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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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BASIC NOTIONS FOR LIE GROUPS I

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Basic Notions for Lie groups

0. Review of differential geometry

A. Manifolds

Definition 1 A n dimensional C^∞ manifold M is a Hausdorff topological space, whose topology has a denumerable basis, and on which is defined

(i) an open covering $\{U_\alpha ; \alpha \in A\}$ of M

(ii) an homeomorphism $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ = open subset of $\mathbb{R}^n$

defined for each $\alpha \in A$

One also assumes that, for all $\alpha, \beta \in A$

(iii) $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a C^∞ map

(hence a C^∞ diffeomorphism between these open sets of $\mathbb{R}^n$)

Remarks 1 The collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha), \alpha \in A\}$ satisfying properties (i) (ii) and (iii) is called a C^∞ n dimensional atlas of M.

2 A pair $(U_\alpha, \varphi_\alpha)$, where U_α is an open set of M and $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ = open subset of $\mathbb{R}^n$ is an homeomorphism, is called a chart of M.
If $p \in U_\alpha$, $\varphi_\alpha(p)$ has coordinates $x^1(p) \dots x^n(p)$ in $\mathbb{R}^n$. The set of functions $\{x^i(p) ; i \leq n\}$ is called a local coordinate system of M, on U_α.

3 A chart (V, ψ) is said to be compatible with the atlas $\mathcal{A}$ if $\mathcal{A} \cup (V, \psi)$ is a C^∞ atlas. Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are equivalent if their union is a C^∞ atlas.

A C^∞ structure on M, of dimension n, is defined by an equivalence class of C^∞ atlases on M (or, equivalently, by the complete atlas $\bar{\mathcal{A}}$ associated to $\mathcal{A}$, which consists of all charts which are compatible with $\mathcal{A}$)

A C^∞ manifold of dimension n is a topological space which is Hausdorff, whose topology has a denumerable basis together with a C^∞ structure of dimension n on M. By abuse of notation, we denote by M the manifold $(M, \bar{\mathcal{A}})$; in definition 1 above, we considered an atlas $\mathcal{A}$ which belongs to the equivalence class, this is enough to define the C^∞ structure

4. Similarly one speaks of a n-dimensional complex manifold M if the atlas $\{(U_\alpha, \varphi_\alpha), \alpha \in A\}$ is such that $\varphi_\alpha(U_\alpha)$ is an open set of $\mathbb{C}^n (\sim \mathbb{R}^{2n})$ and if $\varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U_\alpha \cap U_\beta)$

5 Topological assumptions such as Hausdorff or separability are sometimes abandonned.

Examples ① $\mathbb{R}^n$ with the atlas composed of the unique chart $(\mathbb{R}^n, id.)$

② $S^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$.

Let $U_+ = \{x \in S^n \mid x \neq (0, \ldots, 0, 1)\}$, $U_- = \{x \in S^n \mid x \neq (0, \ldots, 0, -1)\}$

and let $\varphi_\pm(x) = \left(\frac{x_i}{1 \mp x_{n+1}}; 1 \leq i \leq n\right)$ be the stereographic projection

Then $\varphi_- \circ \varphi_+^{-1}$ $\mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ $\xi \longrightarrow \frac{\xi}{\|\xi\|^2}$

So $\mathcal{A} = \{(U_+, \varphi_+) (U_-, \varphi_-)\}$ is C^∞ n-dimensional atlas on S^n

③ U = open subset of the C^∞ manifold M of dimension m. If $\mathcal{A}$ is an atlas of M:

$$\mathcal{A}' = \{(U_\alpha \cap U, \varphi_\alpha|_{U_\alpha \cap U}) \mid (U_\alpha, \varphi_\alpha) \in \mathcal{A}\}$$

is a C^∞, m dimensional atlas of U.

In particular take $M = \mathbb{R}^{m^2} = \{m \times m$ matrices with real entries$\}$ and $U = GL(m, \mathbb{R}) = \{A \mid \det A \neq 0\}$.

