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COLLEGE ON

REPRESENTATION THEORY OF LIE GROUPS

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(1) Harmonic Analysis on Compact Lie Groups
(Peter-Weyl Theorem, Schur Orthogonolity Relations for Irreducible Characters)
(continued)

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These are preliminary lecture notes, intended only for distribution to participants.

Lemma

Let G be a compact Lie group. Then a finite-dimensional representation (V,π) of G is unitarizable.

Proof

Let $(\, , \,)$ be any inner product on V. Define an inner product
$$\langle u, v\rangle = \int_G (\pi(g)u, \pi(g)v)\, dg \qquad u, v \in V$$
then the invariance of dg gives that $\langle \pi(g)u, \pi(g)v\rangle = \langle u, v\rangle$ for all $g \in G$. $\square$

The Regular representation:

G Lie group, Haar measure dx.

On $C_0(G)$ define inner product $(f_1, f_2) = \int_G f_1(x)\overline{f_2(x)}\, dx$.

Left regular (anti-)representation of G, l with $(l_g f)(g_1) = f(g^{-1}g_1)$.

Also if G is unimodular, define the right regular representation r of G, $(r_g f)(g_1) = f(g_1 g)$ and the regular representation τ of $G \times G$ $(\tau(g,h)f)(x) = f(g^{-1}xh)$ i.e. $\tau(g,h)f = l_g r_h f$, $f \in C_0(G)$.

These extend unitarily to the Hilbert space completion $L^2(G)$.

G compact. (V,π) irreducible unitary representation of G.

Take $(V \otimes V, \pi \otimes \pi)$ and define for $u, v \in V$
$$A(u \otimes v)(x) = \langle \pi(x)u, v\rangle \quad \text{then } A \in \mathrm{Hom}_{G\times G}(V \otimes V, L^2(G))$$
by irreducibility, $\operatorname{Ker} A = \{0\}$ so by taking the diagonal subrepresentation of G in $(V \otimes V, \pi \otimes \pi)$ we, that (V,π) can be

regarded as a closed subrepresentation of $(L^2(G), \tau)$

Proposition

Let G be a compact Lie group. Let V be a closed invariant subspace of $L^2(G)$ under l which is irreducible. Then V is finite-dimensional.

Proof

Let f be a unit vector in V and for $v \in L^2(G)$, $Pv = (v,f)f$.

Define operator T on V by $(Tv, w) = \int_G \langle P\pi(g)v, \pi(g)w\rangle\, dg$

for $v, w \in C(G)$. Now $Tv(x) = \int_G k(x,y)v(y)\, dy$

where continuous kernel function $k(x,y) = \int_G \overline{f(gx)}\, f(gy)\, dg$

with $k(x,y) = \overline{k(y,x)}$. Thus T is completely continuous and self-adjoint. Also $T(V) \subseteq V$ and $T|_V \neq 0$ as $(Tf, f) > 0$.

Clearly $\pi(g)T = T\pi(g)$ for $g \in G$. T has a non-zero eigenvalue λ, so irreducibility of π implies that $T|_V = \lambda I$. But then the identity map on V is completely continuous, and it follows that V must be finite-dimensional. $\square$

Let G be a Lie group and (U,π) a representation of G. If $f \in C_0(G)$ define $\pi(f)$ by $(\pi(f)v, w) = \int_G \langle f(g)\pi(g)v, w\rangle\, dg$

which we write as $\pi(f) = \int_G f(g)\pi(g)\, dg$. $\pi(f)$ is continuous $f \in C(G)$.

Small subgroups: A topological group which has a nbd of the identity that doesn't contain any non-trivial subgroups is called a group with no small subgroups. eg Every Lie group has no small subgroups (Proof from the fact that exp is a local diffeo).

<u>Theorem</u>
Let G be a compact Lie group, then G is linear in fact
(i) there exists an isomorphism from G onto a closed subgroup of some $O(m)$
(ii) $U(m)$

Proof
<u>Lemma</u>
If A is a nbd of e in G, there exists a continuous homomorphism π from G into some $O(m)$ s.t. $\ker \pi \subseteq A$.

