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SMR/161 - 22

COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
(4 November - 6 December 1985)

- KIRILLOV THEORY. CHAIN ALGEBRAS.
- CONVOLUTION SEMI-GROUPS OF PROBABILITY MEASURES ON LIE GROUPS.
- DECAY OF PROBABILITY SEMI-GROUPS AT INFINITY REPRESENTATIONS ON BANACH SPACES.

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KIRILLOV THEORY. CHAIN ALGEBRAS

We present a short summary of the Kirillov theory of irreducible unitary representations of nilpotent Lie groups and we discuss the generalized chain algebras which Joe Jenkins and the author used in [2]. Our presentation makes utmost use of [4].

Let $\underline{g}$ be a Lie algebra such that for a positive integer c $[X_0,[X_1,\dots[X_{c-1},X_c]\dots] = 0$ for all $X_0,\dots,X_c \in \underline{g}$. Then $\underline{g}$ is called nilpotent of class $\leq$ c. A nilpotent Lie algebra has a non-zero center $Z(\underline{g}) = \{X: [X,Y]= 0 \text{ for all } Y\in \underline{g}\}$.

The following facts are easy to prove:

Let G be a simply connected Lie group such that the Lie algebra $\underline{g}$ of G is nilpotent. Then the exponential map exp: $\underline{g}$ -----> G is a diffeomorphism. We write G = exp$\underline{g}$ and we identify G and $\underline{g}$ as manifolds, i.e. G = $\underline{R}^n$ and the group multiplication in $\underline{R}^n$ is given by xy = c(x,y), where c(x,y) is given by the Hausdorff-Campbel formula and in the case of a nilpotent Lie algebra it is a polynomial.

We write $\underline{S}(G) = \underline{S}(\underline{g})$ for the spce of the Schwrtz functions on G and we note that the group translation x ----> xy induces a linear homeomorphism on $\underline{S}(G)$.

Let G be a simply connected nilpotent Lie group i.e, a group whose Lie algebra $\underline{g}$ is nilpotent. In 1961 A.A.Kirillov gave the following description of all unitary representations of G.

Let $\underline{g}'$ be the dual space to the linear space $\underline{g}$. We define the co-adjoint action of G on $\underline{g}'$ by

$$\langle X , Ad'_x\sigma\rangle = \langle Ad_x X , \sigma\rangle \quad , x\in G,\ X\in \underline{g}, \sigma\in\underline{g}',$$

and we write

$$O_\sigma = \{ Ad'_x\sigma : x\in G\} .$$

For a σ in $\underline{g}'$ we take a subalgebra $\underline{h}$ of $\underline{g}$ which has the following properties:

(a) $\underline{h}$ is subordinate to σ, i.e. $\langle [X,Y], \sigma \rangle = 0$

(b) $\underline{h}$ is of maximal dimension with respect to (a).

Let $H = \exp \underline{h}$. By (a) the mapping $\chi : x \dashrightarrow e^{i\langle X, \sigma\rangle}$ is a multiplicative character of H. Let π^σ be the induced representation $\pi^\sigma = \mathrm{Ind}_H^G \chi$.

A.A.Kirillov proved that:

(i) Every irreducible unitary representation of G is of the form π^σ

(ii) π^{σ_1} is equivalent to π^σ iff $\sigma_1 \in 0_\sigma$.

Statement (ii) implicitely says that if $\underline{h}_1$ is another algebra which satisfies (a) and (b) for a given σ, then the representations $\mathrm{Ind}_{H_1}^G \chi$ and $\mathrm{Ind}_H^G \chi$ are equivalent.

(iii) Let $f \in \underline{S}(G)$. Then the operator

$$\pi_f^\sigma = \int_G \pi_x^\sigma f(x)\,dx$$

is compact and has finite trace.

We say that σ, or the representation π^σ, is in general position if $\langle Z(\underline{g}), \sigma \rangle \neq 0$.

(iv) There is a subset Λ of $\underline{g}'$ such that Λ intersects every orbit in general position in exactly one point and there exists a measure m called the Plancherel measure, on Λ such that

$$f(0) = \int_\Lambda \mathrm{Tr}\,\pi_f^\sigma \, dm(\sigma) .$$

This implies that $f \in L^2(G)$ iff for m almost all σ $\mathrm{Tr}\pi_{f^* * f}^\sigma$ is finite and the function $\sigma \text{-----} \mathrm{Tr}\pi_{f^* * f}^\sigma$ is integrable.

Then
$$\| f \|_{L^2(G)}^2 = \int \mathrm{Tr}\,\pi_{f^* * f}^\sigma \, dm(\sigma).$$

It seems to me that the best reference for the Kirillov theory is [1] or [5] but for the beginers Kirillov's original paper [3] is perhaps the best.

We are not going to enter the theory in any greater detail, instead we are going to invastigate a very specific, simple but important example. For it we are going to describe all irreducible unitary representations in the general position in terms of the Kirillov theory and in a particularsimplest case we are are going to write down the Plancherel measure.We hope to use this at the end of these talks to invastigate certain Schrödinger operators with polynomial potentials.

Chain algebra. By this we mean a Lie algebra $\underline{g}$ with a basis $X, Y_0, \dots, Y_d$ and the following commutation relations:

$$[X, Y_j] = \begin{cases} Y_{j+1} & , \text{ if } j+1 \leq d \\ 0 & , \text{ if } j+1 > d \end{cases} .$$

Clearly $\underline{g}$ is a nilpotent Lie algebra.

This algebra is isomorphic to the following Lie algebra of operators on $\underline{S}(\underline{R})$. Let $\partial = \frac{d}{dx}$, $M_P f = iPf$, where P is a polynomial of degree d. The chain algebra is isomorphic to the Lie algebra generated by ∂ and M_P because $[\partial, M_P] = M_{\partial P}$ and so the mapping $X \dashrightarrow \partial$ and $Y_0 \dashrightarrow M_P$ extends to an isomorphism.

A generalization of the chain algebra is of importance.

Consider operators $\partial_1, \dots, \partial_n$ and $M_{P_1}, \dots, M_{P_k}$ on $\underline{S}(\underline{R}^n)$, where ∂_j's are partial derivatives and M_{P_j} is the multiplication by a polynomial iP_j. The operators ∂_j and M_{P_j} generate a Lie algebra $\underline{g}$ of operators on $\underline{S}(\underline{R}^n)$ which is of the form

$$\underline{g} = D + V,$$

where $D = \mathrm{lin}\{\partial_1, \dots, \partial_n\}$, $V = \mathrm{lin}\{M_{\partial^\alpha P_j} : j=1,\dots,k;\ \alpha \text{ a multiindex}\}$ V is an ideal in $\underline{g}$ and for d in D ad_d acts on V as a nilpotent operator; also, of course, both D and V are commutative.

The following conditions are easily seen to be equivalent.

(i) There is no non-singular linear transformation T of $\underline{R}^n$ such that the polynomials $P_1 \circ T, \dots, P_k \circ T$ depend on less than n variables.

(ii) $\underline{g}$ has one-dimensional center.

