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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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BASIC STRUCTURE THEORY OF COMPACT LIE GROUPS AND SEMISIMPLE LIE ALGEBRAS OVER IR,C (E. CARTAN)

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G a compact connected abelian Lie group. A compact connected abelian Lie group is a torus ie isomorphic to a direct product of a finite number of circle groups. A maximal torus H of G is a torus st if $H \subseteq A \subseteq G$ and A is a torus, then $H = A$. Any subtorus is contained in a maximal one ($H_1 \subset H_2 \subset \ldots$ gives $\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \ldots$)

Lemma
(i) The maximal tori in G are the maximal connected abelian subgroups.
(ii) H is a maximal torus of G iff $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$.

Proof
(i) Let H be a maximal torus of G. If A is connected abelian containing H, $H \subseteq A \subseteq \bar{A}$ the closure of A; then as $\bar{A}$ is closed, connected and abelian we have $H = \bar{A}$. Also if A is maximal connected abelian then $A = \bar{A}$ so A is compact.
(ii) Follows from (i) using the correspondence $\begin{array}{cc} G & \mathfrak{g} \\ \text{connected subgroups}, & \text{subalgebras} \end{array}$ □

Proposition
(i) Any $g \in G$ lies in a maximal torus.
(ii) Let A be a torus in G and $g \in G$ with $gag^{-1} = a$ for all $a \in A$ then there is a maximal torus in G containing A and g.

Proof
(i) Every $g \in G$ lies on a 1-parameter subgroup and therefore lies in the torus which is the closure of this one parameter subgroup.

(ii) Let B be the closure of the subgroup of G generated by A and g (this is compact abelian) and B_0 the identity component of B. $A \subseteq B_0$ and B_0 is a torus so if $g \in B_0$ we are done. So suppose $g \notin B_0$. Now B/B_0 is a finite abelian group (since it is compact discrete) therefore $g^k \in B_0$ for some $k \in \mathbb{N}$. Also $g^k = \exp \xi$ for some $\xi \in \mathfrak{b}_0$. Let $h = \exp(1/k\, \xi) \in B_0$ and put $h_1 = gh^{-1}$; then $h_1^k = e$. Let $t \in B_0$ be a generator (ie the cyclic subgroup generated by t is dense in B_0) $t = \exp \eta$ for some $\eta \in \mathfrak{b}_0$. Set $h_2 = h_1 \exp(1/k\, \eta)$ and let B_1 be the closure of the cyclic subgroup generated by h_2. Then $B \subseteq B_1$, hence the result follows from (i). □

Cor
For any torus A the centralizer of A in G, $C_G(A)$ is the union of all maximal tori containing A. If H is a maximal torus then $C_G(H) = H$ ie H is a maximal abelian subgroup.

Proof
Clearly if H is a max torus containing A then $H \subseteq C_G(A)$. Also if $g \in C_G(A)$ then A and g lie in a max torus. □

N.B. A maximal abelian subgroup need not be a torus eg in $SO(3)$ the matrices of the form $\begin{pmatrix} \pm 1 & & 0 \\ & \pm 1 & \\ 0 & & \pm 1 \end{pmatrix}$ Klein 4-group form a max abelian subgroup which is not a torus.

$SU(n)$ A maximal torus is $\mathrm{diag}\left(e^{\sqrt{-1}a_1}, \dots\dots, e^{\sqrt{-1}a_n}\right)$,

$\sum_i a_i = 0$.

$SO(3)$ $\begin{pmatrix} \cos b & \sin b & 0 \\ -\sin b & \cos b & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $b \in \mathbb{R}$

$SO(4)$ $\begin{pmatrix} \cos b & \sin b & & \\ -\sin b & \cos b & & 0 \\ & 0 & \cos c & \sin c \\ & & -\sin c & \cos c \end{pmatrix}$

$b, c \in \mathbb{R}$.

The adjoint action and its orbits. cf R. Bott

Recall that for G linear $\mathrm{Ad}\, g\, \xi = g \xi g^{-1}$, $\xi \in \mathfrak{g}$

eg in $SO(3)$ taking the basis of infinitesimal rotations about the x, y, z axes ξ_1, ξ_2, ξ_3 with $[\xi_1, \xi_2] = \xi_3$ and cyclic permutations, the adjoint representation becomes $g\underline{x}$, $g \in SO(3)$, $\underline{x} \in \mathbb{R}^3$. The non-trivial orbits are the 2-spheres centred at the origin, and every sphere cuts a line through 0 in two points.

