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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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BASIC NOTIONS FOR LIE ALGEBRA

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Basic notions for Lie algebras

1 Definitions

A Lie algebra $\mathcal{L}$ (we shall only consider here finite dimensional Lie algebras) over a field k (we shall always assume that k is commutative and has characteristic zero) is a finite dim. vector space $\mathcal{L}$ over k with a bilinear pairing, called the bracket, $[\ ,\ ] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ $(x, y) \to [x, y]$ such that

$$[x, x] = 0$$

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \quad \text{(Jacobi's identity)}$$

A subalgebra of $\mathcal{L}$ is a subspace $\mathcal{L}'$ such that $[\mathcal{L}', \mathcal{L}'] \subset \mathcal{L}'$; it is an ideal if $[\mathcal{L}', \mathcal{L}] \subset \mathcal{L}'$

A map $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ between two Lie algebras is a homomorphism if it is linear and $\phi([x, y]) = [\phi(x), \phi(y)]$ (it is an isomorphism if it is bijective). If $\mathcal{L}_2$ is the Lie algebra of all linear endomorphisms of a vector space V over k, i.e $\mathcal{L}_2 = gl(V)$, a homomorphism $\phi : \mathcal{L}_1 \to gl(V)$ is called a representation of $\mathcal{L}_1$ in V.

A derivation of a Lie algebra $\mathcal{L}$ is a linear map $D . \mathcal{L} \to \mathcal{L}$ such that $D([x, y]) = [Dx, y] + [x, Dy]$ $\forall x, y \in \mathcal{L}$. We denote by $\mathrm{Der}(\mathcal{L})$ the space of derivations; it is a subalgebra of $gl(\mathcal{L})$.

In particular if $x \in \mathcal{L}$ $\mathrm{ad}\, x : \mathcal{L} \to \mathcal{L}$ $y \to [x, y]$ is a derivation of $\mathcal{L}$; such derivations are called inner, all others are called outer.

The map $\mathrm{ad}\ \mathcal{L} \to gl(\mathcal{L})$ is a representation of $\mathcal{L}$ called the adjoint representation

The center of the Lie algebra $\mathcal{L}$ is defined by $Z(\mathcal{L}) = \{ z \in \mathcal{L} \mid [x, z] = 0 \ \forall x \in \mathcal{L} \}$. It is an ideal in $\mathcal{L}$. The derived algebra of $\mathcal{L}$, $\mathcal{L}^{(1)} = [\mathcal{L}, \mathcal{L}]$ is also an ideal in $\mathcal{L}$

If K is any subspace of $\mathcal{L}$, the <u>normalizer</u> of K in $\mathcal{L}$, $N_{\mathcal{L}}(K)$ is defined by $N_{\mathcal{L}}(K) = \{x \in \mathcal{L} \mid [x, K] \subset K\}$; it is a subalgebra of $\mathcal{L}$.

If K is any subset of $\mathcal{L}$, the <u>centralizer</u> of K in $\mathcal{L}$ is the subalgebra defined by $C_{\mathcal{L}}(K) = \{x \in \mathcal{L} \mid [x, K] = 0\}$.

If $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ is a homomorphism of Lie algebras, $\operatorname{Ker} \phi$ is an ideal in $\mathcal{L}_1$, $\operatorname{Im} \phi$ is a subalgebra of $\mathcal{L}_2$.

If $\mathcal{L}$ is a Lie algebra and $\mathcal{J}$ is an ideal of $\mathcal{L}$, the <u>quotient algebra</u> $\mathcal{L}/\mathcal{J}$ is the quotient space with the bracket defined by $[x + \mathcal{J}, y + \mathcal{J}] = [x, y] + \mathcal{J}$.
In particular, if $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ is a homomorphism, $\mathcal{L}_1/_{\operatorname{Ker}\phi}$ is isomorphic to $\operatorname{Im}\phi$.

<u>Example</u>. Let $\mathcal{L} = gl(m, k)$ = set of $m \times m$ matrices with coefficients in k, with the bracket defined by the commutator of matrices, i.e $[A, B] = AB - BA$.

If $\phi : \mathcal{L} \to \mathbb{R}$ $A \to \operatorname{trace} A$, then ϕ is a Lie homomorphism and $\operatorname{Ker} \phi = sl(m, k) = \{A \in gl(m, k) \mid \operatorname{tr} A = 0\}$ is an ideal in $\mathcal{L}$.

If $D(m, k)$ is the set of all diagonal matrices in $\mathcal{L}$; it is a subalgebra ($[D, D] = 0$), but not an ideal ($[\operatorname{diag}(a_1, \dots, a_m), E_{ij}] = (a_i - a_j) E_{ij}$ if E_{ij} is the matrix whose only nonzero coefficient is a one at the intersection of the i^{th} row and j^{th} column and $\operatorname{diag}(a_1, \dots, a_m) = \sum_i a_i E_{ii}$)

The centralizer of $D(m, k)$ in $\mathcal{L}$ is equal to its normalizer and is equal to $D(m, k)$

2. <u>Jordan - Chevalley decomposition</u>

Let k be algebraically closed and V be a finite dimensional vector space over k.

<u>Definitions</u> : An endomorphism $X \in \operatorname{End} V$ is <u>semisimple</u> if X is diagonalizable.
An endomorphism $X \in \operatorname{End} V$ is <u>nilpotent</u> if $X^n = 0$ for some integer n

<u>Proposition 1</u> Any $X \in \operatorname{End} V$ can be written uniquely as $X = X_s + X_n$ where X_s is semisimple, X_n is nilpotent and X_s and X_n commute. There exist polynomials $a(t)$, $b(t)$ without constant term such that $X_s = a(X)$, $X_n = b(X)$.