(iii) For d in D $[d,v]=0$ for all v in V implies $d=0$ and $\{v \in V : [d,v] = 0 \text{ for all d in } D\} = \underline{R}z$, $z \neq 0$.

By a <u>generalized chain algebra</u> we mean a Lie algebra $\underline{g}$

$$\underline{g} = D + V$$

where both D and V are commutative, V is an ideal in $\underline{g}$, D acts on V by ad as nilpotent linear transformations and the equivalant conditions (i) - (iii) are satisfied.

Now let $\underline{g}$ be a fixed generalized chain algebra and let $G = \exp \underline{g}$ We note few simple facts.

$ad_v d = -[d,v] \in V$ for d in D, v in V.

Let $Ad_X = Ad_{\exp X} = \exp ad_X$, $X \in \underline{g}$, We have

(1) $$Ad_v X = X + [v,X] \qquad v \in V,\ X \in \underline{g} .$$

For every v in V $D \ni d \longrightarrow Ad_d v \in V$ is a polynomial map. Consider $\underline{g}' = D' + V'$ and let z be the unique, up to a scalar, vector in V such that $z \neq 0$ and $ad_d z = 0$ for all d in D. Let

$$\underline{g}'_o = \{\sigma \in \underline{g}' : \langle z , \sigma \rangle \neq 0\}.$$

<u>Lemma</u>. <u>If $\sigma \in \underline{g}'_o$, then</u> $O_\sigma = O_\sigma + d'$ <u>for every</u> d' <u>in</u> D' .

<u>Proof</u>. It is sufficient to show that given a d' in D' there is a v in V such that $\langle X , \sigma + d' \rangle = \langle X , Ad'_v \sigma \rangle$ for all X in $\underline{g}$, or, by (1) , that $\langle X , d' \rangle = \langle [v,X] , \sigma \rangle$ for all X in $\underline{g}$, i.e.

$$\langle d , d' \rangle = -\langle ad_d v , \sigma \rangle \text{ for all d in D.}$$

We define $T: V \longrightarrow D'$ by $\langle d, Tv \rangle = -\langle ad_d v , \sigma \rangle$ and we want to prove that T is "onto". Suppose it is not, then for a d_o in D, $d_o \neq 0$, we have

(2) $$0 = \langle d_o, Tv \rangle = -\langle ad_{d_o} v , \sigma \rangle \text{ for all v in V .}$$

But $\{ad_d : d \in D\}$ is a commuting family of nilpotent linear transformations of V which leave $ad_{d_o} V$ invariant, so if $ad_{d_o} V \neq 0$, it contains a non-zero element v_o such that $ad_d v_o = 0$ for all d in D, hence $v_o = az$, for a non-zero scalar a, which is a contradiction, since $\langle z , \sigma \rangle \neq 0$.

The lemma implies that in order to find all the irreducible unitary representations of G in the general position it is sufficient to restrict to functionals σ in V'. Since V is commutative it is subordinate to σ . To show that it is of maximal dimension we argue as in the proof of the lemma. Suppose it is not maximal, then for some d_o (2) holds and this leads to a contradiction, as we have just seen.

Consequently, every representation given, in the Kirillov model, by a functional σ in V'_o is as follows.

The space of the representation is $L^2(D) = L^2(G/V)$ and for a function φ in $L^2(D)$ we have

$$\pi^\sigma_{\exp(d+v)} \varphi(x) = e^{i\langle Ad_{x-d} v, \sigma \rangle} \varphi(x-d)$$

If $d\pi^{\sigma}$ is the representation of the Lie algebra, i.e.

$$d\pi^{\sigma}_{X}\varphi = \frac{d}{dt}\pi^{\sigma}_{\exp tX}\varphi\Big|_{t=0}, \quad (\varphi \in \underline{S}(D)),$$

then for d in D π^{σ}_{d} is the derivative of φ in the direction d (independent of σ in general position) and for v in V

$$d\pi^{\sigma}_{v}\varphi(x) = i\langle Ad_{x}v, \sigma\rangle\varphi(x) .$$

We recall that $\langle Ad_{x}v, \sigma\rangle$ is a polynomial in x. We also note that if we identify v with the polynomial $\langle Ad_{x}v, \sigma\rangle = P_{v}(x)$, then the linear space $\{P_{v}: v \in V\}$ is stable under derivatives.and $P_{Ad_{d}v}(x) = P_{v}(x+d) =: P^{d}_{v}(x)$. It follows that two functionals on V σ and σ_{1} belong to the same orbit iff $\langle P_{v}, \sigma_{1}\rangle = \langle P^{d}_{v}, \sigma\rangle$ for some d in D.

Now, comming back to the begining of the story, if we are given polynomials $P_{1},\dots,P_{k}$ on $\underline{R}^{n}$ which satisfy (i) and the partial derivatives $\partial_{1},\dots,\partial_{n}$, then the natural representation π of the Lie algebra generated by $\partial_{1},\dots,\partial_{n}$ and $M_{P_{1}},\dots,M_{P_{k}}$ on $\underline{S}(\underline{R}^{n})$ is of the form $\pi = \pi^{\sigma}$, where the functional σ on

$$V = \mathrm{lin}\{\partial^{\alpha}P_{j}: j=1,\dots,k; \alpha \text{ a multiindex}\}$$

is $\langle P, \sigma\rangle = P(0)$.

Now we turn to the chain algebra. We see that then $D = \underline{R}X$, $V = \mathrm{lin}\{Y_{o},\dots,Y_{d}\}$ and $\underline{R}Y_{d}$ is the center. A functional on V is then $\sigma = (\sigma_{1},\dots,\sigma_{d})$, where $\sigma_{j} = \langle Y_{j}, \sigma\rangle$. So σ is in general position iff $\sigma_{d} \neq 0$. We have $Ad_{x}Y_{j} =$

$$Ad_{x}Y_{j} = \sum_{k=o}^{d-j}\frac{x^{k}}{k!}Y_{j+k}$$

and so

$$(3)\quad Ad'_{x}\sigma = \Big(\sum_{k=o}^{d}\frac{x^{k}}{k!}\sigma_{k},\dots,\sum_{k=o}^{d-j}\frac{x^{k}}{k!}\sigma_{j+k},\dots,\sigma_{d}x+\sigma_{d-1},\sigma_{d}\Big) .$$

Thus putting $x = -\sigma_{d-1}/\sigma_{d}$ we see that the orbit O_{σ} contains a unique functional σ' with $\sigma'_{d-1} = 0$. Consequently, we may select Λ as

$$\Lambda = \{\sigma = (\sigma_{o},\dots,\sigma_{d-2},0,\sigma_{d}) : \sigma_{d} \neq 0 \quad \sigma_{j} \in \underline{R}\}$$

To find the Plancherel measure we take a function f on G of the form $\alpha(x)\beta(v)$, $\alpha \in \underline{S}(D)$, $\beta \in \underline{S}(V)$ and we see that

$$\pi^{\sigma}_{f}\varphi(y) = \int e^{i\langle Ad_{y-x}v,\sigma\rangle}\beta(v)dv\,\alpha(x)\,\varphi(y-x)dx$$

$$= \int\hat{\beta}(Ad'_{y-x}\sigma)\,\alpha(x)\,\varphi(y-x)dx$$

$$= \int K(y,x)\varphi(x)dx ,$$

where $K(y,x) = \hat{\beta}(Ad'_{x}\sigma)\alpha(y-x)$.