G a compact, connected Lie group. Take an inner product $(\,,\,)$ on the Lie algebra $\mathfrak{g}$ s.t the adjoint reps of G is orthogonal. For $\xi \in \mathfrak{g}$ let

$$\mathfrak{g}_\xi = \{\eta \in \mathfrak{g} \,;\, [\xi \eta] = 0\} = \mathrm{Ker}\, \mathrm{ad}\, \xi \quad \text{the centralizer of } \xi \text{ in } \mathfrak{g},$$

and $\mathfrak{g}^\xi = [\xi\, \mathfrak{g}] = \mathrm{Im}\, \mathrm{ad}\, \xi$. $\mathfrak{g}_\xi$ is the Lie algebra of $G_\xi = \{g \in G \,;\, \mathrm{Ad}\, g\, \xi = \xi\}$ the stabilizer of ξ.

$$\mathfrak{g} = \mathfrak{g}_\xi \oplus \mathfrak{g}^\xi \quad \text{an orthogonal direct sum.}$$

An element $\xi \in \mathfrak{g}$ is said to be regular if $\dim \mathfrak{g}_\xi \leq \dim \mathfrak{g}_\eta$ for all $\eta \in \mathfrak{g}$ ie if the centraliser of ξ has minimum dimension.

Lemma

(i) If $\xi \in \mathfrak{g}$ is a regular element then $\mathfrak{g}_\xi$ is a maximal abelian subalgebra.

(ii) If $\mathfrak{h}$ is maximal abelian then there is $\xi \in \mathfrak{h}$ with $\mathfrak{h} = \mathfrak{g}_\xi$.

Proof

(i) Let $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{g}_\xi$, then for small t, $\mathrm{ad}(\xi + t\eta)$ is a linear isomorphism on $\mathfrak{g}^\xi$; thus $\mathfrak{g}^\xi \subseteq \mathfrak{g}^{\xi+t\eta}$ and taking orthogonal complements $\mathfrak{g}_{\xi+t\eta} \subseteq \mathfrak{g}_\xi$. Now if $\zeta \in \mathfrak{g}_\xi$ with $[\zeta\, \eta] \neq 0$ then $\dim \mathfrak{g}_{\xi+t\eta} < \dim \mathfrak{g}_\xi$ ~~thus~~ $\cancel{\mathfrak{g}_\xi}$

thus $\mathfrak{g}_\xi$ is abelian for ξ regular.

Also if $\mathfrak{g}_\xi \subseteq \mathfrak{h}$ with $\mathfrak{h}$ abelian, then $[\xi\, \zeta] = 0$ for $\zeta \in \mathfrak{h}$.

(ii) Take $\mathfrak{g}_\xi$, $\xi \in \mathfrak{h}$ of minimal dimension, and $\eta \in \mathfrak{h}$ in the proof of (i). □

Conjugacy theorem.

Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$, then for each $\xi \in \mathfrak{g}$ the adjoint orbit $O(\xi) = \mathrm{Ad}\, G\, \xi$ intersects $\mathfrak{h}$.

Proof

$\mathfrak{h} = \mathfrak{g}_\zeta$ for some $\zeta \in \mathfrak{h}$. Define $f : O(\xi) \longrightarrow \mathbb{R}$ by $f(\eta) = (\eta, \zeta)$. Each orbit is compact so f attains a critical value at $\xi \in O(\xi)$ say. For each $\eta \in \mathfrak{g}$ take the curve $\gamma_\eta(t) = f(\mathrm{Ad}(\exp t\eta)\xi)$ through $f(\xi)$, then

$$\gamma_\eta'(0) = 0 \quad \text{ie} \quad 0 = ([\mathfrak{g}\; \xi], \zeta) = (\xi, \mathfrak{g}^\zeta) \quad \text{so} \quad \xi \in \mathfrak{g}_\zeta. \qquad □$$

We call $\mathfrak{g}_\xi$, ξ regular a Cartan subalgebra of $\mathfrak{g}$.

Corollary 1

Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be maximal abelian subalgebras then there exists $g \in G$ s.t $\mathrm{Ad}\, g\, \mathfrak{h}_1 = \mathfrak{h}_2$. The Cartan subalgebras in $\mathfrak{g}$ are the maximal abelian subalgebras (all have the same dimension)

Corollary 2

Any two maximal tori are conjugate ie given H_1, H_2 there exists $g \in G$ s.t $gH_1g^{-1} = H_2$.

Proof

Conjugacy is transitive so we may take $\mathfrak{h}_1 = \mathfrak{g}_{\xi_1}$ with ξ_1 regular. There exists $g \in G$ with $\mathrm{Ad}\, g\, \xi_1 \in \mathfrak{h}_2$, then as $\mathrm{Ad}\, g$ is an automorphism $\mathfrak{h}_2 = \mathrm{Ad}\, g\, \mathfrak{h}_1$ (if $\xi \in \mathfrak{g}_{\xi_1}$ then $[\mathrm{Ad}\, g\, \xi, \mathrm{Ad}\, g\, \xi_1] = \mathrm{Ad}\, g\, [\xi\, \xi_1] = 0$ ie $\mathrm{Ad}\, g\, \mathfrak{g}_{\xi_1} = \mathfrak{g}_{\mathrm{Ad}\, g\, \xi_1}$ and $\mathrm{Ad}\, g\, \xi_1$ is regular)

From $g \exp \xi\, g^{-1} = \exp \mathrm{Ad}\, g\, \xi$ one sees that $gH_1g^{-1} \subseteq H_2$. □

