}$, such that, by setting $U_\lambda = G_\lambda$ for $\lambda \notin \{\lambda_1, \dots, \lambda_n\}$, it turns out that

$$\prod_{\lambda \in \Lambda} U_\lambda \subset U$$

Since $G_{\lambda_1}, \dots, G_{\lambda_n}$ are topological groups, there exist neighborhoods V_{λ_i} and V'_{λ_i} of g_{λ_i} and g'_{λ_i} in G_{λ_i} such that

$$V_{\lambda_i} V'^{-1}_{\lambda_i} \subset U_{\lambda_i} \qquad (i = 1, \dots, n)$$

If we set $V_\lambda = V'_\lambda = G_\lambda$ for $\lambda \notin \{\lambda_1, \dots, \lambda_n\}$ it turns out that the sets

$$V = \prod_{\lambda \in \Lambda} V_\lambda \qquad V' = \prod_{\lambda \in \Lambda} V'_\lambda$$

are neighborhoods of g and g' in G for which

$$VV'^{-1} \subset U. \qquad \square$$

§2. A THEOREM OF PONTRYAGIN AND INVARIANT METRICS.

We have seen that condition T_0 implies condition T_2 on a topological group G. We will prove now the

THEOREM 2.1 (L.S. Pontryagin). Every topological group is a Tychonoff space.

In other words we will prove that, given a closed set $F \neq \emptyset$ and a point $g \notin F$ in a topological group G, there exists a continuous function $f: G \to I$ ($I = [0,1]$) such that $f(g) = 0$, $f(F) = \{1\}$

First of all we will prove a Lemma, whose proof has the same structure of the Urysohn Lemma.

Let I^* be the subset of I whose elements are the rational numbers of type $\frac{m}{2^n}$, with $m, n \in \mathbb{N}$ and $m \leq 2^n$.

LEMMA 2.2 Given any sequence $\{U_\nu\}_{\nu \in \mathbb{N}}$ of neighborhoods of e, in a topological group G, there exists a family $\{V_r\}_{r \in I^*}$ of neighborhoods of e with $V_1 = U_0 \cap U_0^{-1}$ satisfying the following conditions:

i) $V_{\frac{1}{2^n}}$ is symmetric and $V^2_{\frac{1}{2^{n+1}}} \subset V_{\frac{1}{2^n}} \cap U_n$ for $n = 0, 1, 2 \dots$

ii) $V_{\frac{2k}{2^{n+1}}} = V_{\frac{k}{2^n}}$ for $k = 1, \dots, 2^n$, $n = 0, 1, 2, \dots$;

iii) $V_{\frac{2k+1}{2^{n+1}}} = V_{\frac{1}{2^{n+1}}} \cdot V_{\frac{k}{2^n}}$ for $k = 1, \dots, 2^n$, $n = 0, 1, 2, \dots$;

iv) $V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} \subset V_{\frac{m+1}{2^n}}$ for $0 < m \leq 2^n - 1$;

v) $V_r \subset V_{r'}$ if $r < r' \leq 1$, $r, r' \in I^*$.

Proof.

Let us set $W_0 = U_0 \cap U_0^{-1}$, and construct a sequence $\{W_n\}_{n\in\mathbb{N}}$ of symmetric neighborhoods of e, such that

$$W_{n+1}^2 \subset W_n \cap U_n \quad , \text{ for } n=0,1,2,\dots$$

For every $n=0,1,2,\dots$ let us set

$$V_{\frac{1}{2^n}} = W_n$$

and define $V_{\frac{m}{2^n}}$, for $m=1,\dots,2^n$, by ii) and iii).

By relation ii) every V_r depends on r but not on the binary representation of r. Obviously relation i) holds.