Hence

$$\mathrm{Tr}\,\pi^{\sigma}_{f} = \int K(x,x)dx = \alpha(0)\int\hat{\beta}(Ad'_{x}\sigma)dx$$

and

$$\int \mathrm{Tr}\,\pi^{\sigma}_{f}\,dm(\sigma) = \alpha(0)\int\hat{\beta}(Ad'_{x}\sigma)dx\,dm(\sigma).$$

Now if we put

$$(4)\qquad dm(\sigma) = (2\pi)^{-(d+1)/2}\,d\sigma_{o}\dots d\sigma_{d-2}|\sigma_{d}|d\sigma_{d},$$

in virtue of (3) in which $\sigma_{d-1} = 0$ an easy change of the variable shows

$$\int\hat{\beta}(Ad'_{x}\sigma)dx\,dm(\sigma) = (2\pi)^{-(d+1)/2}\int\hat{\beta}(v)dv = \beta(0),$$

and so (4) defines the Planchrel measure.

REFERENCES

[1] P.Bernat et al. Représentations des groupes de Lie résoluble, Paris 1972, Dunod.

[2] Andrzej Hulanicki and Joe Jenkins, Nilpotent Lie groups and eigen-function expansions of Schrödinger operators II,(to appear)

[3] A.A.Kirillov, Unitary representations of nilpotent Lie groups, Uspehi Mat.Nauk 17 (1962), 57-110

[4] H.Leptin, pre-print

[5] L.Pukanszky, Leçons sur les représentations des groupes,Dunod Paris 1967

CONVOLUTION SEMI-GROUPS OF PROBABILITY MEASURES ON LIE GROUPS

G.Hunt theory

We present now the basic facts concerning convolution semi-groups of probability measures on Lie groups. The proofs if any will be very sketchy only.

We start with a very brief survey of the main definitions concerning semi-groups of bounded operators on Banach spaces.

A strongly continuous semi-group of operators on a Banach space B is a family of bounded operators $\{T_t\}_{t>0}$ such that

(i) $T_{s+t} = T_s T_t$,(ii) $\lim_{t\to 0} \|T_t f - f\|_B = 0$ for all f in B.

We note that (ii) implies $\|T_t\| \leqslant C$ for $t \leqslant 1$ and we call the semigroup equicontinuos if $\|T_t\| \leqslant C$ for all t.

Proposition 1. The function $(0,\infty)\ni t \text{ ---- } T_t f \in B$ is continuous for all f in B.

We define the infinitesimal generator of the semi-group $\{T_t\}_{t>0}$ to be a , in general unbounded, operator A such that

(1) $$Af = \lim_{t\to 0} t^{-1}(T_t f - f),$$

and the domain of A is $\underline{D}(A)$ = all the f's for which the limit in the norm exits.

Proposition 2. For every f in $\underline{D}(A)$ the function

$$F:(0,\infty)\ni t \text{ ------ } T_t f \in \mathcal{B}$$

has continuous derivative and

$$\frac{d}{dt}F(t) = AT_t f = T_t Af.$$

Proposition 3. For every f in B

$$f_s = \int_0^s T_t f \, dt \in \underline{D}(A) \quad \text{and} \quad Af_s = T_s f - f .$$

If f $\underline{D}(A)$, then $T_s f - f = \int_0^s T_t Af \, dt$.

Proposition 4. $\underline{D}(A)$ is dense in B and A is a closed operator on $\underline{D}(A)$.

Let A be a closed operator on a Banach space B with a dense domain $\underline{D}(A)$. The resolvent set $\varrho(A)$ is the set of complex numbers λ such that $(\lambda - A)\underline{D}(A) = B$, $\lambda - A$ is 1 - 1 on $\underline{D}(A)$ and $(\lambda - A)^{-1}$ is bounded. We write $R(\lambda,A) = (\lambda - A)^{-1}$ and we call $R(\lambda,A)$ the resolvent of A.

Proposition 5. The set $\varrho(A)$ is open , the function

$$\varrho(A)\ni\lambda \text{ -----}\!> R(\lambda,A) \in B(B)$$

is holomorphic, $R(\lambda,A)$ and $R(\mu,A)$ commute for all λ , μ in $\varrho(A)$.

Proposition 6. If $\{T_t\}_{t>0}$ is an equicontinuos semi-group and A is the infinitesimal generator, then $\varrho(A)$ contains the positive half-line and

$$R(\lambda,A)f = \int_0^\infty e^{-\lambda t} T_t f \, dt .$$

Moreover, if $\|T_t\| \leqslant C$, then $\|(\lambda R(\lambda,A))^n\| \leqslant C$ for all $n,\lambda > 0$.

Proposition 7. If A is an arbitrary closed operator on A with dense domain such that $\varrho(A)$ contains the positive half-line and $\|\lambda R(\lambda,A)\| \leqslant C$ for all $\lambda > 0$, then $\lim_{\lambda\to\infty} \lambda R(\lambda,A)f = f$ for all f in B.

Theorem (Hille-Yosida). Suppose A is a densely defined closed operator A such that $\varrho(A)$ contains the positive half-line and $\|(\lambda R(\lambda,A))^n\| \leqslant C$ for all $n,\lambda > 0$. Then there exists a unique semi-group $\{T_t\}_{t>0}$ such that $\|T_t\| \leqslant C$ and A is the infinitesimal generator of $\{T_t\}_{t>0}$.

Moreover,

$$T_t f = \lim_{\lambda\to\infty} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} (\lambda R(\lambda,A))^n f.$$

Let A be a self-adjoint operator on a Hilbert space $\underline{H}$ and let $Af = \int \lambda dE(\lambda) f$, $f \in \underline{D}(A)$, be its spectral presentation. We say that A is positive if $(Af,f) \geqslant 0$ for all f in $\underline{D}(A)$, then $Af = \int_0^\infty \lambda dE(\lambda) f$

Proposition 8. If A is a self-adjoint positive operator on a Hilbert space H, then $-A$ is the infinitesimal generator of the a semi-group

$$(2) \qquad T_t f = \int_0^\infty e^{-\lambda t} dE(\lambda) f \quad , \; f \in \underline{H},$$

where T_t are hermitian and of norm ≤ 1. Conversely, if $\{T_t\}_{t>0}$ is a semi-group of hermitian operators of norm ≤ 1 on a Hilbert space $\underline{H}$, then the infinitesimal generatorof it is a self-adjoint operator $-A$ such that A is positive and (2) holds.