<u>Proof</u>
Put $B = A \cap A^{-1}$. There is a real valued continuous function f on G satisfying $f(g) < f(e)$ for g in $G \backslash B$. Let V be the smallest closed subspace of $C(G, R)$ containing f and all $l_g f$ for $g \in G$; then V is invariant under l, therefore has finite dimension m. If $g \in G \backslash B$, $(l_g f)(e) = f(g^{-1}) < f(e)$ showing $g \notin \ker \pi$. Moreover the subrepresentation (V, π) of $(L^2(G), l)$ is orthogonal. □

Let A be a nbd of e in G which doesn't contain any non-trivial subgroups. In the Lemma, $\ker \pi$ must be trivial. Now $\pi(G)$ is compact and π^{-1} is continuous on $\pi(G)$. (ii) follows from (i) as $O(m)$ is closed in $U(m)$. □

<u>Theorem</u>
Let $G = [GG]$ and $\pi : G \longrightarrow GL(V)$ a finite-dim, irred! reps. Then $\pi(x)^n = I$ where $n = \dim V$, each x in the centre Z of G.

<u>Proof</u>
By Schur's lemma $\pi(x) = \lambda(x)$ a scalar, $x \in Z$. x is a product of commutators, therefore $\det \pi(x) = 1$ ie $\lambda(x)^n = 1$, $n = \dim V$ □

<u>Cor</u>
If G is a connected matrix group with G semi simple, then every element in the centre of G has finite order.

<u>Proof</u>
Faithful reps $G \xrightarrow{X} GL(n, \mathbb{C})$. X is completely reducible $X = X_1 \oplus \ldots \oplus X_m$. In $GL(n_j, \mathbb{C})$, $X_j(x)^{n_j} = 1$ so x has finite order. □

Example of a non-linear Lie group $\tilde{SL}(2, \mathbb{R})$

$SL(2, \mathbb{R}) \simeq SU(1,1)$

conjugate by $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ $\quad \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ $\quad a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$

parametrized by $b \in \mathbb{C}$, $\arg a$ and space is $S^1 \times \mathbb{C}$
write out multiplication in (b, θ). Use same formula for $b \in \mathbb{C}$, $\theta \in \mathbb{R}$ to get $\tilde{SL}(2, \mathbb{R})$.

Let G be a compact Lie group and $\hat{G}$ the set of equivalence classes of irreducible representations of G (each is finite dimensional and unitarizable). Let for $\nu \in \hat{G}$, (U_ν, Π_ν) be a representative. For $\nu \in \hat{G}$ let A_ν ~~from~~ on $U_\nu^* \otimes U_\nu$ be defined by

$$A_\nu(\lambda \otimes v)(g) = \lambda(\Pi_\nu(g) v) \qquad g \in G$$

The image of A_ν is spanned by the matrix elements of Π_ν and A_ν is an intertwining operator in $\mathrm{Hom}_{G \times G}\left(\underset{\Pi_\nu^* \otimes \Pi_\nu}{U_\nu^* \otimes U_\nu}, \underset{\tau}{L^2(G)} \right)$

<u>Theorem</u>
$\hat{G}$ is countable and there is a Hilbert space direct sum

$$L^2(G) = \sum_{\nu \in \hat{G}}^{\oplus} U_\nu^* \otimes U_\nu \quad \text{with } \tau = \sum_\nu^{\oplus} \Pi_\nu^* \otimes \Pi_\nu$$

<u>Theorem</u>
The matrix elements of the irreducible representations of G are dense in $C(G)$ relative to the uniform norm $\|f\|_\infty = \sup_{x \in G} |f(x)|$

constitute the Peter-Weyl theorem.
Under l one has the primary decomposition $L^2(G) = \sum_{\nu \in \hat{G}}^{\oplus} \Gamma_\nu(G)$ where the intertwining number $i_G(L^2(G), U_\nu) = d(\nu)$ the degree of ν.

