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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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**BASIC NOTIONS FOR**
**LIE GROUPS IV**

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These are preliminary lecture notes, intended only for distribution to participants.

## D. Consequences

**Proposition 1.** Let $\varphi : \mathbb{R} \to G$ be a continuous homomorphism of the real additive group $\mathbb{R}$ into the Lie group $G$. Then $\varphi$ is $C^\infty$.

*Proof.* It is enough to show that $\varphi$ is $C^\infty$ in a neighborhood of $0$ (it follows then that it is $C^\infty$ everywhere by composition with suitable left translations).

Let $(V, \log)$ be a logarithmic coordinates neighborhood of $e$ in $G$ such that $U = \log V$ is spherical. Let $U' = \{ X/2 \,;\, X \in U \}$ and $t_0 > 0$ such that $\varphi(t) \in \exp(U')$ for $t \leq t_0$. Set $Y = \log \varphi(t_0)$. If $X = \log \varphi(\frac{t_0}{n})$, $\exp nX = \varphi(t_0) = \exp Y$ so $X = \frac{1}{n} Y$ as both sides are in $\exp U$ (and thus as $Y \in U'$ in $\exp U'$) by induction on $n$. If $m \leq n$ is a positive integer then $\varphi(\frac{m}{n} t_0) = \exp \frac{m}{n} Y$. Similarly if $m$ is negative. It follows by continuity that $\varphi(t) = \exp \frac{tY}{t_0}$ for $|t| \leq t_0$. Thus $\varphi$ is $C^\infty$. □

**Proposition 2.** Let $f : G \to G'$ be a continuous group homomorphism between the two Lie groups $G$ and $G'$. Then $f$ is a Lie homomorphism.

*Proof.* Again, it is enough to prove that $f$ is $C^\infty$ at $e$. Let $X_1 \cdots X_d$ be a basis of the Lie algebra $\mathfrak{g}$ of $G$. The map $\eta(t_1 \cdots t_d) = \exp t_1 X_1 \cdots \exp t_d X_d$ is $C^\infty$ and $\eta_{*0}$ is non singular, because in logarithmic coordinates $\log \eta(t_1 \cdots t_d) = t_1 X_1 + \cdots + t_d X_d + O(t^2)$. So $\eta$ is a diffeomorphism of a neighborhood $V'$ of $0$ in $\mathbb{R}^n$ onto a neighborhood $U'$ of $e$ in $G$ and $\eta^{-1}$ is $C^\infty$ on $U'$. By proposition 1, the map $f \circ \eta$,

$$f(\eta(t_1 \cdots t_d)) = f(\exp t_1 X_1) \cdots f(\exp t_d X_d), \text{ is } C^\infty$$

As $f = (f \circ \eta) \circ \eta^{-1}$ in $U'$, $f$ is $C^\infty$ in $U'$. □

**Proposition 3.** Let $\varphi : G \to G'$ be a Lie homomorphism. Then the following diagram is commutative

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi_*"] \arrow[d, "\exp_{\mathfrak{g}}"'] & \mathfrak{g}' \arrow[d, "\exp_{\mathfrak{g}'}"] \\ G \arrow[r, "\varphi"'] & G' \end{tikzcd}$$

*Proof.* Let $X \in \mathfrak{g} = T_e G$. The curves $t \to \exp_{\mathfrak{g}'} t(\varphi_* X)$ and $t \to \varphi(\exp_{\mathfrak{g}} tX)$ are one-parameter subgroups of $G'$ whose tangent vector at $e'$ is $\varphi_{*e} X$. Hence, they coincide and $\exp_{\mathfrak{g}'} t(\varphi_* X) = \varphi(\exp_{\mathfrak{g}} tX)$. □