Let G be a Lie group and let $\underline{g}$ be the Lie algebra of it. We identify $\underline{g}$ with the left-invariant differential operators of order one on G. We introduce a norm $\|X\|$on $\underline{g}$ such that if $V = \{X: \|X\| < a\}$, then exp is a diffeomorphism of V onto an open neighbourhood U in G. For X in $\underline{g}$, let X^+ be the correspondig right-invariant differential operator

$$X^+ f(x) = \frac{d}{dt} f(\exp tX \cdot x)\Big|_{t=0} \quad , \qquad f \in C_c^\infty(G).$$

For $x = \exp(x_1 X_1^+ + \ldots + x_n X_n^+)$, where $X_1, \ldots, X_n$ is a basis in G, we write $x = (x_1, \ldots, x_n)$ and let

$$D_j f(x) = \frac{\partial}{\partial x_j} f(x_1, \ldots, x_n) \qquad x \in U \text{ ,(a small enough).}$$

Let $C_o(G)$ be the space of continuos functions on G vanishing at infinity and let $C_\infty(G)$ be the spece of continuous function on G which have limits at infinity. We write

$$C^k_\infty(G) = \{f: X^+_{i_1} \ldots X^+_{i_k} f \in C_\infty(G) \text{ for all } i_1, \ldots, i_k \in (1, \ldots,$$

Let

$$M = \{f \in C^k_\infty(G): f(e) = X_j^+ f(e) = 0 \text{ for } j = 1, \ldots, n\}.$$

Clearly M is a closed subspace of $C^k_\infty(G)$, $k \geq 2$, of finite codimension.

Since for f in $C^\infty(U)$

$$D_j f(x) = \sum_k a_{jk}(x) X_k^+ f(x),$$

where $a_{jk}(e) = \delta_{jk}$, we have

$$(3) \qquad f(x) = \frac{1}{2} \sum_{ij} X_i^+ X_j^+ f(e) x_i x_j + o(\|x\|^2) \quad , \; x \in U.$$

By a semi-group of probability measures on G we mean a family $\{\mu_t\}_{t>0}$ of probability measures such that

$$\mu_s * \mu_t = u_{s+t}$$

and

$$\lim_{t \to 0} \|f * \mu_t - f\|_{C_\infty(G)} = 0 \qquad \text{for f in } C_\infty(G).$$

We see that$\{\mu_t\}_{t>0}$ defines a strongly continuous semi-group $\{T_t\}_{t>0}$ of operators on all C^k (G) by the formula $T_t f = f * \mu_t$. T_t preseves $C_o(G)$ and it is uniquely determind by its action on $C_o(G)$.

We shall use the same letter A to denote the infinitesimal generator on all spaces on which $\{\mu_t\}_{t>0}$ acts by convolution reserving the notation $\underline{D}^k(A)$for the domain of A in $C^k_\infty(G)$. We write $\underline{D}^o(A) = \underline{D}(A)$.

Let $\underline{C}$ be the cone of non-negative functions on G. Since,by proposition 3, $s^{-1}\int_0^s f * \mu_t \, dt \in \underline{D}^k(A)$, if $f \in C^k_\infty(G)$, we see that $\underline{C} \cap \underline{D}^k(A)$ is dense in $\underline{C} \cap C^k_\infty(G)$.

We define a characteristic functional F for a semi-group of probability measures $\{\mu_t\}_{t>0}$ by

$$F: \underline{D}(A) \ni f \longrightarrow Af(e) \; .$$

Proposition 9. F is bounded on $C^2_\infty(G)$.

The proof of proposition 9 is based on the followin classical lemma by Helley.

<u>Lemma (Helley)</u>. Let B be a Banach space, C a cone in B and C_o a dense convex subset of C. Then for every f in C and $\varepsilon > 0$ and functionals $F_1, \dots, F_n$ in B' there exists a g in C_o such that $\langle g , F_j \rangle = \langle f , F_j \rangle$, $j = 1, \dots, n$ and $\| f - g \|_B < \varepsilon$.

<u>Proof of proposition 9</u>. First we note that by Helley's lemma $\underline{C} \cap \underline{D}(A) \cap M$ is dense in $M \cap \underline{C}$ (we take $\underline{D}(A) = C_o$ and $M = C$).

Now we are going to construct so called <u>Hunt's function</u>. This is a function φ in $\underline{D}(A) \cap M \cap \underline{C}$ such that

(i) $X_i^+ X_j^+ \varphi(e) = 2 \delta_{ij}$, $i,j = 1, \dots, n$,

(ii) $\lim_{x \to \infty} \varphi(x) > 0$

(iii) $\varphi(x) > 0$ for $x \neq e$.

Using Helley's lemma we take a function ψ in $\underline{D}(A) \cap M \cap \underline{C}$ which satisfies (i) and $\lim_{x \to \infty} \psi(x) = 1$. There is a compact subset K of G such that $\psi(x) > 1/2$ for $x \notin K$ and also, by (3) , there is a neighbourhood V of e such that $\psi(x) > 0$ for $x \neq e$ and $x \in V$. Now for each x in $K \setminus V$ take $\varphi_x \in \underline{D}(A) \cap M \cap \underline{C}$ which satisfies (i), (ii) and $\varphi_x(x) = 1$ (Helley's lemma again!). Thus $U_x = \{ y : \varphi_x(y) > 1/2 \}$ is open and a finite number of them $U_{x_1}, \dots, U_{x_k}$ cover $K \setminus V$ and so $(k+1)^{-1} (\psi + \varphi_{x_1} + \dots + \varphi_{x_k}) = \varphi$ satisfies (i) - (iii). Of course we can also have $\varphi(x) = \varphi(x^{-1})$.

By (i) and (3) we see that for a $c > 0$ we have $c\, \varphi(x) > \| x \|^2$ for x in U.

Now we are ready to prove that the functional F is continuous. Suppose that $f_n \in \underline{D}(A) \cap M$ and $\| f_n \|_{C^2_\infty(G)} \longrightarrow 0$. Then, by (3), for every $\varepsilon > 0$

$$| f_n(x) | \leqslant \varepsilon \| x \|^2 \quad \text{for } x \in U \text{ and } n > n_o .$$

Consequently, for n_o still larger perhaps,

$$| f_n(x^{-1}) | \leqslant \varepsilon \varphi(x) \qquad x \in G .$$

Thus

$$t^{-1} | f_n * \mu_t(e) | \leqslant t^{-1} \langle | f_n^\sim | , \mu_t \rangle \leqslant t^{-1} \langle \varepsilon \varphi , \mu_t \rangle \leqslant t^{-1} \varepsilon\, \varphi * \mu_t(e),$$

and so

$$\lim_{t \to 0} t^{-1} \langle f_n * \mu_t(e) \rangle \leqslant \varepsilon \langle \varphi , F \rangle ,$$

which shows that

$$\lim_{n \to \infty} \langle f_n , F \rangle = 0 .$$

Therefore F extends uniquely and continuously from $\underline{D}(A) \cap M$ to M. But M is of finite co-dimension in $C^2_\infty(G)$ and is defined on a dense subset $\underline{D}^2(A)$ by

(4) $$\langle f , F \rangle = \lim_{t \to 0} t^{-1} (f * \mu_t(e) - f(e))$$

consequently, F is bounded on $C^2_\infty(G)$ and is given by (4).