The normaliser of a maximal torus H in G, $N_G(H) = \{g \in G ; gHg^{-1} \subseteq H\}$ contains $C_G(H) = H$ as a normal subgroup. This is also the normaliser of $\mathfrak{h}$ in G ie all $g \in G$ st $\mathrm{Ad}\, g\, \mathfrak{h} \subseteq \mathfrak{h}$, then $\mathrm{Ad}\, g$ depends only on the coset gH and $\mathrm{Ad}\, g|_{\mathfrak{h}} = I$ iff $g \in H$. Let $\mathfrak{n}$ be the Lie algebra of $N_G(H)$, then $[\mathfrak{n}\, \mathfrak{h}] \subseteq \mathfrak{h}$ which implies that $\mathfrak{n} = \mathfrak{h}$. Hence, because it is compact discrete, $W(G, H) = N_G(H)/H$ is a finite group called the Weyl group, which is a group of linear transformations of $\mathfrak{h}$ (independent of H, by conjugacy, up to isomorphism)

$W(H) = N(H)/H$ is a finite group called the Weyl group, which can be identified with a group of linear transformations of $\mathfrak{h}$. $O(\xi) \cap \mathfrak{h}$ is an orbit of the Weyl group. (Note that $W(H)$ is independent of [illegible] conjugacy, up to isomorphism.)

Example $U(n)$ the group of unitary matrices with Lie algebra $\mathfrak{u}(n)$ the skew-hermitian matrices. Take $\mathfrak{h}$ the diagonal matrices $h = \sqrt{-1}\,\mathrm{diag}(a_1, \dots, a_n)$ in $\mathfrak{u}(n)$. Let E_{ij} be the (i,j) elementary matrix, one computes $[h, E_{ij}] = \sqrt{-1}(a_i - a_j)E_{ij}$. Put $F_{ij} = (E_{ij} - E_{ji}) + \sqrt{-1}(E_{ij} + E_{ji})$ instead. $\mathfrak{h}$ with $\{F_{ij}\}$ form a basis, then one [illegible] that:

A diagonal ~~matrix~~ [illegible] if and only if all the diagonal elements are distinct and [illegible] consists of the diagonal [illegible] matrices. The Weyl group is the permutations of the diagonal elements. [illegible] put into diagonal form.

$$G = \dot{\bigcup_{g \in C_H}} gHg^{-1}$$ a disjoint union over the conjugacy classes of a maximal torus H in G. Hence any class function on G is determined by its restriction to H, in particular any character is so determined.

<u>Example</u> $SU(2)$, let $h = \begin{pmatrix} e^{\sqrt{-1}a} & 0 \\ 0 & e^{-\sqrt{-1}a} \end{pmatrix}$ $a \in \mathbb{R}$, these form a maximal torus.

Consider the polynomial $P\binom{x}{y} = x^{n-j}y^j$ in U_n. Then

$$(\Pi_n(h)P)\binom{x}{y} = P\left(h\binom{x}{y}\right) = P\left(e^{\sqrt{-1}a}x, e^{-\sqrt{-1}a}y\right)^T = e^{\sqrt{-1}(n-2j)a}\,P\binom{x}{y}$$

Let χ_n be the character of $\Pi_{n/2}$, then on H it is given by

$$\chi_n(h) = \operatorname{trace}\Pi_n(h) = \sum_{j=0}^{n} e^{\sqrt{-1}(n-2j)a} = e^{\sqrt{-1}na}\,\frac{\left(1 - e^{-\sqrt{-1}\,2(n+1)a}\right)}{1 - e^{-\sqrt{-1}\,2a}}$$

$$= \frac{e^{\sqrt{-1}(n+1)a} - e^{-\sqrt{-1}(n+1)a}}{e^{\sqrt{-1}a} - e^{-\sqrt{-1}a}}$$

Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$. $\mathfrak{g}$ is semi-simple if the radical is zero ie no non-zero soluble ideals. $\mathfrak{g}$ is simple if no non-trivial ideals and not of dimension 0 or 1; simple implies semi-simple. $\mathfrak{g}$ is reductive if the adjoint reps is completely reducible. Using the

correspondence	G	$\mathfrak{g}$
	connected subgroups	subalgebras
	normal	ideals

set definitions for G eg G is semi-simple if no connected soluble normal subgroups.

Examples
$GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $U(n)$ are reductive but not semi-simple.
$SL(n,\mathbb{R})$, $SL(n,\mathbb{C})$, $SU(n)$, $SO(n)$ are semi-simple.
The Killing form of $\mathfrak{g}$ is defined by

$$B(\xi,\eta) = \operatorname{tr}(\operatorname{ad}\xi \circ \operatorname{ad}\eta) \qquad \xi,\eta \in \mathfrak{g}.$$

this is invariant under automorphisms of $\mathfrak{g}$.

Cartan's Criterion:
$\mathfrak{g}$ is semi-simple iff $B(\ ,\ )$ is non-degenerate.
Then
$\mathfrak{g}$ is s.s iff it is a direct sum of simple ideals.

$\mathfrak{g}$ is reductive iff $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ where $\mathfrak{z}$ is the center and $\mathfrak{g}' = [\mathfrak{g}\,\mathfrak{g}]$ the derived algebra with $\mathfrak{g}'$ semi-simple.

