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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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BASIC NOTIONS FOR LIE ALGEBRA III

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These are preliminary lecture notes, intended only for distribution to participants.

8 Theorem of Weyl

Definition. Let $\mathcal{L}$ be a Lie algebra and $\rho: \mathcal{L} \to gl(V)$ be a representation of $\mathcal{L}$.
The representation ρ is called irreducible if the only subspaces of V which are invariant under ρ are V and $\{0\}$.
The representation ρ is called completely reducible (or semisimple) if V is the direct sum of invariant subspaces V_i, and the restriction of ρ to V_i, $\rho|_{V_i}$, is irreducible for all i. Equivalently, the representation ρ in V is completely reducible if any invariant subspace W of V has a complement W' (i.e. $V = W + W'$, $W \cap W' = \{0\}$) which is invariant under ρ.

Theorem (Weyl). Let $\mathcal{L}$ be a semisimple Lie algebra over k. Then all finite dimensional representations of $\mathcal{L}$ are completely reducible.

Proof. Let ρ be a finite dimensional representation of $\mathcal{L}$ in V and let W be a subspace of V invariant under ρ. Assume $W \neq 0$ and $W \neq V$.

Choose $\overline{W}$ a complementary subspace of W in V and let π be the projection of V onto W parallel to $\overline{W}$.
If A is any projection of V onto W (i.e. $A: V \to V$ is linear, $A^2 = A$ and $A(V) = W$) then $\operatorname{Ker} A$ is a complementary subspace of W in V, this subspace is invariant under ρ if and only if $\rho(X)A - A\rho(X) = 0$ $\forall X \in \mathcal{L}$.

We shall modify π to have a projection A satisfying $[\rho(X), A] = 0$ for all $X \in \mathcal{L}$. Let $F = \{C \in \operatorname{End} V \mid C(V) \subset W \text{ and } C(W) = 0\}$, then F is a nonzero vector space (because $W \neq 0$ and $W \neq V$). A is a projection of V onto W if and only if $A = \pi - C$ for some $C \in F$.
So we want to find $C \in F$ so that $[\rho(X), C] = [\rho(X), \pi]$ $\forall X \in \mathcal{L}$.

Define $\sigma: \mathcal{L} \to gl(F)$ $X \to \sigma(X)$ where $\sigma(X) \cdot D = [\rho(X), D]$
(Clearly $\sigma(X)D \in F$ $\forall X \in \mathcal{L}$ $D \in F$ so $\sigma(X)$ indeed belongs to $gl(F)$).
The map $\sigma: \mathcal{L} \to gl(F)$ is a representation of $\mathcal{L}$ into F.

The element $\theta(X) = [\rho(X), \pi]$ belongs to F for any $X \in \mathcal{L}$, indeed $\pi(V) = W$ and $\pi w = w$ for any $w \in W$.

The map $\theta: \mathcal{L} \to F$ is linear. Considering the Chevalley cohomology of $\mathcal{L}$ with values in F associated to σ, we have

$$\delta\theta(X,Y) = \sigma(X)\,\theta(Y) - \sigma(Y)\,\theta(X) - \theta([X,Y])$$
$$= [\rho(X)[\rho(Y),\pi]] - [\rho(Y)[\rho(X),\pi]] - [\rho([X,Y]),\pi] = 0$$

so θ is a 1-cocycle. Since $H^1(\mathcal{L},\sigma) = 0$ by the first Whitehead lemma, there is an element $C \in F$ so that $\theta = \delta C$ i.e. so that

$$\theta(X) = [\rho(X),\pi] = \sigma(X)\cdot C = [\rho(X), C] \qquad \square$$

Remark. This theorem reduces the study of arbitrary representation of a semisimple Lie algebra to the study of its irreducible representations.

Proposition. Let $\mathcal{L} \subset \mathfrak{gl}(V)$ be a semisimple Lie algebra (V is finite dimensional). Then $\mathcal{L}$ contains the semisimple and nilpotent parts of its elements and the abstract Jordan decomposition coincide with the usual Jordan decomposition.