A distribution $F \in C_c^\infty(G)'$ is called <u>dissipative</u> if for every real-valued function f in $C_c^\infty(G)$ such that $\max \{ f(x) : x \in G \} = f(e)$ we have $\langle f , F \rangle \leqslant 0$.

By propositon 9 (4) defines a distribution (in fact of order $\leqslant 2$) which is dissipative because , since μ_t is a probability measure $f * \mu_t(e) \leqslant \max \{ f(x) : x \in G \}$.

<u>Proposition 10</u>. If F is a dissipative distribution, then for every neighbourhood V of e

$$F = F_V + \mu_V ,$$

where $\operatorname{supp} F_V \subset \bar{V}$ and μ_V is a positive bounded measure supported on V^c.

Let A_F be the convolution operator defined on $C_c^\infty(G)$ with values in $C_o(G)$ given by the formula

$$A_F f = f * F_V^\sim + f * \mu_V^\sim = f * F^\sim$$

We note that the definition does not depend on the choice of V and that A_F admits the closure $\overline{A}_F$.

<u>Theorem (G.Hunt)</u>. Let F be a dissipative distribution. There exists a unique semi-group of probability measures $\{\mu_t\}_{t>0}$ on G such that the infinitesimal generator of $\{\mu_t\}_{t>0}$ on $C_o(G)$ is $\overline{A}_F$.

<u>Proof</u>. For the proof of this theorem we assume that all functions are real valued .

Since F is dissipative, for $\lambda > 0$ we verify that

(i) $\|\lambda f\|_{C_o(G)} \leq \|\lambda f - A_F f\|_{C_o(G)}$

(ii) $(\lambda - A_F)C_c^\infty(G)$ is dense in $C_o(G)$.

In fact, to see (i) we note that both sides of (i) are invariant under left translations, so we may assume that $\|f\|_{C_o(G)} = f(e)$. Then $\|f\|_{C_o(G)} = f(e) \leq f(e) - A_F f(e) \leq \|f - A_F f\|_{C_o(G)}$.
To prove (ii) we suppose that for a bounded measure μ

$$\langle \lambda f - A_F f , \mu \rangle = 0 \quad \text{for all f in } C_c^\infty(G).$$

Hence

$$\langle f^\sim * \mu , \lambda\delta_e - F^\sim \rangle = 0$$

and so

$$\lambda f^\sim * \mu(e) = \langle f^\sim * \mu , F^\sim \rangle,$$

whence, since F is dissipative, $f^\sim * \mu(e) = 0$ if $\max\{f^\sim * \mu(x) : x \in G\} = f^\sim * \mu(e)$. Hence, translating $f^\sim$ on the left and multiplying by -1, if necessary, we see that $f^\sim * \mu(e) = 0$ for all f in $C_c^\infty(G)$, whence $\mu = 0$, and (ii) is proven.

(i) and (ii) show that the closure $\overline{A}_F$ of A_F has the property that for $\lambda > 0$ $\lambda \in \varrho(\overline{A}_F)$ and $R(\lambda, \overline{A}_F)$ has the norm equal to λ^{-1}.

Thus the assumptions of Hille-Yosida theorem are satisfied and so there exists a semi-group $\{T_t\}_{t>0}$ of contractions on $C_o(G)$ whose infinitesimal generator is $\overline{A}_F$. Since A_F commutes with left translations, so does $\overline{A}_F$ and also $R(\lambda, \overline{A}_F), \lambda > 0$. Another use of dissipativity of F shows that $R(\lambda, \overline{A}_F)$ maps non-negative functions in $C_o(G)$ onto non-negative functions and so, by the formula at the end of Hille-Yosida theorem, so do the operators T_t $t > 0$. From these we easily conclude that $T_t f = f * \mu_t$, where μ_t is a probability measure.

Suppose now that ϑ_t is another semi-group of probability measures such that

$$\lim_{t \to 0} t^{-1}(f * \vartheta_t(e) - f(e)) = \langle f , F \rangle, \; f \in C_c^\infty(G).$$

Let H be the infinitesimal generator of $\{\vartheta_t\}_{t>0}$.
By translating from the left we see that for every f in $C_c^\infty(G)$

$$\lim_{t \to 0} t^{-1}(f * \vartheta_t(x) - f(x)) = \langle {}_x f , F \rangle \text{ for every } x \in G. \tag{5}$$

Let H' be the operator whose domain consists of all the functions f in $C_o(G)$ for which the limit in (5) exists for every x in G.
Of course H' contains H. Now for a $\lambda > 0$ we verify that $\lambda - H'$ is 1 - 1 on $\underline{D}(H')$. In fact, if $(\lambda - H')f = 0$, $f \neq 0$, multiplying by -1 if necessary we may assume that $\max\{f(x) : x \in G\} = f(x_o) > 0$, whence $(\lambda - H')f(x_o) = \lambda f(x_o) - H'f(x_o) > 0$, since by (5), $H'f(x_o) \leq 0$, which is a contradiction. Hence $\lambda - H'$ maps $\underline{D}(H')$ onto $C_o(G)$ and $\lambda - H$ maps $\underline{D}(H)$ onto $C_o(G)$ both are 1 - 1 and they agree on $\underline{D}(H)$, whence $\underline{D}(H) = \underline{D}(H')$ i.e. $H = H'$.
On the other hand we see by (5) that $A_F \subset H'$ and so $\overline{A}_F \subset H' = H$.
Thus , since for a $\lambda > 0$ $\lambda - \overline{A}_F$ maps $\underline{D}(\overline{A}_F)$ onto $C_o(G)$ and is 1 - 1 we see that $\overline{A}_F = H$ and hence by Hille-Yosida theorem $\mu_t = \vartheta_t$.

Examples of dissipative distributions. Let X be a vector field on an open set U containing the identity e of G. Then

$$f \dashrightarrow Xf(e) \quad \text{and} \quad f \dashrightarrow X^2 f(e)$$

are dissipative distributions. In fact, for a system of coordinates around e we have $Xf(x) = \sum_j a_j(x) D_j f(x)$ with $a_j(e) = 1$. Consequently,

$$X^2 f(x) = \sum_{i,j} a_i(x) a_j(x) D_i D_j f(x) + \sum_{i;j} a_i(x) D_i a_j(x) D_j f(x$$

whence, if $f(e) = \max\{f(x): x \in U\}$, then $D_j f(e) = 0$ and $X^2 f(e) = \sum_{i,j} D_i D_j f(e) \leq 0$.

Every convex combination of dissipative distributions is dissipative.

If $X \in \underline{g}$, then for $F: f \dashrightarrow Xf(e)$ we have $Xf(x) = f * F^{\sim}(x)$, and similarily for $F: f \dashrightarrow \partial f(e)$ where ∂ is any element in the envelloping algebra of $\underline{g}$. Thus we arrive to the following

Proposition II. Let $A = X_o + X_1^2 + \ldots + X_k^2$, where $X_o, \ldots, X_k$ are elements of the Lie algera $\underline{g}$ of a Lie group G. Then the closure of A (as an operator on $C_c^{\infty}(G)$) is the infinitesimal generator of a semi-group of probability measures on G.