<u>Proof</u> : One first show that if $p(t)$ is the characteristic polynomial of X, i.e $p(t) = \det(tI - X)$, then $p(X) = 0$. Indeed, as V is finite dimensional there exists a smallest integer k' so that $1, X, \ , X^{k'}$ are linearly dependent, $\sum_{j=0}^{k'} b_j X^j = 0$ with $b^{k'} = 1$. Write $d(t) = \sum_{j=0}^{k'} b_j t^j = \prod_{i=1}^{s} (t - c_i)^{t_i}$
Set $W_j = \prod_{\substack{i=1 \\ i \neq j}}^{s} (X - c_i)^{t_i}$, $V_j = W_j(V)$. Then $(X - c_j)^{t_j} = 0$ on V_j but no smaller power is zero, let $v_{t_j} \in V_j$ so that $(X - c_j)^{t_j - 1} v_{t_j} \neq 0$ and set $v_{t-1} = (X - c_j) v_t$
Then $v_1 \cdots v_{t_j}$ are linearly independent and $X v_r = c_j v_r + v_{r-1}$ $(1 < r \leq t_j)$
Extending the v_i's to a basis of V, X takes the form

$$\left(\begin{array}{c|c} \begin{matrix} c_j & * \\ 0 & c_j \end{matrix} & * \\ \hline 0 & X' \end{array}\right)$$

Hence $p(t) = (t - c_j)^{t_j} \det(tI - X')$, so that $d(t)$ divides $p(t)$ and, hence, $p(X) = 0$

• Let us write $p(t) = \prod_{i=1}^{k} (t - a_i)^{r_i}$
The polynomials $f_1(t) = (t - a_1)^{r_1}$ and $p'(t) = \prod_{i=2}^{k} (t - a_i)^{r_i}$ are relatively prime so there exist polynomials $g(t)$ and $q(t)$ so that

$$g(t) f_1(t) + q(t) p'(t) = 1 . \qquad (*)$$

Set $M_1 = \operatorname{Ker} (X - a_1)^{r_1}$, $M_2 = \operatorname{Ker} p'(X)$; then $q(X) p'(X) V \subset M_1$ and $g(X) f_1(X) \subset M_2$ (because $p(X) = 0$). As $V = (g(X) f_1(X) + q(X) p'(X)) V$ (by $(*)$)
$V = M_1 + M_2$ . . the sum is direct ([illegible]

The characteristic polynomial of $X|_{M_1}$ is equal to $(t-a_1)^{\dim M_1}$, with $\dim M_1 \leq r_1$ (by the same reasoning as above) and the characteristic polynomial of $X|_{M_2}$ is not dividable by $(t-a_1)$. So $\dim M_1 = r_1$, $q(X)p'(X)$ is an projection on M_1 and the characteristic polynomial of $X|_{M_2}$ is $p'(t)$.

By induction V is the direct sum of the spaces

$$V_i = \operatorname{Ker}(X-a_i)^{r_i}$$

and there is a polynomial p_i of X which is a projection of V onto V_i.

Define X_S by $X_S|_{V_i} = a_i \, \mathrm{Id}$ and $X_N = X - X_S$. In a suitable basis $X = \begin{pmatrix} \begin{matrix} a_1 & * \\ 0 & a_1 \end{matrix} & 0 \\ 0 & \begin{matrix} a_k & * \\ 0 & a_k \end{matrix} \end{pmatrix}$

Remark that $X_S = \sum a_i \, p_i(X)$ and that one can choose a polynomial without constant terms so that $X_S = a(X)$.

The unicity of the decomposition follows from the fact that if

$$X_S + X_N = X_{S'} + X_{N'}$$

then $X_S - X_{S'} = X_N - X'_N$. As all those endomorphisms commute, $X_S - X'_S$ is both semisimple and nilpotent, hence zero.

Remark If $A \subset B \subset V$ are subspaces and X maps B into A, then so do X_S and X_N as they can be written as polynomials in X without constant term.

Lemma 1 If $X = X_S + X_N$ is the so called Jordan decomposition described in proposition 1, then $\operatorname{ad} X = \operatorname{ad}(X_S) + \operatorname{ad}(X_N)$ is the Jordan decomposition of $\operatorname{ad} X$.

Proof $\operatorname{ad} X_N$ is nilpotent because $(\operatorname{ad} X_N)^r Y$ is a combination of $X_N^s \, Y \, X_N^t$ with $t+s=r$. Similarly $\operatorname{ad} X_S$ is semisimple and $[\operatorname{ad} X_S, \operatorname{ad} X_N] = \operatorname{ad}[X_S, X_N] = 0$. □

Lemma 2 Let k be arbitrary ($\operatorname{char} k = 0$) and $x \in \operatorname{End}(V)$ where V is a finite dimensional vector space over k. For any $y \in \operatorname{End}(V)$, denote by $N(y)$ its kernel (- or null space $N(y) = \{v \in V \mid y\,v = 0\}$) and by $R(y)$ its range. Then V is the direct sum of N and R where

$$N = \bigcup_{s \geq 1} N(x^s) \qquad R = \bigcap_{s \geq 1} R(x^s)$$

Proof. Assume first that k is algebraically closed. Then, using the Jordan Chevalley decomposition there is a basis in which

$$x = \begin{pmatrix} \begin{matrix} a_1 & * \\ 0 & a_1 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} a_n & * \\ 0 & a_n \end{matrix} \end{pmatrix}$$

where the a_i's are the eigenvalues.

clearly $R = \bigcup_{a_i \neq 0} V_{a_i}$ where V_{a_i} is the generalized eigenspace associated to a_i, and $N = V_0$, hence the result.

