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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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BOREL - WEIL THEOREM

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1. Complex manifolds.

Let Ω be an open set in $\mathbb{C}^n$. A map $\Phi : \Omega \longrightarrow \mathbb{C}^m$ is said to be holomorphic if for $i = 1,\ldots,m$, the function $p_i \circ \Phi : \Omega \longrightarrow \mathbb{C}$ is a holomorphic (complex valued) function, where $p_i : \mathbb{C}^m \longrightarrow \mathbb{C}$ denotes the i^{th} projection (i.e., if $\Phi = (f_1,\ldots,f_m)$ then the functions f_i one holomorphic).

Let Ω_1 and Ω_2 be two open sets in $\mathbb{C}^n$. A map $\Phi : \Omega_1 \longrightarrow \Omega_2$ is said to be a holomorphic diffeomorphism if Φ is bijective and both Φ and Φ^{-1} are holomorphic.

Let M be a Hausdorff topological space which we assume to be paracompact. Suppose that we are given an open cover $\{U_\alpha\}_{\alpha \in I}$ of M and for each α a homeomorphism

$$\phi_\alpha : U_\alpha \longrightarrow \phi_\alpha(U_\alpha)$$

onto an open set $\phi_\alpha(U_\alpha)$ of $\mathbb{C}^n$ such that whenever $U_\alpha \cap U_\beta \neq \phi$ the map

$$\phi_\alpha \circ \phi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

is a holomorphic diffeomorphism. Then we say that M has a structure of a complex manifold or simply that M is a complex manifold (of complex dimension n). The family of "charts" (U_α, ϕ_α) is called an atlas. We shall assume that the atlas we have is a maximal atlas in the sense that we can not add more charts to the atlas still preserving the compatibility on the overlaps. Any atlas is contained in a unique maximal atlas.

Let M be a complex manifold. A function $f : M \longrightarrow \mathbb{C}$ is said to be holomorphic if for each α the function $f \circ \phi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \longrightarrow \mathbb{C}$ is holomorphic.

Let M and N be complex manifolds and $\Phi : M \longrightarrow N$ be a map. We say that Φ is holomorphic if Φ is continuous and the following condition is satisfied : let $\{(U_\alpha,\phi_\alpha)\}$ be an atlas for M and $\{(V_\beta,\psi_\beta)\}$ an atlas for N; then for each (α,β) with $W = U_\alpha \cap \Phi^{-1}(V_\beta) \neq \phi$ the map

$$\psi_\beta \circ \Phi \circ \phi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \longrightarrow \psi_\beta(V_\beta)$$

is holomorphic.

If M and N are complex manifolds, a holomorphic diffeomorphism $\Phi : M \longrightarrow N$ is bijective mapping that Φ and Φ^{-1} are holomorphic.

If M (resp. N) is a complex manifold of dimension m (resp. n) then $M \times N$ is in a natural way a complex manifold of dimension $m + n$.

2. Holomorphic vector bundles.

Let M be a complex manifold of dimension n. A holomorphic vector bundle of rank on over M is a complex manifold E of dimension $m + n$ together with a holomorphic map $\pi : E \longrightarrow M$ such that the following two conditions are satisfied :

(i) For each $x \in M$, $\pi^{-1}(x)$ has the structure of an m-dimensional vector space over $\mathbb{C}$.

(ii) For each $x \in M$ there exists an (open) neighbourhood U of x and a holomorphic diffeomorphism

$$\tau : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^m$$

such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \tau\ } & U \times \mathbb{C}^m \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p_U} & \\ & U & \end{array}$$

commutes and such that the induced map $\tau_y : \pi^{-1}(y) \longrightarrow y \times \mathbb{C}^m = \mathbb{C}^m$ is a bijective linear map for each $y \in U$. ($p_U : U \times \mathbb{C}^m \longrightarrow U$ is the natural projection onto U).

The pair (U,τ) will be called a local triviliasation of E.