Another important class of dissipative distributions is obtained by subordination. Let $\{\mu_t\}_{t>0}$ be a semi-group of probability measures on G and let A be the infinitesimal generator of $\{\mu_t\}_{t>0}$. For $0 < a < 1$ we define

$$-|A|^a f = c\int t^{-1-a}(f * \mu_t - f)\, dt \qquad f \in \underline{D}(A),\ c > 0.$$

We see that $f \dashrightarrow -|A|^a f(e)$ is a dissipative distribution and so the closure of $-|A|^a$ is the infinitesimal generator of a semi-group of probability measures on G.

There are very many books on the theory of semi-groups in Banach spaces. I would recommend E.B.Davies, One-parameter semigroups, Academic Press 1980.

DECAY OF PROBABILITY SEMI-GROUPS AT INFINITY

REPRESENTATIONS ON BANACH SPACES

Let G be a locally compact group. We say that a function ϕ is submultiplicative, if (i) ϕ is locally bounded, (ii) $\phi(x^{-1}) = \phi(x)$, (iii) $\phi(x) \geq 1$, $\phi(xy) \leq \phi(x)\phi(y)$. We say that a submultiplicative function ω is a polynomial weight if

$$\omega(xy) \leq C(\omega(x) + \omega(y))$$

Let G be compactly generated and let $U = U^{-1}$ be a compact set of generators of G, then

$$\tau_U(x) = \min\{\, n: x \in U^n\}$$

is subadditive, i.e. instead of (iii) it satisfies (iii)' $\tau(x) \geq 0$ and $\tau(xy) \leq \tau(x)\tau(y)$, and $\omega(x) = (1 + \tau_U(x))^a$, $a > 0$ is a polynomial weight.

It is easy to verify that every submultiplicative function on G is dominated by $e^{C\tau_U(x)+C}$.

Let M(G) be the space of bounded (complex valued) measures on G. For a submultiplicative function ϕ we define

$$M_\phi = \{\mu \in M(G): \int \phi \, d|\mu| = \|\mu\|_{M_\phi} < \infty\}.$$

For $\mu, \nu \in M$ we have

$$\langle \phi, \mu * \nu \rangle \leq \langle \phi, \mu \rangle \langle \phi, \nu \rangle$$

which shows that M_ϕ is a Banach algebra with involution ($\mu^*(X) = \mu(X^{-1})^-$). Let m be the right invariant Haar measure on G. We define

$$L^1_\phi = \{\, f \in L^1(m): fm \in M_\phi\} .$$

L^1_ϕ is a Banach $*$-subalgebra of M .

Proposition 1. Let $\{\mu_t\}_{t>0}$ be a semi-group of probability measures on a locally compact group G. Let A be the infinitesimal generator of $\{\mu_t\}_{t>0}$ considered on $L^1(m)$, where m is the right invariant Haar measure. Let ϕ be a submultiplicative function on G. Suppose that for a (single) non-zero non-negative function f in $\underline{D}(A)$ we have $\int f\phi \, dm = a$ and $\int |Af|\phi \, dm = b$, then

$$\int \phi \, d\mu_t \leq abe^{tbc},$$

where $c = (\int f \phi^{-1} dm)^{-1}$.

Proof. We note first that if ψ is a submutiplicative function and $f \geq 0$ is such that $\langle f, \psi \rangle = \int f\psi \, dm < \infty$, we have

(1) $$\langle f, \psi^{-1} \rangle \psi(x) \leq f * \psi(x) \leq \langle f, \psi \rangle \psi(x).$$

In fact, (ii) and (iii) imply

$$\psi(x)\psi(y)^{-1} \leq \psi(y^{-1}x) \leq \psi(y)\psi(x)$$

whence, since $f*\psi(x) = \int f(y)\psi(y^{-1}x) \, dm(y)$, (1) follows.

Now for the submultiplicative function ϕ and a positive integer n we define $\phi_n = \min\{n, \phi(x)\}$. Clearly, ϕ_n is submultiplicative and bounded. We define

$$h_n(t) = \langle f*\mu_t, \phi_n \rangle = \langle \mu_t, f^{\sim} * \phi_n \rangle.$$

Since $f \in \underline{D}(A)$, $Af \in L^1(m)$ and, by proposition 2, section II,

$$\frac{d}{dt} f*\mu_t = Af*\mu_t.$$

Hence $h_n'(t) = \langle Af*\mu_t, \phi_n \rangle$ and so,

$$\begin{aligned} h_n'(t) &\leq \langle |Af|*\mu_t, \phi_n \rangle \\ &\leq \langle \mu_t, \phi_n \rangle \langle |Af|, \phi_n \rangle \\ &\leq \langle \mu_t, f^{\sim} * \phi_n \rangle b \langle f^{\sim}, \phi_n^{-1} \rangle^{-1} \\ &\leq h_n(t) b \langle f^{\sim}, \phi^{-1} \rangle^{-1} \\ &\leq bch_n(t). \end{aligned}$$

Consequently,

$$h_n(t) \leq h_n(0) e^{tbc},$$

i.e.

$$\langle f*\mu_t, \phi_n \rangle \leq \langle f, \phi \rangle e^{tbc}.$$

Finally, for all n we have

$$\langle \mu_t, \phi_n \rangle \langle f*\mu_t, \phi_n \rangle \langle f, \phi_n^{-1} \rangle^{-1} \leq \langle f, \phi \rangle \langle f, \phi^{-1} \rangle^{-1} e^{tbc} = abe^{tbc}$$

which completes the proof of proposition 1.

Proposition 2. Let $\{\mu_t\}_{t>0}$ be a semi-group of probability measures on a Lie group G. Let A be the infinitesimal generator of $\{\mu_t\}_{t>0}$ and let $\underline{D}_1(A)$ be the domain of A in $L^1(m)$. Given a submutiplicative function ϕ suppose that for a subset M of $\underline{D}_1(A)$ containing non-zero non-negative functions and dense in L^1_ϕ we have $\langle Af, \phi \rangle < \infty$ for f in M. Then $\{\mu_t\}_{t>0}$ defines a strongly continuous semi-group of operators on L^1_ϕ and the domain of A in L^1_ϕ contains M.

Proof. By proposition 1, we have $\langle \mu_t, \phi \rangle < C$ for $t \in (0,T)$, $T > 0$. Consequently the operators

$$L^1_\phi \ni f \dashrightarrow f*\mu_t \in L^1_\phi$$

are well-defined and uniformly bounded for every fixed T and $t < T$, because

$$\langle f*\mu_t, \phi \rangle < C\langle f, \phi \rangle.$$

Thus it is sufficient to prove that

$$\langle |f*\mu_t - f|, \phi \rangle = O(t) \text{ as } t \to 0 \text{ fot f in M.}$$

Let $\phi_n(x) = \min\{\phi(x), n\}$. For f in M we have

$$f*\mu_t - f = \int_0^t (Af)*\mu_s \, ds$$

and so

$$\langle |f*\mu_t - f|, \phi_n \rangle \leq \int_0^t \langle |Af|*\mu_s, \phi_n \rangle ds \leq t \sup_{s<t} \langle \mu_s, \phi_n \rangle \langle |Af|, \phi \rangle$$

$$tC\langle |Af|, \phi \rangle,$$

which completes the proof.

