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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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# Supplement to "HARMONIC ANALYSIS ON COMPACT LIE GROUPS"

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These are preliminary lecture notes, intended only for distribution to participants.

G linear.

$$\begin{aligned}\mathrm{Ad}\, g\, \xi = da_g(\xi) &= \frac{d}{dt}\, a_g(\exp t\xi)\Big|_{t=0} \\ &= \frac{d}{dt}\, g \exp t\xi\, g^{-1}\Big|_{t=0} \\ &= \frac{d}{dt}\, \exp t g\xi g^{-1}\Big|_{t=0} \\ &= g\xi g^{-1} \quad , g \in G\end{aligned}$$

M connected. If M is orientable, can partition the $n$-forms without zeros into 2 classes ie if $\beta, \gamma$ are non-zero $n$-forms $\gamma = f\beta$ for a function $f$. Say that $\beta \sim \gamma$ if $f > 0$, ~~so either $\beta\sim$~~ if not then $\beta \sim -\gamma$ ie there are two equivalence classes. Choosing one of these classes is called an orientation of M. The forms in this class are called oriented forms.

A chart $(U, \phi)$ at $x$ is called oriented if $dx^1\wedge\ldots\ldots\wedge dx^n$ is oriented ie if $\beta$ is an oriented $n$-form on M and $\beta|_U = g\, dx^1\wedge\ldots\ldots\wedge dx^n$, then $g > 0$ on $U$. Now if $(V, \psi)$ is another chart at $x$ one has under change of coordinates

$$dy^1\wedge\ldots\ldots\wedge dy^n = \det\left(\frac{\partial \underline{y}}{\partial \underline{x}}\right) dx^1\wedge\ldots\ldots\wedge dx^n$$

so if this chart is also oriented we have $\det\left(\frac{\partial \underline{y}}{\partial \underline{x}}\right) > 0$.

Lebesgue measure on $\mathbb{R}^n$ transforms

$$dy^1\ldots\ldots dy^n = \left|\det \frac{\partial \underline{y}}{\partial \underline{x}}\right| dx^1\ldots\ldots dx^n$$ on change of coordinates. These two formulae agree on oriented charts.

Suppose $\beta \in \Lambda^n(M)$ with supp $\beta \subseteq U$, $(U,\phi)$ oriented so $\beta\big|_U = f\, dx^1\wedge\ldots\ldots\wedge dx^n$ on $U$ with supp $f \subseteq U$ supp $f = \overline{\{x ; f(x) \neq 0\}}$. Can regard $f\circ\phi^{-1}$ as a compactly supported function on $\mathbb{R}^n$. Define

$$\int_M \beta = \int_{\mathbb{R}^n} f\circ\phi^{-1}\, dx^1\ldots\ldots dx^n$$ rhs independent of oriented chart.

For any $\beta$ cover M with oriented charts and use partition of unity argument.

Finite dimensional representation

$$\Pi : G \longrightarrow GL(V) \quad \text{with differential}$$
$$\Pi_* : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

$$\Pi_*(\xi) = \frac{d}{dt}\,\Pi(\exp t\xi)\Big|_{t=0}$$ is the infinitesimal generator of the 1-parameter subgroup; and $\Pi_* \in \mathfrak{g}^* \otimes \mathfrak{gl}(V)$.

Now w.r.t a basis of $V$ each matrix element of $\Pi$ may be differentiated by the exterior derivative $d$ to get a 1-form on $G$ ie $d\Pi$ is a $\mathfrak{gl}(V)$ valued 1-form on $G$ ie $d\Pi \in \Omega^1(G) \otimes \mathfrak{gl}(V)$. The left invariant 1-forms $\Omega^1_\ell(G)$ may be identified with $\mathfrak{g}^*$, by $\omega$ left invariant 1-form $\omega(\xi) = \omega(e)\,\xi(e)$, $e$ the identity element of $G$ and $\Omega^1(G)$ may be identified with $\mathfrak{X}(G) \longrightarrow C^\infty(G)$ by $\omega$ a 1-form, $\omega(\xi)(g) = \omega(g)\,\xi(g)$ $\xi$ a vector field on $G$.

If $f \in C^\infty(G)$, we have $df(\xi)(g) = \xi f(g) = \frac{d}{dt} f(g \exp t\xi)\Big|_{t=0}$, $\xi \in \mathfrak{g}$

Thus $d\Pi(\xi)(e) = \Pi_*(\xi)$, $\xi \in \mathfrak{g}$ and more generally

$$d\Pi(\xi)(g) = \Pi(g)\,\Pi_*(\xi), \quad g \in G$$



Matrix group $G$. $\quad G \xrightarrow{X} GL(n,\mathbb{C})$

Let $\Omega = X^{-1}dX$. Regard $\Omega$ as a $\mathfrak{gl}(n,\mathbb{C})$ valued 1-form ie $\Omega \in \Omega^1(G) \otimes \mathfrak{gl}(n,\mathbb{C})$.

<u>Lemma</u>

Under left translations, $\ell_g^* \Omega = \Omega$, $g \in G$ ie $\Omega$ is a matrix of left invariant 1-forms.

