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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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# BASIC NOTIONS FOR LIE ALGEBRA IV

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These are preliminary lecture notes, intended only for distribution to participants.



# Root space decomposition - Weyl group

## (1°) Cartan Subalgebras

In this paragraph the field $k$ is $\mathbb{R}$ or $\mathbb{C}$

Recall that if $\mathcal{L}$ is a Lie algebra over $k$, a <u>Cartan subalgebra</u> $\mathfrak{h}$ (C.S.A) is a subalgebra which is nilpotent and equal to its own normalizer in $\mathcal{L}$

<u>Lemma 1</u>. Let $\mathcal{L}$ be a Lie algebra over $k$, $\mathfrak{h}$ a CSA of $\mathcal{L}$

Then 1) $\mathfrak{h}$ is maximal nilpotent

2) if $k = \mathbb{R}$, $\mathfrak{h}^{\mathbb{C}}$ is a CSA of $\mathcal{L}^{\mathbb{C}}$

3) $\xi(X) = \det (\operatorname{ad} X)_{\mathcal{L}/\mathfrak{h}}$ for $X \in \mathfrak{h}$ is a polynomial function on $\mathfrak{h}$ that does not vanish identically.

<u>Proof</u> 1) Let $\mathfrak{n} \supset \mathfrak{h}$. Then $\mathfrak{h} \to \operatorname{End}(\mathfrak{n}/\mathfrak{h}) \mid X \to \operatorname{ad} X|_{(\mathfrak{n}/\mathfrak{h})}$ is a representation of $\mathfrak{h}$ by nilpotent endomorphisms. So $\exists\, N \in \mathfrak{n} \setminus \mathfrak{h}$ such that $\operatorname{ad} X . N \in \mathfrak{h}$ $\forall X \in \mathfrak{h}$ (Engel's thm) so $N \in$ Normalizer of $\mathfrak{h}$ in $\mathcal{L}$ which is impossible

2) clearly if $\mathfrak{m}$ is the normalizer of $\mathfrak{a}$ in $\mathcal{L}$, $\mathfrak{m}^{\mathbb{C}}$ is the normalizer of $\mathfrak{a}^{\mathbb{C}}$ in $\mathcal{L}^{\mathbb{C}}$.

3) In view of 2 we may assume $k = \mathbb{C}$. If $\xi_{\mathfrak{h}} \equiv 0$, 0 must be a weight of the representation $\rho : \mathfrak{h} \to \operatorname{End}(\mathcal{L}/\mathfrak{h})$. The restriction of $\rho$ to the generalized weight space corresponding to 0 is thus given by nilpotent endomorphisms and, by Engel's theorem, $\exists\, N \in \mathcal{L} \setminus \mathfrak{h}$ so that $\operatorname{ad} X . N \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. which is a contradiction

**Definition** Let $\det(\mathrm{ad}\,X - tI) = \sum t^k p_k(X)$. Then $p_k$ is a polynomial function on $\mathfrak{g}$. The smallest integer $r$ so that $p_r$ is not identically zero is called the rank of the Lie algebra $\mathfrak{g}$, denoted $\mathrm{rk}(\mathfrak{g})$.
An element $X \in \mathfrak{g}$ is said to be regular if $p_r(X) \neq 0$ where $r = \mathrm{rk}(\mathfrak{g})$.

**Remark** For any $X \in \mathfrak{g}$, let $\nu(X)$ be the multiplicity of the root $0$ in the characteristic equation of $\mathrm{ad}\,X$. Then $\nu(X) = \mathrm{rk}(\mathfrak{g})$ if and only if $X$ is regular.
We denote by $\mathfrak{g}'$ the set of regular elements of $\mathfrak{g}$.

**Theorem:** Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$ ($\mathbb{R}$ or $\mathbb{C}$). For $X \in \mathfrak{g}$, define

$$\mathfrak{h}_X = \{ Y \in \mathfrak{g} \mid (\mathrm{ad}\,X)^s(Y) = 0 \text{ for some integer } s \geq 1 \}$$

Then (i) for any regular $X$, $\mathfrak{h}_X$ is a CSA of $\mathfrak{g}$ and $\dim \mathfrak{h}_X = \mathrm{rk}(\mathfrak{g})$

(ii) any Cartan subalgebra of $\mathfrak{g}$ is of the form $\mathfrak{h}_X$ for some regular $X \in \mathfrak{g}$

(iii) Let $G$ be the adjoint group of $\mathfrak{g}$, i.e. $G$ is the connected subgroup of $Gl(\mathfrak{g})$ whose Lie algebra is $\mathrm{ad}\,\mathfrak{g}$. There are finitely many CSA's $\mathfrak{h}_1 \dots \mathfrak{h}_k$ such that any CSA of $\mathfrak{g}$ is conjugate to one of the $\mathfrak{h}_i$ through an element of $G$. If $\Bbbk = \mathbb{C}$, all CSA are mutually conjugate under $G$.

**Proof** (i) $\mathfrak{h}_X$ is a subalgebra, because

$$(\mathrm{ad}\,X)^s [Z,Y] = \sum_{t=0}^{s} \frac{s!}{t!(s-t)!} \left[ (\mathrm{ad}\,X)^t Z, (\mathrm{ad}\,X)^{s-t} Y \right]$$

It is nilpotent. Indeed $\mathfrak{g} = \mathfrak{h}_X + \mathfrak{m}$ where $\mathfrak{m} = \bigcap_{s \geq 1} \mathrm{Range}((\mathrm{ad}\,X)^s)$ (cf lemma B2, 2) and $\mathfrak{h}_X$ and $\mathfrak{m}$ are stable under $\mathrm{ad}\,Z$ for $Z \in \mathfrak{h}_X$ ( $[Z, \mathrm{ad}\,X^s Y] = (-1)^s [\mathrm{ad}\,X^s Z, Y] + Y'$ where $Y' \in \mathrm{Range}\ \mathrm{ad}\,X$ )
Let $\mathfrak{h}' = \{ Y \in \mathfrak{h}_X$ so that $\det(\mathrm{ad}\,Y)_{\mathfrak{g}/\mathfrak{h}} \neq 0 \}$.



For any $Y \in \mathfrak{h}'$, $\mathrm{ad}\,Y|_{\mathfrak{m}}$ is invertible so $\mathfrak{h}_Y \subset \mathfrak{h}_X$; since $X$ is regular $\dim \mathfrak{h}_X = \mathrm{rk}(\mathfrak{g}) \leq \nu(Y) = \dim \mathfrak{h}_Y$ so $\mathfrak{h}_Y = \mathfrak{h}_X$ and $\mathrm{ad}\,Y|_{\mathfrak{h}_X}$ is nilpotent. As $\mathfrak{h}'$ is dense in $\mathfrak{h}_X$ by lemma 1, $\mathrm{ad}\,Y|_{\mathfrak{h}_X}$ is nilpotent for all $Y \in \mathfrak{h}_X$ and $\mathfrak{h}_X$ is nilpotent.

$\mathfrak{h}_X$ is clearly equal to its normalizer $\mathfrak{n}$ because $Y \in \mathfrak{n}$ implies $[Y,X] \in \mathfrak{h}$ so $\mathrm{ad}\,X^s\, Y = 0$ and $Y \in \mathfrak{h}$.

Note that $\dim \mathfrak{h}_X = \nu(X) = \mathrm{rk}(\mathfrak{g})$.