~~G unimodular, type I.~~
(U, Π) unitary. For $f \in C_0(G)$ (define $\Pi(f) = \int_G f(g) \Pi(g)\, dg$)
$\Pi(f)$ is
extension of Π to the group algebra $L^1(G)$, which is a Banach algebra under convolution $*$

$$(f * h)(g) = l_g(f) h\ (g)$$

For ~~each~~ $u \in U$ define $T_u(f) = \Pi(f) u$ then

$$T_u\, l_g(f) = \Pi(l_g f) u = \Pi(g) \Pi(f) u = \Pi(g) T_u f$$

G compact
For each $u \in U$, T_u intertwines l and Π. Thus if Π is irreducible, it must be a subrepresentation of l.

We now give an alternative proof of the fact that Π_ν, $\nu \in \hat{G}$ is finite-dimensional for G compact, and the Peter-Weyl theorem. In fact we get these at once using ~~some~~ elliptic operator theory.

G compact Lie group. Then $\mathfrak{g}$ is reductive ie

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$$

the direct sum of the center and the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}'$ is semi-simple. To see this let $\langle , \rangle$ be any inner product on $\mathfrak{g}$ then $(\xi, \eta) = \int_G \langle \mathrm{Ad}g\, \xi, \mathrm{Ad}g\, \eta \rangle\, dg \qquad \xi, \eta \in \mathfrak{g}$

defines an inner product such that the adjoint representation of G is orthogonal. The adjoint reps of $\mathfrak{g}$ is then skew; thus if $\mathfrak{a}$ is an ideal so is $\mathfrak{a}^\perp$. Hence

$$\mathfrak{g} = \mathfrak{a}_1 \oplus \dots\dots \oplus \mathfrak{a}_p \quad \text{with } \mathfrak{a}_i \text{ simple or 1-dim.}$$

[illegible] $\mathfrak{u}(n) = i\mathbb{R}I \oplus \mathfrak{su}(n)$.

The Killing form $B(\xi,\eta) = \operatorname{tr}(\operatorname{ad}\xi \circ \operatorname{ad}\eta)$ on $\mathfrak{g}$ is negative semi-definite and -ve definite on $\mathfrak{g}'$ for: if $\xi \in \mathfrak{g}$, then $\operatorname{ad}\xi$ is skew-symmetric relative to $(\,,\,)$ so $(\operatorname{ad}\xi)^2$ is symmetric with nonpositive eigenvalues thus $\operatorname{tr}(\operatorname{ad}\xi)^2 \leq 0$. If $\operatorname{tr}(\operatorname{ad}\xi)^2 = 0$, then $\operatorname{ad}\xi = 0$.

Let $\{\eta_i\}$ be an orthonormal basis of $\mathfrak{g}$ w.r.t $(\,,\,)$. Define $\Omega_G = -\sum_i \eta_i^2$ in $\mathfrak{U}(\mathfrak{g})$ (indep of ortho basis)

Ω_G is called the Casimir element and lies in the center of $\mathfrak{U}(\mathfrak{g})$. ($\mathfrak{U}(\mathfrak{g})$: the tensor algebra of $\mathfrak{g}$ modulo the two sided ideal generated by $\{\xi\otimes\eta - \eta\otimes\xi - [\xi\eta]\,;\, \xi,\eta\in\mathfrak{g}\}$. Any repn ϕ of $\mathfrak{g}$ extends to $\mathfrak{U}(\mathfrak{g})$) If (V,π) is finite-dim,

$$\pi(g)\, d\pi(\xi)\, \pi(g)^{-1} = d\pi(\operatorname{Ad}g\,\xi)$$

and so

$$\pi(g)\, d\pi(\Omega_G)\, \pi(g)^{-1} = d\pi(\Omega_G), \quad g \in G.$$

Thus if π is irreducible, by Schur's lemma, $d\pi(\Omega_G)$ is a constant.

G unimodular. The differentials of the left and right regular representation of G are dl, dr where

$$dl(\xi)_g f = \frac{d}{dt} f(\exp -t\xi\, g)\Big|_{t=0}, \quad dr(\xi)_g f = \frac{d}{dt} f(g \exp t\xi)\Big|_{t=0}$$

for $\xi \in \mathfrak{g}$, $g \in G$ and $f \in C_0(G)$. Also have the differential of the regular representation $d\tau$. We have

$$-dl(\operatorname{Ad}g\,\xi)_g f = dr(\xi)_g f$$

and for G compact, $dl(\Omega_G) = dr(\Omega_G)$. $dl(\Omega_G), dr(\Omega_G), d\tau(\Omega_G)$ is an intertwining operator for l, r, τ resp.