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a nilpotent subalgebra that is its own normalizer. ~~Any two~~ If $\mathfrak{g}$ is complex any two are conjugate by an inner automorphism; if $\mathfrak{g}$ is real there are finitely many conjugacy classes.

$\operatorname{Aut}(\mathfrak{g})$ the automorphisms is a closed subgroup of $GL(\mathfrak{g})$ with Lie algebra $\operatorname{Der}(\mathfrak{g})$ the derivations of $\mathfrak{g}$ ($D[\xi\eta] = [D\xi,\eta]+[\xi,D\eta]$)
$\operatorname{ad}\mathfrak{g}$ is a subalgebra of $\operatorname{Der}(\mathfrak{g})$, and to it there corresponds the ~~Every derivation~~ inner automorphisms $\operatorname{Int}\mathfrak{g}$. If G is connected, $\operatorname{Int}\mathfrak{g} = \operatorname{Ad}G$.

Every derivation of a semi-simple Lie algebra is inner, then $\operatorname{Int}\mathfrak{g}$ is the identity component of $\operatorname{Aut}\mathfrak{g}$.

For $\mathfrak{g}$ semi-simple, a Cartan subalgebra $\mathfrak{h}$ is a maximal abelian subalgebra consisting of semi-simple elements. Th[illegible] $\mathfrak{h}$.

Let $\mathfrak{g}$ be real reductive with Cartan subalgebra $\mathfrak{h}$. Then $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}_1$ with $\mathfrak{h}_1$ a Cartan subalgebra of $\mathfrak{g}'$. The adjoint reps of $\mathfrak{g}_{\mathbb{C}}$ restricted to $\mathfrak{h}$ has the dec^n

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha\in R} \oplus\, \mathfrak{g}^{\alpha}$$

where $\mathfrak{g}^{\alpha} = \{\xi \in \mathfrak{g};\ [\eta\,\xi] = \alpha(\eta)\xi,\ \forall \eta \in \mathfrak{h}\}$.

the root system $R = \{\alpha \in \mathfrak{h}^*;\ \alpha \neq 0 \text{ and } \mathfrak{g}^{\alpha} \neq 0\}$. Define $\mathfrak{g}^{\lambda} = 0$ if $\lambda \notin R$. Then

(i) If $\alpha,\beta \in R$ with $\alpha+\beta \neq 0$, then $[\mathfrak{g}^{\alpha}\mathfrak{g}^{\beta}] = \mathfrak{g}^{\alpha+\beta}$

(ii) $\mathfrak{g}^{\alpha} \perp \mathfrak{g}^{\beta}$ w.r.t $B(\ ,\ )$, $\mathfrak{h} \perp \mathfrak{g}^{\alpha}$ for all α.

(iii) $B(\xi,\eta) = \sum_{\alpha} n_{\alpha}\,\alpha(\xi)\,\alpha(\eta)$, $n_{\alpha} = \dim \mathfrak{g}^{\alpha}$

(iv) Each root vanishes on $\mathfrak{z}$. If all roots vanish on an element $\xi \in \mathfrak{h}_1$ then ξ is [illegible]. The roots span $\mathfrak{h}_1^*$.

v) The Killing form is non-degenerate on $\mathfrak{h}_1$; If $\alpha \in R$ then so is $-\alpha$.

If $\alpha\in R$ from (V) there is a unique τ_α in $\mathfrak{h}_{\mathbb{C}}$ with $B(\tau_\alpha,\xi)=\alpha(\xi)$ for all $\xi\in\mathfrak{h}$

Lemma
$[\mathfrak{g}^{\alpha}\,\mathfrak{g}^{-\alpha}]=\mathbb{C}\,\tau_\alpha$ in fact if $\xi\in\mathfrak{g}^{\alpha}$, $\eta\in\mathfrak{g}^{-\alpha}$ then $[\xi\,\eta]=B(\xi,\eta)\tau_\alpha$.

Proof
$[\xi\,\eta]\in\mathfrak{h}_{\mathbb{C}}$. If $\zeta\in\mathfrak{h}$, $B([\xi\,\eta],\zeta)=B(\xi,[\eta\,\zeta])=\alpha(\zeta)B(\xi,\eta)$ $\square$

Choosing $\varepsilon_\alpha\in\mathfrak{g}^{\alpha}$, $\varepsilon^{\alpha}\in\mathfrak{g}^{-\alpha}$ with $B(\varepsilon_\alpha,\varepsilon^{\alpha})=1$ we see that $\{\tau_\alpha,\varepsilon_\alpha,\varepsilon^{\alpha}\}$ spans a subalgebra $\mathfrak{g}^{(\alpha)}$ of $\mathfrak{g}_{\mathbb{C}}$ isomorphic to $A_1=sl(2,\mathbb{C})$. Recall that with $\tau=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\varepsilon_+=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\varepsilon_-=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $[\tau\,\varepsilon_+]=2\varepsilon_+$, $[\tau\,\varepsilon_-]=-2\varepsilon_-$ and $[\varepsilon_+\,\varepsilon_-]=\tau$. The eigenvalues of τ in any representation form a chain of integers running in steps of 2 from a maximum $+r$ to a minimum $-r$. The α string of roots through β is $\beta+t\alpha$ $t\in\mathbb{R}$, only finitely many occur as $\beta\in R$; denote by $\mathfrak{g}^{\alpha}_{\beta}$ the direct sum of the $\mathfrak{g}^{\beta+t\alpha}$