**Proposition 4.** Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.

*Proof.* If $\varphi : G \to G'$ is an isomorphism so is $\varphi_* : \mathfrak{g} \to \mathfrak{g}'$. Conversely, if $\varphi_* : \mathfrak{g} \to \mathfrak{g}'$ is an isomorphism and if $U, U'$ are logarithmic coordinate systems of $G$ and $G'$ such that $N = \exp U$, $N' = \exp U'$ satisfy the hypotheses of C.B.H theorem and $\varphi_* N \subset N'$ then one defines

$$\varphi : U \to U' \quad x \to \varphi(x) = \exp \varphi_*(\log x)$$

Clearly $\varphi(\exp X) = \exp \varphi_* X$ $\forall X \in N$ and

$$\varphi(\exp X \exp Y) = \varphi(\exp \mu(X,Y)) = \exp \varphi_* \mu(X,Y)$$

$$\varphi(\exp X) \cdot \varphi(\exp Y) = \exp \varphi_* X \exp \varphi_* Y = \exp \mu'(\varphi_* X, \varphi_* Y) = \exp \varphi_* \mu(X,Y)$$

thus $\varphi$ is a local homomorphism. □

**Proposition 5.** Let $G$ be a Lie group, $H$ a Lie subgroup of $G$, $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Then

(i) $\exp_{\mathfrak{h}}(X) = \exp_{\mathfrak{g}}(X)$ for all $X \in \mathfrak{h}$

(ii) $\mathfrak{h} = \{ X \in \mathfrak{g} \mid$ the curve $t \to \exp tX$ is a continuous curve in $H \}$.

*Proof.* Let $i : H \to G$ be the inclusion map. Then $i_* (= Id)$ is a Lie homomorphism. By proposition 3 $\exp_{\mathfrak{h}}(X) = \exp_{\mathfrak{g}}(X)$. Proposition 2 implies (ii). □

Remark 1. If $H_1$ and $H_2$ are two Lie subgroups of a Lie group which coincide as topological groups then $H_1 = H_2$ as Lie groups. This was already stated in (1.D prop 4). It is an easy consequence of proposition 4.

Theorem. Let $G$ be a Lie group and $H$ be a subgroup of $G$ which is closed as a subset of $G$. Then there exists a unique manifold structure on $H$ such that $H$ is an embedded Lie subgroup of $G$. If $\mathfrak{h}$ (resp $\mathfrak{g}$) is the Lie algebra of $H$ (resp. $G$) then

$$\mathfrak{h} = \{ X \in \mathfrak{g} \mid \exp tX \in H \text{ for all } t \in \mathbb{R} \}$$

Proof. Let $\mathfrak{h}$ denote the subset of $\mathfrak{g}$ given by $\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp tX \in H, \forall t \in \mathbb{R}\}$. We shall prove that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$

(i) If $X \in \mathfrak{h}$, $sX \in \mathfrak{h}$ obviously

(ii) If $X, Y \in \mathfrak{h}$, $X+Y \in \mathfrak{h}$ because $H$ is closed, and by CBH formula,

$$\left(\exp \frac{t}{n} X \ \exp \frac{t}{n} Y\right)^n = \exp\left(t(X+Y) + \frac{t^2}{2n}[X,Y] + O\left(\frac{1}{n^2}\right)\right) \text{ for a fixed } t$$

(iii) If $X, Y \in \mathfrak{h}$, $[X,Y] \in \mathfrak{h}$ because $H$ is closed and, again by CBH formula:

$$\left(\exp -\frac{t}{m} X \ \exp -\frac{t}{m} Y \ \exp \frac{t}{m} X \ \exp \frac{t}{m} Y\right)^{m^2} = \exp\left(t^2[X,Y] + O\left(\frac{1}{m}\right)\right) \text{ for a fixed } t$$

Let $H^*$ be the connected Lie subgroup of $G$ with Lie algebra $\mathfrak{h}$. Clearly $H^* \subset H$. If $N$ is a neighborhood of $e$ in $H^*$, it is a neighborhood of $e$ in $H$ with the relative topology.