If k is not algebraically closed, we go to its closure $\bar{k}$. Let $V^c = V \otimes_k \bar{k}$ and consider x as an endomorphism of V^c; we will denote it x^c. Then $(x^s)^c = (x^c)^s$ and for any $y \in \operatorname{End}(V)$

$$N(y^c) = (N(y))^c \qquad R(y^c) = (R(y))^c$$

So V^c is the direct sum of N^c and R^c, hence the result. □

3 Nilpotent Lie algebras

A. Engel's theorem

Definition Let $\mathcal{L}$ be a Lie algebra over k. One defines a sequence of ideals $\mathcal{L}^n$ of $\mathcal{L}$ by

$\mathcal{L}^0 = \mathcal{L}$ $\quad \mathcal{L}^1 = [\mathcal{L},\mathcal{L}]$ $\quad \mathcal{L}^2 = [\mathcal{L},\mathcal{L}^1]$ $\quad \mathcal{L}^{i+1} = [\mathcal{L},\mathcal{L}^i]$ ...

$\mathcal{L}$ is called <u>nilpotent</u> if $\mathcal{L}^n = 0$ for some n. It is <u>abelian</u> if $\mathcal{L}^1 = 0$.

Examples . Any abelian Lie algebra $\mathcal{L}$ is nilpotent. In particular the set $D(m,k)$ of all $m\times m$ diagonal matrices with coefficients in k

. Let $\mathcal{N}(m,k)$ be the set of all $m\times m$ strictly upper triangular matrices (i.e $a_{ij}=0$ if $i\geq j$) with coefficients in k. Any Lie subalgebra of $\mathcal{N}(m,k)$ is nilpotent

. Let $\mathcal{L} = h_m$ be the Heisenberg algebra. $h_m = \mathbb{R}^{2m}\times\mathbb{R}$ with the bracket $[(y,a),(y',a')] = F(y,y')$ where F is a non degenerate skewsymmetric 2-form on $\mathbb{R}^{2m}$. $\mathcal{L}^2 = 0$ so $\mathcal{L}$ is nilpotent

Proposition 1 Let $\mathcal{L}$ be a Lie algebra

(a) If $\mathcal{L}$ is nilpotent, so are all subalgebras and homomorphic images of $\mathcal{L}$

(b) If $\mathcal{L}/Z(\mathcal{L})$ is nilpotent, so is $\mathcal{L}$

(c) If $\mathcal{L}$ is nilpotent $Z(\mathcal{L}) \neq 0$.

The proofs are direct consequences of the definition

Proposition 2 If $\mathcal{L}$ is nilpotent, $\operatorname{ad} x$ is a nilpotent endomorphism of $\mathcal{L}$ for all $x\in\mathcal{L}$.

Proof If $\mathcal{L}$ is nilpotent $\mathcal{L}^n = \underbrace{[\mathcal{L},[\mathcal{L}, \ldots [\mathcal{L},\mathcal{L}]]]}_{n} = 0$ so $\operatorname{ad} x^n = 0$ $\forall x\in\mathcal{L}$. □

We shall see that the converse is true

Remarks (1°) If $x\in gl(V)$ is a nilpotent endomorphism, then so is $\operatorname{ad} x$.

(2°) The reverse is false. a matrix x can be ad-nilpotent (i.e $\operatorname{ad} x$ is nilpotent) without being nilpotent (ex identity matrix!)

Theorem 1 Let A be a subalgebra of $gl(V)$, V finite dim. vector space of k. If A consists of nilpotent endomorphisms and $V\neq 0$, there exists a nonzero vector $v\in V$ for which $A.v = 0$.

Proof. Use induction on $\dim A$; the case $\dim A = 1$ follows from the fact that $\operatorname{Ker} x \neq \emptyset$ if x is nilpotent. Let $K \subsetneq A$ be any subalgebra; K acts on A/K via ad, thus, by induction, there exists $x+K\neq K$ so that $[K, x+K]\subset K$; thus $x\notin K$ and $x\in N_A(K)$. If K is a maximal proper subalgebra of A, $N_A(K) = A$, i.e. K is an ideal in A. Clearly $\dim A/K = 1$ (because the inverse image in A of a 1-dim. subspace is a subalgebra containing K). Thus $A = K + kx$. By induction $W = \{v\in V \mid Kv = 0\}$ is non zero; as K is an ideal, W is stable under A. The endomorphism $x\in A\setminus K$ is nilpotent on W, hence there exists a vector $w\in W$ so that $x.w = 0$ □

Corollary If A is a subalgebra of $gl(V)$ consisting of nilpotent endomorphisms, there exists a flag (V_i) in V (i.e subspaces V_i such that $\dim V_i = i$ and $V_{i-1}\subset V_i$ $\forall i$) stable under A, with $AV_i\subset V_{i-1}$ $\forall i\geq 1$.

In other words, there exists a basis of V relative to which the matrices of A are upper triangular.