Remark: $p_r(Y) = \xi_{\mathfrak{h}_X}(Y) = \det(\mathrm{ad}\,Y)_{\mathfrak{g}/\mathfrak{h}_X}$ for any $Y \in \mathfrak{h}_X$

as can been seen in choosing a basis of $\mathfrak{h}$ where $\mathrm{ad}\,Y$ is strictly upper diagonal and a complementary basis in $\mathfrak{m}$.
So $Y \in \mathfrak{h}_X$ is regular if and only if $\xi_{\mathfrak{h}_X}(Y) \neq 0$.

(ii). If $\mathfrak{h}$ is a CSA and $X \in \mathfrak{h}$ is a regular element then $\mathfrak{h} = \mathfrak{h}_X$.
Indeed $\mathfrak{h}_X$ is nilpotent so $\mathfrak{h}_X \subseteq \mathfrak{h}$ because $\mathfrak{h}$ is maximal nilpotent but since $\mathfrak{h}$ is nilpotent $\mathfrak{h} \subset \mathfrak{h}_X$ as $\mathrm{ad}\,X|_{\mathfrak{h}}$ is nilpotent.

• Let $\mathfrak{h}' = \{ Y \in \mathfrak{h} \mid \xi_{\mathfrak{h}}(Y) \neq 0 \}$. By lemma 1, $\mathfrak{h}'$ is a dense open subset of $\mathfrak{h}$. We shall see that $\mathfrak{h}'$ contains a regular element $X$ of $\mathfrak{g}$ so that $\mathfrak{h} = \mathfrak{h}_X$.
Consider $\psi : G \times \mathfrak{h} \to \mathfrak{g}$ $(g, X) \mapsto g(X)$
Identifying the tangent spaces to $G$, $\mathfrak{h}$ and $\mathfrak{g}$ with $\mathrm{ad}\,\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{g}$ we get

$$\psi_{*(g,X)}(\mathrm{ad}\,Y, Z) = \psi_{*(g,X)}(\mathrm{ad}\,Y, 0) + \psi_{*(g,X)}(0, Z)$$

$$= \frac{d}{dt} \psi(g \exp t\,\mathrm{ad}\,Y, X)\Big|_{t=0} + \frac{d}{dt} \psi(g, X + tZ)\Big|_{t=0}$$

$$= g([Y,X] + Z)$$

So $\psi_{*(g,X)}$ is surjective if and only if $\mathrm{ad}\,X$ is an invertible endomorphism of $\mathfrak{g}/\mathfrak{h}$ i.e. $\xi_{\mathfrak{h}}(X) \neq 0$. So $\psi(G \times \mathfrak{h}')$ is an open subset of $\mathfrak{g}$ because $\psi$ is a submersion at each of its point. Since $\mathfrak{g}' = \{ X \in \mathfrak{g} \mid X \text{ is regular} \}$ is dense.

$\mathcal{L}' \cap \Psi(G \times \mathfrak{h}')$ is not empty. Since $\mathcal{L}'$ is $G$-invariant because $\det(\operatorname{ad} X - tI)$ is $G$-invariant, $\mathcal{L}' \cap \mathfrak{h}'$ is not empty and there is a regular element $X$ of $\mathcal{L}'$ which is in $\mathfrak{h}'$ thus in $\mathfrak{h}$; hence $\mathfrak{h} = \mathfrak{h}_X$

(iii) Let $\mathcal{L}_1 \ldots \mathcal{L}_k$ be the connected components of $\mathcal{L}'$

Remark: The fact that there is a finite number of them results from a theorem of Whitney (Ann. of Math 66 (1957) 545-556). If $k = \mathbb{C}$ the fact that there is only 1 component results from: if $V$ is a vector space of finite dimension over $\mathbb{C}$ and if $f$ is a polynomial on $V$ then $V' = \{v \in V \mid f(v) \neq 0\}$ is connected. (Indeed, let $v, v' \in V'$ and let $g(t) = f(tv + (1-t)v')$ where $t \in \mathbb{C}$, then $g$ is a polynomial on $\mathbb{C}$ and $g \neq 0$. Let $Z$ be the finite set of zeros of $g$ and $W = \{tv + (1-t)v' \;\; t \in \mathbb{C} \setminus Z\}$, $W$ is connected – being the continuous image of $\mathbb{C} \setminus Z$ which is connected – contains $v, v'$ and is included in $V'$ so $V'$ is connected)

Let $X_i \in \mathcal{L}_i$ $(1 \leq i \leq k)$ be arbitrary. Since $G$ is connected, each $\mathcal{L}_i$ is invariant under $G$. For $X \in \mathcal{L}_i$ let $\mathfrak{h}_X^+$ be the connected component of $\mathfrak{h}_X \cap \mathcal{L}'$ that contains $X$ and let $\mathcal{V}_X = \Psi(G \times \mathfrak{h}_X^+)$.
As $\mathfrak{h}_X^+$ is open in $\mathfrak{h}_X \cap \mathcal{L}'$ and as $\Psi$ is a submersion (on $G \times (\mathfrak{h}_X \cap \mathcal{L}')$ because $\mathfrak{h}_X \cap \mathcal{L}' = \{Y \in \mathfrak{h}_X \mid \varepsilon_{\mathfrak{h}_X}(Y) \neq 0\}$) by points (1) and (2), $\mathcal{V}_X$ is a connected open subset of $\mathcal{L}'$.
Thus $\mathcal{V}_X \subseteq \mathcal{L}_i$. If $Z \in \mathfrak{h}_X^+$, $\mathfrak{h}_Z^+ = \mathfrak{h}_X^+$ and $\mathcal{V}_Z = \mathcal{V}_X$. If $X, Y \in \mathcal{L}_i$ are such that $\mathcal{V}_X \cap \mathcal{V}_Y \neq \emptyset$, $Y$ is the image by $G$ of an element $Z \in \mathfrak{h}_X^+$ and $\mathcal{V}_Y = \mathcal{V}_Z = \mathcal{V}_X$. In other words two members of $\{\mathcal{V}_X, X \in \ \}$ are either disjoint, either identical but as they are all open and $\mathcal{L}_i$ is connected $\mathcal{V}_X = \mathcal{L}_i$ for all $X \in \mathcal{L}_i$. Thus $\mathcal{L}_i = \mathcal{V}_{X_i}$ for $1 \leq i \leq r$.
If $\mathfrak{h}$ is a CSA, $\mathfrak{h} = \mathfrak{h}_X$ where $X$ is regular, hence $X \in \mathcal{L}_i$ for some $i$ so $X \in \mathcal{V}_{X_i}$ and there is a $g \in G$ so that $X \in g(\mathfrak{h}_{X_i}^+)$ and $\mathfrak{h} = g(\mathfrak{h}_{X_i})$.
This shows the theorem, defining $\mathfrak{h}_i = \mathfrak{h}_{X_i}$ for $1 \leq i \leq r$. □.