Let us prove iv)

If $m=2k$ ($k=1,2,\dots,2^{n-1}$) we have, by ii), iii),

$$V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} = V_{\frac{1}{2^n}} V_{\frac{2k}{2^n}} = V_{\frac{1}{2^n}} V_{\frac{k}{2^{n-1}}} = V_{\frac{2k+1}{2^n}} = V_{\frac{m+1}{2^n}}$$

which proves iv) when m is even. If $m=2k+1$ ($k=0,1,2,\dots,2^{n-1}-1$) we have

$$(1) \qquad V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} = V_{\frac{1}{2^n}} V_{\frac{2k+1}{2^n}} = V_{\frac{1}{2^n}} V_{\frac{1}{2^n}} V_{\frac{k}{2^{n-1}}} \subset$$
$$\subset V_{\frac{1}{2^{n-1}}} V_{\frac{k}{2^{n-1}}}$$

by iii) and i), whereas by ii)

$$(2) \qquad V_{\frac{m+1}{2^n}} = V_{\frac{2k+2}{2^n}} = V_{\frac{k+1}{2^{n-1}}}.$$

If $n=1$, relation iv) becomes

$$V_{\frac12} V_{\frac12} \subset V_{\frac22}$$

which coincides with i), since by ii) $V_{\frac22} = V_1 = W_0 \subset U_0$.

By induction, suppose we have proved relation iv) for $n-1$, i.e. that for $1 \le k \le 2^{n-1}-1$

$$V_{\frac{1}{2^{n-1}}} V_{\frac{k}{2^{n-1}}} \subset V_{\frac{k+1}{2^{n-1}}}.$$

Since iv) has already been proved for m even, let us consider $m=2k+1$. Condition $0<m\le 2^n-1$ is equivalent to $0\le k\le 2^{n-1}-1$. From (1), (3) and (2) it follows that

$$V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} = V_{\frac{1}{2^n}} V_{\frac{2k+1}{2^n}} \subset V_{\frac{1}{2^{n-1}}} V_{\frac{k}{2^{n-1}}} \subset V_{\frac{k+1}{2^{n-1}}} = V_{\frac{m+1}{2^n}},$$

and iv) is completely proved.

Finally, let us prove relation v). If we set $r = \frac{m}{2^n}$ and $r' = \frac{m'}{2^n}$, for suitable $n\ge 1$ and $1\le m\le 2^n$, $1\le m'\le 2^n$, condition $r<r'$ is equivalent to $m'\le 2^n$ and $m+1\le m'$.

By iv) and by the symmetry of $V_{\frac{1}{2^n}}$ it turns out that

$$V_{\frac{m}{2^n}} \subset V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} \subset V_{\frac{m+1}{2^n}}.$$

If $m+1=m'$ the assertion is proved. Otherwise it is enough to repeat the above argument □

Let us now prove Theorem 2.1. Notice at first that it is enough to prove the existence of f in the case in which $g=e$. Let, thus, F be a closed set of G not containing e. By applying the above Lemma to the sequence $\{U_n\}_{n\in\mathbb{N}}$ with $U_n = G\setminus F$ for all $n\in\mathbb{N}$, let us construct the family $\{V_r\}_{r\in I^*}$, and define

$f : G \to I = [0,1]$, by

$$\begin{cases} f(x) = 0 & \text{if } x \in V_r \text{ for all } r \in I^* \\ f(x) = 1 & \text{if } x \notin V_1, \\ f(x) = \sup\{r \in I^* \mid x \notin V_r\}. \end{cases}$$

Since $e \in V_r$ for every $r \in I^*$, we have that $f(e) = 0$. Moreover, since $V_1 = U_0 \cap U_0^{-1} \subset G \setminus F$, it is $f(x) = 1$. The fact that $f(G) \subset I$ is also clear; hence the last thing to prove is that f is continuous.

Let us fix $\varepsilon > 0$, and let p be a positive integer such that $\frac{1}{2^p} < \varepsilon$.

