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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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INDUCED REPRESENTATIONS

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Lecture 1

Induced Representations.

The representation theory of compact Lie groups has been described in considerable detail before. The theory of roots ~~of~~ and weights has ~~a~~ enabled one to give a satisfactory classification of irreducible unitary representations of such groups. On the other hand (connected) abelian Lie groups being products of copies of $\mathbb{R}$ and the circle groups, one can describe all their irreducible unitary representations - which are nothing but characters - satisfactorily. However even the simplest non compact non-abelian connected Lie groups present difficulties which are not encountered in the compact or the abelian situation. One reason for this is no doubt the fact that for compact or abelian groups all irreducible unitary representations are finite dimension[al]. In fact even exhibiting irreducible unitary representation[s] of non-compact non abelian groups is often not so simple. In the present lecture we will do precisely this for some connected Lie groups with specially simple structures. The constructions we will make arise out of single technique viz. the method of induced representation, a technique that enables us to define a representation of a group making use of a representation of a (lower dimensional) subgroup

Before we go to the definition of induced representation we need some preliminary comments on homogeneous spaces of locally compact groups.

Let G be a locally compact group and $H \subset G$ a closed sub-group. Let $\chi : H \to \mathbb{C}^*$ be a continuous homomorphism of H in $\mathbb{C}^*$. Let $C_c(G, H; \chi)$ denote the vector space of all continuous functions $f : G \to \mathbb{C}$ such that

$$f(gh) = \chi(h) f(g)$$

for all $g \in G$ and $h \in H$ and whose support is contained in a subset of the form KH with K compact.

We denote by $C_c(G)$ the space of all continuous complex valued functions on G with compact support. For $f \in C_c(G)$, let $I_\chi f$ be the continuous function on G given by

$$I_\chi f(g) = \int_H f(gh)\chi(h^{-1})\, dh$$

where dh is a left-invariant Haar measure on H. It is easy to check that $I_\chi f \in C_c(G, H, \chi)$ and that I_χ is a linear map over $\mathbb{C}$. Somewhat less evident is

Lemma 1. I_χ is surjective

Proof. Let $u \in C_c(G)$ and $f \in C_c(G, H; \chi)$. Then $uf \in C_c(G)$ and we have $I_\chi(uf)(x) = \int_H u(xh) f(xh)\chi(h^{-1})\, dh = \int_H u(xh)\, dh\, f(x)$

ie $I_\chi(uf) = I_1(u).f$ where 1 denotes the trivial character on H.

To prove the lemma we have only to choose u such that $I_1(u) \equiv 1$ on support f for a given f. To see that such a choice of u is possible, fix a compact set $K \subset G$ such that $\operatorname{Supp} f \subset KH$. Let φ be a continuous function in $C_c(G)$ with $\varphi \equiv 1$ on K. Then $\psi = I_1(\varphi)$ is a right H-invariant function with support compact modulo H and such that $\psi(x) > 0$ for all $x \in KH$. Now $\psi \in C_c(G, H; 1)$ ie ψ is a continuous function on G/H with compact support. Let ψ' be a continuous function on G/H with compact support such that $\psi'\psi \equiv 1$ on KH/H. Clearly $\psi' \in C_c(G, H; 1)$ and we have $I_1(\psi'\varphi) = \psi' I_1(\varphi)$ is identically 1 on KH. We may then take for u the function $\psi'\varphi$.

Lemma 2. Let Δ_G (resp Δ_H) denote the modular character on G (resp H). Then for $f \in C_c(G)$ if $I_{\Delta_G^{-1}\Delta_H} f = 0$, then

$$\int_G f(g)\, dg = 0$$

Proof. Let $\varphi \in C_c(G)$. Then we have

$$0 = \int_G \varphi(g)\, dg \int_H f(gh)\, \Delta_G(h)\, \Delta_H(h)^{-1}\, dh$$

$$= \int_H dh\, \Delta_G(h)\, \Delta_H(h^{-1}) \int_G f(gh)\, \varphi(g)\, dg \quad \text{(Fubini's theorem)}$$

$$= \int_H dh\, \Delta_G(h)\, \Delta_H(h^{-1}) \int_G f(g)\, \varphi(gh^{-1})\, \Delta_G(h^{-1})\, dg \quad \text{(Definition of } \Delta_G)$$

$$= \int_G f(g)\, dg \int_H \varphi(gh^{-1})\, \Delta_H(h^{-1})\, dh$$

$$= \int_G f(g)\, dg\, c \int_H \varphi(gh)\, dh \quad \text{(for some } c>0) .$$

Choose now φ such that $\int_H \varphi(gh)\, dh$ is identically 1 for all $g \in$ Support f. The lemma follows.