Remark. The last part follows from the first: $\operatorname{ad} X = \operatorname{ad} X_S + \operatorname{ad} X_N = \operatorname{ad} S + \operatorname{ad} N$ is a unique decomposition and ad is injective because $\mathcal{L}$ is semisimple.
For a proof see Humphreys "Introduction to Lie algebras and representation theory" p. 29

Corollary. If $\mathcal{L}$ is semisimple and $\phi: \mathcal{L} \to \mathfrak{gl}(V)$ is a finite dimensional representation of $\mathcal{L}$ and if $X = S + N$ is the abstract Jordan decomposition of $X \in \mathcal{L}$, then $\phi(X) = \phi(S) + \phi(N)$ is the usual Jordan decomposition of $\phi(X)$.

For a proof see Humphreys p. 30

9 The Levi decomposition

A. Semi direct products

Definition. Let $\mathcal{L}, \mathcal{P}$ be two Lie algebras over k and let σ be a representation of $\mathcal{L}$ on the vector space $\mathcal{P}$ such that $\sigma(X)$ is a derivation of $\mathcal{P}$ for all $X \in \mathcal{L}$. The semidirect product of $\mathcal{P}$ with $\mathcal{L}$ relative to σ, denoted by $\mathcal{P} \times_\sigma \mathcal{L}$ is the Lie algebra defined as follows:

(i) $\mathcal{P} \times_\sigma \mathcal{L}$, as a vector space is $\mathcal{P} \times \mathcal{L} = \{(v,x) \mid v \in \mathcal{P}, x \in \mathcal{L}\}$

(ii) The Lie bracket is given by:

$$[(v,x),(v',x')] = ([v,v'] + \sigma(x')v - \sigma(x)v', [x,x'])$$

Lemma. Let $\mathcal{L}$ be a Lie algebra over k, $\mathcal{J}$ an ideal of $\mathcal{L}$ and $\mathcal{P}$ a subalgebra of $\mathcal{L}$ such that $\mathcal{J} + \mathcal{P} = \mathcal{L}$, $\mathcal{J} \cap \mathcal{P} = \{0\}$. Then $\mathcal{L} \cong \mathcal{J} \times_\sigma \mathcal{P}$ where $\sigma: \mathcal{P} \to \operatorname{Der} \mathcal{J}$ is given by $\sigma(X)Y = -[X,Y]$

Proof. The isomorphism is given by $\sigma: (X,Y) \to X + Y$, $X \in \mathcal{J}$, $Y \in \mathcal{P}$

B. Levi subalgebras

Let $\mathcal{L}$ be a Lie algebra over k, $\mathcal{R}$ its radical (i.e. its maximal solvable ideal). Then $\mathcal{L}/\mathcal{R}$ is semisimple. A Levi subalgebra of $\mathcal{L}$ is a subalgebra $\mathcal{S}$ of $\mathcal{L}$ such that $\mathcal{L} = \mathcal{S} + \mathcal{R}$ as a vector space and $\mathcal{S} \cap \mathcal{R} = 0$. Remark that a Levi subalgebra of $\mathcal{L}$ is isomorphic to $\mathcal{L}/\mathcal{R}$ and thus is necessarily semisimple.

Lemma. Let $\mathcal{L}$ be a Lie algebra, $\mathcal{R}$ its radical. If $\mathcal{J}$ is an ideal such that $\mathcal{L}/\mathcal{J}$ is semisimple, then $\mathcal{R} \subseteq \mathcal{J}$. If π is a homomorphism of $\mathcal{L}$ onto a Lie algebra $\mathcal{L}'$ then $\pi(\mathcal{R})$ is the radical of $\mathcal{L}'$.

Proof. Let $\tau: \mathcal{L}' \to \mathcal{L}/\mathcal{I}$ be the projection map. Then $\tau(\mathcal{R})$ is solvable, hence zero and so $\mathcal{R} \subseteq \mathcal{I}$.

If $\mathcal{R}'$ is the radical of $\mathcal{L}'$, as π induces a homomorphism of $\mathcal{L}/\mathcal{R}$ onto $\mathcal{L}'/\pi(\mathcal{R})$ and as $\mathcal{L}/\mathcal{R}$ is semisimple, so is $\mathcal{L}'/\pi(\mathcal{R})$.
By the above $\pi(\mathcal{R}) \supset \mathcal{R}'$ and the converse is obvious. □.