Prop^n
(i) The values $\beta(\tau_\alpha)$ for $\alpha,\beta\in R$ are integers (called Cartan integers and are denoted by $a_{\beta\alpha}$) there are non-negative integers t', t'' s.t the α-string of roots through β is unbroken with $-t'\le t\le t''$, t integer.

$$a_{\beta\alpha}=\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}=t'-t''$$

(ii) For $\alpha,\beta\in R$, $w_\alpha(\beta)=\beta-a_{\beta\alpha}\alpha$ again lies in R.

(iii) If $\beta-\alpha\notin R$ then $a_{\beta\alpha}\le 0$

(iv) For each $\alpha\in R$, $\dim\mathfrak{g}^{\alpha}=1$. A multiple $r\alpha$ with $r\in\mathbb{C}$ belongs to R iff $r=\pm1$.

Proof
(i) $A_1=\mathfrak{g}^{(\alpha)}$ acts on $\mathfrak{g}^{\alpha}_{\beta}$ via the adjoint rep^n. The eigenvalues of $\operatorname{ad}\tau_\alpha$ are $\beta(\tau_\alpha)+2t$.

(ii) From (+) (iii) $t'=0$ in (+)

(iv) Consider the vector space spanned by $\varepsilon_{-\alpha}$, τ_α and all $\mathfrak{g}_{t\alpha}$ for $t=1,2,\ldots$; this is invariant under $\operatorname{ad}\varepsilon_\alpha$, $\operatorname{ad}\varepsilon_{-\alpha}$. Then $\operatorname{ad}\tau_\alpha=\operatorname{ad}[\varepsilon_\alpha\,\varepsilon^{\alpha}]=[\operatorname{ad}\varepsilon_\alpha\,\operatorname{ad}\varepsilon^{\alpha}]$ has zero trace, but the trace is $\alpha(\tau_\alpha)(-1+n_\alpha+2n_{2\alpha}+\ldots)$. Hence $n_\alpha=1$, $n_{2\alpha}=n_{3\alpha}=\ldots=0$.

If $\beta=r\alpha$ is a root, integrality of $\beta(\tau_\alpha)$ etc. implies that r is ±1 or ±2; but ±2 has just been excluded. $\square$

Let $\mathfrak{h}_{\mathbb{R}}$ be the real span of the τ_α, $\alpha\in R$; this is a real form of $\mathfrak{h}_{\mathbb{C}}$ on which the Killing form is +ve definite and we have an isometry

$$(\mathfrak{h}^{*}_{\mathbb{R}},\langle\,,\rangle)\longrightarrow(\mathfrak{h}_{\mathbb{R}},B(\,,))$$
$$\lambda\longmapsto\tau_\lambda$$

Vector Figure
Let V be a real finite-dim vector space with an inner product $\langle\,,\rangle$. A vector figure is a finite non-empty subset R in V not containing 0, with $V=\operatorname{Span}\{R\}$ and satisfying

(i) For $\alpha,\beta\in R$ $\quad a_{\beta\alpha}=\dfrac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is an integer.

(ii) $\quad w_\alpha(\beta)=\beta-a_{\beta\alpha}\alpha$ is also in R

(iii) If $\alpha\in R$ and a real multiple $r\alpha$ is also in R then $r=\pm1$.

the rank of R is the dimension of V. The reflection of V in the hyperplane orthogonal to α is given by

$$w_\alpha(\lambda) = \lambda - \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$$

the (finite) group W of isometries of V generated by the w_α, $\alpha \in R$ is called the Weyl group of R. R is simple if it is not the union of two non-empty subsets that are orthogonal to each other. Any R is expressed uniquely as the union of simple ones.

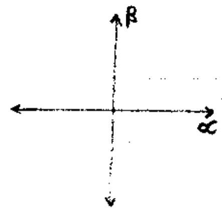
Vector figures of rank 2.

Let R be a vector figure and take $\alpha, \beta \in R$. Now $a_{\alpha\beta} a_{\beta\alpha} = \frac{4\langle\alpha,\beta\rangle^2}{|\alpha|^2|\beta|^2} = 4\cos^2\theta$ where θ is the angle between α & β. The $a_{\beta\alpha}$'s being integers, the possible values of $4\cos^2\theta$ are $0,1,2,3,4$. We suppose that $a_{\beta\alpha} \leq 0$ (or replace β by $w_\alpha(\beta)$) and $|\alpha| \leq |\beta|$; note that $a_{\beta\alpha}/a_{\alpha\beta} = |\beta|^2/|\alpha|^2$. We obtain the following table

$a_{\alpha\beta}$	$a_{\beta\alpha}$	θ	$\lvert\beta\rvert^2/\lvert\alpha\rvert^2$
-1	-1	$2\pi/3$	1
-1	-2	$3\pi/4$	2
-1	-3	$5\pi/6$	3
0	0	$\pi/2$	?