If $x \in X$, we will denote $\pi^{-1}(x)$ by E_x and call it the fibre over x. Let $E^* = \coprod_{x \in M} E_x^*$, where E_x^* is the dual of the vector space E_x. Then E^* has a natural structure of a vector bundle and E^* will be called the dual bundle of E. One can also define the tensor product $E \otimes F$ of two holomorphic vector bundles in an obvious way and this is a holomorphic vector bundle whose rank is the product of ranks of E and F.

A holomorphic vector bundle of rank 1 will be called a line bundle.

If M is a complex manifold, then $M \times \mathbb{C}^m$ as the structure of a holomorphic vector bundle of rank m. This bundle is called the trivial bundle of rank m over M.

Remark 2.1. Let M be a complex manifold and $x \in M$. Let $\mathcal{O}_x$ be the ring of (germs of) holomorphic functions at x (i.e. defined in a neighbourhood of x.) A $\mathbb{C}$-linear map $L : \mathcal{O}_x \longrightarrow \mathbb{C}$ satisfying $L(f \cdot g) = L(f)g(x)+f(x)L(g)$ is called a (holomorphic) tangent vector x. If we denote by T_x the holomorphic tangent space at x, then $\coprod_{x \in M} T_x$ forms an n-dimensional holomorphic

bundle, the tangent bundle $T(M)$. A section of $T(M)$ is called a holomorphic vector field.

Let E be a (holomorphic) vector bundle over M. A section of E over U is a holomorphic map $\sigma : U \longrightarrow E$ such that $(\pi \circ \sigma)(x) = x$ for each $x \in U$. The sections of E over U form a vector space over $\mathbb{C}$, if we define $(\lambda_1\sigma_1+\lambda_2\sigma_2)(x) = \lambda_1\sigma_1(x)+\lambda_2\sigma_2(x)$, where $\lambda_1,\lambda_2 \in \mathbb{C}$ and σ_1,σ_2 are sections over U. A section of the trivial bundle over U can be identified with a holomorphic function $U \longrightarrow \mathbb{C}^m$ (In fact if $\sigma(x) = (x,f(x))$, $f(x) \in \mathbb{C}$, σ is identified with f).

We shall denote by $\Gamma(E)$ the vector space of sections of E over M.

Theorem 2.1. Let E be a holomorphic vector bundle on a compact complex manifold. Then the sections of E over M form a finite dimensional vector over $\mathbb{C}$.

Sketch of proof : Let (U_α,τ_α) be local trivilisations of E such that $\{U_\alpha\}$ form a finite open covering of M. Let V_α be a shrinking of U_α, that is V_α is an open cover such that $\bar{V}_\alpha \subset U_\alpha$. Let Γ be the space of sections of E over M. Let $s \in \Gamma$. Over each U_α the section s is given by m holomorphic functions $(f_1^\alpha,\ldots,f_m^\alpha)$. Let $b_{\alpha,i} = \sup_{\bar{V}_\alpha}|f_i^\alpha|$ and put $\|s\| = \max_{i,\alpha} b_{\alpha,i}$. This defines the structure of a normed linear space on Γ. By using Montel's theorem one sees that the unit ball in this normed linear space is relatively compact. By a well-known result in functional analysis this implies that Γ is finite dimensional.

Exercise. Prove that a holomorphic function on a compact connected complex manifold is constant.