Proof. By induction. Let $v\in V$ be so that $Av = 0$. Set $V' = kv$ and $V^1 = V/V'$. The action of A induces an action on V^1 given by nilpotent endomorphisms. □

Theorem 2 (Engel) Let $\mathcal{L}$ be a Lie algebra. If $\operatorname{ad} x$ is nilpotent for all $x\in\mathcal{L}$, then $\mathcal{L}$ is nilpotent.

Proof. We may assume $\mathcal{L}\neq 0$. Then $\operatorname{ad}\mathcal{L}\subset gl(\mathcal{L})$ satisfies the hypothesis of theorem 1. Thus, there exists $x\in\mathcal{L}$ so that $[y,x] = 0$ $\forall y$, i.e. $Z(\mathcal{L})\neq 0$. $\mathcal{L}/Z(\mathcal{L})$ has smaller dimension and consists of ad-nilpotent elements; by induction it is nilpotent and by 3.A. proposition 1, $\mathcal{L}$ is nilpotent. □

B. Representation of nilpotent Lie algebras

Let $\mathcal{L}$ be a nilpotent Lie algebra and ρ be a representation of $\mathcal{L}$ in a finite-dimensional vector space V over k.

<u>Definition</u>. A linear function $\lambda : \mathcal{L} \to k$ is a <u>weight</u> of ρ if there exists $v \neq 0$ in V and an integer $m(v) \geq 1$ such that

$$(\rho(X) - \lambda(X) I)^m v = 0 \qquad \text{for all } X \in \mathcal{L}.$$

In this case, the set of all such v's and zero form a vector space, called the <u>weight space</u> of ρ corresponding to λ and denoted $V_{\rho,\lambda}$

<u>Notation</u> If W is an invariant subspace of V under ρ (i.e $\rho(X)W \subset W \ \ \forall X \in \mathcal{L}$), we denote by $\rho|_W$ the restriction of $\rho(X)$ to W for any $X \in \mathcal{L}$.

<u>Remark</u> The representation ρ consists of nilpotent endomorphisms if and only if $V = V_{\rho,0}$.

<u>Theorem</u> Let $\mathcal{L}$ be a nilpotent Lie algebra, ρ a representation of $\mathcal{L}$ in a finite dimensional vector space V over k. Assume k to be <u>algebraically closed</u>. Then, the weight subspaces of ρ corresponding to distinct weights are linearly independent, each weight subspace is invariant under ρ and if $\lambda_1 \dots \lambda_r$ are all distinct weights of ρ, $V = \bigoplus_{i=1}^{r} V_{\rho,\lambda_i}$ the sum being direct.

<u>Proof</u>. If $\dim \mathcal{L} = 1$ the result follows from 2 proposition 1: if $T \in \mathrm{End}(V)$ then $V = \oplus V_i$; $V_i = \mathrm{Ker}\,(T - \lambda_i I)^{m_i}$ where λ_i $(i = 1, \dots, k)$ are the distinct eigenvalues with multiplicities m_i, i.e the characteristic polynomial of T is given by $(t-a_1)^{m_1} \cdots (t-a_k)^{m_k}$.

. If $\lambda_1, \dots, \lambda_r$ are distinct weights, choose X_0 so that the $\lambda_i(X_0)$ are all distinct. Then $V_{\rho,\lambda_i} \in \mathrm{Ker}\,(T - \lambda_i(X_0))^{m_i}$ where $T = \rho(X_0)$ and m_i is the multiplicity of $\lambda_i(X_0)$ in $\rho(X_0)$. So the V_{ρ,λ_i} are linearly independent.

. By induction, one has for any $X, Y \in \mathcal{L}$

$$\rho(Y) \ (\rho(X) - \alpha I)^q - (\rho(X) - \alpha I)^q \rho(Y) = -\sum_{i=0}^{q-1} (\rho(X) - \alpha I)^i \rho([X,Y]) (\rho(X) - \alpha I)^{q-i-1}$$

Again by induction, there exist integers c_{qi} so that

$$\rho(Y) \ (\rho(X) - \alpha I)^q - (\rho(X) - \alpha I)^q \rho(Y) = \sum_{i=0}^{q-1} c_{qi} \ \rho(\mathrm{ad}X^{i+1} Y)(\rho(X) - \alpha I)^{q-i-1}$$

Let $v \in V_{\rho,\lambda_i}$ and t be such that $(\rho(X) - \lambda_i(X)I)^t v = 0 \ \ \forall X \in \mathcal{L}$. Then, for $q \geq t$

$$-(\rho(X) - \lambda_i(X)I)^q \rho(Y) v = \sum_{i=q-t}^{q-1} c_{qi} \ \rho(\mathrm{ad}X^{i+1} Y) \ (\rho(X) - \alpha I)^{q-i-1} v$$

Since $\mathcal{L}$ is nilpotent, $\mathrm{ad}X^i Y = 0$ for sufficiently large i and so, for q large enough the right hand side is zero. Thus each space V_{ρ,λ_i} is stable under ρ.

. We show that $V = \oplus V_{\rho,\lambda_i}$ by induction on $\dim \mathcal{L}$.

Consider a subspace M of codimension 1 in $\mathcal{L}$ so that $[\mathcal{L}, \mathcal{L}] \subset M$. Clearly M is a subalgebra of $\mathcal{L}$ and, by induction, if ρ' is the restriction of ρ to M and $\beta_1 \dots \beta_j$ are the weights of ρ': $V = \oplus V_{\rho',\beta_j}$

By the formulas above each V_{ρ',β_j} is invariant under ρ.