<u>Proof of [illegible]</u>

Consider the spectral decomposition of the 'Laplacian' $d\tau(\Omega_G)$. From general theory of an elliptic, essentially self-adjoint differential operator on a compact orientable manifold one has

$$L^2(G) = \sum_i{}^{\oplus}\, E_i$$

a countable orthogonal direct sum of eigenspaces of $d\tau(\Omega_G)$. each E_i consists of smooth functions and is finite-dimensional. Now under τ each subrepresentation (E_i, τ) is completely reducible. Hence we deduce 1°.

Suppose that U, V are equivalent $G\times G$ representations of $L^2(G)$ that are irreducible, we show that $U = V$. Let $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ be orthonormal bases of U and V s.t the corresponding matrix repns are the same. Set $F(x,y) = \sum_i e_i(x)\overline{f_i(y)}$, $x,y \in G$. τ is unitary so $F(gxh^{-1}, gyh^{-1}) = F(x,y)$ for $g,h \in G$; in particular $F(e, g^{-1}) = F(g, e)$. Let $Sf(g) = f(g^{-1})$ for $f \in C(G)$. S extends to a unitary operator on $L^2(G)$. It is clear that if W is an invariant subspace of $L^2(G)$ then so is SW; also if $\overline{W} = \{\bar f \,;\, f\in W\}$ then $\overline{W}$ is invariant. Now

$$\sum_i \overline{f_i(e)}\, e_i(g) = F(g,e) = F(e,g^{-1}) = \sum_i e_i(e)\overline{f_i(g^{-1})} = \sum_i e_i(e)\,\overline{Sf_i(g)}$$

and

thus $U \cap \overline{\mathfrak{F}V} \neq 0$. Since U is irreducible $U \subseteq \overline{\mathfrak{F}V}$. But $\dim U = \dim \overline{\mathfrak{F}V}$ so $U = \overline{\mathfrak{F}V}$. The same argument applies if we take $U = V$ to get $V = \overline{\mathfrak{F}V}$. Hence $U = \overline{\mathfrak{F}V} = V$.

We now show that there is a unit vector $u \in U$ so that $\tau(g,g)u = u$ for all $g \in G$. Define a map A of U into $\mathbb{C}^n$, $A(c_1 f_1 + \dots + c_n f_n) = (c_1, \dots, c_n)$. Let for each $(g,h) \in G \times G$, $P(g,h)$ be the linear operator on $\mathbb{C}^n$ given by $A(\tau(g,h)f) = P(g,h)Af$. Let $T(x,y) = (f_1(xy^{-1}), \dots, f_n(xy^{-1}))$ from $G \times G$ into $\mathbb{C}^n$; then $T(gx, gy) = \overline{P(g,h)}\, T(x,y)$. Then take v in the direction $A^{-1}T(e,e)$.

It now follows that $(U, \tau) = (U_\nu^* \otimes U_\nu, \pi_\nu^* \otimes \pi_\nu)$ under A_ν and each $\nu \in \hat{G}$ occurs. Hence we deduce the result. □

The P.W theorem is very general. It says that understanding the representations of G is equivalent to understanding $L^2(G)$, but does not tell us what the π_ν's are $\hat{G}$ etc.

□ If H is compact, abelian show that an irreducible reps of H is one-dimensional.

S' $\quad \Delta = -\frac{d^2}{dx^2}$, x coord fn.

spectral decomposition $\{n^2, e^{inx}\}_{n=-\infty}^{\infty}$ Classical Fourier expansion.