<u>Proof</u>

$$(\ell_g^*\Omega)(g_1)\,\xi(g_1) = \Omega(gg_1)\,\ell_{g*}\xi(g_1) = \Omega(gg_1)\,\xi(gg_1)$$
$$= X(g_1)^{-1}X(g)^{-1}\,dX(gg_1)\,\xi(gg_1)$$

Now $dX(gg_1)\,\xi(gg_1) = X(g)\,dX(g_1)\,\xi(g_1)$, $g_1 \in G$, $\xi \in \mathfrak{g}$ $\quad \square$

$\mathbb{R}$ addition   left invariant 1-form is $c\,dx$

$\mathbb{R}\setminus\{0\}$ multiplication   $l_a : x \longmapsto ax$   $\dfrac{dx}{x} \longmapsto \dfrac{a\,dx}{ax} = \dfrac{dx}{x}$

affine group of $\mathbb{R}$   $x \longmapsto ax + b$

$$l_{(g,h)}(a,b)(x) = g(ax+b)+h = gax + gb + h$$

1-forms   $f(a,b)\,da + g(a,b)\,db$

left invariant 1-forms have basis $\dfrac{da}{a}$, $\dfrac{db}{a}$

left invariant 2-form   $\dfrac{da \wedge db}{a^2}$

$SL(2,\mathbb{R})$   $X = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, $X^{-1} = \begin{pmatrix} z & -x \\ -y & w \end{pmatrix}$

$dX = \begin{pmatrix} dw & dx \\ dy & dz \end{pmatrix}$   $\Omega = \begin{pmatrix} z\,dw - x\,dy & z\,dx - x\,dz \\ -y\,dw + w\,dy & -y\,dx + w\,dz \end{pmatrix}$

$\det X = 1$   $wz - xy = 1$

$z\,dw + w\,dz - x\,dy - y\,dx = 0$ ie $\operatorname{tr}\Omega = 0$

basis for left invariant 1-forms is

$\omega_1 = z\,dw - x\,dy$, $\omega_2 = z\,dx - x\,dz$, $\omega_3 = -y\,dw + w\,dy$

left invariant 2-forms have basis $\omega_1 \wedge \omega_2$, $\omega_1 \wedge \omega_3$, $\omega_2 \wedge \omega_3$

left invariant 3-form is $\omega_1 \wedge \omega_2 \wedge \omega_3$

$$= z\,dw \wedge dx \wedge dy + x\,dw \wedge dy \wedge dz$$



$SU(2)$   $X = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}$   $|x|^2 + |y|^2 = 1$, $X^{-1} = \begin{pmatrix} \bar{x} & -y \\ \bar{y} & x \end{pmatrix}$

$dX = \begin{pmatrix} dx & dy \\ -d\bar{y} & d\bar{x} \end{pmatrix}$, $\Omega = \begin{pmatrix} \bar{x}\,dx + y\,d\bar{y} & \bar{x}\,dy - y\,d\bar{x} \\ \bar{y}\,dx - x\,d\bar{y} & \bar{y}\,dy + x\,d\bar{x} \end{pmatrix}$

$x\bar{x} + y\bar{y} = 1$ ie $x\,d\bar{x} + \bar{x}\,dx + \bar{y}\,dy + y\,d\bar{y} = 0$, $\operatorname{tr}\Omega = 0$;

also $\Omega + \bar{\Omega}^T = 0$.

$SL(n,\mathbb{R})$   $X = (x_{ij})$, $(\operatorname{adj} X)_{ij} = X_{ij}$; $\det X = 1 = \sum_j x_{ij} X_{jk} = \delta_{ik}$

$0 = d(\det X) = \sum_{i,j} dx_{ij} X_{ji} = \operatorname{tr}(X^{-1} dX)$

left invariant 1-forms spanned by those in $\Omega$ with $\operatorname{tr}\Omega = 0$.

$SO(n)$   $X^T X = I$, $(dX)^T X + X^T dX = 0$ ie $\Omega + \Omega^T = 0$

$U(n)$   $\Omega + \bar{\Omega}^T = 0$

$SP(n)$   $\Omega J + J\Omega^T = 0$

Maurer - Cartan equations $d\Omega = -\Omega \wedge \Omega$

Proof

$$0 = d(XX^{-1}) = dX\,X^{-1} + X\,dX^{-1} \quad \text{ie} \quad dX^{-1} = -X^{-1}dX\,X^{-1}$$

$$d(X^{-1}dX) = dX^{-1} \wedge dX = -X^{-1}dX \wedge X^{-1}dX \qquad \square$$

affine group of $\mathbb{R}$ $\quad X = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$

$$dX = \begin{pmatrix} da & db \\ 0 & 0 \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{pmatrix}$$

$$\Omega_\ell = X^{-1}dX = \begin{pmatrix} a^{-1}da & a^{-1}db \\ 0 & 0 \end{pmatrix}$$

$$\Omega_r = \cancel{dX\,dX}\; dX\,X^{-1} = \begin{pmatrix} a^{-1}da & -a^{-1}b\,da + db \\ 0 & 0 \end{pmatrix}$$

left invariant 1-forms have basis $\dfrac{da}{a}, \dfrac{db}{a}$

2-form $\dfrac{da \wedge db}{a^2}$

right invariant 1-forms have basis $\dfrac{da}{a}, \dfrac{b\,da}{a} - db$

2-form $\dfrac{da \wedge db}{a}$ not unimodular

$$\Omega_r X = -\Omega_\ell X^{-1}$$

Proof

$$\Omega_\ell X^{-1} = X\,dX^{-1} = -XX^{-1}dX\,X^{-1} = -\Omega_r X \qquad \square$$



Poincaré group $\quad X = \begin{pmatrix} A & \underline{x} \\ 0 & 1 \end{pmatrix}, \quad A \in SO(3)$

$$X^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\underline{x} \\ 0 & 1 \end{pmatrix} \qquad dX = \begin{pmatrix} dA & d\underline{x} \\ 0 & 0 \end{pmatrix}$$

$$\Omega_\ell = \begin{pmatrix} A^{-1}dA & A^{-1}d\underline{x} \\ 0 & 0 \end{pmatrix}, \quad \Omega_r = \begin{pmatrix} dA.A^{-1} & -dA.A^{-1}\underline{x} + d\underline{x} \\ 0 & 0 \end{pmatrix}$$