**Theorem 2** Let $\mathfrak{g}$ be a semisimple Lie algebra over $k = \mathbb{R}$ or $\mathbb{C}$.
A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a Cartan subalgebra if and only if

(i) $\mathfrak{h}$ is maximal abelian

(ii) $\operatorname{ad} H$ is semisimple for any $H \in \mathfrak{h}$

In this case the restriction of $\beta$ to $\mathfrak{h} \times \mathfrak{h}$ is non degenerate

(Remark: an endomorphism $x \in \operatorname{End}(V)$ where $V$ is a finite dimensional vector space over $k$ is semisimple iff $\begin{cases} x \text{ is diagonalisable if } k = \mathbb{C} \\ x \text{ in } V^{\mathbb{C}} \text{ is diagonalisable over } \mathbb{C} \text{ if } k = \mathbb{R} \end{cases}$)

__Proof__. Clearly it is enough to show this when $k = \mathbb{C}$, because $\mathfrak{h}$ is maximal abelian in $\mathfrak{g}$ if and only if $\mathfrak{h}^{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{h}^{\mathbb{C}}$ is a CSA of $\mathfrak{g}^{\mathbb{C}}$ if $\mathfrak{h}$ is a CSA of $\mathfrak{g}$ and $\beta_{\mathfrak{g}^{\mathbb{C}}}(X, Y) = \beta_{\mathfrak{g}}(X, Y)$ when $X, Y \in \mathfrak{g}$ and $\beta_{\mathfrak{g}^{\mathbb{C}}}$ is the trace computed on a vector space over $\mathbb{C}$.

- We first prove that $\beta|_{\mathfrak{h} \times \mathfrak{h}}$ is non degenerate. Let $X$ be a regular element of $\mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{h}_X$. Let $X = S + N$ be the abstract Jordan decomposition of $X$ (cf B 6), then $[S, X] = 0$ (because $[S, N] = 0$) so $S \in \mathfrak{h}_X = \mathfrak{h}$.
Since $\operatorname{ad} S$ is the semisimple component of $\operatorname{ad} X$, they both have the same characteristic polynomial and $S$ is regular, so $\mathfrak{h} = \mathfrak{h}_S$ and as $S$ is semisimple $\mathfrak{h}$ is the centralizer of $S$ in $\mathfrak{g}$. If $\mathfrak{m} = \operatorname{ad} S\, \mathfrak{g}$, then $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where the sum is direct (cf B 2 Lemma 2 for a semisimple endomorphism). If $H \in \mathfrak{h}$ and $Y \in \mathfrak{g}$

$$\beta(H, [S, Y]) = \beta([H, S], Y) = 0$$

so that $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal relatively to $\beta$. As $\beta$ is nondegenerate, $\beta|_{\mathfrak{h} \times \mathfrak{h}}$ is non degenerate.

- We show now that any $H \in \mathfrak{h}$ is ad-semisimple

Let $N \in \mathfrak{h}$ be such that ad $N$ is nilpotent. There is a basis of $\mathfrak{g}$ (over $\mathbb{C}$) with respect to which $\{\mathrm{ad}\, H,\ H \in \mathfrak{h}\}$ is upper triangular (indeed it is a solvable subalgebra of $\mathfrak{gl}(\mathfrak{g})$), since ad $N$ is nilpotent its matrix has zeros on the diagonal. So $\mathrm{tr}(\mathrm{ad}\, N, \mathrm{ad}\, H) = 0 \ \forall H$ and $N = 0$ as $\beta$ is nondegenerate.
Let $\mathfrak{h}'$ be the set of regular elements of $\mathfrak{h}$. If $X = S + N \in \mathfrak{h}'$, $[X, N] = 0$ and $N \in \mathfrak{h}_X = \mathfrak{h}$ so $N = 0$ and $\mathfrak{h}'$ consists of ad semisimple elements.
Since $\mathfrak{h}'$ is open in $\mathfrak{h}$ we can find a basis for $\mathfrak{h}$ of elements in $\mathfrak{h}'$. As elements in $\mathfrak{h}'$ commute (because $\mathfrak{h} = \mathfrak{h}_X = \{Y,\ [X,Y] = 0,\ Y \in \mathfrak{g}\}\ \forall X \in \mathfrak{h}'$) and are ad-semisimple, the elements of $\mathfrak{h}$ commute and are ad-semisimple.
Since $\mathfrak{h}$ is maximal nilpotent, it must a fortiori be maximal abelian.

Conversely, let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$ such that ad $H$ is semisimple for any $H \in \mathfrak{h}$; so $\{\mathrm{ad}_{\mathfrak{g}} H \mid H \in \mathfrak{h}\}$ can be simultaneously diagonalize and there exists a subspace $\mathfrak{q}$ of $\mathfrak{g}$, invariant under ad $\mathfrak{h}$ so that $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, $\mathfrak{h} \cap \mathfrak{q} = 0$.
If $Y \in N_{\mathfrak{g}}(\mathfrak{h})$, $Y = H + Y'$ where $H \in \mathfrak{h}$, $Y' \in \mathfrak{q}$ and $[Y, H'] = [Y', H'] \in \mathfrak{q} \cap \mathfrak{h}$ for all $H' \in \mathfrak{h}$ so $Y$ is in the centralizer of $\mathfrak{h}$ and, as $\mathfrak{h}$ is maximal abelian, $Y \in \mathfrak{h}$. Thus $\mathfrak{h}$ is a Cartan subalgebra □



## (2°) Representations of $\mathfrak{sl}(2, \mathbb{C})$

We consider the standard basis of $\mathfrak{sl}(2, \mathbb{C})$ given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $[H, X] = 2X \quad [H, Y] = -2Y \quad [X, Y] = H$

<u>Theorem</u> Let $\rho$ be a finite dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ in a complex vector space $V$. Then $\rho(H)$ is semisimple, its eigenvalues are integers and have multiplicity one. Moreover there exists an integer $j \geq 0$ and a basis $\{v_0, \dots, v_j\}$ for $V$ such that

$$\rho(H) v_p = (j - 2p) v_p \qquad 0 \leq p \leq j$$
$$\rho(X) v_0 = 0 \qquad \rho(X) v_p = p(j - p + 1) v_{p-1} \qquad 1 \leq p \leq j$$
$$\rho(Y) v_j = 0 \qquad \rho(Y) v_p = v_{p+1} \qquad 0 \leq p \leq j - 1$$

Conversely for any integer $j \geq 0$, there is a representation of $\mathfrak{sl}(2, \mathbb{C})$ in a complex vector space of dimension $(j+1)$ given by the formulas above

<u>Proof</u>. As $H$ is ad-semisimple and $\mathfrak{sl}(2, \mathbb{C})$ is semisimple, $\rho(H)$ is semisimple. For any $\lambda \in \mathbb{C}$ let $V_\lambda = \{v \in V \mid \rho(H) v = \lambda v\}$.
Clearly

$$\begin{cases} \rho(X) V_\lambda \subset V_{\lambda+2} & \text{because } [\rho(H), \rho(X)] = 2\rho(X) \\ \rho(Y) V_\lambda \subset V_{\lambda-2} & \text{because } [\rho(H), \rho(Y)] = -2\rho(Y) \end{cases}$$

As $V$ is finite dimensional there is an eigenvalue $j$ of $\rho(H)$ such that $(j+2)$ is not an eigenvalue of $\rho(H)$. Let $v_0 \in V_j$, and define $v_s = (\rho(Y))^s v_0$. We have seen that $\rho(H) v_s = (j - 2s) v_s$; so, as $V$ is fin. dim. there must be an $s \geq 1$ such that $v_s = 0$. Let $m$ be chosen so that $v_p \neq 0$ for $0 \leq p \leq m$ and $v_{m+1} = 0$.

One can show, by induction, that for any integer $p \geq 0$

$$X Y^{p+1} = Y^{p+1} X + (p+1) Y^p (H - p)$$
$$Y X^{p+1} = X^{p+1} Y - (p+1) X^p (H + p)$$

thus for $p \geq 1$

$$\rho(X) v_p = \rho(X) (\rho(Y))^p v_0 = p(j - p + 1) \quad v_{p-1}$$

So, the subspace spanned by $v_0, \ldots, v_m$ is invariant under $\rho$ hence is $V$ as $V$ is irreducible.