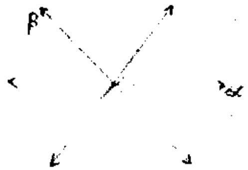
Then there exist up to congruence exactly the following four v.f.s of rank 2

$A_1 \oplus A_1$



$\alpha \perp \beta$. any ratio $|\alpha| : |\beta|$

A_2

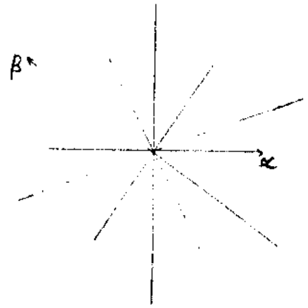


All 6 vectors have the same norm. angle between adjacent vectors is $\pi/3$.

B_2

$|\beta| : |\alpha| = \sqrt{2}$, angle between adjacent vectors is $\pi/4$.
8 roots

G_2



$|\beta| : |\alpha| = \sqrt{3}$, angle is $\pi/6$. 12 roots.

Let R be a vector figure. Choose an element f in V^* that doesn't vanish at any α in R and define an order $\lambda \geq \mu$ if $f(\lambda) \geq f(\mu)$, $\lambda, \mu \in V$. This ~~defines~~ divides R into the two subsets R^+, R^- of positive and negative elements. α in R^+ is simple if it is not the sum of two roots in R^+. Let $F = \{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots in R.

Prop:
(i) If α, β are distinct in F then $\langle \alpha, \beta \rangle \leq 0$.
(ii) F is a linearly independent set.
(iii) Every vector of R^+ is a linear combination of the simple vectors with non-negative integral coefficients, therefore $l = \operatorname{rank} R$.

Proof. Exercise $\square$

Fundamental System
in V with $\langle , \rangle$ is a basis F st for any two α_i, α_j in F,

$$\frac{2\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = a_{ji}$$ is a non-positive integer (can only be $0, -1, -2, -3$)

(a_{ij}) Cartan matrix. $a_{ii} = 2$.

F splits uniquely into mutually orthogonal simple (not decomposable ones). Let $W(F)$ be the Weyl group of F. F determines R by $R = W(F) \cdot F$, then $W(R) = W(F)$.

Theorem

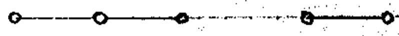
Classification of s.s. Lie algebras over $\mathbb{C}$ $\longleftrightarrow$ Classification of vector figures $\longleftrightarrow$ Classification of fundamental systems

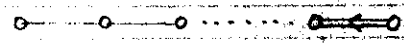
simple $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra — simple R — simple F

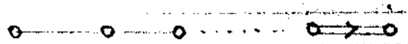
A fundamental system F is determined by a Coxeter-Dynkin diagram: for any two vectors α_i, α_j the corresponding vertices are connected by $a_{ij} a_{ji}$ $(= 0, 1, 2, 3)$ edges. In particular, if $\langle \alpha_i, \alpha_j \rangle = 0$ then there is no edge. If there are 2 or 3 edges, we draw an arrow pointing from the smaller of $|\alpha_i|, |\alpha_j|$ to the larger one.

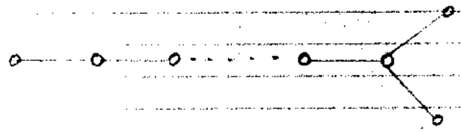
The simple fundamental systems F (up to congruence) are the following:

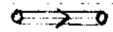
Rank

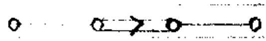
A_l  $l = 1, 2, 3, \dots$

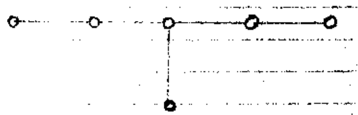
B_l  $l = 2, 3, 4, \dots$

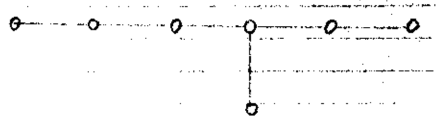
C_l  $l = 3, 4, 5, \dots$

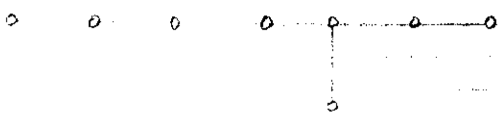
D_l  $l = 4, 5, 6, \dots$

G_2  2

F_4  4

E_6  6

E_7  7

E_8  8

For $\lambda \in \mathfrak{h}^*$ from (V) there is a unique $\mathfrak{z}_\lambda$ in $\mathfrak{h}_{\mathbb{C}}$ s.t. $B(\mathfrak{z}_\lambda, \mathfrak{z}) = \lambda(\mathfrak{z})$ for all $\mathfrak{z} \in \mathfrak{h}$. With $\langle \lambda, \mu \rangle = (\mathfrak{z}_\lambda, \mathfrak{z}_\mu)$ we have an isometry

$$(\mathfrak{h}^*_{i\mathbb{R}}, \langle\,,\rangle) \longrightarrow (\mathfrak{h}_{i\mathbb{R}}, B(\,,))$$
$$\lambda \longmapsto \mathfrak{z}_\lambda$$

where $\mathfrak{h}_{i\mathbb{R}}$ is the real span of the $\mathfrak{z}_\alpha$.