3. The Complex projective space and the tautological line bundle.

Consider the set $\mathbb{P}^n$ of 1-dimensional vector subspaces of $\mathbb{C}^{n+1}$. We shall put a structure of an n-dimensional complex manifold on $\mathbb{P}^n$. Note that $\mathbb{P}^n$ can be identified with the quotient space of $\mathbb{C}^{n+1}-(0)$ by the action of $\mathbb{C}^*$ given by $(\lambda,v)\mapsto\lambda v$, $\lambda\in\mathbb{C}^*$, $v\in\mathbb{C}^{n+1}-0$. (Two points $(Z_1,\ldots,Z_{n+1})$ and $(Z_1',\ldots,Z_{n+1}')$ in $\mathbb{C}^{n+1}-0$ are equivalent if there exists $\lambda\in\mathbb{C}^*$ with $\lambda Z_i = Z_i'$ for $i=1,\ldots,n+1$). We endow $\mathbb{P}^n$ with the quotient topology. For fixed i consider the open set U_i of points $\underline{a}\in\mathbb{P}^n$ whose representatives one of the form $(Z_1,\ldots,Z_i,\ldots,Z_{n+1})$ with $Z_i\neq 0$. Then the map $\varphi_i : U_i \to \mathbb{C}^n$,

$$\underline{a}\to\left(\frac{Z_1}{Z_i},\ldots\frac{Z_{i-1}}{Z_i},\frac{Z_{i+1}}{Z_i},\ldots,\frac{Z_{n+1}}{Z_i}\right)\in\mathbb{C}^n$$

gives a chart. These charts cover $\mathbb{P}^n$ and are related by holomorphic functions on overlaps. In fact

$$\varphi_i\circ\varphi_j^{-1} : \varphi_j(U_i\cap U_j) = \{\zeta\in\mathbb{C}^n \text{ with } \zeta_i\neq 0\}\to\mathbb{C}^n$$

is given by

$$(\zeta_1,\ldots,\zeta_n)\mapsto\left(\frac{\zeta_1}{\zeta_i},\ldots,\frac{1}{\zeta_i},\ldots,\frac{\zeta_n}{\zeta_i}\right).$$

In particular $\mathbb{P}^1$ is the Riemann sphere $\mathbb{C}\cup\infty$.

If V is a finite dimensional vector space over $\mathbb{C}$ of dimension $(n+1)$ then the space $\mathbb{P}(V)$ of 1-dimensional subspaces of V has a

natural structure of a complex manifold of dimension n.

If to every 1-dimensional subspace ξ of V we associate the vector space ξ itself as the fibre over ξ we get a line bundle on $\mathbb{P}(V)$, called the tautological line bundle on $\mathbb{P}(V)$. More formally, consider the subset $\mathcal{O}(-1)$ of $\mathbb{P}^n\times V$ consisting of pair (ξ,v) with $v\in\xi$ (recall that ξ is a subspace of V and v is an element of V); then $(\mathcal{O}(-1),\pi)$, where π is the restriction to $O(-1)$, of the projection of $\mathbb{P}^n\times V$ to $\mathbb{P}^n$, forms a line bundle. The dual line bundle of $\mathcal{O}(-1)$ will be denoted $\mathcal{O}(1)$ and called the hyperplane bundle on $\mathbb{P}(V)$. We will denote by $\mathcal{O}$ the trivial line bundle on $\mathbb{P}^n$ and for an integer $k>0$ by $\mathcal{O}(k)$ the k-fold tensor product of $\mathcal{O}(1)$.

Proposition 3.1. We have

1) $\Gamma(\mathcal{O})\approx\mathbb{C}$

2) $\Gamma(\mathcal{O}(k) = S^k(V^*)$,

where $S^k(V^*)$ denotes the space of homogeneous polynomials of degree k on V.

Sketch of proof. We may assume $V=\mathbb{C}^{n+1}$, $n\geq 1$. We verify that a section of $\mathcal{O}(n)$ is given by a holomorphic function $f : \mathbb{C}^{n+1}-0\to\mathbb{C}$ satisfying $f(\lambda Z)=\lambda^n f(Z)$, $\lambda\in\mathbb{C}^*$. By a well known result in several complex variables, a holomorphic function of $\mathbb{C}^{n+1}-0$ has a unique holomorphic extension to $\mathbb{C}^{n+1}$ and we denote this extension still by f. If $\alpha=(\alpha_1,\ldots,\alpha_{n+1})$,