Since $v_{m+1} = \rho(Y) v_m = 0$ we have $0 = \rho(X)(\rho(Y))^{m+1} v_0 = (m+1)(j-m) v_m$ which implies $m = j$. Therefore $j$ is an integer $\geq 0$, $\dim V = j+1$ and $\rho$ is given by the above relations. Conversely, a straightforward calculation shows that $hH + xX + yY \longrightarrow h\rho(H) + x\rho(X) + y\rho(Y)$ is a representation of $\mathfrak{sl}(2,\mathbb{C})$ in a vector space $V$ over $\mathbb{C}$ with a chosen basis $\{v_0, \ldots, v_j\}$ and with $\rho$ given by the formulas; this representation is irreducible as any nonzero invariant subspace $W$ is spanned by the $v_p$'s it contains (because of the invariance under $\rho(H)$) and if $s$ is the smallest integer so that $v_s \in W$ then $s = 0$ (if $s > 0$ $\rho(X) v_s$ is a nonzero multiple of $v_{s-1}$ and lies in $W$) so $v_0 \in W$ and $V = W$.

Corollary  Let $\rho$ be a representation of $\mathfrak{sl}(2,\mathbb{C})$ in a finite dimensional vector space $V$. Then $\rho(H)$ is semisimple, its eigenvalues are all integers and each nonzero one occurs along with its negative an equal number of times.

The endomorphisms $\rho(X)$ and $\rho(Y)$ are nilpotent. As $\mathfrak{sl}(2,\mathbb{C})$ is semisimple, $V$ can be decomposed (cf Weyl) as a direct sum of irreducible subspaces (i.e $V = \bigoplus_i W^i$ where the $W^i$'s are invariant under $\rho$ and $\rho|_{W^i}$ is irreducible) and the number of summands is equal to $\dim V_0 + \dim V_1$ (i.e the sum of the multiplicities of the eigenvalues 0 and 1 for $H$) where $V_j$ is the eigenspace of $\rho(H)$ for the eigenvalue $j$.



## 3° Roots and root space decomposition

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra.

For $\lambda \in \mathfrak{h}^*$ let $\quad \mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H,X] = \lambda(H) X \text{ for all } H \in \mathfrak{h}\}$

An element $\lambda \in \mathfrak{h}^*$ is called a <u>root</u> if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. We write $\Phi$ for the set of roots.

Since $\mathfrak{h}$ is the centralizer of any regular element in $\mathfrak{h}$, clearly $\mathfrak{g}_0 = \mathfrak{h}$.

Since ad $H$ is semisimple for $H \in \mathfrak{h}$, the weight subspaces are the root subspaces and one has

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

the sum being direct. This is called the <u>root space decomposition</u> of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

<u>Lemma 1</u>  Let $\alpha, \alpha' \in \mathfrak{h}^*$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] \subset \mathfrak{g}_{\alpha+\alpha'}$. If $X \in \mathfrak{g}_\alpha$ for $\alpha \neq 0$ then ad $X$ is nilpotent. If $\alpha, \alpha' \in \mathfrak{h}^*$ and $\alpha + \alpha' \neq 0$, $\beta(\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}) = 0$ where $\beta$ is the Killing form of $\mathfrak{g}$.

<u>Proof</u>  Since ad $H$ is a derivation of $\mathfrak{g}$ $\;[H,[X,Y]] = [[H,X],Y] + [X,[H,Y]]$.

So if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{\alpha'}$, $\;[H,[X,Y]] = (\alpha(H) + \alpha'(H))[X,Y]$.

On the other hand $\;\beta([H,X],Y) = \alpha(H)\beta(X,Y) = -\beta(X,[H,Y]) = -\alpha'(H)\beta(X,Y)$

for $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{\alpha'}$ and $H \in \mathfrak{h}$. ∎

As $\beta$ is nondegenerate on $\mathfrak{h} \times \mathfrak{h}$, to any $\alpha \in \mathfrak{h}^*$ corresponds a unique vector in $\mathfrak{h}$, denoted $H_\alpha$ so that $\quad \alpha(H) = \beta(H_\alpha, H)$ for all $H \in \mathfrak{h}$.

Lemma 2 (i) The set of roots, $\phi$, span $\mathfrak{h}^*$

(ii) If $\alpha \in \phi$, then $-\alpha \in \phi$ and $\beta$ is a nondegenerate pairing of $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$.
If $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ then $[X,Y] = \beta(X,Y) H_\alpha$

(iii) If $\alpha \in \phi$, there exists $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and $T_\alpha \in \mathfrak{h}$ so that $\{X_\alpha, Y_\alpha, T_\alpha\}$ span a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, the isomorphism sending $X_\alpha$ to $X$, $Y_\alpha$ to $Y$ and $T_\alpha$ to $T$ and $T_\alpha = \frac{2H_\alpha}{\beta(H_\alpha, H_\alpha)}$

Proof (i) If $\phi$ does not span $\mathfrak{h}^*$, there exists $H \in \mathfrak{h}$, $H \neq 0$ so that $\alpha(H) = 0\ \forall \alpha \in \phi$. Hence $[H, \mathfrak{g}_\alpha] = 0\ \forall \alpha \neq 0$ but as $H$ is abelian, $[H, \mathfrak{g}] = 0$ which is impossible as $\mathfrak{g}$ is semisimple.

(ii) is obvious as $\beta$ is nondegenerate and by lemma 1 $\beta(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\beta = -\alpha$. We have $\beta(H, [X,Y]) = \beta([H,X], Y) = \alpha(X)\beta(X,Y) = \beta(X,Y)\beta(H, H_\alpha)$ for all $H \in \mathfrak{h}$, $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$. So $[X,Y] = \beta(X,Y) H_\alpha$.

(iii) If $\alpha \in \phi$, we first remark that $\alpha(H_\alpha) = \beta(H_\alpha, H_\alpha) \neq 0$.
Indeed, if $\alpha(H_\alpha) = 0$, $[H_\alpha, \mathfrak{g}_\alpha] = 0 = [H_\alpha, \mathfrak{g}_{-\alpha}]$ so the Lie algebra $S$ spanned by $H_\alpha$, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ is a solvable subalgebra of $\mathfrak{g}$. Thus $ad_{\mathfrak{g}} s$ is nilpotent for any $s \in [S,S]$, in particular $s = H_\alpha$. So $ad_{\mathfrak{g}} H_\alpha$ is both semisimple and nilpotent hence zero and $H_\alpha$ is in the center of $\mathfrak{g}$ which is impossible.

Then, we take $X_\alpha \neq 0 \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ so that $\beta(X,Y) = \frac{2}{\beta(H_\alpha, H_\alpha)}$ and $T_\alpha = \frac{2H_\alpha}{\beta(H_\alpha, H_\alpha)}$. It is clearly isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.



Lemma 3 (i) If $\alpha \in \phi$, $\dim \mathfrak{g}_\alpha = 1$. In particular the algebra spanned by $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$

(ii) If $\alpha \in \phi$ the only scalar multiples of $\alpha$ which are roots are $\alpha$ and $-\alpha$

(iii) If $\alpha, \alpha' \in \phi$, then $\alpha'(T_\alpha) \in \mathbb{Z}$ and $\alpha' - \alpha'(T_\alpha)\alpha$ is a root.
The numbers $\alpha'(T_\alpha)$ are called Cartan integers.