		Rank	Dimension
A_l	$sl(l+1, \mathbb{C})$	$l \geqslant 1$	$l(l+2)$
B_l	$so(2l+1, \mathbb{C})$	$l \geqslant 2$	$l(2l+1)$
C_l	$sp(l, \mathbb{C})$	$l \geqslant 3$	$l(2l+1)$
D_l	$so(2l, \mathbb{C})$	$l \geqslant 4$	$l(2l-1)$
G_2		2	14
F_4		4	52
E_6		6	78
E_7		7	133
E_8		8	248

$\mathfrak{g}$ real semi-simple. To find a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, take 'the' compact real form $\mathfrak{k}$ of $\mathfrak{g}_{\mathbb{C}}$ then a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ to get $\mathfrak{h}_{\mathbb{C}}$.

$A_l = sl(l+1, \mathbb{C})$. Compact real form is $su(l+1)$.
Take $\mathfrak{h}$ consisting of $\mathfrak{z} = \sqrt{-1}\,\mathrm{diag}(a_1, \ldots\ldots, a_{l+1})$
the roots are α_{ij}, $i \neq j$ with $\alpha_{ij}(\mathfrak{z}) = \sqrt{-1}(a_i - a_j)$, $\mathfrak{z} \in \mathfrak{h}$.

Define an order by the element $-\sqrt{-1}\,\mathrm{diag}(l+1, l, \ldots, 2, 1)$ i.e. the positive roots are α_{ij}, $i<j$ and the fundamental system is $\{\alpha_{12}, \alpha_{23}, \ldots, \alpha_{l,l+1}\}$. Consider 'simple' root chains, the only

non-trivial ones are obtained by adding two adjacent simple roots, the Cartan integer is -1. Hence we obtain the C-D diagram A_l. The reflection w_{ij} is $(\alpha_{i,j}, \alpha_{j,i})$. $W(R)$ is generated by the 2-cycles $(i, i+1)$ $i = 1, \ldots\ldots, l-1$ i.e. is $\mathrm{Symm}(l+1)$.

$B_l = so(2l+1, \mathbb{C})$. Compact real form is $so(2l+1)$

Take $\mathfrak{h}$ consisting of $\mathfrak{z} = \mathrm{diag}\left(\begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}, \ldots\ldots, \begin{pmatrix} 0 & b_l \\ -b_l & 0 \end{pmatrix}\right)$, $b_i \in \mathbb{R}$
Roots are $\pm\alpha_i$, $\pm\beta_{ij}$, $i<j$, $\pm\gamma_{ij}$ $i<j$
where $\alpha_i(\mathfrak{z}) = -\sqrt{-1}\,b_i$, $\beta_{ij}(\mathfrak{z}) = -\sqrt{-1}(b_i - b_j)$, $\gamma_{ij}(\mathfrak{z}) = -\sqrt{-1}(b_i + b_j)$
Define order by $\sqrt{-1}\,\mathfrak{z}$ with $b_1 = l, b_2 = l-1, \ldots\ldots, b_{l-1} = 2, b_l = 1$.
Maximal root, $\gamma_{12} = \beta_{12} + 2(\beta_{23} + \ldots\ldots + \beta_{l-1,l} + \alpha_l)$
Positive roots are α_i, β_{ij} $i<j$, γ_{ij} $i<j$. $F = \{\beta_{12}, \beta_{23}, \ldots\ldots, \beta_{l-1,l}, \alpha_l\}$.
Consider non-trivial strings of simple roots. Can add two consecutive simple roots $\beta_{i-1,i}$ and $\beta_{i,i+1}$ Cartan integer -1 or can add α_l to $\beta_{l-1,l}$ twice Cartan integer is -2. These are the only possibilities. $w_{l-1,l} : \alpha_l \longmapsto \alpha_{l-1}$
$W(R)$ is the group of sign changes and $\quad w_l : \beta_{il} \longmapsto \alpha_i$
permutations on $\{1, \ldots, l\}$; order is $2^l\, l!$

Propn
G compact Lie group. Then $\mathfrak{g}$ is reductive. The Killing-form is negative semi-definite and negative definite on $\mathfrak{g}'$.

Proof
Recall that there is an inner product $(,)$ on $\mathfrak{g}$ s.t. the adjoint repn of G is orthogonal; then ad is skew thus if $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ so is $\mathfrak{a}^{\perp}$. Hence $\mathfrak{g} = \mathfrak{a}_1 \oplus \dots \oplus \mathfrak{a}_r$ with $\mathfrak{a}_i$ simple or 1-dim.