④ $M \times N$, where M and N are C^∞ manifolds of dimension m and n. If $\mathcal{A}$ (resp $\mathcal{A}'$) is an atlas of M (resp N) take for atlas of $M \times N$:

$$\tilde{\mathcal{A}} = \{(U \times V, \varphi \times \psi) \mid (U, \varphi) \in \mathcal{A}, (V, \psi) \in \mathcal{A}'\}$$

It is C^∞ and of dimension $m + n$.

In particular take $S^1 \times \cdots \times S^1$ (m times); this is the m-dimensional torus T^m.

Definition 2 A continuous map $f: M \longrightarrow N$ between C^∞ manifolds is said to be $\boxed{C^\infty}$ if for each $p \in M$ there exist charts (U, φ) of M at p and (V, ψ) of N at $f(p)$ such that (i) $f(U) \subset V$ and (ii) $\psi \circ f \circ \varphi^{-1}$ $\varphi(U) \longrightarrow \psi(V)$ is a C^∞ map

This definition is clearly independent of the

If the C^∞ map f is bijective and f^{-1} is also C^∞, it is a *diffeomorphism*.

If $N = \mathbb{R}$, the C^∞ map f is called a C^∞ *function*. The set of C^∞ functions on M forms an associative algebra (over $\mathbb{R}$), it is denoted $C^\infty(M, \mathbb{R})$.

B Tangent vectors - Tangent bundle

Let M be a m-dimensional C^∞ manifold; let $p \in M$. A C^∞ *curve* in M, *through* p, is a C^∞ map $\sigma : (-\epsilon, \epsilon)(\subset \mathbb{R}) \longrightarrow M$ $(\epsilon > 0)$ such that $\sigma(0) = p$.

If σ is a C^∞ curve through p, it defines a linear map $L_\sigma : \mathscr{C}^\infty(M, \mathbb{R}) \longrightarrow \mathbb{R}$

$$L_\sigma(f) = \frac{d}{dt} f(\sigma(t))\Big|_{t=0}$$

Such a linear map is called a *tangent vector at* p.

Definition 1 The *tangent space* to M at p, denoted $T(M)_p$, is the set of all L_σ with σ a C^∞ curve in M through p.

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Let (U, ψ) be a chart of M at p and let $\{x^i ; i \leq m\}$ be the corresponding coordinate system. If $\{e_i ; i \leq m\}$ is the standard basis of $\mathbb{R}^m$, let:

$$\sigma_i(t) = \psi^{-1}(\psi(p) + t e_i) \qquad (t \text{ small})$$

$$L_{\sigma_i} f \underset{not}{=} \frac{\partial}{\partial x^i}\Big|_p \cdot f \qquad (f \in \mathscr{C}^\infty(M, \mathbb{R}))$$

If σ is a C^∞ curve through p,

$$L_\sigma f = \sum_{i=1}^{m} \frac{d(x^i \circ \sigma)}{dt}(0) \cdot \frac{\partial f}{\partial x^i}(p)$$

Hence $(TM)_p$ is a vector space spanned by the vectors $\frac{\partial}{\partial x^i}\big|_p$ $(i \leq n)$; one checks that these vectors are linearly independent and thus form a basis.

Remark. $\frac{\partial}{\partial x^i}\big|_p f = \frac{d}{dt}(f \circ \sigma_i(t))_0 = \frac{d}{dt}\big(f \circ \psi^{-1} \circ \psi \circ \sigma_i(t)\big)(0)$
$= \frac{d}{dt}(f \circ \psi^{-1})(\psi(p) + t e_i)(0) = \frac{\partial(f \circ \psi^{-1})}{\partial x^i}(\psi(p))$ (where this denotes the usual partial derivative)

Definition 2. The *tangent bundle* to M, denoted $T(M)$, is the C^∞, $2m$-dimensional manifold:

- $T(M) = \bigcup_{p \in M} (TM)_p$
- $\pi : T(M) \longrightarrow M : L_\sigma (\in (TM)_p) \longrightarrow p$ is called the *canonical projection*

. If $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ is a C^∞ atlas of M, with coordinate systems $(x^i_{(\alpha)} ; i \le m)$, the atlas of TM is,

$$\{(\pi^{-1}U_\alpha, \psi_\alpha) \mid \alpha \in A\} \text{ where } \psi_\alpha(v\,(\in (TM)_p)) = (\varphi_\alpha(p), v^i_{(\alpha)}\ (i \le m))$$

if $v = \sum_{i=1}^{m} v^i_{(\alpha)} \frac{\partial}{\partial x^i_{(\alpha)}}$.