Let Y be an element of $\mathcal{L} \setminus M$, then, under $\rho(Y)$ each V_{ρ',β_j} decomposes into

$$V_{\rho',\beta_j} = V_1^{(j)} \oplus \cdots \oplus V_{n_j}^{(j)}$$

where $V_k^{(j)}$ is the generalized eigenspace for $\rho(Y)|_{V_{\rho',\beta_j}}$ for the eigenvalue $\lambda_k^{(j)}$.

Again each $V_i^{(j)}$ is invariant under ρ' hence under ρ.

The weight of ρ on $V_i^{(j)}$ is given by $\begin{cases} \alpha_i^j(X) = \beta_j(X) & \text{if } X \in M \\ \alpha_i^j(Y) = \lambda_i^{(j)}(Y) \end{cases}$

Clearly $V = \oplus V_i^{(j)}$ where the sum is direct, each $V_i^{(j)}$ is invariant under ρ and $V_i^{(j)}$ is a weight space corresponding to the weight $\alpha_i^{(j)}$.

The last statement follows from:

$$(\rho(X + kY) - \alpha_i^j(X + kY))^q = \sum_{t \geq 0}^{q} (\rho(X) - \alpha_i^j(X))^t k^{q-t} (\rho(Y) - \alpha_i^j(Y))^{q-t} \frac{q!}{t!(q-t)!}$$

$$+ \sum_{i=1}^{q-1} c_{ilq} \ (\rho(X) - \alpha_i^j(X))^{\ell} \rho(\underbrace{[\,[\quad [\,,\,]}_{X \text{ and } Y\text{'s in number } i+1}) \ (\rho(Y) - \alpha_j^i(Y))^{q-i-\ell}$$

where c_{ilq} are real numbers, as the right hand side vanishes on $v \in V_i^{(j)}$ for large enough q. □

C Nilpotent Lie groups

Definition: A Lie group is said to be nilpotent if its Lie algebra is nilpotent.

Theorem: Let G be a connected nilpotent Lie group, $\mathfrak{g}$ its Lie algebra.

(i) there exists a polynomial map $P : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$\exp X \cdot \exp Y = \exp P(X,Y) \qquad \forall X, Y \in \mathfrak{g}$$

(ii) the map $\exp : \mathfrak{g} \to G$ is surjective

Proof We know by Campbell-Baker-Hausdorff formula that

$$\exp X \cdot \exp Y = \exp \mu(X,Y) \qquad \forall X, Y \in N \text{ neighborhood of } 0 \text{ in } \mathfrak{g}$$

with $\mu(X,Y) = \sum_{n \geq 1} C_n(X,Y)$ $\qquad C_n : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}^{n-1} = [\mathfrak{g},[\mathfrak{g} \quad [\mathfrak{g},\mathfrak{g}]]]$

so that the series is a finite sum when G is nilpotent.

Consider $P : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ $\quad P(X,Y) = \sum C_n(X,Y)$; this is a polynomial map and, by analyticity the formula $\exp X \cdot \exp Y = \exp P(X,Y)$ is valid everywhere. Define $G' = \exp \mathfrak{g}$, it is a subgroup of G and it contains an open neighborhood of e in G, thus G' is open and thus closed in G, as G is connected $G = G'$ and exp is surjective □

Definition: An endomorphism u of a finite dim vector space V is said to be unipotent if $u - I$ is nilpotent ; it is then invertible and u^{-1} is also unipotent.

Theorem 2 Any connected simply connected nilpotent Lie group is isomorphic to a closed subgroup of $Gl(V)$ consisting of unipotent endomorphisms, for some finite dimensional vector space V.

For a proof see Varadarajan p. 199.

4 Solvable Lie algebras

A. Definitions

Definition Let $\mathcal{L}$ be a Lie algebra over k. One defines the descending sequence of ideals $\mathcal{L}^{(n)}$ of $\mathcal{L}$ by $\mathcal{L}^{(0)} = \mathcal{L}$; $\mathcal{L}^{(1)} = [\mathcal{L}, \mathcal{L}]$, $\mathcal{L}^{(2)} = [\mathcal{L}^{(1)}, \mathcal{L}^{(1)}]$

$\mathcal{L}^{(i+1)} = [\mathcal{L}^{(i)}, \mathcal{L}^{(i)}]$

A Lie algebra is solvable if $\mathcal{L}^{(n)} = 0$ for some n.

Remark that, for any integer n $\quad \mathcal{L}^{(n)} \subset \mathcal{L}^{n}$

Examples
- Any nilpotent Lie algebra is solvable
- Let $\tau(m,k) = \eta(m,k) + \mathcal{D}(m,k)$ be the set of all $m \times m$ upper triangular matrices (i.e. $a_{ij} = 0$ if $i > j$) with coefficients in k. Any Lie algebra contained in $\tau(m,k)$ is solvable.

Proposition: Let $\mathcal{L}$ be a Lie algebra

(a) If $\mathcal{L}$ is solvable, so are all subalgebras and homomorphic images of $\mathcal{L}$

(b) If $\mathfrak{I}$ is a solvable ideal of $\mathcal{L}$ such that $\mathcal{L}/\mathfrak{I}$ is solvable, then $\mathcal{L}$ itself is solvable.

(c) If $\mathfrak{I}$ and $\mathfrak{I}'$ are solvable ideals of $\mathcal{L}$, so is $\mathfrak{I} + \mathfrak{I}'$.

The proof is a direct consequence of the definitions.