Corollary The linear form μ defined by $\mu(f) = \int_G f(g)\, dg$ on $C_c(G)$ factors through $I_{\Delta_G^{-1}\Delta_H}$ to define a positive linear functional again denoted μ of $C_c(G, H; \Delta_G^{-1}\Delta_H)$ in $\mathbb{C}$.

Now let $\rho : G \longrightarrow \mathbb{R}^+$ be the (unique) ~~character~~ continuous homomorphism of G in $\mathbb{R}^+$ defined by

$$\Delta_G^{-1}\Delta_H(x) = \rho(x)^2$$

for all $x \in G$. Suppose now that π is a unitary representation of H on a Hilbert space $\mathbb{H}$. We define then a new Hilbert space $I(\mathbb{H})$ as follows. ~~An element of $I(\mathbb{H})$ is a measurable function $f : G \longrightarrow \mathbb{H}$ such that~~

~~(i) $f(gh) = \rho(h)\pi(h)^{-1} f(g)$ and~~

Let $C_c(G, H; \rho, \pi)$ denote the space of all continuous $\mathbb{H}$-valued functions on G satisfying the following condition

$$f(gh) = \rho(h)\, \pi(h)^{-1} f(g)$$

for $g \in G$ and $h \in H$; then the function $g \longmapsto \langle f(g), f'(g) \rangle_{\mathbb{H}}$ $f, f' \in C_c(G, H, \rho, \pi)$ which we denote $\langle f, f' \rangle_{\mathbb{H}}$ in the sequel, belongs to $C_c(G, H; \Delta_G^{-1}\Delta_H)$. It follows that

$$(f, f') \longmapsto \mu(\langle f, f' \rangle_{\mathbb{H}})$$

defines a Pre-Hilbert structure on $C_c(G, H; \rho, \pi)$. The linear functional μ on $C_c(G, H; \Delta_G^{-1}\Delta_H)$ extends to a wider class functions just as the continuous linear functional on the space of continuous functions on a compact space defines a Borel measure. and we can characterise the Hilbert space completion $I(\mathbb{H})$ of $C_c(G, H; \rho, \pi)$ above as the space ~~of square summable~~ measurable functions ~~in G~~ for G with values in $\mathbb{H}$ satisfying $f(gh) = \rho(h)\pi(h)^{-1} f(g)$ and which a "square summable" for μ. On $I(\mathbb{H})$ we can define a unitary representation I_π of G as follows:

For $f \in C_c(G, H; \rho, \pi)$ and $x \in G$,

$$(I_\pi(x) f)(g) = f(x^{-1}g).$$

That I_π is a unitary representation follows from the left G-invariance of μ — which is immediate from the left invariance of the Haar measure. I_π is called the induced representation of the representation π of H.

In the rest of the talk we will give a number of examples of this construction which yield interesting representation of non compact non-abelian groups. But we begin with the simple situation of finite groups.

(1) Let G be a finite group and $H \subset G$ a subgroup. Evidently Δ_G and Δ_H are trivial. Let $\mathbb{C}[G]$ (resp $\mathbb{C}[H]$) denote the group algebra of G (resp H). Then H has a natural structure of a $\mathbb{C}[H]$-module. On the other hand $\mathbb{C}[G]$ is $\mathbb{C}[H]$-bimodule. The right $\mathbb{C}[H]$-module structure on $\mathbb{C}[G]$ can be made into a left-module structure via the involution $h \mapsto h^{-1}$, $h \in H$. Then $\mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], H)$ is a vector space on which $\mathbb{C}[G]$ acts on the left through the standard left action of $\mathbb{C}[G]$ on itself. It is easy to see that this is precisely the induced ~~rep~~ representation of π. (The decomposition of an induced representation into irreducible components is described by the Frobenius Reciprocity theorem; this last theorem holds more generally for compact groups)

(2) The group $B = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{R}^*, b \in \mathbb{R} \right\}$ This solvable group is the semi direct product of its diagonal subgroup D (which is isomorphic to $\mathbb{R}^*$) with the normal subgroup $N = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \middle| u \in \mathbb{R} \right\}$ which is isomorphic to $\mathbb{R}$. It is not difficult to see that $\Delta_B \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = |a|$, $a \in \mathbb{R}^*$, $b \in \mathbb{R}$. The modular character Δ_D is of course trivial. Let χ be a character on D. Then the representation space for I_χ consists of (measurable) functions $f : B \to \mathbb{C}$ satisfying the following two conditions

(i) $f\left(x \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = |t| \cdot \chi(t)^{-1} f(x)$

(ii) $f_0 = f|_N$ belongs to $L^2(N)$.