$\alpha_i \geq 0$, $\alpha_i \in \mathbb{Z}$ is a multi-index and $D^\alpha = (\frac{\partial}{\partial Z_1})^{\alpha_1} \cdots (\frac{\partial}{\partial Z_{n+1}})^{\alpha_{n+1}}$, then $(D^\alpha f)(\lambda Z) = \lambda^{k-|\alpha|} D^\alpha f(Z)$, where $|\alpha| = \alpha_1 + \ldots + \alpha_{n+1}$, $\lambda \in \mathbb{C}^*$, $Z \neq 0$. If If $|\alpha| > k$, it follows, letting $\lambda \to 0$ that $D^\alpha f(Z) = 0$. On the other hand if $|\alpha| < k$, $D^\alpha f(0) = 0$. It follows by the Taylor series expansion at 0, that f is homogeneous polynomial of degree k.

Remark 3.2. Consider the case $V = \mathbb{C}^2$. One knows that all irreducible representations of the Lie algebra $s\ell(2)$ are given by the natural representations of $s\ell(2)$ on $S^k(V)$. By Proposition 3.1, $S^k(V^*)$ is the space of sections of a line bundle on $\mathbb{P}^1$. Thus all irreducible representations are realised on the dual of the space of sections of line bundles on $\mathbb{P}^1$. A generalization of this result to arbitrary complex semi-simple algebras will be the Borel-Weil theorem.

4. Line bundles and mappings into a projective space.

Let L be a line bundle on a complex manifold M. We shall say that L is generated by its sections if the evaluation map $\Gamma(L) \to L_x$ is surjective for every $x \in X$, where L_x is the fibre of L at x (This is equivalent to requiring that given $x \in X$ there exists a (holomorphic) section s of L with $s(x) \neq 0$).

Proposition 4.1. Let M be a compact (connected) complex manifold and L a line bundle on M generated by its sections. Then there exists a canonical holomorphic map

$$\varphi_L : M \to \mathbb{P}(\Gamma(L)^*)$$

where $\Gamma(L)^*$ is the dual space of the space $\Gamma(L)$ of sections of L. (Note that by Theorem 2.1, $\Gamma(L)$ is finite dimensional).

Proof : Let $x \in M$. By hypothesis, the evaluation mapping at x, $\Gamma(L) \to L_x$ is surjective. Hence the dual mapping $L_x^* \to \Gamma(L)^*$ is injective and we define $\varphi_L(x) \in \mathbb{P}(\Gamma(L)^*)$ to be the image. (The space of sections vanishing at x is of codimension 1 in $\Gamma(L)$ and $\varphi_L(x)$ is the subspace of $\Gamma(L)^*$ orthogonal to this space).

Let $s_1, \ldots, s_{N+1}$ be a basis of $H^0(X,L)$ and let s_i be given, with respect to a trivialisation of L in a neighbourhood of x, by the function f_i. Then it is easily checked that φ_L is given locally by $Z \to (f_1(Z), \ldots, f_{N+1}(Z))$ in terms of the homogeneous coordinates with respect to the dual basis $s_1^*, \ldots, s_{N+1}^*$. This proves that φ_L is holomorphic.

Proposition 4.2. With the hypothesis and notation of Proposition 4.1, let $p : \Gamma(L)^* - 0 \to \mathbb{P}(\Gamma(L)^*)$ be the canonical projection. Then $p^{-1}(\varphi_L(M))$ generates the vector space $\Gamma(L)^*$.

Proof. Let W be the vector space generated by $p^{-1}(\varphi_L(M))$. To prove that $W = \Gamma(L)^*$ it is enough to show that if $s \in \Gamma(L)$ and $\langle s,w \rangle = 0$, for every $w \in W$, then $s = 0$. (Hence $\langle\ ,\ \rangle$ denotes the pairing between $\Gamma(L)$ and $\Gamma(L)^*$). If $\langle s,w \rangle = 0$ for all $w \in W$, then in particular, for every $w \in \varphi_L(x) = L_x^*$, $\langle s,w \rangle = 0$. But for $w \in L_x^*$, $\langle s,w \rangle = \langle s(x),w \rangle$ where the pairing on the right is given by the duality between L_x and L_x^*. Hence $s(x) = 0$ for every x in M, that is $s = 0$.