If $\xi \in \mathfrak{g}$, then $(\mathrm{ad}\,\xi)^2$ is symmetric with non-positive eigenvalues thus $\mathrm{tr}\,(\mathrm{ad}\,\xi)^2 \leq 0$. If $\mathrm{tr}\,(\mathrm{ad}\,\xi)^2 = 0$, then $\mathrm{ad}\,\xi = 0$. □

$\mathfrak{g}$ complex. A real form is a real subalgebra $\mathfrak{g}_r$ s.t. $\mathfrak{g}_{r\,\mathbb{C}} = \mathfrak{g}$.
A real Lie algebra $\mathfrak{g}$ is said to be compact if the Killing-form is -ve definite (cannot be +ve definite). Reason for def^n is:

Propn
Let $\mathfrak{g}$ be compact and G a connected Lie group with Lie algebra $\mathfrak{g}$ then G is compact.

Proof
$\mathfrak{g}$ is semi-simple so $\mathrm{Ad}\,G = \mathrm{Int}\,\mathfrak{g}$. $\mathfrak{g}$ is isomorphic to $\mathrm{Der}\,\mathfrak{g}$ (by $\xi \longmapsto \mathrm{ad}\,\xi$) so $\mathrm{Ad}\,G$ also has Lie algebra $\mathfrak{g}$.

Recall that given a real Lie algebra $\mathfrak{g}$ there is a unique connected, simply-connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$. If G also has Lie algebra $\mathfrak{g}$ then $G \cong \tilde{G}/D$ where D is discrete normal (so lies in the center). Now if G_1 is compact and has Lie algebra $\mathfrak{g}$, then $\tilde{G}$ is compact; hence D is finite and G is compact. □

A compact connected Lie group is semi-simple iff it has finite center.
If $\mathfrak{g}$ is real semi-simple, $\mathfrak{h}$ a Cartan subalgebra, then there is a compact real form $\mathfrak{g}_c$ of $\mathfrak{g}_{\mathbb{C}}$ containing $\sqrt{-1}\,\mathfrak{h}_{\mathbb{R}}$. Any two compact real forms are conjugate by an inner automorphism. Then from conjugacy of Cartan subalgs in the compact case get conjugacy in complex case. Fails in real case eg $SL(2,\mathbb{R})$

Weyl's Theorem
Every finite-dimensional representation of a semi-simple Lie group G is completely reducible.

Proof
The fact that $G_{\mathbb{C}}$ has a compact subgroup K with $\mathfrak{k}$ a real form of $\mathfrak{g}_{\mathbb{C}}$ □

Structure of compact, connected Lie groups:

Theorem
Every compact connected Lie group G is isomorphic to $Z_0 \times \tilde{G}_1 \times \dots \times \tilde{G}_m / E$ where the $\tilde{G}_i$ are compact simple, simply-connected Lie groups and E is a finite subgroup of the center of the product.

Need to find the ~~compact~~ simple simply-connected Lie groups and their centres: G compact simple connected, $G \cong \tilde{G}/E$

$\mathfrak{g}_{\mathbb{C}}$	A_l	B_l	C_l	D_l	G_2	F_4	E_6	E_7	E_8
Center of $\tilde{G}$	$\mathbb{Z}_{l+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$, l even; $\mathbb{Z}_4$, l odd	$\{0\}$	$\{0\}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\{0\}$
$\tilde{G}$	$SU(l+1)$	$\tilde{SO}(2l+1)$ Spin(2l+1)	$Sp(l)$	$\tilde{SO}(2l)$ Spin(2l)					

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<u>Weyl's Theorem</u>

Every finite-dimensional representation of a semi-simple Lie group G is completely reducible.

<u>Proof.</u>

The ~~the~~ fact that $G_{\mathbb{C}}$ has a compact subgroup K with $\mathfrak{k}$ a real form of $\mathfrak{g}_{\mathbb{C}}$ $\square$

Structure of compact, connected Lie groups:

<u>Theorem</u> cf J. Price

Every compact connected Lie group G is isomorphic to one of the form $Z_0 \times \tilde{G}_1 \times \ldots \times \tilde{G}_m / E$ where the $\tilde{G}_i$ are compact simple, simply-connected Lie groups and E is a finite subgroup of the center of the product.

Z_0 a torus

Need to find the ~~compact~~ simple simply-connected Lie groups and their centres: G compact simple connected, $G \cong \tilde{G}/E$

$\mathfrak{g}_{\mathbb{C}}$	A_ℓ	B_ℓ	C_ℓ	D_ℓ	G_2	F_4	E_6	E_7	E_8
Center of $\tilde{G}$	$\mathbb{Z}_{\ell+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$, ℓ even; $\mathbb{Z}_4$ ℓ odd	$\{0\}$	$\{0\}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\{0\}$
$\tilde{G}$	$SU(\ell+1)$	$\widetilde{SO}(2\ell+1)$ Spin$(2\ell+1)$	$Sp(\ell)$	$\widetilde{SO}(2\ell)$ Spin(2ℓ)					

Spinor group

$\pi_1(SO(n)) = \mathbb{Z}_2,\ n \geq 3$