C. Differential of a map

Definition 1 Let $f : M \to N$ be a smooth map, the differential of f is the map, denoted $f_* : TM \to TN$ such that

$$f_{*p} L_\sigma = L_{f\circ\sigma}$$

If one considers a chart (U, φ) of M at p, and a chart (V, ψ) of N at $f(p)$, such that $f(U) \subset V$ and if one denotes x^i $(i \le m)$ (resp y^a $(a \le n)$) the local coordinates in U (resp in V) one gets:

$$f_{*p}\left(\frac{\partial}{\partial x^i}\right) = \sum_{a=1}^{n} \frac{\partial}{\partial x^i}(y^a \circ f)\frac{\partial}{\partial y^a}$$

Thus f_{*p} is linear. Observe that if $g \in C^\infty(N, \mathbb{R})$, $X_p \in (TM)_p$,

$$(f_{*p} X_p)\, g = X_p(g \circ f)$$

Remarks 1. Let $M \xrightarrow{f} N \xrightarrow{g} P$ be smooth maps. Then:

$$(g\circ f)_* = g_* \circ f_*$$

2. If σ is a smooth curve through p,

$$L_\sigma = \sigma_{*0}\frac{\partial}{\partial t} \qquad (t = \text{usual coordinate in } \mathbb{R})$$

By abuse of notation we shall often write $L_\sigma = \frac{d\sigma}{dt}(0)$.

Proposition 1 (inverse function theorem) Let $f : M \to N$ be a smooth map; let $x_0 \in M$. Assume $f_{*x_0} : M_{x_0} \to N_{f(x_0)}$ is a linear isomorphism. Then there exist a neighborhood ω of x_0 in M and a neighborhood $\tilde{\omega}$ of $f(x_0)$ in N such that $f|\omega : \omega \to \tilde{\omega}$ is a diffeomorphism.

Proposition 2. (constant rank theorem) Let $f : M \to N$ be a smooth map. The rank of f at x_0 $(\in M)$, denoted $(rk\, f)(x_0)$ is the rank of f_{*x_0}. The function $M \to \mathbb{N}$ $x \to (rk\, f)(x)$ is lower semi continuous. If this function is constant, and has value p $(\le \min\{\dim M = m, \dim N = n\})$. There exist a chart (U, φ) around any point $x_0 \in M$, and a chart (V, ψ) around $f(x_0)$ such that:

(i) $f(U) = V$

(ii) if x^i $(i \leq m)$ (resp y^α $(\alpha \leq n)$) are the local coordinates in U (resp V)

$$\psi \circ f \circ \varphi^{-1}(x^1 \ldots x^p x^{p+1} \ldots x^m) = (x^1 \ldots x^p \underbrace{0 \ldots 0}_{(n-p)\text{ times}})$$

D <u>Submanifolds</u>

Let $i : N \longrightarrow M$ be a smooth injective map such that, for all $p \in N$, $i_{*p} : (TN)_p \longrightarrow (TM)_{i(p)}$ is injective. Then the pair (N, i) is called a <u>smooth submanifold</u> of M. In what follows we will identify N with $i(N)$ and consider a submanifold as a subset of M; i is then the canonical injection.

<u>Remark</u> N has in general a finer topology than the one induced by M; an example of this situation will be described later. When the topology of N coincides with the induced topology, the submanifold is said to be <u>embedded</u>.