Indeed, if it were not, $\exists\ c_k \to e$ in $G$, $c_k \in H \setminus N$. Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ ($\mathfrak{m}$ complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$); $\exists$ nghds $U$ of $o$ in $\mathfrak{h}$, $U'$ of $o$ in $\mathfrak{m}$, ($U'$ bounded) and $V$ of $e$ in $G$ such that $\exp U \subset N$

$U' \times U \to V$ $\quad (A,B) \to \exp A \exp B$ is a diffeomorphism onto $V$

then $c_k = \exp A_k \exp B_k$ $\quad A_k \neq 0$, $\lim A_k = 0$, $\exp A_k \in H$

As $A_k \neq 0$, one introduces an integer $r_k$ such that $r_k A_k \in U'$, $(r_k+1)A_k \notin U'$

The sequence $(r_k A_k)$ has a convergent subsequence (still denoted $(r_k A_k)$) which converges to an element $A \in \mathfrak{m}$.
Since $(r_k+1)A_k \notin U'$ and $\lim A_k = 0$, $A$ lies on the boundary of $U'$, hence is not zero.
If $p, q$ are integers $\quad p r_k = s_k q + t_k \quad$ where $0 \leq t_k < q$
thus $\exp \frac{p}{q} A = \lim_k \exp \frac{p r_k}{q} A_k = \lim_k (\exp A_k)^{s_k} \in H$.
By continuity $\exp tA \in H$ $\forall t \in \mathbb{R}$ so $A \in \mathfrak{h}$, hence a contradiction as $A \in \mathfrak{m}$ and $A \neq 0$.

Thus, taking $N = H^*$, $H^*$ is open in $H$ so $H_0$ — and thus $H$ — has an analytic structure compatible with the relative topology of $G$ so that $H$ is a Lie subgroup of $G$.

Examples of uses of the last theorem

Any closed subgroup $H$ of $Gl(n, \mathbb{K})$ — $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ — is a Lie subgroup and its Lie algebra is given by $\mathfrak{h} = \{X \in \mathfrak{gl}(n, \mathbb{R} \text{ or } \mathbb{C}) \mid \exp tX \in H \text{ for all } t \in \mathbb{R}\}$

① • $Sl(n, \mathbb{R}) = \{A \in Gl(n,\mathbb{R}) \mid \det A = 1\}$ is a Lie group
$\mathfrak{sl}(n, \mathbb{R}) = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \operatorname{tr} X = 0\}$ is its Lie algebra

② Let $I_{p,q} = \begin{pmatrix} -I_p & \\ & I_q \end{pmatrix} \in Gl(p+q, \mathbb{K})$ where $I_r$ is the identity matrix in $Gl(r, \mathbb{K})$

Let $O_{p,q} = \{A \in Gl(p+q, \mathbb{R}) \mid {}^tA\, I_{p,q}\, A = I_{p,q}$ where ${}^t$ is the transpose$\}$; it is a Lie group.

Its Lie algebra is given by $\mathfrak{o}(p,q) = \{X \in \mathfrak{gl}(p+q, \mathbb{R}) \mid {}^tX\, I_{p,q} + I_{p,q} X = 0\}$.

In particular $O_{0,n} = O(n, \mathbb{R})$ is the orthogonal group.
Similarly one defines $O(n, \mathbb{C}) = \{A \in Gl(n, \mathbb{C}) \mid {}^tA A = Id\}$

## 3 Simply connected Lie groups

<u>Definition</u> A $C^\infty$ map $\pi : M \to M'$ between two manifolds is a <u>covering</u> if $\pi$ is $C^\infty$, surjective, $M$ is a connected manifold and each point $y \in M'$ has a neighborhood $V$ whose inverse image under $\pi$ is a disjoint union of open sets in $M$, each diffeomorphic to $V$ under $\pi$.

<u>Theorem 1</u> Each connected Lie group has a simply connected connected covering space which is itself a Lie group, such that the covering map is a Lie group homomorphism.