1) Let $x \in G$ be such that $0 < f(x) < 1$, and let m, n be two positive integers such that $n > p+1$, $m > 3$ and

(4) $$\frac{m-2}{2^n} < f(x) < \frac{m}{2^n}.$$

It turns out that

(5) $$x \in V_{\frac{m}{2^n}} \setminus V_{\frac{m-2}{2^n}}.$$

If $y \in V_{\frac{1}{2^n}} x$, we have

$$y \in V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} \subset V_{\frac{m+1}{2^n}}$$

$$yx^{-1} \in V_{\frac{1}{2^n}},$$

and hence, by the symmetry of $V_{\frac{1}{2^n}}$, $xy^{-1} \in V_{\frac{1}{2^n}}$, i.e.

$$x \in V_{\frac{1}{2^n}} y.$$

If $y \in V_{\frac{m-3}{2^n}}$, then the fact that

$$x \in V_{\frac{1}{2^n}} V_{\frac{m-3}{2^n}} \subset V_{\frac{m-2}{2^n}}$$

contradicts (5). Therefore $y \notin V_{\frac{m-3}{2^n}}$, and hence

$$y \in V_{\frac{m+1}{2^n}} \setminus V_{\frac{m-3}{2^n}}$$

whence

(6) $$\frac{m-3}{2^n} \leq f(y) \leq \frac{m+1}{2^n}.$$

By (4) and (6), it follows that, for every $y \in V_{\frac{1}{2^n}} x$,

$$|f(x) - f(y)| \leq \frac{4}{2^n} = \frac{1}{2^{n-2}} \leq \frac{1}{2^p} < \varepsilon.$$

2) If $f(x) = 0$, let $n > p$ and let $y \in V_{\frac{1}{2^n}} x$. Since $x \in V_{\frac{1}{2^n}}$, it turns out that

$$y \in V_{\frac{1}{2^n}}^2 \subset V_{\frac{1}{2^{n-1}}} \subset V_{\frac{1}{2^p}}$$

and hence

$$f(y) \leq \frac{1}{2^p} < \varepsilon$$

3) If $f(x) = 1$, let us choose $n > p$, and let $y \in V_{\frac{1}{2^n}} x$.

By the symmetry of $V_{\frac{1}{2^n}}$ it follows that, as before,

$$x \in V_{\frac{1}{2^n}}\, y .$$

If $y \in V_{\frac{m}{2^n}}$ with $0 < m \leq 2^n - 2$, then

$$x \in V_{\frac{1}{2^n}} V_{\frac{m}{2^n}} \subset V_{\frac{m+1}{2^n}}$$

and then

$$f(x) \leq \frac{m+1}{2^n} < 1 .$$

Therefore $y \notin V_{\frac{m}{2^n}}$, for every integer such that $0 < m < 2^n - 2$, and then

$$f(y) \geq 1 - \frac{2}{2^n}$$

whence

$$|f(x) - f(y)| = 1 - f(y) < \frac{2}{2^n} \leq \frac{1}{2^p} < \varepsilon .$$

Theorem 2.1 is now proved. □

Since, for a topological space, Tychonoff implies regular, from Theorem 2.1 it follows

COROLLARY 2.3. A topological group is a regular space.

By the Urysohn's Theorem.
All topological groups with countable base are metrizable

We will obtain this last fact directly, by proving the following

THEOREM 2.4 (G. Birkhoff - S. Kakutani). Let G be a topological group. If e has a countable fundamental system of neighborhoods, then the topology of G is defined by a distance d, which is left-invariant with respect to the action of the group, i.e. a distance such that

$$d(gx, gy) = d(x, y)$$

for all $x, y, g \in G$.

Proof.

Let $\{W_n\}_{n \in \mathbb{N}}$ be a countable fundamental system of neighborhoods of e. If we set

$$U_n = \bigcap_{i=1}^{n} W_i \qquad \text{for all } n = 0, 1, 2, \dots$$

we obtain a fundamental system of neighborhoods of e, $\{U_n\}_{n \in \mathbb{N}}$, such that

$$U_n \supset U_{n+1} \qquad \text{for all } n = 0, 1, 2, \dots$$

Let $\{V_r\}_{r \in I^*}$ be a family of neighborhoods satisfying all the conditions of Lemma 2.2.; The neighborhoods $V_r V_r^{-1}$ are all symmetric. Let us define a function

$$f : G \times G \to \mathbb{R}^+$$

by setting

1) $f(x, y) = 0$ if $y \in x V_r V_r^{-1}$ for all $r \in I^*$
2) $f(x, y) = \sup\{ r \in I^* \mid y \notin x V_r V_r^{-1} \}$.