Example: $\mathcal{L} = gl(n,k)$ $\quad \mathcal{I} = k\,Id$ $\quad \mathcal{L}/\mathcal{I} \simeq sl(n,k)$ which is semisimple.
So rad $(gl(n,k)) = k\,Id$

Theorem. Any Lie algebra can be written as the semi-direct product of its radical $\mathcal{R}$ with a semisimple Lie algebra $\mathcal{S}$.

Proof. We only have to show that $\mathcal{L}$ admits at least one Levi subalgebra.
We do it by induction on $\dim \mathcal{R}$. If $\dim \mathcal{R} = 0$, $\mathcal{L}$ itself is a Levi subalgebra.
Let $\dim \mathcal{R} \geq 1$. We study 2 cases, assuming the existence of Levi subalgebras for any Lie algebra whose radical has a dimension smaller than $\dim \mathcal{R}$.
Case 1: $[\mathcal{R},\mathcal{R}] \neq 0$. Consider $\mathcal{L}' = \mathcal{L}/[\mathcal{R},\mathcal{R}]$ and $\pi: \mathcal{L} \to \mathcal{L}'$ the canonical projection. Then $\pi(\mathcal{R}) = \mathcal{R}'$ is the radical of $\mathcal{L}'$. By the induction hypothesis, $\mathcal{L}'$ admits a Levi subalgebra $\mathcal{S}'$. Define $\mathcal{S}_0 = \pi^{-1}(\mathcal{S}')$. Then $\mathcal{L} = \mathcal{S}_0 + \mathcal{R}$ and $[\mathcal{R},\mathcal{R}] = \mathcal{S}_0 \cap \mathcal{R}$. As $[\mathcal{R},\mathcal{R}]$ is a solvable ideal in $\mathcal{S}_0$ and $\mathcal{S}_0/[\mathcal{R},\mathcal{R}] \simeq \mathcal{S}'$ which is semisimple, we have (cf lemma above) $[\mathcal{R},\mathcal{R}] = \text{rad}(\mathcal{S}_0)$.
As $\mathcal{R}$ is solvable, $\dim [\mathcal{R},\mathcal{R}] < \dim \mathcal{R}$ and again by induction $\mathcal{S}_0 = [\mathcal{R},\mathcal{R}] + \mathcal{S}$ where $\mathcal{S}$ is a Levi subalgebra of $\mathcal{S}_0$.
Clearly $\mathcal{L} = \mathcal{R} + \mathcal{S}$ and $\mathcal{R} \cap \mathcal{S} = 0$ so $\mathcal{S}$ is a Levi subalgebra of $\mathcal{L}$.
Case 2: $[\mathcal{R},\mathcal{R}] = 0$ so $\mathcal{R}$ is abelian. Let $\mathcal{L}_1 = \mathcal{L}/\mathcal{R}$ and $\pi: \mathcal{L} \to \mathcal{L}_1$ the natural onto homomorphism. Select a linear map $\mu: \mathcal{L}_1 \to \mathcal{L}$ so that $\pi \circ \mu$ is the identity.
For any $X_1 \in \mathcal{L}_1$ let $\rho(X_1) = \text{ad}\, X|_{\mathcal{R}}$ where $X \in \mathcal{L}$ is such that $\pi(X) = X_1$.
(the definition makes sense as $\mathcal{R}$ is abelian so $\text{ad}\, Y|_{\mathcal{R}} = 0$ if $Y \in \mathcal{R}$).