Corollary Any Lie algebra $\mathcal{L}$ has a unique maximal solvable ideal (i.e one which contains any larger solvable ideal)

Definition The radical of a Lie algebra is its maximal solvable ideal. It is denoted by rad $(\mathcal{L})$.

A Lie algebra $\mathcal{L}$ is semisimple if $\mathcal{L} \neq 0$ and rad$(\mathcal{L}) = 0$.

We shall see in § 9 that any Lie algebra splits into its radical and a semisimple part.

B. Lie's theorem

Theorem Let $\mathcal{L}$ be a solvable subalgebra of $gl(V)$ (V finite dim. vector space over k). Assume k to be algebraically closed. If $V \neq 0$, then V contains a common eigenvector for all the endomorphisms in $\mathcal{L}$.

<u>Proof</u> One uses induction on $\dim \mathcal{L}$, the case $\dim \mathcal{L} = 1$ being obvious (an endomorphism has an eigenvector when k is algebraically closed).

– One locates an ideal K of codimension one: since $\mathcal{L}$ is solvable and $\mathcal{L} \neq 0$, $[\mathcal{L}, \mathcal{L}] \neq \mathcal{L}$ and one takes for K the inverse image of any subspace of codimension one in $\mathcal{L}/_{[\mathcal{L},\mathcal{L}]}$ which is abelian, clearly K is an ideal as $K \supset [\mathcal{L}, \mathcal{L}]$.

– Use induction: there exists a common eigenvector $v \in V$ for K. Define $\lambda : K \to k$ such that $x.v = \lambda(x) v \;\; \forall x \in K$ and let $W = \{ w \in V \mid x.w = \lambda(x) w \;\; \forall x \in K \}$. So $W \neq 0$.

(Remark: if $K = 0$, $\mathcal{L}$ is abelian of dimension one.)

– Show that W is invariant under $\mathcal{L} = K + kz$.

Let $w \in W$, $x \in \mathcal{L}$; we want to see that $y . x w = \lambda(y) x w \;\; \forall y \in K$.

One has $y . x w = [y, x] w + x y w = \lambda(y) x w + \lambda([y, x]) w$ (*)

• We prove that $\lambda([y, x]) = 0 \;\; \forall x \in \mathcal{L}, y \in K$. We fix $x \in \mathcal{L}, w \in W$.

Let $n > 0$ be the smallest integer for which $w, xw, \dots, x^n w$ are linearly dependent. Let $W_i = \text{span}\{ w, xw, \dots, x^{i-1} w \}$, $W_0 = 0$.

Each $y' \in K$ leaves W_i invariant (cf *). Relative to the given basis, $y' \in K$ is represented on W_i by an upper triangular matrix with diagonal entries equal to $\lambda(y')$. So $tr_{W_n}(y') = n \lambda(y')$. In particular if $y' = [x, y]$ where $x \in \mathcal{L}$, $y \in K$, as both x and y stabilize W_n, $[x, y]$ acts as the commutator of endomorphisms and $tr_{W_n}([x, y]) = n \lambda([x, y]) = 0$.

– Take any eigenvector v of z in W; it is an eigenvector for all the endomorphisms in $\mathcal{L}$. □

<u>Corollary 1</u> (Lie's theorem) Let $\mathcal{L}$ be a solvable subalgebra of $gl(V)$ (V: finite dim. vector space over k), k algebraically closed. Then $\mathcal{L}$ stabilizes some flag in V. In other words, the matrices of $\mathcal{L}$ relative to a suitable basis are upper triangular.

The proof follows by induction on $\dim V$.

<u>Corollary 2</u> A Lie algebra $\mathcal{L}$ is solvable if and only if its derived algebra $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$ is nilpotent.

<u>Proof</u> If $\mathcal{L}'$ is nilpotent, clearly $\mathcal{L}$ is solvable. For the reverse, we first assume k to be algebraically closed. If $\mathcal{L}$ is solvable, so is the algebra $\{ \text{ad}\, x \,;\, x \in \mathcal{L} \} \subset gl(\mathcal{L})$. By Lie's theorem, there exists a flag in $\mathcal{L}$ stabilized by $\mathcal{L}$, i.e. a chain of ideals in $\mathcal{L}$ or a basis of $\mathcal{L}$ in which the matrices of $\text{ad}\, \mathcal{L}$ are upper triangular. As $\text{ad}\, [\mathcal{L}, \mathcal{L}] = [\text{ad}\, \mathcal{L}, \text{ad}\, \mathcal{L}]$, the matrices of $\text{ad}\, \mathcal{L}'$ are strictly upper triangular, hence nilpotent. By Engel's theorem $\mathcal{L}'$ is nilpotent.

If k is not algebraically closed, we denote by $\bar{k}$ its algebraic closure, by $\mathcal{L}^c = \mathcal{L} \otimes_k \bar{k}$. Then $\mathcal{L}^c$ is solvable if and only if $[\mathcal{L}^c, \mathcal{L}^c] = [\mathcal{L}, \mathcal{L}]^c$ is nilpotent. But $\mathcal{L}$ is nilpotent (resp. solvable) if and only if $\mathcal{L}^c$ is nilpotent (resp. solvable). □

5 Killing form and Cartan's Criterion

A. Killing form

The <u>Killing form</u> β of a Lie algebra $\mathcal{L}$ is a symmetric bilinear form on $\mathcal{L}$ defined by

$$\beta(X,Y) = \operatorname{Tr} \operatorname{ad}X \operatorname{ad}Y \qquad \forall X,Y \in \mathcal{L}$$

This form is invariant in the sense that

$$\beta([X,Y],Z) + \beta(Y,[X,Z]) = 0 \qquad \forall X,Y,Z \in \mathcal{L}$$

(This results from the fact that $\operatorname{Tr}(A\,B\,C) = \operatorname{Tr}(C\,A\,B)$ for any $A\,B\,C \in \operatorname{End}(\mathcal{L})$.)