Since for all $g \in G$, the fact that

$$y \notin x V_r V_r^{-1}$$

is equivalent to

$$gy \notin gx V_r V_r^{-1},$$

it turns out that

$$f(gx, gy) = f(x, y)$$

for all $g \in G$. Since $V_r V_r^{-1}$ is symmetric, the condition

$$y \notin x V_r V_r^{-1}$$

is equivalent to

$$x \notin y V_r V_r^{-1}.$$

Hence

$$f(x,y) = f(y,x).$$

If $f(x,y)=0$ we have by definition

$$x^{-1}y \in V_{\frac{1}{2^n}} V_{\frac{1}{2^n}}^{-1} \subset V_{\frac{1}{2^n}}^2$$

for all $n = 1, 2, 3, \ldots$. Since $V^2_{\frac{1}{2^n}} \subset V_{\frac{1}{2^{n-1}}} \cap U_{n-1}$ if $n>0$, (by 1) of Lemma 2.2) and since

$$\bigcap_{n\in\mathbb{N}} U_n = \{e\},$$

we have $x^{-1}y = e$. In conclusion, if $f(x,y)=0$, we get $x=y$, and conversely.

If we set

$$d(x,y) = \sup_{z\in G} |f(x,z) - f(y,z)|$$

it turns out that

(7) $$d(x,y) \le 2, \quad d(x,y) = d(y,x)$$
$$d(x,y) \ge f(x,y) \ge 0.$$

Obviously $d(x,y)=0$ if, and only if, $x=y$. Moreover

$$d(x,y) = \sup_{z\in G} |f(x,z)-f(y,z)| \le \sup_{z\in G} |f(x,z)-f(v,z)| +$$
$$+ \sup_{z\in G} |f(v,z)-f(y,z)| = d(x,v) + d(v,y),$$

for all $v \in G$. The distance d is invariant:

$$d(gx,gy) = \sup_{z\in G} |f(gx,z)-f(gy,z)| = \sup_{z\in G} |f(x,g^{-1}z)-f(y,g^{-1}z)|$$
$$= \sup_{z\in G} |f(x,z)-f(y,z)| = d(x,y).$$

We have to prove, now, that the invariant distance d defines on G a topology which is equivalent to the original one. By the invariance of d, it is enough to consider the neighborhoods of e.

Let us show, first of all, that, if $x \in V_{\frac{1}{2^{n+1}}}$ $(n \ge 1)$ then

$$d(x,e) \le \frac{1}{2^{n-1}}.$$

For any $z \in G$, let us calculate $|f(x,z) - f(z,e)|$. There exists an entire number m, with $1 \le m \le 2^{n+1}$ such that

(8) $$\frac{m-2}{2^{n+1}} < f(z,e) < \frac{m}{2^{n+1}}.$$

The left inequality implies that

(9) $$f(x,z) \ge \frac{m-3}{2^{n+1}}.$$

In fact, if $f(x,z) < \frac{m-3}{2^{n+1}}$ for some $m \ge 3$, i.e. if

$$zx^{-1} \in V_{\frac{m-3}{2^{n+1}}} V^{-1}_{\frac{m-3}{2^{n+1}}},$$

then

$$z \in x V_{\frac{m-3}{2^{n+1}}} V^{-1}_{\frac{m-3}{2^{n+1}}} \subset V_{\frac{1}{2^{n+1}}} V_{\frac{m-3}{2^{n+1}}} V^{-1}_{\frac{m-3}{2^{n+1}}} \subset V_{\frac{m-2}{2^{n+1}}} V^{-1}_{\frac{m-3}{2^{n+1}}} \subset$$

$$\subset V_{\frac{m-2}{2^{n+1}}} V^{-1}_{\frac{m-2}{2^{n+1}}} .$$

and hence $f(z,e) \le \frac{m-2}{2^{n+1}}$.