E. <u>Vector fields</u>

<u>Definition 1.</u> A <u>smooth vector field</u> X on M is a smooth map $X : M \longrightarrow TM$ such that $\pi \circ X = id_M$ (π = canonical projection $TM \to M$). The set $\mathcal{X}(M)$ of smooth vector fields on M has a natural structure of:

- real vector space
- $\mathcal{C}^\infty(M, \mathbb{R})$ module
- real Lie algebra; the commutator $[X, Y]$ of 2 vector fields is defined by:

$$[X\,Y]\,f = X(Yf) - Y(Xf) \qquad (f \in \mathcal{C}^\infty(M,\mathbb{R}))$$

The <u>commutator</u> or <u>Lie bracket</u> is bilinear, antisymmetric and obeys Jacobi identity:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

In a local coordinate system $(x^i; i \leq m)$ defined on U one has:

$$[X,Y]^i = \sum_j X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}$$

if $X_{|U} = \sum_i X^i \frac{\partial}{\partial x^i}$ and $Y_{|U} = \sum_i Y^i \frac{\partial}{\partial x^i}$.

<u>Proposition 1</u> Let $f : M \to N$ be a smooth map; let X, Y be smooth vector fields on M; let $\tilde{X}, \tilde{Y}$ be smooth

vector fields on N such that, for any $x \in M$, $f_{*x} X = \tilde{X}_{f(x)}$ and $f_{*x} Y = \tilde{Y}_{f(x)}$. Then, for any $x \in M$,

$$f_{*x}[X, Y] = [\tilde{X}, \tilde{Y}]_{f(x)}$$

Proof Let $g \in \mathcal{E}^\infty(N, \mathbb{R})$, then

$$\begin{aligned}(f_{*x}[X,Y])g &= [X,Y]_x(g\circ f) = X_x(Y(g\circ f)) - Y_x(X(g\circ f)) \\ &= X_x(((f_*Y)g)\circ f) - Y_x(((f_*X)g)\circ f) \\ &= (f_{*x}X)(\tilde{Y}g) - (f_{*x}Y)(\tilde{X}g) \\ &= [\tilde{X}, \tilde{Y}]_{f(x)} g\end{aligned}$$

Proposition 2. Let X be a smooth vector field on M; let $x_0 \in M$ and assume $X_{x_0} \neq 0$. Then there exist a chart (U, φ) of M at x_0 such that in this chart $X_{|U} = \frac{\partial}{\partial x^1}$.

Definition 2 Let X be a smooth vector field on M; let $x_0 \in M$ and let $t_0 \in \mathbb{R}$. The Cauchy problem for (X, x_0, t_0) is the search of a curve $\gamma \in \mathcal{E}^\infty(I, M)$ $(I \subset \mathbb{R})$ such that:

(i) $t_0 \in I$ and $\gamma(t_0) = x_0$

(ii) $\gamma_{*t}(\frac{\partial}{\partial t}) = X_{\gamma(t)}$ for all $t \in I$.

Proposition 3 The Cauchy problem for (X, x_0, t_0) admits a solution. Furthermore if γ_i ($\in \mathcal{E}^\infty(I_i, M)$; $i=1,2$) are 2 solutions of the Cauchy problem, they coincide on $I_1 \cap I_2$.

Definition 3 Let $\{\gamma_\alpha \in \mathcal{E}^\infty(I_\alpha, M) \mid \alpha \in A\}$ be the set of all solutions of the Cauchy problem for $(X, x_0, 0)$. Let $I = \bigcup_{\alpha\in A} I_\alpha$ and let $\gamma : I \longrightarrow M$ be defined by $\gamma_{|I_\alpha} = \gamma_\alpha$. Then $\gamma \in \mathcal{E}^\infty(I, M)$ is the maximal solution of the Cauchy problem relative to $(X, x_0, 0)$, it is unique.

Definition 4 A vector field X is said to be complete if for all $x_0 \in M$, the maximal solution of the Cauchy problem relative to $(X, x_0, 0)$ is defined on $\mathbb{R}$.

Definition 5 A curve γ $(a, b) \longrightarrow M$ is said to go to infinity in b if for all compact $K \subset M$, there exists $\epsilon > 0$ such that $\gamma(t) \notin K$, for all $b > t > b - \epsilon$.

Proposition 4 If the maximal solution of the Cauchy problem γ relative to $(X, x_0, 0)$ is defined on (a, b), then

δ tends to infinity in a unit 6. In particular if M is a compact manifold, any smooth vector field on M is complete.

Definition 6 Let X be a complete vector field on M. If $t \in \mathbb{R}$, define the map

$$\varphi_t : M \longrightarrow M : x \longrightarrow \gamma_x(t)$$

where γ_x denotes the maximal solution of the Cauchy problem relative to $(X, x, 0)$.