<u>Examples</u> 1) If $\mathcal{L} = \mathbb{R}$ $\quad [X,Y]=0 \ \forall X,Y \in \mathcal{L}$ and $\beta \equiv 0$.

2) If $\mathcal{L} = \mathfrak{gl}(m,\mathbb{R})$, let E_{ij} be the matrix with a 1 at the intersection of the i^{th} row and the j^{th} column and zeros elsewhere. As $(\operatorname{ad}A)(B) = AB - BA$ we have

$$(\operatorname{ad}A \operatorname{ad}A' E_{ij})_{kl} = \sum_r A_{kr} A'_{ri} \delta_{jl} - A_{ki} A'_{jl} - A'_{ki} A_{jl} + \sum_r \delta_{ki} A'_{jr} A_{rl}$$

Thus

$$\beta(A,A') = 2m \operatorname{Tr}(AA') - 2 \operatorname{Tr}A \operatorname{Tr}A' \qquad \forall A,A' \in \mathfrak{gl}(m,k).$$

<u>Lemma</u>: If $\mathcal{I}$ is an ideal in $\mathcal{L}$, the Killing form of $\mathcal{I}$ viewed as a Lie algebra is equal to the restriction to $\mathcal{I}$ of the Killing form of $\mathcal{L}$.

The proof is a consequence of the fact that $\operatorname{ad}\mathcal{I}$ maps $\mathcal{L}$ into $\mathcal{I}$. □

<u>Example</u> If $\mathcal{L} = \mathfrak{sl}(m,k) = \{X \in \mathfrak{gl}(m,k) \mid \operatorname{tr} A = 0\}$, $\mathcal{L}$ is an ideal in $\mathfrak{gl}(m,k)$ because $[\mathfrak{gl}(m,k), \mathfrak{gl}(m,k)] \subset \mathfrak{sl}(m,k)$. By the lemma above

$$\beta(A,A') = 2m \operatorname{tr}(AA') \qquad \forall A,A' \in \mathfrak{sl}(m,k)$$

B Cartan's Criterion for solvability

<u>Lemma</u>: Let A, B be two subspaces of $\mathfrak{gl}(V)$ where V is a finite dim. vector space over k. Assume k to be algebraically closed.
Set $M = \{y \in \mathfrak{gl}(V) \mid [y,B] \subset A\}$. Suppose $x \in M$ satisfies $\operatorname{Tr}(xy) = 0$ for all $y \in M$. Then x is nilpotent.

<u>Proof</u> Let $x = x_s + x_n$ be the Jordan decomposition of x.
Fix a basis $v_1 \dots v_m$ of V relative to which x_s has matrix $\operatorname{diag}(a_1, \dots, a_m)$.
Let E be the vector subspace of k over the prime field $\mathbb{Q}$, spanned by the eigenvalues $a_1 \dots a_m$. We want to show that $x_s = 0$, thus that $a_i = 0 \ \forall i$.
It is clearly enough to show that $E^* = 0$. Let $f : E \to \mathbb{Q}$ be a linear function, and let $y = \operatorname{diag}(f(a_1) \dots f(a_m)) \in \mathfrak{gl}(V)$. One has (with the notations of 5A example 2)

$$\operatorname{ad}x_s(E_{ij}) = (a_i - a_j) E_{ij} \qquad \operatorname{ad}y(E_{ij}) = (f(a_i) - f(a_j)) E_{ij}$$

Let $r(t)$ be a polynomial without constant term so that

$$r(a_i - a_j) = f(a_i) - f(a_j) \qquad \text{for all pairs } i,j$$

This is possible as f is linear. Clearly $\operatorname{ad}y = r(\operatorname{ad}x_s)$.
By Jordan decomposition $\operatorname{ad}x_s$ is a polynomial in $\operatorname{ad}x$ without constant term, so $\operatorname{ad}y$ is a polynomial in $\operatorname{ad}x$ without constant term.
As $\operatorname{ad}x$ maps B into A, so does $\operatorname{ad}y$ and $y \in M$.
Then $\operatorname{tr}(xy) = 0 = \sum a_i f(a_i)$; this is a $\mathbb{Q}$-linear combination of elements of E, applying f one gets $\sum f(a_i)^2 = 0$ so $f(a_i) = 0 \ \forall i$ and $f = 0$. □

<u>Theorem</u> (Cartan's criterion) Let $\mathcal{L}$ be a subalgebra of $\mathfrak{gl}(V)$ (V finite dim.). Suppose that $\operatorname{tr}(xy) = 0 \quad \forall x \in [\mathcal{L},\mathcal{L}], \ y \in \mathcal{L}$. Then $\mathcal{L}$ is solvable.