<u>Lemma</u> Schur Orthogonality relations.
Let (U,π), (V,π_1) be irreducible representations of G a compact Lie group. Then for $u, v \in U$ and $u_1, v_1 \in V$

$$\int_G \langle \pi(g)u, v\rangle \overline{\langle \pi_1(g)u_1, v_1\rangle}\, dg = 0 \quad \text{if } \pi \not\simeq \pi_1$$
$$= \frac{1}{\dim U} \langle u, u_1\rangle \overline{\langle v, v_1\rangle} \quad \text{if } \pi \simeq \pi_1 \qquad \dagger$$

<u>Proof</u>
If $\pi \not\simeq \pi_1$, then it follows from P.W that $(f,h) = 0$ for f, h a matrix element of π, π_1. So suppose $\pi \simeq \pi_1$; the LHS of $\dagger$ defines a $G \times G$ inner product on $U^* \otimes U$ under $\pi^* \otimes \pi$, and thus equals a positive constant c times the usual tensor product inner product $\langle u, u_1\rangle \overline{\langle v, v_1\rangle}$, it remains to compute c. Let $\{u_i\}$ be an orthonormal basis of U and $f_{ij}(g) = \langle \pi(g)u_j, u_i\rangle$ $g \in G$, then the f_{ij} define up to a constant factor an orthonormal basis of the matrix elements of (U, π). Thus since π is unitary $\sum_{i,j} f_{ij}(g)\overline{f_{ij}(g)} = r$ a constant; but with $g = e$, $\sum_{i,j} |f_{ij}(e)|^2 = \dim U$, and $\dim U = \sum_{i,j} \int_G f_{ij}(g)\overline{f_{ij}(g)}\, dg = c(\dim U)^2$, hence $c = \frac{1}{\dim U}$ □

<u>Corollary</u>
Let χ_U, χ_V be the characters of U, V. Then

$$\int_G \chi_U(g)\overline{\chi_V(g)}\, dg = 0 \quad \text{if } U \text{ and } V \text{ are not equivalent}$$
$$= 1 \quad \text{if } U \text{ and } V \text{ are equivalent.}$$

Proof.
If $U \neq V$ this follows from $(f, h) = 0$, f, h a matrix element of Π, Π_1 resp.
If $U = V$ then $\chi_U = \chi_V$ and $\chi_U = \sum_i f_{ii}$; and from

$$\int_G f_{ii}(g)\,\overline{f_{jj}(g)}\,dg = \delta_{ij}\frac{1}{\dim U}$$ so running over i, j gives the result. □

(U, Π), (U_1, Π_1) finite-dimensional. Suppose $\chi_\Pi = \chi_{\Pi_1}$.

$$U = \sum_i n_i U_i, \qquad U_1 = \sum_i m_i U_i.$$

$$\chi_\Pi = \sum_i n_i \chi_i, \qquad \chi_{\Pi_1} = \sum_i m_i \chi_i.$$

From the orthonormality relations, $n_i = (\chi_\Pi, \chi_i) = (\chi_{\Pi_1}, \chi_i) = m_i$ each i.
Hence χ_Π determines Π up to equivalence.

Let $P_\nu : L^2(G) \longrightarrow U_\nu \otimes U_\nu^*$ (where $B_\nu : U_\nu \otimes U_\nu^* \to C(G)$ is $B_\nu(u \otimes u^*)(g) = u^*(\Pi_\nu(g)^{-1}u)$) be the orthogonal projection, then [illegible] Π_ν

$$(P_\nu f)(x) = d(\nu)\int_G \overline{\chi_\nu(g)}\, f(g^{-1}x)\,dg,$$ the Fourier transform.

The χ_ν, $\nu \in \hat{G}$ form a complete orthonormal set for the class functions on G.

(V, Π) finite-dim. Recall that Π is irreducible iff $d\Pi$ is, (G connected).
For G compact, to determine $\hat{G}$ we find all finite-dim irreducible representations of $\mathfrak{g}_{\mathbb{C}}$ and then see which of these are differentials of reps. of G. We now illustrate this for $SU(2)$, $SO(3)$. In fact we need to determine the type A_1 representations for the general theory.

Representations of $sl(2, \mathbb{C})$. cf. J.A. Wolf

The 3-dim real Lie algebras $su(2)$, $so(3)$ and $sl(2, \mathbb{R})$ each complexify to $sl(2, \mathbb{C})$, type A_1; for taking a basis ξ_1, ξ_2, ξ_3 of $su(2)$ with $[\xi_1, \xi_2] = \xi_3$ + cyclic permutations we have $ad : su(2) \longrightarrow so(3)$ an isomorphism, as $su(2)$ has no center.