From the right inequality, in (8), it follows that

$$z \in V_{\frac{m}{2^{n+1}}} V^{-1}_{\frac{m}{2^{n+1}}}$$

and then

$$x^{-1}z \in V_{\frac{1}{2^{n+1}}} V_{\frac{m}{2^{n+1}}} V^{-1}_{\frac{m}{2^{n+1}}} \subset V_{\frac{m+1}{2^{n+1}}} V^{-1}_{\frac{m}{2^{n+1}}} \subset V_{\frac{m+1}{2^{n+1}}} V^{-1}_{\frac{m+1}{2^{n+1}}}$$

whence

(10) $$f(x,z) < \frac{m+1}{2^{n+1}} .$$

Therefore (8) implies (9) and (10), and

$$|f(x,z) - f(z,e)| < \frac{3}{2^{n+1}} < \frac{1}{2^{n-1}} .$$

If $D(e,\rho)$ is the open disk centered at e, with radius $\rho > 0$, for the distance d, then it turns out that, for $n \ge 1$

$$V_{\frac{1}{2^{n+1}}} \subset D(e, \frac{1}{2^{n-1}}) ,$$

i.e. the topology of G is finer than the topology induced by d on G.

Let U be a neighborhood of e for the original topology of G, and let U_n be such that

$$U_n \subset U .$$

If $d(e,x) < \frac{1}{2^{n+1}}$, it follows that $f(x,e) < \frac{1}{2^{n+1}}$ and that

$$x \in V^2_{\frac{1}{2^{n+1}}} \subset V_{\frac{1}{2^n}} \cap U_n \subset U .$$

Therefore $D(e, \frac{1}{2^{n+1}}) \subset U$, which implies that the topology defined on G by the distance d is finer than the topology of G. In conclusion, the two topologies are equivalent. □

REMARKS. If the topology of G is metrizable, then e has a countable fundamental system of neighborhoods.

With analogous considerations to those used above, one can construct, on the topological group G - which satisfies the hypotheses of Theorem 2.4 - a right-translations invariant distance.

COROLLARY 2.5 *Let G be a topological group whose identity element has a countable fundamental system of neighborhoods. For every closed subgroup H of G, the quotient group G/H, with the quotient topology, is metrizable.*

Let d be the invariant distance constructed above, on G. If we set

$$\delta(xH, yH) = \operatorname{Inf}_{h,k \in H} (d(xh, yk)),$$

δ defines a distance on G/H which generates the quotient topology. The proof is left as an exercise.

EXAMPLE. If G is (the additive group of) $\mathbb{R}$ with the Euclidean topology, then the hypotheses of Theorem 2.4 are satisfied. The distance defined by the absolute value $(x,y) \mapsto |x-y|$ is clearly invariant. On the other hand, by (7), this distance cannot be obtained with the procedure used in the proof of Theorem 2.4. One can prove, as an exercise, that with the choice

$$V_n = \left] -\frac{1}{2^n}, \frac{1}{2^n} \right[$$

one gets the invariant distance

$$d(x,y) = \min(|x-y|, 1).$$

§3. CONNECTEDNESS.

Let G be a topological group.

PROPOSITION 3.1 *The connected component, G_1, of the identity element e of G is a closed normal subgroup of G. Moreover, for all $g \in G$, gG_1 is the connected component of g in G.*

Proof.

For all $x \in G_1$, $x^{-1}G_1$ is connected and contains e. Therefore $x^{-1}G_1 \subset G_1$ for all $x \in G_1$ and hence $G_1^{-1}G_1 \subset G_1$, $G_1^{-1}G_1 = G_1$, i.e. G_1 is a subgroup of G. Similarly, for every $g \in G$, gG_1g^{-1} is connected and contains e, then

$$gG_1g^{-1} \subset G_1$$

i.e. G_1 is a normal subgroup of G.