5. Complex Lie groups.

Let G be a group which has a structure of a complex manifold. We say that G is a complex Lie group if the map $G \times G \longrightarrow G$, given by $(x,y) \mapsto xy^{-1}$ is holomorphic. The left invariant holomorphic vector fields on G form a complex Lie algebra Lie (G). We shall assume that G is connected. G is said to be semi-simple if its Lie algebra is semi-simple.

Examples. $GL(n,\mathbb{C})$ is a complex Lie group. $SL(n,\mathbb{C})$ is a complex semi-simple Lie group. If V is a finite dimensional vector space over $\mathbb{C}$, then the group $GL(V)$ of linear automorphisms of V is a complex Lie group.

A holomorphic homomorphism $\rho : G \longrightarrow GL(V)$ is said to be a (holomorphic) representation on V. A (holomorphic) homomorphism into $\mathbb{C}^*$ will be called a character.

If $\varphi : G_1 \longrightarrow G_2$ is a holomorphic homomorphism then φ induces a homomorphism of Lie (G_1) into Lie (G_2). Conversely if G_1 is simply connected, a homomorphism of Lie (G_1) into Lie (G_2) is induced by a homomorphism of G_1 into G_2.

6. Homogeneous vector bundles and induced representations.

Let G be a complex Lie group. If H is a closed holomorphic subgroup then the left coset space G/H has a natural structure of a complex manifold. Let $\rho : H \longrightarrow \mathrm{Aut}\ (V)$ be a finite dimensional holomorphic representation of H. We will associate to ρ a holomorphic vector bundle on G/H and G will operate on E_ρ as vector bundle automorphisms

in such a way that the action of an element $g \in G$ on E_ρ projects into the natural action of g on G/H. The bundle E_ρ will be called the homogeneous vector bundle associated with ρ.

Consider on $G \times V$ the action on the right by H given by

$$(g,v)\cdot h = (g\cdot h, \rho(g)^{-1}h),\ g \in G,\ v \in V,\ h \in H.$$

The quotient space of $G \times V$ by this action of H is in a natural way a holomorphic vector bundle E_ρ on G/H. The action of G (on the left) on $G \times V$ given by $g'(g,v) = (g'g,v)$, $g',g \in G$, $v \in V$ descends to an action of G on E_ρ and G acts on E_ρ as vector bundle automorphisms. As such, G also acts on the space of sections of E_ρ.

This action of G on $\Gamma(E_\rho)$ can be made explicit as follows. One first checks that a section of E_ρ can be identified with a holomorphic function $f : G \longrightarrow V$ satisfying

$$f(gh) = \rho(h)^{-1}f(g), \quad \text{for} \quad g \in G,\ h \in H.$$

With this identification the action $\mathrm{Ind}(\rho) = \underline{\rho}$ of G on $\Gamma(E_\rho)$ is is given by :

$$\underline{\rho}(g)f(x) = f(g^{-1}x), \text{ for } x \in G .$$

If G/H is compact we thus obtain a finite dimensional representation $\mathrm{Ind}(\rho)$ of G on $\Gamma(E_\rho)$. The representation $\mathrm{Ind}(\rho)$ will be called the representation induced by ρ.

Remark 6.1. Note that if $\underline{e}$ denotes the coset (H), then the group H acts on the fibre of E_ρ at $\underline{e}$ and we thus get a representation of H on $(E_\rho)_{\underline{e}}$.