Let ϕ be an irreducible representation of $sl(2, \mathbb{C})$ on a complex vector space V. $\phi : sl(2, \mathbb{C}) \longrightarrow gl(V)$.
Take basis $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathcal{E}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathcal{E}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
with $[J\,\mathcal{E}_+] = 2\mathcal{E}_+$, $[J\,\mathcal{E}_-] = -2\mathcal{E}_-$ and $[\mathcal{E}_+\,\mathcal{E}_-] = J$.
Suppose that J has an eigenvector ie

$$V_\lambda = \{ v \in V ;\ \phi(J)v = \lambda v \},\quad \lambda \in \mathbb{C}$$

is non-zero for some $\lambda = \lambda_0$. Now

$$J\mathcal{E}_+ = [J\mathcal{E}_+] + \mathcal{E}_+ J, \quad J\mathcal{E}_- = [J\mathcal{E}_-] + \mathcal{E}_- J \quad \text{so}$$

$$\mathcal{E}_+ V_\lambda \subseteq V_{\lambda+2}, \quad \mathcal{E}_- V_\lambda \subseteq V_{\lambda-2}$$ $\mathcal{E}_\pm$ ladder operators.

By irreducibility,
$$V = \sum_{n=-\infty}^{\infty} V_{\lambda_0 + 2n},$$ algebraic direct sum.

In particular, if $\dim V < \infty$ then $\phi(J)$ is semi-simple and $\phi(\mathcal{E}_+)$, $\phi(\mathcal{E}_-)$ are nilpotent.

Now suppose that V is a 'highest weight' module ie there is a highest weight vector, that is a non-zero v in V_λ such that $\mathcal{E}_+ v = 0$; this is of course the case when

dim $V < \infty$. Then we take $\lambda_0 = \lambda$, $v_0 = v$ and define
$$v_n = \mathcal{E}_-^n v_0 \in V_{\lambda_0 - 2n}.$$
By induction $\mathcal{E}_+ \mathcal{E}_-^n = \mathcal{E}_-^n \mathcal{E}_+ + n\,\mathcal{E}_-^{n-1}(\mathcal{J} - n + 1)$ in $U(\mathfrak{g})$, $n \in \mathbb{N}$

(φ of $\mathfrak{g}$ extends to a representation of the universal enveloping algebra $U(\mathfrak{g})$, which is the tensor algebra of $\mathfrak{g}$ modulo the two-sided ideal generated by $\{\xi\otimes\eta - \eta\otimes\xi - [\xi\eta];\ \xi,\eta\in\mathfrak{g}\}$)

thus by irreducibility, V has basis $\{v_0, v_1, \ldots\}$ and $\mathfrak{g}$ acts by
$$\mathcal{E}_- v_n = v_{n+1},\quad \mathcal{J} v_n = (\lambda_0 - 2n)v_n \text{ and } \mathcal{E}_+ v_n = n(\lambda_0 - n+1)v_{n-1}.$$

Case 1 λ_0 is an integer ≥ 0.

Then $\mathcal{E}_+ v_n = 0$ for $n = \lambda_0 + 1 > 0$. If $v_n \neq 0$ then $\{v_n, v_{n+1}, \ldots\}$ spans an invariant subspace, contradicting irreducibility. Thus V has finite dimension $\lambda_0 + 1$ with basis $\{v_0, v_1, \ldots, v_{\lambda_0}\}$. It is customary to write $\lambda_0 = 2J$, J zero or a ½ positive integer. The eigenvalues of $\mathcal{J}$ are the integers
$2J, 2(J-1), \ldots, -2(J-1), -2J$ called weights; each weight occurs with multiplicity 1.

Case 2 λ_0 is not an integer ≥ 0.

Then $\mathcal{E}_+ v_n \neq 0$ for all $n \geq 1$ (by induction) so dim $V = \infty$. These can be constructed: V is the space of $SO(2)$-finite vectors in an irreducible unitary representation Π_{λ_0} of $SL(2,\mathbb{R})$ (first) when λ_0 is a ~~neg~~ negative integer; for $\lambda < -1$ it ~~has~~ is

Construction of case 1 representations: The irreducible representations of $SU(2)$, $SO(3)$.