<u>Proof</u> Any connected manifold $M'$ has a connected simply connected covering $\tilde{M}$ and the covering map $\pi$ is $C^\infty$. For any map $\alpha : N \to M$, $\alpha$ smooth, there exists a unique smooth map $\tilde{\alpha}$ $N \to \tilde{M}$ such that $\begin{cases} \tilde{\alpha}(x) = y \in \pi^{-1}(\alpha(x)) \\ \alpha = \pi \circ \tilde{\alpha} \end{cases}$ if $N$ is simply connected.
Consider such a $\tilde{G}$, covering of the Lie group $G$.
Consider $\alpha . \tilde{G} \times \tilde{G} \to G$ $\quad \alpha(\tilde{\sigma}, \tilde{\sigma}') = \pi(\tilde{\sigma})\, \pi(\tilde{\sigma}')^{-1}$. Choose $\tilde{e} \in \pi^{-1}(e)$. There exists a unique $\tilde{\alpha} : \tilde{G} \times \tilde{G} \to \tilde{G}$ such that $\pi \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(\tilde{e}, \tilde{e}) = \tilde{e}$. For $\tilde{\sigma}$ and $\tilde{\tau}$ in $G$, we define $\tilde{\tau}^{-1} = \tilde{\alpha}(e, \tilde{\tau})$ $\quad \tilde{\sigma}\tilde{\tau} = \tilde{\alpha}(\tilde{\sigma}, \tilde{\tau}^{-1})$. This makes $\tilde{G}$ into an abstract group. As $\tilde{\alpha}$ is $C^\infty$, it makes $\tilde{G}$ into a Lie group. $\square$

<u>Theorem 2</u> If $G$ and $G'$ are Lie groups, if $G$ is simply connected, if $\mathcal{G}$, $\mathcal{G}'$ are their Lie algebras and if $\psi_* \cdot \mathcal{G} \to \mathcal{G}'$ is a Lie algebra homomorphism, then there is a unique Lie group homomorphism $\varphi : G \to G'$ such that $\psi_* = \varphi_*$.

<u>Corollary</u>. There is a bijective correspondence between Lie algebras of Lie groups and connected simply connected Lie groups.

<u>Theorem 3</u> Let $G$ be a connected Lie group, $\tilde{G}$ its universal covering. Then the fundamental group $\pi_1(G)$ is isomorphic to $\pi^{-1}(e)$, which is a subgroup of $\tilde{G}$. If $\Gamma$ is any discrete central subgroup of $\tilde{G}$ then $\tilde{G}/\Gamma$ is a Lie group and the natural map $p \cdot \tilde{G} \to \tilde{G}/\Gamma$ is a covering. Then $\pi_1(\tilde{G}/\Gamma) \simeq \Gamma$

For proofs see for example F. Warner p 98

## 4 Homogeneous Spaces

<u>Theorem 1</u> Let $H$ be a closed subgroup of a Lie group $G$, and let $G/H$ be the set $\{gH ; g \in G\}$ of left cosets modulo $H$. Let $\pi : G \to G/H$ $g \to \pi(g) = gH$ denote the natural projection. Then $G/H$ has a unique $C^\infty$ manifold structure such that

(a) $\pi$ is smooth

(b) there exists local smooth sections of $G/H$ in $G$, i.e. if $gH \in G/H$, there is a neighborhood $W$ of $gH$ and a smooth map $\sigma : W \to G$ such that $\pi \circ \sigma = id$.

Then, the map $G \times G/H \to G/H$ $\quad (g_1, g_2H) \to g_1 g_2 H$ is smooth.

(Proof: see Warner p 120).