(iv) If $\alpha, \gamma, \alpha+\gamma \in \phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\gamma] = \mathfrak{g}_{\alpha+\gamma}$

(v) If $\alpha, \gamma \in \phi$, $\gamma \neq \pm\alpha$, let $r, s$ be the largest integers for which $\gamma - r\alpha \in \phi$ and $\gamma + s\alpha \in \phi$. Then $\gamma + j\alpha \in \phi$ for all $-r \leq j \leq s$ and $\gamma(T_\alpha) = r - s$

Proof (i)(ii) For each pair of roots $\alpha, -\alpha$ let $S_\alpha$ be the Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ constructed in lemma 2.

The space $M$ spanned by $\mathfrak{h}$ and all $\mathfrak{g}_{c\alpha}$ $(c \in \mathbb{C})$ is stable under $ad\, S_\alpha$ hence we have a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $M$. Thus

• the weights of $T_\alpha$ on $M$ are $0$ and $c\alpha(T_\alpha)$ (when $\mathfrak{g}_{c\alpha} \neq 0$). Thus all $c$ occurring must be integral multiples of $1/2$, as the weights must integers and $\alpha(T_\alpha) = 2$.
Consider $\{H \in \mathfrak{h} \mid \alpha(H) = 0\} = \operatorname{Ker}\alpha$; it is a subspace of codimension 1 in $\mathfrak{h}$ complementary to $\mathbb{C}T_\alpha$. Taken together, $\operatorname{Ker}\alpha$ and $S_\alpha$ exhaust the occurrences of the weight $0$ for $T_\alpha$ in $M$ and $M = \operatorname{Ker}\alpha \oplus S_\alpha \oplus M'$ where $M'$ is invariant under $S_\alpha$. So the only even weight for $T_\alpha$ are $0$, $+2$ and $-2$. This proves that $2\alpha$ is not a root, hence twice a root is never a root, so half a root is never a root and $1$ is not a weight of $T_\alpha$ on $M$. So the only weights of $T_\alpha$ on $M$ are even, $M = \mathfrak{h} + S_\alpha$. In particular $\dim \mathfrak{g}_\alpha = 1\ \forall \alpha \in \phi$, and the only multiples of $\alpha \in \phi$ which are in $\phi$ are $+\alpha$ and $-\alpha$.

(iii - iv) We now look at the representation of $S_\alpha$ by the adjoint action on the space $K = \sum_{i \in \mathbb{Z}} \mathfrak{g}_{\gamma + i\alpha}$ where $\alpha$ and $\gamma \in \phi$, $\gamma \neq \pm\alpha$. We have seen that each $\mathfrak{g}_{\gamma+i\alpha}$ is one-dimensional and no $\gamma + i\alpha$ can equal $0$.

The weights of $T_\alpha$ on $K$ are given by $\{\gamma(T_\alpha) + 2i$ for $i \in \mathbb{Z}$ such that $\gamma + i\alpha \in \phi\}$.

Clearly, the weights are all even or all odd and $K$ is irreducible.

So $\gamma(T_\alpha) \in \mathbb{Z}$ and, with the notations of iv, $\gamma + j\alpha \in \phi$ $(-r \leq j \leq s)$. As $(\gamma - r\alpha)(T_\alpha) = -(\gamma + s\alpha)(T_\alpha)$, $\gamma(T_\alpha) = r - s$. In particular $\gamma - \gamma(T_\alpha)\alpha \in \phi$.

As ad $X_\alpha$ maps $L_\gamma$ onto $L_{\gamma+\alpha}$ if $\alpha, \gamma, \alpha+\gamma \in \phi$, one has $[\mathfrak{g}_\alpha, \mathfrak{g}_\gamma] = \mathfrak{g}_{\alpha+\gamma}$.

Lemma 4 Let $\alpha \in \phi$. Then $\beta(H_\alpha, H_\alpha)$ is a positive rational number.

Proof We know that $\beta(H_\alpha, H_\alpha) \neq 0$ and we have

$$\beta(H_\alpha, H_\alpha) = \mathrm{Tr}(\mathrm{ad} H_\alpha, \mathrm{ad} H_\alpha) = \sum_{\gamma \in \phi} (\gamma(H_\alpha))^2 = \frac{1}{4} \sum_{\gamma \in \phi} (\gamma(T_\alpha))^2 (\beta(H_\alpha, H_\alpha))^2 \quad \square$$

Remark We can transfer the Killing form on $\mathfrak{h}^*$ and define

$$\langle \alpha, \gamma \rangle = \beta(H_\alpha, H_\gamma) \quad \text{for all } \alpha, \gamma \in \mathfrak{h}^*$$

We have seen that $\langle \alpha, \gamma \rangle$ is rational for any $\alpha, \gamma \in \phi$ (cf lemma 3, iii and lemma 4).

Proposition Let $\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha \in \phi} \mathbb{R} H_\alpha$. Then $\mathfrak{h}_{\mathbb{R}}$ spans $\mathfrak{h}$, $\dim_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}} = \ell$ where $\ell = \mathrm{rk}(\mathfrak{g}) = \dim_{\mathbb{C}} \mathfrak{h}$ and the Killing form is positive definite on $\mathfrak{h}_{\mathbb{R}}$. Each root is real-valued on $\mathfrak{h}_{\mathbb{R}}$.

Proof $\mathfrak{h}_{\mathbb{R}}$ spans $\mathfrak{h}$ by lemma 2, i. Its dimension (as a real vector space) is $\ell = \dim_{\mathbb{C}} \mathfrak{h}$. Indeed $\dim \mathfrak{h}_{\mathbb{R}} \geq \dim_{\mathbb{C}} \mathfrak{h}$ as $\mathfrak{h}_{\mathbb{R}}$ spans $\mathfrak{h}$.

We have seen that $\alpha(H)$ is real for any $\alpha \in \phi$, $H \in \mathfrak{h}_{\mathbb{R}}$, as $\alpha(H_\gamma)$ is real for any $\alpha, \gamma \in \phi$.



If $\mathfrak{h}$ is spanned over $\mathbb{C}$ by $H_{\alpha_1} \dots H_{\alpha_\ell}$ where $\alpha_i \in \phi$ then if $H \in \mathfrak{h}_{\mathbb{R}}$:

$$H = \sum_{i=1}^{\ell} c_i H_{\alpha_i} \quad \text{and} \quad \alpha_j(H) = \sum c_i \, \beta(H_{\alpha_i}, H_{\alpha_j})$$

The matrix $(\beta(H_{\alpha_i}, H_{\alpha_j}))$ is invertible because $\beta$ is non degenerate on $\mathfrak{h} \times \mathfrak{h}$, and its entries are real; as $\alpha_j(H)$ is also real for any $j$ the $c_i$'s are real and $\mathfrak{h}_{\mathbb{R}}$ is spanned by $H_{\alpha_1} \dots H_{\alpha_\ell}$ over $\mathbb{R}$.

The Killing form is positive definite as

$$\beta(H, H) = \sum_{\gamma \in \phi} \gamma(H)^2 \qquad \square$$

## (4) Significance of the root pattern

Theorem Let $\mathfrak{g}, \mathfrak{g}'$ be two complex semisimple Lie algebras, $\mathfrak{h}$ (resp $\mathfrak{h}'$) CSA of $\mathfrak{g}$ (resp of $\mathfrak{g}'$), $\phi, \phi'$ the corresponding root systems and

$$\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha \in \phi} \mathbb{R} H_\alpha \qquad \mathfrak{h}'_{\mathbb{R}} = \sum_{\alpha' \in \phi'} \mathbb{R} H_{\alpha'}$$

Suppose $\varphi : \mathfrak{h}_{\mathbb{R}} \to \mathfrak{h}'_{\mathbb{R}}$ is linear, onto, one-to-one and its transpose ${}^t\varphi$ maps $\phi$ (viewed as sitting in $\mathfrak{h}^*_{\mathbb{R}}$ as each root is real valued on $\mathfrak{h}_{\mathbb{R}}$) onto $\phi'$.