The following proposition has been proved in [1].

Proposition 2. Suppose $\{\mu_t\}_{t>0}$ is a semi-group of probability measures on a Lie group G. Let A be the infinitesimal generator of $\{\mu_t\}_{t>0}$ and suppose that for a polynomial weight ω $\int \omega d\mu_t < \infty$ for all t. Then for every $0 < a < 1$ there is a $\alpha > 0$ such that for every non-negative function f in $C_c(G)$ we have $\langle -|A|^a f , \omega^\alpha \rangle < +\infty$.

Let G be a Lie group and let B be a Banach space. By a representation of G on B we mean a homomorphism

$$\pi : G \longrightarrow \underline{B}(B)$$

of G into bounded operators on B such that for every ξ in B the function $G \ni x \longrightarrow \pi_x \xi \in B$ is continuous. We note that

$$\|\pi_{xy}\| \le \|\pi_x\| \|\pi_y\| ,$$

whence, if

$$\phi(x) = \max\{\|\pi_x\| , \|\pi_{x^{-1}}\| , 1\}$$

then ϕ is a submultiplicative function.

For a measure μ in M_ϕ we write

$$\pi_\mu \xi = \int \pi_x \xi d\mu(x) .$$

Of course $\|\pi_\mu\| \le \|\mu\|_{M_\phi}$. Also $\pi_{\mu * \nu} = \pi_\mu \pi_\nu$.

If $\{f_j\}$ is a bounded approximate identity in L^1_ϕ, we have

$$\lim_j \|\pi_{f_j} \xi - \xi\|_B = 0 .$$

We define the Garding space of π as the image of the map

$$C_c^\infty(G) \otimes B \ni f \otimes \xi \longrightarrow \pi_f \xi \in B$$

and we denote it by B_o. In other words $B_o = \mathrm{lin}\{\pi_f \xi : f \in C_c^\infty(G), \xi \in B\}$.

Let U be a distribution with compact support on G. We define a (unbounded) operator $\underline{\pi}_U$ as follows.

$$\underline{D}(\underline{\pi}_U) = \{\xi \in B : \text{there is } \eta \in B \ \pi_{f*U} = \pi_f \eta \text{ for all } f \in C_c^\infty(G)\} .$$

Putting an approximate identity f_j in place of f we see that η is defined uniquely by U and ξ and so we put $\underline{\pi}_U \xi = \eta$.

By definition, we have $\pi_{f*U}\xi = \pi_f \underline{\pi}_U \xi$ for f in $C_c^\infty(G)$. We also easily verify that $\underline{\pi}_U$ is a closed operator: if $\xi_n \in \underline{D}(\underline{\pi}_U)$, $\xi_n \longrightarrow \xi$ and $\underline{\pi}_U \xi_n \longrightarrow \eta$, then for all f in $C_c^\infty(G)$

$$\pi_{f*U}\xi = \lim \pi_{f*U}\xi_n = \lim \pi_f \underline{\pi}_U \xi_n = \pi_f \eta ,$$

whence $\xi \in \underline{D}(\underline{\pi}_U)$ and $\underline{\pi}_U \xi = \eta$.

Since for all $f, g \in C_c^\infty(G)$ and $\xi \in B$ we have $\pi_{g*U}\pi_f \xi = \pi_g \pi_{U*f}\xi$, $\pi_f \xi \in \underline{D}(\underline{\pi}_U)$ and so $B_o \subset \underline{D}(\underline{\pi}_U)$. Let π_U be the closure of the restriction of $\underline{\pi}_U$ to B_o. We have $\pi_U \pi_f \xi = \pi_{U*f}\xi$ for $\xi \in B$, $f \in C_c^\infty(G)$. So in the case $U*f = f*U$ we have $\pi_U = \underline{\pi}_U$. In fact, if $\{f_j\}$ is an approximate identity in $C_c^\infty(G)$, then for $\xi \in \underline{D}(\underline{\pi}_U)$ we have

$\pi_{f_j}\xi \longrightarrow \xi$ $\underline{\pi}_U \pi_{f_j}\xi = \pi_{f_j}\underline{\pi}_U \xi \longrightarrow \underline{\pi}_U \xi$, so since π_U is closed, $\xi \in D(\pi_U)$.

Example. Let B one of the follwing spaces: $L^p(m)$, L^p_ϕ, where ϕ is a submultiplicative function, $C_o^\infty(G)$. Let π be the right regular representation, i.e.

$$\pi_x f(y) = f(yx) .$$

If U is a distribution with compact support, then for every $g \in B$ $g * \tilde{U}$ is a well-difeined distribution.

We have

$$\underline{D}(\underline{\pi}_U) = \{g \ B : g * \tilde{U} \in B\} .$$

In fact, if $g \in \underline{D}(\underline{\pi}_U)$, we have $\pi_f g = g * \check{f}$, for f in $C_c^\infty(G)$ and so $g * \tilde{U} * \tilde{f} = k * \tilde{f}$ for some k in B and all f in $C_c(G)$. Hence, for h in $C_c^\infty(G)$, $\langle g * \tilde{U} * \tilde{f} , h \rangle = \langle k * \tilde{f} , h \rangle$, i.e. $\langle g * \tilde{U} , h * f \rangle = \langle k , h * f \rangle$, which shows that $g * U^{\check{}} = k$ in the sense of distributions. The converse inclusion is still simpler.

On the other hand π_U is the closure of the operator $f \dashrightarrow f * U^{\check{}}$ for f in $C_c^\infty(G)$.

Now our aim is to use the remarks above to study the following situation.

Let $\{\mu_t\}_{t>0}$ be a semi-group of probability measures on a Lie group G and let π be a representation of G on a Banach space B. Let A be the infinitesimal generator of $\{\mu_t\}_{t>0}$ on $C_\infty(G)$. By Hunt's theory, $C_c^\infty(G)$ is contained in $\underline{D}(A)$ and $Af = f * F^{\sim}$, where F is the dissipative distribution defined by $\langle f, F\rangle = Af(e)$, $f \in C_c^\infty(G)$. Also for a compact neighbourhood **V of** e we have $F = F_V + \mu_V$, where F has support in V and μ_V is a bounded measure, which is non-negative and vanishes on V.

Since F_V has compact support, we define π_{F_V} and $\underline{\pi}_{F_V}$. If $\langle \mu_V, \phi\rangle < \infty$, where ϕ is the submultiplicative function $\phi = \max\{\|\pi_x\|, \|\pi_{x^{-1}}\|, 1\}$, we write $\pi_F = \pi_{F_V} + \pi_{\mu_V}$ and $\underline{\pi}_F = \underline{\pi}_{F_V} + \pi_{\mu_V}$, where, of course, $\pi_{\mu_V} = \int \pi_x \, d\mu_V(x)$.