Proposition 5 Let X be a complete vector field on M. Then the set of maps φ_t form a one parameter group of diffeomorphisms of M (i.e. $\varphi_t \circ \varphi_{t'} = \varphi_{t+t'}$)

Proposition 6 Let X be a complete vector field and let φ_t be the corresponding one parameter group of diffeomorphisms of M. Let Y be a vector field and let $x \in M$. Then:

$$[X, Y]_x = \frac{d}{dt}\left((\varphi_{-t*} Y)(x)\right)_{t=0}$$

F. Frobenius theorem.

Definition 7 Let M be a smooth manifold of dimension m; a p-dimensional distribution $\mathcal{D}$ on M is a correspondence which associates to each point $x \in M$, a p-dimensional subspace of $(TM)_x$. The p-dimensional distribution $\mathcal{D}$ is said to be smooth (C^∞) if for each $x \in M$, there exist a neighborhood ω of x and p smooth vector fields X_i $(i \le p)$ on ω such that for each $y \in \omega$, $\mathcal{D}_y$ = linear span of $\{X_{iy}, i \le p\} \underset{not}{=} > X_{1y} \dots X_{py} <$

A vector field X is said to belong to $\mathcal{D}$ if for all $x \in M$, $X_x \in \mathcal{D}_x$.

A smooth p-dimensional distribution $\mathcal{D}$ on M is said to be integrable if for all vector fields X, Y which belong to $\mathcal{D}$, $[X, Y]$ belongs to $\mathcal{D}$.

Definition 8 Let $\mathcal{D}$ be a smooth p-dimensional distribution on M. An integral manifold of $\mathcal{D}$ is a submanifold (N, i) of M such that, for any y

in N, $i_* N_y = \mathcal{D}_{i(y)}$. A <u>maximal integral manifold</u> of $\mathcal{D}$ is a connected integral manifold of $\mathcal{D}$ which is not a proper subset of any other connected integral manifold of $\mathcal{D}$.

<u>Proposition</u> 1. Let $\mathcal{D}$ be a p-dimensional smooth integrable (or involutive) distribution on M; let $x_0 \in M$. Then there exists a unique maximal connected integral manifold of $\mathcal{D}$ containing x_0. Furthermore any connected integral manifold of $\mathcal{D}$ containing x_0 is contained in the maximal one.

<u>Proposition</u> 2. Let $f : M \longrightarrow N$ be a smooth map, let $\mathcal{D}$ be a smooth integrable distribution on N; let (V, i) be an integral manifold of $\mathcal{D}$. Assume that $f(M) \subset i(V)$. Let $\tilde{f}$ be the unique map $M \longrightarrow V$ such that $i \circ \tilde{f} = f$. Then $\tilde{f}$ is a smooth map.

<u>References.</u>

- For remark ② p 3 see S Lang "<u>Introduction to differentiable manifolds</u>" 1967, p 16
- For proposition ① p 8 see R Narasimhan "<u>Analysis on real and complex manifolds</u>" 1968, p 14 theorem 1.3.2 and page 17 corollary 1.3.9 and remark 1.3.11.
- For proposition ② p 8 see previous reference p.18, theorem 1.3.14 or P. Malliavin "<u>Géométrie différentielle intrinsèque</u>" 1972 p.45, theorem 5.3.
- For proposition ② p.11 see P Malliavin in the previous reference p 293 (appendix 1)
- For proposition ③ p.13 see P Malliavin p 96, theorem 2.2.2
- For proposition ④ p.13 see P. Malliavin, p 98, proposition 2.3.2
- For proposition ⑤ p.14 see P Malliavin p.99, theorem 2.4.3 (and remark above) and p 100, theorem 2.4.7.
- For proposition ⑥ p 14 see P. Malliavin p.107, theorem 3.3.2
- For proposition ① p.16 see F. Warner "Foundations of differentiable manifolds and Lie groups" 1971, p 42 theorem 1.60 and p 48 theorem 1.64.
- For proposition ② p 16 see previous reference, p 47 theorem 1.62.