<u>Definition 1</u> Manifolds of the form $G/H$ where $G$ is a Lie group, $H$ a closed subgroup and the manifold structure is the unique one satisfying ⓐ and ⓑ of the above theorem are called <u>homogeneous manifolds</u>

<u>Definition 2</u> Let $G$ be a Lie group and $M$ be a $C^\infty$ manifold. A ($C^\infty$ left) <u>action</u> of $G$ on $M$ is a $C^\infty$ map, which we denote by $\cdot$

$$\cdot : G \times M \to M \quad (g, \ ) \to g \cdot x$$

such that
$$e \cdot x = x \quad \forall x \in M$$
$$g_1 \cdot (g_2 \cdot x) = g_1 g_2 \cdot x \qquad \forall g_1, g_2 \in G, \forall x \in M.$$

An action is said to be <u>transitive</u> if $\forall x, y \in M$, $\exists g \in G$ such that $g \cdot x = y$

<u>Theorem 2</u> If $\cdot$ is a transitive $C^\infty$ left action of the Lie group $G$ on the manifold $M$ and if $p \in M$, let $G_p = \{g \in G \mid g \cdot p = p\}$ be the stabilizer of $p$ in $G$. Then $M$ is diffeomorphic with $G/G_p$

(Proof see Warner p 123).

<u>Examples of homogeneous manifolds</u>

① Let $\{e_i ; 1 \leq i \leq n\}$ be the canonical basis of $\mathbb{R}^n$. Each matrix $A \in Gl(n, \mathbb{R})$ determines a linear transformation on $\mathbb{R}^n$ : $A(e_i) = \sum_j A_{ji} e_j$ -

This defines an action of $Gl(n,\mathbb{R})$ on $\mathbb{R}^n$.

Restricting this action to $O(n,\mathbb{R}) = \{A \in Gl(n,\mathbb{R}) \mid {}^tA\,A = Id$ where ${}^t$ is the transpose$\}$ acting on $S^{n-1} = \{x \in \mathbb{R}^n \mid \sum_i (x_i)^2 = 1\}$, one has a natural smooth left action $O(n) \times S^{n-1} \to S^{n-1}$ which is transitive.

If $p = (0,\dots 0,1) \in S^{n-1}$ the stabilizer of $p$ in $O(n)$ is isomorphic to the set of matrices $A \in O(n)$ such that $A = \begin{pmatrix} * & \begin{matrix}0\\ \vdots \\ 1\end{matrix}\end{pmatrix}$, i.e to $O(n-1)$.

Hence $S^{n-1} \cong O(n)/O(n-1)$

By a similar argument, $S^{n-1}$ is diffeomorphic to $SO(n)/SO(n-1)$ where $SO(n) = \{A \in O(n) \mid \det A = 1\}$

② In a similar way $S^{2n-1}$ is diffeomorphic to $U(n)/U(n-1)$ or $SU(n)/SU(n-1)$ where $U(n) = \{A \in Gl(n,\mathbb{C}) \mid {}^t\bar{A}A = Id\}$ and $SU(n) = \{A \in U(n) \mid \det A = 1\}$.

③ Let $V$ be a real vector space of dimension $d$. Let $H_k(V)$ be the set of all $k$-dimensional subspaces of $V$. If we choose a basis $v_1 \cdots v_d$ for $V$, the group $O(d)$ acts naturally by matrix multiplication and one has a map $\eta\ O(d) \times H_k(V) \to H_k(V)$

Let $P_0$ be the $k$-dim vector space spanned by $v_1 \cdots v_k$. Then the stabilizer of $P_0$ in $O(d)$ is the subgroup $H = \{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B\end{array}\right)\ A \in O(k)\ B \in O(d-k)\}$

so $H$ is a closed subgroup of $O(d)$ isomorphic to $O(k) \times O(d-k)$

We make $H_k(V)$ into a $(d-k)k$ smooth manifold by requiring the bijective map $\sigma: O(d)/O(k)\times O(d-k) \to H_k(V)$ : $A(O(k)\times O(d-k)) \to AP_0$ to be a diffeomorphism. (This structure does not depend on the chosen basis)

$H_k(V)$ is called the <u>Grassmann manifold</u> of $k$-planes in $V$