<u>Proof</u> It is enough to show that $[\mathcal{L},\mathcal{L}]$ is nilpotent (cf 4 B cor 2) or, by Engel's theorem (3.A thm 2) that $\operatorname{ad}x$ is nilpotent for any $x \in [\mathcal{L},\mathcal{L}]$.
If k is algebraically closed, one uses lemma 1 with $V = \mathcal{L}$, $A = [\mathcal{L},\mathcal{L}]$, $B = \mathcal{L}$ so $\mathcal{L} \subset M = \{x \in \mathfrak{gl}(V) \mid [x,\mathcal{L}] \subset [\mathcal{L},\mathcal{L}]\}$. To apply it, one needs $\operatorname{tr}(xy) = 0 \ \forall x \in [\mathcal{L},\mathcal{L}], y \in M$. We know it is true if $y \in \mathcal{L}$ by hypothesis and

$$\operatorname{tr}([x,x']y) = \operatorname{tr}(xx'y - x'xy) = \operatorname{Tr}([x'y].x]$$

so if $x, x' \in \mathcal{L}$ and $y \in M$ the last term is zero as $[x',y] \in [\mathcal{L},\mathcal{L}]$ by definition of M.
So indeed $\operatorname{tr}(x,y) = 0 \ \forall x \in [\mathcal{L},\mathcal{L}], y \in M$ and $\operatorname{ad}x$ is nilpotent for any $x \in [\mathcal{L},\mathcal{L}]$.
If k is not algebraically closed, we go to its closure and the hypothesis is still verified. □

Corollary. A Lie algebra $\mathcal{L}$ is solvable if

$$\beta([X,Y],Z)=0 \qquad \forall X,Y,Z\in\mathcal{L}$$

Proof. We apply the above theorem to the adjoint representation. If $\beta([\mathcal{L},\mathcal{L}],\mathcal{L})=0$, $\operatorname{ad}\mathcal{L}$ is solvable. As $\operatorname{Ker}\operatorname{ad}=Z(\mathcal{L})$ is abelian and thus solvable, $\mathcal{L}$ is solvable (cf 4A prop 1)

C Cartan's Criterion for semisimplicity

Lemma. A nonzero Lie algebra is semisimple (i.e. $\operatorname{Rad}\mathcal{L}=0$) if and only if $\mathcal{L}$ has no nonzero abelian ideals

Proof. If R is a nonzero solvable ideal, the last term in the derived series $R, R^{(1)}, R^{(2)}, \dots, R^{(k)}$ $(R^{(k+1)}=0)$ is abelian, so if $\mathcal{L}$ has no nonzero abelian ideal, it has no nonzero solvable ideal and $\operatorname{rad}(\mathcal{L})=0$.
Conversely, any abelian ideal must be in the radical, so if $\operatorname{rad}(\mathcal{L})=0$ there is no nonzero abelian ideals $\square$

Theorem. Let $\mathcal{L}$ be a nonzero Lie algebra. Then $\mathcal{L}$ is semisimple if and only if its Killing form is nondegenerate (i.e. $\beta(X,Y)=0\ \forall X\in\mathcal{L} \Rightarrow Y=0$).

Proof. Let $S=\{Y\in\mathcal{L} \mid \beta(X,Y)=0\ \forall X\in\mathcal{L}\}$
If $\mathcal{L}$ is semisimple we want to show that $S=0$. We know that $\operatorname{tr}\operatorname{ad}X\operatorname{ad}Y=0$ $\forall Y\in S$ $X\in[S,S]\subset Y$. Hence by Cartan's criterion S is solvable so $S\subset\operatorname{Rad}(\mathcal{L})=0$
Conversely, suppose $S=0$. To prove that $\mathcal{L}$ is semisimple, it is enough to show that every abelian ideal $\mathfrak{J}$ of $\mathcal{L}$ is included in S.
Suppose $X\in\mathfrak{J}$, $Y\in\mathcal{L}$. Then $\operatorname{ad}X\operatorname{ad}Y$ maps $\mathcal{L}$ into $\mathfrak{J}$ and $(\operatorname{ad}X\operatorname{ad}Y)^2$ maps $\mathcal{L}$ into $[\mathfrak{J},\mathfrak{J}]=0$, hence $\operatorname{ad}X\operatorname{ad}Y$ is nilpotent.
Thus $\operatorname{tr}(\operatorname{ad}X\operatorname{ad}Y)=\beta(X,Y)=0$ and $\mathfrak{J}\subset S$

Example. If $\mathcal{L}=\mathfrak{sl}(n,k)=\{A\in\mathfrak{gl}(n,k) \mid \operatorname{tr}A=0\}$, $\beta(A,A')=2n\operatorname{tr}AA'$ so β is non degenerate and $\mathfrak{sl}(n,k)$ is semisimple

D. Trace form

If ρ is a finite dimensional representation of a Lie algebra $\mathcal{L}$, one defines the trace form β^ρ associated to ρ by:

$$\beta^\rho(X,Y)=\operatorname{tr}\rho(X)\rho(Y) \qquad \forall X,Y\in\mathcal{L}$$

Again β^ρ is invariant in the sense that

$$\beta^\rho([X,Y],Z)+\beta^\rho(Y,[X,Z])=0 \qquad \forall X,Y,Z\in\mathcal{L}$$

Lemma. If $\mathcal{L}$ is semisimple, the trace form β^ρ is non degenerate on $(\ker\rho)^\perp$ where $\perp$ is the orthogonal space in $\mathcal{L}$ relative to the Killing form β

Proof. On $P=(\ker\rho)^\perp$, ρ is injective. If S is the radical of β^ρ on P, then $\operatorname{tr}\rho(X)\rho(Y)=0$ $\forall X\in S$, $Y\in[S,S]\subset P$.
Hence $\rho(S)$ is solvable thus S is solvable. So $S\subset\operatorname{Rad}(\mathcal{L})=0$.