We claim that this representation of H is equivalent to ρ. In fact let q be the canonical map $G \times V \longrightarrow E_\rho$. Consider the isomorphism $\psi : V \longrightarrow (E_\rho)_e$ = fibre of E_ρ at e, given by $v \longmapsto q(e,v)$ for $v \in V$, where e is the identity element of G. If $h \in H$, then the image of $\rho(h)v$ under ψ is $q(e,\rho(h)v) = q(h\cdot h^{-1},\rho(h)v) = q(h,v)$. But the action of h on E_ρ is induced by the map $(g,v) \longrightarrow (hg,v)$ and hence $h\cdot q(e,v) = q(h,v)$. Thus ψ commutes with the actions of H on V and E_ρ. This proves the claim.

We will be concerned in the sequel only with the case where ρ is one dimensional, that is, where ρ is a character χ.

7. Homogeneous line bundles and representations of semi-simple groups : Borel-Weil Theorem.

Let G be a complex semi-simple Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and Φ the system of roots with respect to $\mathfrak{h}$. We set

$$\mathfrak{n}^+ = \sum_{\alpha > 0} \mathfrak{g}^\alpha, \quad \mathfrak{n}^- = \sum_{\alpha < 0} \mathfrak{g}^\alpha, \quad \alpha \in \Phi .$$

Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We denote by $\mathfrak{b}$ the subalgebra $\mathfrak{h} \oplus \mathfrak{n}^+$. Let H (resp. B) be the subgroup of G corresponding to $\mathfrak{h}$ (resp. $\mathfrak{b}$). Then H and B are closed complex subgroups of G and $H \simeq (\mathbb{C}^*)^\ell$. (Any conjugate of B is called a Borel subgroup of G). Moreover the complex manifold G/B is compact. In fact, for a suitable maximal compact subgroup K, we have $G/B = K/T$, where T is the maximal torus of K.

Example. Let $G = SL(n,\mathbb{C})$. By a flag in $\mathbb{C}^n$ we mean sequence of subspaces $L_1,\dots,L_n$ of $\mathbb{C}^n$ such that $L_1 \subset L_2 \dots \subset L_n = \mathbb{C}^n$, with $\dim L_i = i$. Let B be the upper triangular subgroup of $SL(n)$ consisting of matrices in $SL(n)$ whose elements below the diagonal are zero. Note that $SL(n)$ acts on the flags in an obvious way. It is clear see that G/B = space F of flags in $\mathbb{C}^n$. It is easy to see that $SU(n)$ acts transitively on F.

Let ρ be a representation of G on a finite dimensional vector space V. If ξ is a 1-dimensional subspace left invariant by B, we will call ξ a primitive subspace and the character of B given by the representation on ξ, a dominant weight. It is known that every finite dimensional representation has a primitive subspace ξ and that on the G-subspace spanned by ξ the representation of G is irreducible. Moreover if ρ is irreducible there is a unique primitive subspace and the corresponding dominant weight will be called the highest weight of ρ.

Theorem 7.1. Let χ be a character of B and let L denote the homogeneous line bundle on G/B associated to the homomorphism $\chi^{-1} : B \longrightarrow \mathbb{C}^*$. Assume that L has a non-zero section. Then the representation on $\Gamma(L)^*$, dual to the induced representation of G on $\Gamma(L)$, is irreducible and has highest weight χ.

Corollary 7.2. The induced representation on $\Gamma(L)$ is irreducible.

Proof of Theorem 7.1. The action of an element $g \in G$ on E_ρ as a line bundle automorphism projects into the natural action of g on G/H. Since