Then $\varphi$ can be extended to an isomorphism $\tilde{\varphi}$ of $\mathfrak{g}$ onto $\mathfrak{g}'$.

For a proof see Helgason p 173.

Remark This theorem shows that a semisimple Lie algebra over $\mathbb{C}$ is determined - up to isomorphism - by means of a Cartan subalgebra and the corresponding pattern of roots.

## 5 Simple roots.

consider $\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha\in\phi} \mathbb{R} H_\alpha$ and define $\mathfrak{h}'_{\mathbb{R}} = \{H \mid H \in \mathfrak{h}_{\mathbb{R}} \text{ and } \alpha(H) \neq 0 \ \forall \alpha\in\phi\}$

Definition 1 A connected component of $\mathfrak{h}'_{\mathbb{R}}$ is called a Weyl chamber of $\mathfrak{h}$; such a $P$ is convex and open.

Remark that $\alpha \in \Delta$ is either strictly positive or strictly negative on $P$.

Let $\phi_P^+$ denote the set of all roots which are positive on $P$.

Definition 2 An element $\alpha \in \phi_P^+$ is called simple if it cannot be written as $\gamma+\delta$ where $\gamma, \delta \in \phi_P^+$.

We denote by $\Delta_P$ the set of simple roots corresponding to $P$.

Proposition 1 If $\Delta_P = \{\alpha_1, \dots \alpha_\ell\}$ then

(i) if $\alpha \in \phi_P^+$ $\quad \alpha = \sum_{i=1}^{\ell} n_i \alpha_i \quad n_i \geq 0 \ \forall i$ ($n_i$ integers)

(ii) $\langle \alpha_j, \alpha_i \rangle = \alpha_j(H_{\alpha_i}) \leqslant 0 \quad \forall i \neq j$

(iii) $\alpha_1, \dots, \alpha_\ell$ is a basis for $\mathfrak{h}^*_{\mathbb{R}}$. (Equivalently, $H_{\alpha_1} \dots H_{\alpha_\ell}$ is a basis for $\mathfrak{h}_{\mathbb{R}}$).

Proof i) If $\alpha \in \Delta_P$, it is clear, if not $\alpha = \gamma+\delta$ where $\gamma, \delta \in \phi_P^+$ and one proceed by induction

(ii) $\alpha_i - \alpha_j \notin \phi$, indeed, if $\alpha_i-\alpha_j \in \phi$, $\alpha_i = (\alpha_i - \alpha_j) + \alpha_j$ $\quad \alpha_j = (\alpha_j - \alpha_i) + \alpha_i$ so if $\alpha_i - \alpha_j \in \phi$, one of $\{\alpha_i, \alpha_j\}$ can be written as the sum of two elements in $\phi_P^+$ and would not be simple

As $\alpha_i - r\alpha_j + \dots + \alpha_i + s\alpha_j$ is the $\alpha_j$-string through $\alpha_i$ with $r-s = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$, the above implies that $r-s \leqslant 0$.



iii) Clearly $\{\alpha_1 \dots \alpha_\ell\}$ span $\mathfrak{h}^*_{\mathbb{R}}$, as $\phi$ does and as we have i). We have to show that they are linearly independent.

Suppose not. Then there is a $k'$ such that $H_{\alpha_{k'}} = \sum_{j<k'} a_j H_{\alpha_j}$

Set $H_1 = \sum_{a_j > 0} a_j H_{\alpha_j}$ $\qquad H_2 = \sum_{a_j \leqslant 0} a_j H_{\alpha_j}$

Clearly $H_1 \neq 0$ as $\alpha_{k'}$ is positive, so $\beta(H_1, H_1) > 0$.

As $\beta(H_1, H_2) = \sum_{a_j>0} \sum_{a_k \leqslant 0} a_k a_j B(H_{\alpha_j}, H_{\alpha_k}) \geqslant 0$ by ii)

we get $\beta(H_{\alpha_{k'}}, H_1) = \beta(H_1, H_2) + \beta(H_1, H_1) > 0$.

On the other hand $\beta(H_{\alpha_{k'}}, H_1) = \sum_{a_j>0} a_j \beta(H_{\alpha_j}, H_{\alpha_{k'}}) \leqslant 0$ by ii

hence a contradiction. $\square$

Proposition 2 If $\alpha \in \phi_P^+$ and $\alpha \notin \Delta_P$, then $\alpha - \beta \in \phi$ for some $\beta \in \Delta_P$.

Proof If $(\alpha, \beta) \leqslant 0$ for all $\beta \in \Delta_P$, then $\Delta_P \cup \{\alpha\}$ is a set of linearly independent elements of $\mathfrak{h}^*_{\mathbb{R}}$ (by the above reasoning) which is impossible. So $(\alpha, \beta) > 0$ for some $\beta \in \Delta_P$; and $\alpha - \beta \in \phi$ (if $r-s > 0$ if $\alpha - r\beta, \dots, \alpha + s\beta$ is the $\beta$ string through $\alpha$)

Remark that $\alpha - \beta \in \phi_P^+$ as $\alpha-\beta$ is a combination of simple roots with at least one positive coefficients (hence all coefficients are nonnegative because $\phi = \phi_P^+ \cup (-\phi_P^+)$); the existence of this positive coefficient results from the fact that $\beta$ is not proportional to $\alpha$.

Corollary Each $\gamma \in \phi_P^+$ can be written as $\alpha_1 + \dots + \alpha_k$, where the $\alpha_i$ are simple, not necessarily distinct) in such a way that each partial sum $\alpha_1 + \dots + \alpha_i$ $(i = 1, \dots, k)$ is a root.

## 6. Weyl group

In $\mathfrak{h}_{\mathbb{R}}^*$, any root $\alpha$ defines a reflection $\sigma_\alpha$, whose reflecting is $P_\alpha = \{\gamma \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \gamma, \alpha \rangle = 0\}$, given by:

$$\sigma_\alpha(\delta) = \delta - \frac{2(\delta,\alpha)}{(\alpha,\alpha)}\alpha$$

Remark that $\sigma_\alpha$ sends $\phi$ to $\phi$ for any $\alpha \in \phi$

Definition The Weyl group $W$ is the group generated by the $\sigma_\alpha$'s.
Equivalently one can see the Weyl group acting on $\mathfrak{h}_{\mathbb{R}}$ as the group generated by $\tilde{\sigma}_\alpha$ for $\alpha \in \phi$ where $\tilde{\sigma}_\alpha(H) = H - \frac{2\alpha(H)}{\alpha(H_\alpha)} H_\alpha$ (one uses here the isomorphism between $\mathfrak{h}_{\mathbb{R}}^*$ and $\mathfrak{h}_{\mathbb{R}}$, given by the Killing form $\beta$).