The map $\rho: X_1 \to \rho(X_1)$ is a representation of $\mathcal{L}_1$, which is semisimple, in $\mathcal{R}$ and
$$\rho(X_1) = \text{ad}\,\mu(X_1)|_{\mathcal{R}} \qquad \forall X_1 \in \mathcal{L}_1$$
For $X, Y \in \mathcal{L}_1$ let
$$\theta(X,Y) = [\mu(X),\mu(Y)] - \mu([X,Y])$$
Clearly $\theta: \mathcal{L}_1 \times \mathcal{L}_1 \to \mathcal{R}$ $\qquad (\text{if } \pi\,\mu([X,Y]) = [X,Y] - \pi[\mu(X),\mu(Y)])$
So $\theta: \mathcal{L}_1 \times \mathcal{L}_1 \to \mathcal{R}$ is a 2-cochain. It is a 2-cocycle because
$$\delta\theta(X,Y,Z) = \underset{XYZ}{S} \{\rho(X)\,\theta(Y,Z) - \theta([X,Y],Z)\}$$
$$= \underset{XYZ}{S} \{ [\mu(X),[\mu(Y),\mu(Z)]] - [\mu(X),\mu([Y,Z])]$$
$$- [\mu([X,Y]),\mu(Z)] + \mu([[X,Y],Z]) \} = 0$$

Thus, by Whitehead second lemma, there is a linear map $\nu: \mathcal{L}_1 \to \mathcal{R}$ such that $\theta = \delta\nu$, i.e.
$$\theta(X,Y) = \rho(X)\,\nu(Y) - \rho(Y)\,\nu(X) - \nu([X,Y])$$
which means
$$[\mu(X),\mu(Y)] - \mu([X,Y]) = [\mu(X),\nu(Y)] - [\mu(Y),\nu(X)] - \nu([X,Y])$$

Let us define
$$\lambda: \mathcal{L}_1 \to \mathcal{L} \qquad \lambda = \mu - \nu$$
As $\pi \circ \lambda$ is the identity, λ is a linear injection which is a homomorphism of $\mathcal{L}_1$ into $\mathcal{L}$ (indeed $[\nu(X),\nu(Y)] = 0$ because $[\mathcal{R},\mathcal{R}] = 0$)
If $\mathcal{S} = \lambda(\mathcal{L}_1)$, $\mathcal{S}$ is a subalgebra of $\mathcal{L}$ and $\mathcal{L} = \mathcal{S} + \mathcal{R}$ $\quad \mathcal{S} \cap \mathcal{R} = 0$,
hence $\mathcal{S}$ is a Levi subalgebra of $\mathcal{L}$ □.

Definition. Such a decomposition $\mathcal{L} = \mathcal{R} + \mathcal{S}$ with $\mathcal{R} \cap \mathcal{S} = 0$ is called a <u>Levi decomposition</u> of $\mathcal{L}$.

Example. A Levi decomposition of $gl(n,k)$ is given by
$$gl(n,k) = k\,I + sl(n,k)$$
where any matrix A is written as $A = \frac{\text{tr}\,A}{n} I + \left(A - \frac{\text{tr}\,A}{n} I\right)$.

Lemma. If $\mathcal{S}$ is a Levi subalgebra of $\mathcal{L}$, then it is also a Levi subalgebra of $[\mathcal{L},\mathcal{L}]$ and $[\mathcal{L},\mathcal{L}] = [\mathcal{R},\mathcal{L}] + \mathcal{S}$ is a Levi decomposition of $[\mathcal{L},\mathcal{L}]$.

Proof As $\mathcal{L} = \mathfrak{R} + \mathfrak{s}$ and $\mathfrak{s}$ is semisimple, $[\mathcal{L}, \mathcal{L}] = [\mathfrak{R}, \mathcal{L}] + [\mathfrak{s}, \mathfrak{s}] = [\mathfrak{R}, \mathcal{L}] + \mathfrak{s}$. Note that $[\mathfrak{R}, \mathcal{L}]$ is the radical of $[\mathcal{L}, \mathcal{L}]$ as $\mathfrak{s}$ is semisimple and $[\mathfrak{R}, \mathcal{L}] \subset \mathfrak{R}$ is a solvable ideal. □.

C. The Malcev theorem

If $\mathcal{L}$ is any finite dimensional Lie algebra and D is a nilpotent derivation of A, then $\exp D = 1 + \sum_{s \geq 1} \frac{1}{s!} D^s$ is a well defined automorphism of $\mathcal{L}$.