$SU(2)$ matrices $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix}$ $|a|^2 + |b|^2 = 1$, $a, b \in \mathbb{C}$.

The map $g \longmapsto (\mathrm{Re}\,a, \mathrm{Im}\,a, \mathrm{Re}\,b, \mathrm{Im}\,b)$ is a homeomorphism onto the 3-sphere, so the group is compact and simply-connected. The adjoint representation $Ad: SU(2) \longrightarrow SO(3)$ covering is surjective (as $Ad(\exp \xi) = e^{ad\,\xi}$ and the exponential map for a compact, connected Lie group is surjective) and $\mathrm{Ker}\,Ad = \{\pm I\}$ the center of $SU(2)$.

Ex If G is connected show that the kernel of the adjoint rep^n of G is the center.

Let $U = \mathbb{C}[XY]$ the vector space of polynomials in two variables X and Y. For $n \in W$, let U_n be the subspace of homogeneous polynomials of degree n.

$GL(2,\mathbb{C})$ acts on U by $g.p$ where
$$(g.p)\begin{pmatrix}X\\Y\end{pmatrix} = p\left(g^T\begin{pmatrix}X\\Y\end{pmatrix}\right)$$
so with $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $g.X = aX + cY$, $g.Y = bX + dY$
$$g.\ X^a Y^b = (g.X)^a (g.Y)^b$$
Each U_n is an invariant subspace of dimension $n+1$. The differential gives an action of $\mathfrak{gl}(2\mathbb{C})$. The action of

$sl(2,\mathbb{C})$ on U is given by

$$E_+ = X\partial_Y\ , \quad E_- = Y\partial_X\ , \quad J = X\partial_X - Y\partial_Y$$

($(\exp tJ)^T = \exp tJ^T$ so $(J.P)\binom{X}{Y} = P\left(J^T\binom{X}{Y}\right)$, use product rule to get E_+, E_- and $E_+E_- = X\partial_Y\, Y\partial_X = X\partial_X + XY\partial_{YX}$, $E_-E_+ = Y\partial_Y + YX\partial_{XY}$ interchanging X & Y)

Write $n = 2J$; from $(*)$ one sees that

Π_J on U_{2J} is irreducible of dimension $2J+1$

and so from case 1,

$$SU(2)^\wedge = \{\Pi_J\ ;\ J = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots\}$$

Ex Find the weight vectors.

$$\begin{tikzcd} SU(2) \arrow[r, "\Pi"] \arrow[dr, "Ad"'] & GL(V) \\ & SO(3) \arrow[u, dashed] \end{tikzcd} \qquad V \text{ finite-dim}$$

Any action of $SO(3)$ on V gives one of $SU(2)$ by $g.v = Ad(g)v$. Also

Π factors through Ad iff $\Pi(\ker Ad) = \{I\}$ iff $\Pi(-I) = I$

Π_J iff $P\binom{X}{Y} = P\binom{-X}{-Y}$ for each $P \in U_{2J}$

iff $2J$ is even ie J is an integer.

$$SO(3)^\wedge = \{\Pi_J\ ;\ J = 0, 1, 2, \ldots\}$$

Ex Using complete reducibility and case 1 show that

$$\Pi_J \otimes \Pi_K = \Pi_{J+K} \oplus \Pi_{J+K-1} \oplus \Pi_{J+K-2} \oplus \cdots\cdots \oplus \Pi_{|J-K|}$$

Clebsch-Gordan series.

For $J = l \in \mathbb{Z}$ restrict Π_l to $g' = \begin{pmatrix} \cos\theta & \sqrt{-1}\sin\theta \\ \sqrt{-1}\sin\theta & \cos\theta \end{pmatrix}$

and let $a^l_{mn}(\theta)$ be the m,n matrix element (w.r.t. an orthonormal basis)

then $\quad P^l_{mn}(\cos\theta) = a^l_{mn}(\theta)$ with P^l_{00} the Legendre polynomial

P^l_{m0} " " function