Lemma 1 Let $\alpha$ be simple. Then $\sigma_\alpha$ permutes the positive roots other than $\alpha$

Proof If $\delta \in \phi_P^+$, $\delta \neq \alpha$, $\delta = \sum_{\gamma \in \Delta} m_\gamma \gamma$, where $m_\gamma$ is positive for at least one $\gamma \neq \alpha$. □

Corollary 1 If $\delta = \frac{1}{2}\sum_{\beta \in \phi_P^+} \beta$, then $\sigma_\alpha(\delta) = \delta - \alpha$ for every $\alpha \in \Delta_P$.

Lemma 2 Let $\alpha_1 \cdots \alpha_t \in \Delta_P$ (not necessarily distinct). If $\sigma_{\alpha_1} \cdots \sigma_{\alpha_{t-1}}(\alpha_t)$ is negative, then for some index $1 \leq s < t$ $\sigma_{\alpha_1} \cdots \sigma_{\alpha_t} = \sigma_{\alpha_1} \cdots \sigma_{\alpha_{s-1}} \sigma_{\alpha_{s+1}} \cdots \sigma_{t-1}$

Proof Let $\begin{cases} \gamma_i = \sigma_{\alpha_{i+1}} \cdots \sigma_{\alpha_{t-1}}(\alpha_t) & \text{for } 0 \leq i \leq t-2 \\ \gamma_{t-1} = \alpha_t \end{cases}$

As $\gamma_0 \in -\phi_P^+$ and $\gamma_{t-1} \in \phi_P^+$, there exists a smallest $s$ so that $\gamma_s \in \phi_P^+$ and $\sigma_s(\gamma_s) = \gamma_{s-1} \in -\phi_P^+$. By lemma 1, $\gamma_s = \alpha_s$.
As $\sigma_{\tau(\alpha)} = \tau \sigma_\alpha \tau^{-1}$ we have $\sigma_{\gamma_s} = \sigma_{\alpha_{s+1}} \cdots \sigma_{\alpha_{t-1}} \sigma_t \sigma_{\alpha_{t-1}} \cdots \sigma_{\alpha_{s+1}}$. □

Corollary 2 If $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ is an expression of $\sigma \in W$ in terms of reflections corresponding to simple roots with $t$ as small as possible, $\sigma(\alpha_t) < 0$.

Remark When $\sigma \in W$ is written as $\sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ ($\alpha_i \in \Delta_P$, $t$ minimal) the expression is called reduced and one defines the length of $\sigma$, relative to $\Delta$, $\ell(\sigma)$, to be the minimal number $t$. If $m(\sigma)$ is the number of positive roots $\alpha$ for which $\sigma(\alpha) < 0$, $\ell(\sigma) = m(\sigma)$. By definition we put $\ell(1) = 0$.

Proof If $\ell(\sigma) = 0$ it is clear. If it is true for all $\sigma$ with $\ell(\sigma) < \ell(\sigma)$, write $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ in reduced form. Then, by corollary 2, $\sigma(\alpha_t) < 0$ and, by lemma 1, $m(\sigma\sigma_{\alpha_t}) = m(\sigma) - 1$ on the other hand $\ell(\sigma\sigma_{\alpha_t}) = \ell(\sigma) - 1$ hence the result by induction, $m(\sigma\sigma_{\alpha_t}) = \ell(\sigma\sigma_{\alpha_t})$. □
We shall see in the following theorem that it is always possible to write $\sigma$ as above.

Theorem Let $\phi, P, \Delta_P$ be as above. Let $W$ be the Weyl group

ⓐ $W$ acts transitively on Weyl chambers (seen as connected component of $\mathfrak{h}_{\mathbb{R}}'^*$)

ⓑ $W$ is generated by the $\sigma_\alpha$, $\alpha \in \Delta_P$

ⓒ If $s \in W$ and $sQ = Q$ for some Weyl chamber, then $s = \mathrm{Id}$.

Proof. ⓐ Take $\gamma \in \mathfrak{h}_{\mathbb{R}}'^*$ (i.e. $\langle \gamma, \alpha \rangle \neq 0 \ \forall \alpha \in \phi$, $\gamma \in \mathfrak{h}_{\mathbb{R}}^*$)
Choose $\sigma \in W'$ = group generated by $\sigma_\alpha$ for $\alpha \in \Delta_P$, so that $\langle \sigma(\gamma), \delta \rangle$ is as big as possible. Then, for all $\alpha \in \Delta_P$
$\langle \sigma(\gamma), \delta \rangle \geqslant \langle \sigma_\alpha \sigma(\gamma), \delta \rangle = \langle \sigma(\gamma), \sigma_\alpha(\delta) \rangle = \langle \sigma(\gamma), \delta - \alpha \rangle$
thus $\langle \sigma(\gamma), \alpha \rangle \geqslant 0 \ \forall \alpha \in \Delta_P$ so $\sigma(\gamma) \in P$

ⓑ Let us first show that if $\alpha$ is any root, there exists $\sigma \in W'$ so that $\sigma(\alpha) \in \Delta_P$. Because of a) it is enough to show that $\alpha$ is in a "basis" (i.e. a simple system corresponding to a Weyl chamber $P'$). For this take $\gamma'$ so that $\langle \gamma', \alpha \rangle = \varepsilon > 0$ and $|\langle \gamma', \beta \rangle| > \varepsilon \ \forall \beta \neq \alpha$, then $\alpha$ is in the basis of the Weyl chamber containing $\gamma'$

To show that $W = W'$, we have to prove that $\sigma_\alpha \in W'$ for any $\alpha \in \phi$. If $\tilde{\sigma} \in W'$ is chosen so that $\gamma = \tilde{\sigma}(\alpha) \in \Delta_P$ then $\sigma_\gamma = \sigma_{\tilde{\sigma}(\alpha)} = \tilde{\sigma} \sigma_\alpha \tilde{\sigma}^{-1}$ so $\sigma_\alpha = \tilde{\sigma}^{-1} \sigma_\gamma \tilde{\sigma} \in W'$.

(c) If $\sigma(\Delta) = \Delta$ and $\sigma \neq id$, $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ because $W = W'$; if this is a reduced expression, $\sigma(\alpha_t) < 0$ which is impossible, so $\sigma = Id$.

## 7. Cartan matrix.

Let $\phi$ be the system of roots of a semisimple complex Lie algebra, $P \subset \mathfrak{h}^*_{\mathbb{R}}$ a Weyl chamber and $\Delta_P$ the related system of simple roots. Let $\Delta_P = (\alpha_1, \ldots, \alpha_\ell)$. The matrix $\left(\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}\right)$ is called the Cartan matrix.

Remark: This matrix is independent of the choice of $P, \Delta_P$, thanks to the theorem above ($W$ acts transitively on the Weyl chambers); it is determined up to the chosen ordering.

Lemma. The Cartan matrix determines $\phi$ up to isomorphism.

Proof. This results from the fact that any $\beta \in \phi$ is conjugate under $W$ to a simple root, and that $W$ is generated by $\sigma_\alpha$ where $\alpha$ is a simple root, each $\sigma_\alpha$ being defined by the Cartan integers (see Humphreys for details, p 55).

The Dynkin diagram is a graph having $\ell$ vertices (representing the simple roots $\alpha_1 \cdots \alpha_\ell$), the $i$th vertex being joined to the $j$th vertex by $\frac{4\langle\alpha_i,\alpha_j\rangle^2}{\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle}$ edges. Whenever a double or triple edge occurs, one adds an arrow pointing to the shorter of the two roots.

Remark $-\frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}+1$ is the length of the $\alpha_j$ string through $\alpha_i$ when $\alpha_i$ $\alpha_j$ are simple (because $\alpha_i - \alpha_j \notin \phi$ so $r = 0$.)