G acts transitively on G/H and the line bundle has a non-zero section, it follows that the sections of L generate L. Let $\varphi : G/B \longrightarrow \mathbb{P}(\Gamma(L)^*)$ be the associated holomorphic map (Proposition 4.1). If $\underline{e} \in G/B$ is the point corresponding to the coset B, then $\varphi(\underline{e}) = L^*_{\underline{e}}$, where $L_{\underline{e}}$ is the fibre at $\underline{e}$ (see the proof of Proposition 4.1). By Remark 6.1, B acts on the fibre $L_{\underline{e}}$ by the character χ^{-1}; we see that under the dual representation B leaves the subspace $L^*_{\underline{e}}$ invariant and acts by the character χ. Thus $\varphi(\underline{e})$ is a primitive subspace with dominant weight χ. Since $\varphi(G/B) = G(\varphi(\underline{e}))$ we see, using Proposition 4.2, that $\Gamma(L)^*$ is generated by $G(\varphi(\underline{e}))$. Hence by the result on $\mathfrak{g}$-modules generated by primitive elements recalled above, the dual representation is irreducible and its highest weight is clearly χ.

<u>Theorem 7.3</u>. Let ρ be a finite dimensional (holomorphic) irreducible representation of G. Then ρ is equivalent to the dual of the induced representation of G on a homogeneous line bundle L on G/B.

<u>Proof</u>. Let ξ be a primitive subspace of the representation space V and χ be the corresponding dominant weight. Consider the map $G/B \longrightarrow \mathbb{P}(V)$ given by $g \longmapsto g\xi$. For $\lambda \in V^*$ consider the function f_λ on G given by $f_\lambda(g) = \langle\lambda, \rho(g)v\rangle$. For $b \in B$ we have

$$f_\lambda(gb) = \langle\lambda,\rho(gb)v\rangle = \langle\lambda,\rho(g)\rho(b)v\rangle = \langle\lambda,\chi(b)\rho(g)v\rangle$$

$$= \chi(b)\, f_\lambda(g).$$

Let L be the line bundle associated to the character $b \longmapsto \chi^{-1}(b)$. Then,

since $f_\lambda(gb) = \chi(b)f_\lambda(g)$, we see that f_λ is a section of L (see § 6). Thus we have a linear map $\psi : V^* \longrightarrow \Gamma(L)$ given by $\lambda \longmapsto f_\lambda$. The map ψ is non-zero, because the G-span of ξ is the whole of V, as ρ is an irreducible representation. We shall show that ψ commutes with the representation ρ^* on V^* and the induced representation, Ind, on $\Gamma(L)$; this will prove the theorem by Schur's lemma, as the induced representation on $\Gamma(L)$ is irreducible (Corollary 7.2). Now for $g', g \in G$

$$\mathrm{Ind}(g')(f_\lambda(g)) = f_\lambda(g'^{-1}g) = \langle\lambda,\rho(g'^{-1}g)\underline{e}\rangle$$

$$= \langle\lambda,\rho(g'^{-1})\rho(g)\underline{e}\rangle$$

$$= \langle{}^t\rho(g'^{-1})\lambda,\rho(g)\underline{e}\rangle$$

which proves that $(\mathrm{Ind}\, g')\cdot\psi(\lambda) = \psi(\rho^*(g')\lambda)$ where ρ^* is the representation dual to ρ. Q.E.D.

Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of and Φ^+ a system of positive roots. Let ω be a dominant integral form on $\mathfrak{h}$, that is, $\omega(H_\alpha)$ is a non-negative integer for each $\alpha \in \Phi^+$. Let G be a simply connected complex semi-simple group with $\mathfrak{g}$ as its Lie algebra. Then it is known that ω is the differential of a character χ_0 of the subgroup H corresponding to $\mathfrak{h}$. We extend χ_0 to a character χ of B by putting $\chi(u) = 1$ for $u \in U$ where U is the subgroup corresponding to $\mathfrak{n}^+$. With this notation we have

<u>Theorem 7.4</u>. Let G be a simply connected semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let ω be a dominant integral linear form on $\mathfrak{g}$ and χ the

corresponding character on the Borel subgroup B. Let L be the homogeneous line bundle on G/B associated to the character χ^{-1}. Then the representation of G on $\Gamma(L)^*$ dual to the induced representation on $\Gamma(L)$ is irreducible and the associated representation of $\mathfrak{g}$ on $\Gamma(L)^*$ is irreducible with highest weight ω.