Let $G_{(g)}$ be the connected component of $g \in G$. Since $g \in gG_1$ and gG_1 is connected, it turns out $gG_1 \subset G_{(g)}$. Similarly, one can prove that $g^{-1}G_{(g)} \subset G_1$, and hence that $G_{(g)} = gG_1$ □

LEMMA 3.2 *If U is an open neighborhood of e in G_1, U generates the subgroup G_1*

Proof.

Let V be an open symmetric neighborhood of e such that $V \subset U$ and let G' be the subgroup generated by V. Every element of G' is is the product of a finite number of elements of V. Thus G' is open and then also closed. Since $U \subset G_1$ it turns out that $G' \subset G_1$, and being G_1 connected we obtain $G' = G_1$ □

COROLLARY 3.3. If G is connected, every neighborhood of e generates G.

If the connected component of every point in G is the point itself, the group G is called totally disconnected (or of 0-dimension). Obviously G is totally disconnected if, and only if, the connected component of e is the set $\{e\}$. Moreover it is clear that, if G is totally disconnected, all the connected subsets of G do not contain more than one point.

Example Let R be the additive group of real numbers, with the topology of the real line, and let Q be the additive group of rational numbers. The subgroup Q endowed with the induced topology is a totally disconnected topological group. On the other hand, every neighborhood of 0 in Q generates all of Q. Hence Corollary 3.3 holds not only for connected groups.

LEMMA 3.4 If every neighborhood of e in G contains an open subgroup of G, then G is totally disconnected.

Proof.

Let U be a neighborhood of e and let H be an open subgroup of G such that $H \subset U$. Since H is also closed, $\complement H = G \setminus H$ is open, and then G is the union of the two non-empty, disjointed sets H and $\complement H$. Therefore the connected component of the identity element is contained in H, and hence in U. By the

fact that U is arbitrary, one can conclude that the connected component of e is $\{e\}$.

THEOREM 3.5. If G_1 is the connected component of e, the quotient group G/G_1, with the quotient topology, is totally disconnected.

Proof.

Let us denote by π the natural projection $\pi: G \to G/G_1$. We have to prove that the connected component C of the identity element of G/G_1 is a single point.

Let us suppose that this is not the case, i.e. that G_1 is a proper subset of $\pi^{-1}(C)$. In this case $\pi^{-1}(C)$ is not connected, that is, there are two open sets A and B in G such that $A \cap \pi^{-1}(C) \neq \emptyset$, $B \cap \pi^{-1}(C) \neq \emptyset$ with

$$A \cap B \cap \pi^{-1}(C) = \emptyset$$

$$\pi^{-1}(C) = (A \cap \pi^{-1}(C)) \cup (B \cap \pi^{-1}(C)) = (A \cup B) \cap \pi^{-1}(C).$$

The last formula implies that

(1) $\quad C = (\pi(A) \cap C) \cup (\pi(B) \cap C)$

being obviously $(\pi(A) \cap C) \neq \emptyset$, $(\pi(B) \cap C) \neq \emptyset$.

If $a \in A \cap \pi^{-1}(C)$, then

$$aG_1 = (A \cap aG_1) \cup (B \cap aG_1).$$

Since $A \cap aG_1 \neq \emptyset$ and aG_1 is connected, it turns out that $B \cap aG_1 = \emptyset$, $aG_1 \subset A$, i.e. $\pi^{-1}(\pi(a)) \subset A$ for all $a \in A \cap \pi^{-1}(C)$, and hence

$$A \cap \pi^{-1}(C) = \pi^{-1}(\pi(A) \cap C).$$

Similarly we can prove that

$$B \cap \pi^{-1}(C) = \pi^{-1}(\pi(B) \cap C),$$

it follows that $(\pi(A) \cap C) \cap (\pi(B) \cap C) = \emptyset$. By (1) this is a contradiction, since $\pi(A) \cap C$ and $\pi(B) \cap C$ are both non-empty, and $\pi(A)$, $\pi(B)$ are open (see Proposition 1.10) $\square$