Examples (1) $a_n = \mathfrak{sl}(n+1, \mathbb{C}) = \{A \in \mathfrak{gl}(n+1,\mathbb{C}),\ \mathrm{tr}\, A = 0\}$

Let $H_i = E_{ii} - E_{i+1\, i+1}$ $\quad 1 \le i \le n$ and $\mathfrak{h} = \Sigma\, \mathbb{C} H_i$

Then $\mathfrak{h}$ is a Cartan subalgebra of $a_n$

The root space decomposition is given by

$$a_n = \mathfrak{h} + \sum_{i \neq j} \mathbb{C} E_{ij}$$

where the root on $\mathbb{C}E_{ij}$ is given by $e_i - e_j$ where $e_i(H) = \mathrm{tr}(E_{ii} H)$ Ind

$$[H, E_{ij}] = (e_i(H) - e_j(H))\, E_{ij}$$

So $\phi = \{e_i - e_j \quad i \neq j\}$

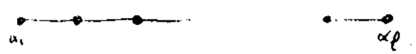
An example of "basis" is given by $\Delta = \{\alpha_i = e_i - e_{i+1},\ 1 \le i \le \ell\}$.

It corresponds to the Weyl chamber containing elements $H \in \mathfrak{h}_{\mathbb{R}}$ so that $e_i(H) > e_j(H)$ when $i < j$

The Cartan matrix is given by

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ \vdots & & & & & \vdots \\ & & & & & -1 \\ 0 & \cdots & & 0 & -1 & 2 \end{pmatrix}$$

And the Dynkin diagram is




(2) $C_n = \mathfrak{sp}(n, \mathbb{C}) = \{A \in \mathfrak{gl}(2n, \mathbb{C}) \mid {}^tA \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A = 0\}$

Let $H_i = E_{ii} - E_{n+i,n+i}$ $\quad 1 \le i \le n$, $\quad \mathfrak{h} = \sum_{i=1}^n \mathbb{C} H_i$ is a CSA

define $e_j(H_i) = \delta_{ij}$, $e_j \in \mathfrak{h}^*_{\mathbb{R}}$

The root space decomposition is given by

$$C_n = \mathfrak{h} + \sum_{i \le j} \mathbb{C}\,(E_{n+i,j} + E_{n+j,i}) + \sum_{i \le j} \mathbb{C}\,(E_{i,n+j} + E_{j,n+i}) + \sum_{i \neq j} \mathbb{C}\,(E_{ij} - E_{n+j,n+i})$$

Indeed $A \in sp(n,\mathbb{C})$ iff $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ with $A_2 = {}^tA_2$ ; $A_3 = {}^tA_3$ ; ${}^tA_1 + A_4 = 0$ and the roots are given by

$$[H, E_{m+i,j} + E_{n+j,i}] = -(e_i(H)+e_j(H))(E_{m+i,j}+E_{n+j,i})$$

$$[H, E_{i,n+j} + E_{j,n+i}] = (e_i(H)+e_j(H))(E_{i,n+j}+E_{j,n+i})$$

$$[H, E_{ij} - E_{n+j,n+i}] = (e_i(H)-e_j(H))(E_{ij}-E_{n+j,n+i}) \quad i \neq j$$

So $\phi = \{\pm 2e_i, \ \pm e_i \pm e_j \quad i,j = 1,\dots,n\}$

A basis is given by $\Delta = \{e_1-e_2, \dots, e_{n-1}-e_n, 2e_n\}$.

The Cartan matrix is then given by

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & & & 0 \\ -1 & 2 & -1 & & & & \vdots \\ 0 & -1 & 2 & & & & \vdots \\ \vdots & & & \ddots & & & \\ \vdots & & & & 2 & -1 & 0 \\ & & & & -1 & 2 & -1 \\ 0 & \cdots & & & 0 & -2 & 2 \end{pmatrix}$$

And the Dynkin diagram is

Indeed, the $\alpha_i$ root strings through $\alpha_j$ for $\alpha_i, \alpha_j \in \Delta$ are ...

$(e_k - e_{k+1})$ , $(e_k - e_{k+1} + e_{k+1} - e_{k+2}) = e_k - e_{k+2}$ for $1 \le k \le n-2$

or

$2e_m$ $\quad 2e_m + (e_{n-1}-e_m) = e_m + e_{n-1}$ $\quad 2e_m + 2(e_{n-1}-e_m) = 2e_{m-1}$.

$e_{n-1}-e_m$ $\quad e_{m-1}-e_n+2e_n = e_{n-1}+e_m$

and $-\frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}+1$ is the length of the $\alpha_j$ string through $\alpha_i$



One can build a complex semisimple Lie algebra, knowing its Cartan matrix

Proposition: Let $\mathfrak{g}$ be a semisimple complex Lie algebra, $\mathfrak{h}$ a CSA, $\phi$ the associated root system, $\Delta_\phi$ a set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Choose $x_i \in \mathfrak{g}_{\alpha_i}$, $y_i \in \mathfrak{g}_{-\alpha_i}$ so that $[x_i, y_i] = \frac{2H_{\alpha_i}}{\langle\alpha_i,\alpha_i\rangle} = t_i$ for any $\alpha_i \in \Delta$. Then $\mathfrak{g}$ is generated by $\{x_i, y_i, t_i\}$ with the relations

(1) $[t_i, t_j] = 0$

(2) $[x_i, y_i] = t_i \qquad [x_i, y_j] = 0$ if $i \neq j$

(3) $[t_i, x_j] = \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle} x_j \qquad [t_i, y_j] = -\frac{2\langle\alpha_j,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} y_j$

(4) $(\mathrm{ad}\, x_i)^{-\frac{2\langle\alpha_j,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}+1}(x_j) = 0$

(5) $(\mathrm{ad}\, y_i)^{-\frac{2\langle\alpha_j,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}+1}(y_j) = 0$.

Proof. The fact that $\mathfrak{g}$ is generated by $\{x_i, y_i, t_i\}$ results from the corollary to proposition 2 in §5. The relations (1) $\longrightarrow$ (5) are direct consequences of the properties of roots. □

Theorem (Serre) Fix a root system $\phi$ with basis $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Let $\mathfrak{g}$ be the Lie algebra generated by $3\ell$ elements $\{x_i, y_i, t_i ; 1 \le i \le \ell\}$ subject to the relations (1),(2) (3) (4) and (5) given above. Then $\mathfrak{g}$ is a finite dimensional Lie algebra with CSA spanned by the $t_i$ and corresponding root system $\phi$.

For a proof see Humphreys p. 99.

Corollary. Let $\mathfrak{g}, \mathfrak{g}'$ be semisimple Lie algebras with respective CSA's $\mathfrak{h}, \mathfrak{h}'$ and root systems $\phi, \phi'$. Let an isomorphism $\phi \to \phi'$ be given, sending a basis $\Delta$ to a basis $\Delta'$ and denote by $\pi : \mathfrak{h} \to \mathfrak{h}'$ the associated isomorphism. For each $\alpha \in \Delta$ ($\alpha' \in \Delta'$) select $x_\alpha \neq 0 \in \mathfrak{g}_\alpha$ ($x'_{\alpha'} \in \mathfrak{g}'_{\alpha'}$). Then, there exists a unique isomorphism $\tilde{\pi} : \mathfrak{g} \to \mathfrak{g}'$ extending $\pi$ and sending $x_\alpha$ to $x'_\alpha$.