Indeed, if $D^n = Id$, $\exp D[X, Y] = \sum_{s \geq 0} \sum_{k=0}^{s} \left[\frac{1}{k!} D^k X, \frac{1}{(s-k)!} D^{s-k} Y \right]$
$= [\exp D X, \exp D Y]$

and everything is well defined as all sums are finite. The homomorphism $\exp D$ is clearly invertible by exhibiting its inverse $1 + \sum_{s \geq 1} \frac{(-1)^s}{s!} D^s$

<u>Definition</u> If $X \in \mathcal{L}$ is such that $\operatorname{ad} X$ is nilpotent, the automorphism $\exp(\operatorname{ad} X)$ is called an <u>inner automorphism</u>

<u>Lemma</u> If $\mathcal{L}$ is a Lie algebra over $\mathbb{k}$ and $\mathfrak{R}$ its radical, then $[\mathfrak{R}, \mathcal{L}]$ consists of ad-nilpotent elements (i.e. $\operatorname{ad} X$ is nilpotent $\forall X \in [\mathfrak{R}, \mathcal{L}]$)

<u>Proof</u> If D is a derivation of $\mathfrak{R}$, we construct $\mathfrak{R}' = \mathfrak{R} \times \mathbb{k}$ with the bracket $[(X, a), (Y, b)] = ([X, Y] + a DY - b DX, 0)$. $\mathfrak{R}'$ is solvable so $[\mathfrak{R}', \mathfrak{R}']$ is nilpotent. As $\mathfrak{R} \times \{0\}$ is an ideal in $\mathfrak{R}'$, $[\mathfrak{R}', \mathfrak{R}'] \cap (\mathfrak{R} \times \{0\})$ is nilpotent. So, $\forall X \in \mathfrak{R}$, $\forall D$ derivation of $\mathfrak{R}$, $[(X, 0), (0, 1)] = (-DX, 0)$ is nilpotent. This means that $\operatorname{ad} DX$ is nilpotent in $\mathfrak{R}$, thus also in $\mathcal{L}$. In particular $\operatorname{ad}[Y, X]$ is nilpotent for all $Y \in \mathcal{L}$, $X \in \mathfrak{R}$. □

<u>Remark</u> If $\mathcal{L} = \mathfrak{R} + \mathfrak{s}$ is a Levi decomposition and G is the group of automorphisms of $\mathcal{L}$ generated by $\exp \operatorname{ad} Z$ for $Z \in [\mathfrak{R}, \mathcal{L}]$, then $\mathfrak{R}$ is stabilized by G and so $g \cdot \mathfrak{s} = \mathfrak{s}'$ is a Levi subalgebra of $\mathcal{L}$.

<u>Theorem</u> (Malcev) Let $\mathcal{L}$ be a Lie algebra, $\mathfrak{R}$ its radical, G the group generated by $\exp \operatorname{ad} [\mathfrak{R}, \mathcal{L}]$ and $\mathfrak{s}_1, \mathfrak{s}_2$ be two Levi subalgebras of $\mathcal{L}$. Then there exists $g \in G$ such that $g \cdot \mathfrak{s}_1 = \mathfrak{s}_2$

For a proof see Varadarajan

10. A Lie group of a Lie algebra

<u>Definition 1</u> Let A, B be two Lie groups and s be a Lie homomorphism of B into the group of Lie automorphisms of A.

One defines $A \times_s B$, the <u>semidirect product of</u> A <u>with</u> B <u>relative to</u> s, to be the Lie group on the manifold $A \times B$ with multiplication given by

$$(a_1, b_1)(a_2, b_2) = (a_1 \, s(b_1)(a_2), b_1 b_2)$$

<u>Lemma 1</u> The Lie algebra $\mathfrak{g}$ of $G = A \times_s B$ is given by the semidirect product of the Lie algebras $\mathfrak{a}$ of A and $\mathfrak{b}$ of B relative to σ, where σ is the differential of the map $\tau: B \to Gl(\mathfrak{a})$ $b \to \tau(b) = s(b)_*$