Claim I_χ is irreducible

Proof. The representation space for I_χ has been identified with $L^2(N) \cong L^2(\mathbb{R})$ above. Let $\mathcal{F}$ denote the Fourier Transform and $I'_\chi = \mathcal{F} \cdot I_\chi \cdot \mathcal{F}^{-1}$. We know that any unitary representation ρ of G has a natural extension (again denoted ρ) to the algebra $L^1(G)$ (under convolution). Using the fact $\mathcal{F}(\varphi * \psi) = \mathcal{F}(\varphi) \cdot \mathcal{F}(\psi)$ one sees that for $\varphi \in L^1(\mathbb{R})$, $I'_\chi(\varphi)(f) = \hat{\varphi} \cdot f$. Thus if an operator T on the representation space $L^2(\mathbb{R})$ of I'_χ commutes with all of $I'_\chi(N)$, it commutes with all operators on $L^2(\mathbb{R})$ of the form $f \mapsto \hat{\varphi} f$, $\varphi \in L^1(\mathbb{R})$, $f \in L^2(\mathbb{R})$. Such an operator T we assert is itself of the form $Tf = uf$ with $u \in L^\infty(\mathbb{R})$. Let $\varphi \in C_c(\mathbb{R})$ be such that $\varphi = \psi * \psi'$; $\psi, \psi' \in C_c(\mathbb{R})$; then $\hat{\varphi} = \hat{\psi}\,\hat{\psi}' \in L^1$ so that T would commute with the operator M_φ

$$M_\varphi f = \varphi f$$

Thus T commutes with $M_{\psi * \psi'}$ for ψ, ψ' with compact support. We can now choose a sequence ψ_n' in $C_c(\mathbb{R})$ such that $\psi * \psi_n'$ converges uniformly to ψ. It is then easily seen that the operator $M_{\psi * \psi_n'}$ converges to M_ψ in the strong sense so that M_ψ commutes with T for all $\psi \in C_c(\mathbb{R})$. Suppose now that $K \subset \mathbb{R}$ is any compact set and u_n, $0 \le u_n \le 1$ a sequence of functions in $C_c(\mathbb{R})$ converging to the characteristic function 1_K of K and with $\operatorname{Supp} u_n \subset K'$, a compact set (independent of n), then $u_n f$ converges $1_K f$ for all $f \in L^2(\mathbb{R})$. We have thus

$$T(1_K f) = \lim_{n\to\infty} T(u_n f) = \lim_{n\to\infty} u_n T(f) = 1_K T(f)$$

Let $\psi_K = T(1_K)$. Then for compact sets K, L with $L \supset K$

we have $\psi_K = T(1_K) = T(1_K \cdot 1_L) = 1_K \psi_L$ so that $\psi_K = \psi_L$ almost everywhere on K.

Suppose next that $T: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded operator which commutes with all of $I'_\chi(B)$. We know from the arguments given above that T is necessarily of the form M_ψ with $\psi \in L^\infty(\mathbb{R})$. (this makes use of the fact that T commutes with $I'_\chi(N)$). Next if we use the fact that $M_\psi (= T)$ commutes $I'_\chi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ we find that we have for $f \in L^2(\mathbb{R})$, $x \in \mathbb{R}$ and $t \in \mathbb{R}^*$

$$\psi(x)\,\chi(t)\,|t|\,f(t x) = \psi(tx)\,\chi(t)\,f(t x)$$

leading us to the conclusion

$$\psi(tx) = \psi(x)$$

for almost all $x \in \mathbb{R}$ and $t \in \mathbb{R}^*$. Evidently this means that ψ is constant almost everywhere i.e. T is a scalar. Thus I'_χ (and hence I_χ) is irreducible.