On the other hand, suppose that $\langle \mu_t, \phi\rangle < C$ for $t \in (0,1)$. The we write

$$\pi_{\mu_t} = \int \pi_x \, d\mu_t(x) .$$

<u>Proposition 3</u>. The following are equivalent:

(i) For a non-negative function $f \neq 0$ in $C_c^\infty(G)$ $\langle |Af|, \phi\rangle < \infty$.

(ii) $\langle \mu_V, \phi\rangle < \infty$.

(iii) $\langle \mu_t, \phi\rangle < C$ for $t \in (0,1)$.

Also (iii) implies

(iv) $\{\pi_{\mu_t}\}_{t>0}$ is a strongly continuous semi-group on B.

Moreover, the infinitesimal generator of $\{\pi_{\mu_t}\}_{t>0}$ π_A is equal to $\pi_F = \underline{\pi}_F$.

<u>Proof.</u> The plan of the proof is as follows. First we show that (i) and (ii) are equivalent and then that (iii) implies (iv). Proposition 1 shows that (i) implies (iii). Then we show that (iii) implies (ii) and finally from (iii) and (ii) we derive (iv).

We have $Af = f * F_V^{\sim} + f * \mu_V^{\sim}$. Since f and F_V have compact support there is a compact neighbourhood U of e such that $\operatorname{supp} f * F_V^{\sim} \cap \operatorname{supp} f * \mu_U^{\sim} = \emptyset$ whence $|\langle |Af|, \phi\rangle - (\langle f * F_V^{\sim}, \phi\rangle + \langle f * \mu_U^{\sim}, \phi\rangle)| \leq \langle f * (\mu_V^{\sim} - \mu_U^{\sim}) * f, \phi\rangle$ Consequently, $\langle |Af|, \phi\rangle$ is finite iff $\langle f * \mu_U^{\sim}, \phi\rangle = \langle \mu_U^{\sim}, f * \phi\rangle$ is finite. But, since ϕ is submultiplicative $c_1 f * \phi(x) \leq \phi(x) \leq c_2 f * \phi(x)$, whence $\langle \mu_U, \phi\rangle < \infty$ iff $\langle |Af|, \phi\rangle < \infty$, i.e. (i) and (ii) are equivalent.

Now assume (iii) is satisfied. To show (ii) we take a non-zero non-negative function f in $C_c^\infty(G)$ and we write

$$(2) \qquad f * \mu_t - f = \int_0^t Af * \mu_s \, ds = \int_0^t f * F_V^{\sim} * \mu_s \, ds + \int_0^t f * \mu_V^{\sim} * \mu_s \, ds .$$

Hence

$$\int_0^t \langle f * \mu_V^{\sim} * \mu_s, \phi\rangle \, ds \leq \langle |f * \mu_t - f|, \phi\rangle + \int_0^t \langle |f * F_V^{\sim}| * \mu_s, \phi\rangle \, ds$$

$$\leq \langle f, \phi\rangle + \langle \mu_t, \phi\rangle \langle f, \phi\rangle + t \langle |f * F_V^{\sim}|, \phi\rangle < \infty .$$

Thus for almost all $s < t$ $\langle f * \mu_V^{\sim} * \mu_s, \phi\rangle$ is finite and thus (ii) follows, since ϕ is submultiplicative and so

$$\langle f * \mu_V^{\sim} * \mu_s, \phi\rangle \geq \langle \mu_V, \phi\rangle \langle f, \phi^{-1}\rangle \langle \mu_s, \phi^{-1}\rangle .$$

Now we write (2) again for arbitrary f in $C_c^\infty(G)$ and by (ii) and (iii) we see that

$$(3) \qquad \langle |f * \mu_t - f|, \psi\rangle \leq Ct \quad \text{for } t < 1 .$$

Since, by (iii) $\|\pi_{\mu_t}\| \leq C$ for $t \leq 1$ and by (3) for ξ in B

$$\|\pi_{\mu_t} \pi_f \xi - \pi_f \xi\|_B \leq \langle |f * \mu_t - f|, \phi\rangle \|\xi\|_B \quad ;$$

we see that $\{\pi_{\mu_t}\}_{t>0}$ is a strongly continuous semi-group on B and also that the Garding space B_o is contained in the domain of the infinitesimal generator π_A of the semi-group $\{\pi_{\mu_t}\}_{t>0}$. Also $\pi_F \subset \pi_A$. We are going to show now that $\pi_A = \pi_F$. First we prove that for $\lambda > 0$ $(\lambda - \pi_F)B_o$ is dense in B. Suppose that for ξ' in B'

$$0 = \langle(\lambda - \pi_F)\pi_f\xi \ , \xi'\rangle = \int(\lambda f(x) - F\, f(x))\langle\pi_x\xi,\xi'\rangle\, dx$$

for all ξ in B and f in $C_c^\infty(G)$. Let $\psi_\xi(x) = \langle\pi_x\xi,\xi'\rangle$. We have $\psi_\xi(x) \leq C\ \phi(x)$. By proposition 2 $T_t: f \dashrightarrow \tilde{\mu}_t * f$ form a strongly continuous semi-group on L^1_ϕ whose generator on $C_c^\infty(G)$ (densee in L^1_ϕ) is $f \dashrightarrow F * f$. Hence $\langle(\lambda f - F*f), \psi_\xi\rangle = 0$ for all $f \in C_c^\infty(G)$ implies $\xi' = 0$.

Hence, since π_F is closed, $(\lambda - \pi_F)\underline{D}(\pi_F) = B$. But $\lambda - \pi_A$ is 1 - 1 on $\underline{D}(\pi_A)$ and $(\lambda - \pi_A)\underline{D}(\pi_A) = B$, whence $\underline{D}(\pi_F) \subset \underline{D}(\pi_A)$ implies $\underline{D}(\pi_F) = \underline{D}(\pi_A)$ i.e; $\pi_F = \pi_A$.

What remains to be shown is $\pi_A = \underline{\pi}_F$. Since $\pi_A = \pi_F \subset \underline{\pi}_F$, it is sufficient to prove that for $\lambda > 0$ $\lambda - \underline{\pi}_F$ is 1 - 1 on $\underline{D}(\underline{\pi}_F)$. Suppose for $\xi \in \underline{D}(\underline{\pi}_F)$ we have for all $\xi' \in B'$

(4) $\quad 0 = \langle(\lambda\pi_f - \pi_f\underline{\pi}_F)\xi \ , \xi'\rangle = \int(\lambda f(x) - f\, F(x))\langle\pi_x\xi \ ,\xi'\rangle\, dx.$

But $f \dashrightarrow f * F$ is the infinitesimal generator of the strongly continuous semi-group $f \dashrightarrow f * \tilde{\mu}_t$ on L^1_ϕ restricted to $C_c^\infty(G)$. Consequently (4) implies that $\langle\pi_x\xi \ , \xi'\rangle = 0$, for all ξ' in B', i.e. $\xi = 0$ and this completes the proof of proposition 3.