<u>Proof</u> For any $a \in A$, $b \in B$ define $a' = (a, e_B)$ $b' = (e_A, b)$, $A' = A \times \{e_B\}$, $B' = \{e_A\} \times B$. Then $b' a' b'^{-1} = s_b(a')$. A' is a closed normal subgroup of G of Lie algebra $\mathfrak{a}' \simeq \mathfrak{a}$, B' is a closed subgroup with Lie algebra $\mathfrak{b}' \simeq \mathfrak{b}$. If $\tau(b) = s(b)_*$, $s(b)(\exp X) = \exp \tau_{(b)}(X)$ for $X \in \mathfrak{a}$ and $(\tau_b(X))' = Ad_G \, b' X'$. Thus $(\sigma(Y) \cdot X)' = [Y', X']$ $(Y \in \mathfrak{b}, X \in \mathfrak{a})$ and, as $\mathfrak{a}' + \mathfrak{b}' = \mathfrak{g}$ $\mathfrak{a}' \cap \mathfrak{b}' = 0$, the map $(X, Y) \to X' + Y'$ is a Lie algebra homomorphism (we have denoted by $X \to X'$ the isomorphism of $\mathfrak{a}$ onto $\mathfrak{a}'$ and similarly for $\mathfrak{b} \simeq \mathfrak{b}'$). □

<u>Lemma 2</u> Let $\mathfrak{g} = \mathfrak{a} \times_\sigma \mathfrak{b}$ be a given semidirect product of Lie algebras, and let A (resp B) be a connected simply connected Lie group with Lie algebra $\mathfrak{a}$ (resp $\mathfrak{b}$)

Then there exists a Lie homomorphism s of B into the group of the automorphisms of A so that the group $G = A \times_s B$ has $\mathfrak{g}$ as Lie algebra

Proof σ is a representation of $\mathfrak{b}$ in $\mathfrak{a}$ such that $\sigma(Y)$ is a derivation of $\mathfrak{a}$ $\forall Y \in \mathfrak{b}$. Since B is simply connected, there is a representation $\bar{\sigma} : B \to Gl(\mathfrak{a})$ such that $\bar{\sigma}_* = \sigma$; each $\bar{\sigma}(b)$ is an automorphism of $\mathfrak{a}$. As A is simply connected, there exists an automorphism of A, $s(b)$, such that $s(b)_* = \bar{\sigma}(b)$. One can check that the map s is a Lie homomorphism and define $G = A \times_s B$. Clearly $\mathfrak{g}$ is its Lie algebra. □

Theorem Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. Then there exists a connected simply connected Lie group G whose Lie algebra is isomorphic to $\mathfrak{g}$

Proof. We know that $\mathfrak{g} = \mathfrak{R} + \mathfrak{s}$, $\mathfrak{R} \cap \mathfrak{s} = \{0\}$ where $\mathfrak{R}$ is its radical (which is an ideal) and $\mathfrak{s}$ a Levi subalgebra; hence $\mathfrak{g}$ is the semi direct product of $\mathfrak{R}$ and $\mathfrak{s}$. From the above, it is enough to consider the two following cases $\mathfrak{g}$ semisimple and $\mathfrak{g}$ solvable.

If $\mathfrak{g}$ is semisimple, the adjoint representation is faithful (i.e. $ad X = 0 \Rightarrow X = 0$) so $\mathfrak{g}$ can be viewed as sitting in $gl(\mathfrak{g})$. Let G' be the corresponding connected Lie subgroup of $Gl(\mathfrak{g})$ and G its universal covering. G is a connected simply connected Lie group with Lie algebra $\mathfrak{g}$.

If $\mathfrak{g}$ is solvable, one proves the theorem by induction on $\dim \mathfrak{g}$. For $\dim \mathfrak{g} = 1$, it is trivial. If not, take a subspace $\mathfrak{a}$ of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a}$ and $\dim \mathfrak{g}/\mathfrak{a} = 1$. Let $\mathfrak{b}$ be a 1-dimensional subspace of $\mathfrak{g}$ complementary to $\mathfrak{a}$. Since $\mathfrak{a}$ is an ideal, $\mathfrak{g}$ is the semidirect product of $\mathfrak{a}$ and $\mathfrak{b}$ relative to σ, where $\sigma(X)Y = [Y, X]$ $\forall X \in \mathfrak{b}, Y \in \mathfrak{a}$, hence the result by lemma 2 and induction □