We record for future the following result essentially proved above

<u>Lemma</u> Let G be a locally compact group and $\hat{G}$ its dual. For $\varphi \in L^\infty(G)$ let $M_\varphi: L^2(G) \to L^2(G)$ be the operator $M_\varphi(f) = \varphi \cdot f$, $f \in L^2$. Then if an operator $T: L^2(G) \to L^2(G)$ commutes with $M_{\hat{u}}$ for all $u \in L^1(\hat{G})$, $T = M_\varphi$ for some $\varphi \in L^\infty(G)$

The induced representation construction thus yields irreducible (necessarily infinite dimensional) representations of the solvable group B. ~~At a later point we will show that all irreducible representations of B which are non-trivial on N are necessarily of this form~~. It must be noted that in general it is far from clear that a locally compact group need have any non-trivial irreducible unitary representation at all. That there are indeed lots of non-trivial irreducible representation will be a consequence of one of the beautiful results we will establish later. For now we will give some further examples.

3) The Heisenberg Group. Let G be a locally compact abelian group. In $L^2(G)$, G operates by translations for $f \in L^2$, $g \in G$, $Lgf \in L^2$ is defined by

$$Lgf(\chi) = f(g^{-1}x), \; \chi \in G.$$

On the other hand $\hat{G}$ also operates on $L^2(G)$: For $\chi \in \hat{G}$ and $f \in L^2(G)$, $M_\chi f \in L^2$ is defined

$$M_\chi f(g) = \chi(g)\, f(g), \; g \in G.$$

Consider now the subgroups $L(G)$ and $M(\hat{G})$ in the group U of unitary operators on $L^2(G)$. Let $g \in G$ and $\chi \in \hat{G}$. Then for $f \in L^2(G)$

$$\begin{aligned}(Lg\, M_\chi)(f)(x) &= \{Lg(M_\chi f)\}\,(x)\\ &= (M_\chi f)(g^{-1}x)\\ &= \chi(g^{-1})\,\chi(x)\, f(g^{-1}x)\end{aligned}$$

while
$$\begin{aligned}(M_\chi Lg)(f)(x) &= (M_\chi(Lgf))(x)\\ &= \chi(x)\, Lgf(x)\\ &= \chi(x)\, f(g^{-1}x)\end{aligned}$$

so that

$$Lg\, M_\chi(f) = \chi(g)^{-1}\, M_\chi Lgf$$

or equivalently

$$(M_\chi^{-1}\, Lg^{-1}\, M_\chi\, Lg)f = \chi(g)f$$

Let π denote the scalar unitary operators. The above equation shows that

$$L(G)\cdot M(\hat{G})\cdot \mathbb{T}$$

is a group. Clearly the map

$$G \times \hat{G} \times \mathbb{T} \longrightarrow U$$

given by $(g,\chi,t) \longrightarrow Lg\, M_\chi(t\cdot 1)$ is a continuous bijection onto a subgroup of U equipped with the strong topology. The group structure on $G \times \hat{G} \times \mathbb{T}$ is given by the composition map

$$\begin{aligned}&(g,\chi,t)\cdot(g',\chi',t')\\ &\quad = (gg', \chi\chi', tt'\chi(g')).\end{aligned}$$

The group $G \times \hat{G} \times \mathbb{T}$ is called the Heisenberg group associated to G and we will denote it by $H(G)$. We have defined above a natural representation σ of the Heisenberg group $H(G)$.

<u>Claim</u>. This representation is irreducible,

Consider the isometric isomorphism of $L^2(G)$ on $L^2(\hat{G})$ given by the Fourier transform under this isomorphism the convolution operator $f \longrightarrow \varphi * f$, $\varphi \in L^1(G)$ on $L^2(G)$ go over to the operator $M_{\hat{\varphi}}$ given by

$$f \longrightarrow \hat{\varphi} f, \; f \in L^2(\hat{G}) \; .$$

Let T be a bounded operator on $L^2(\hat{G})$ commuting with all the operators $\hat{\sigma}(H(G))$ where $\hat{\sigma}$ is the representation on $L^2(\hat{G})$ obtained from σ on $L^2(G)$ through the Fourier transform isometry. Then T commutes with all the $M_{\hat{\varphi}}$, $\varphi \in L^1(G)$. As was seen above, this means that (T commutes with M_u for all $u \in C_c(\hat{G})$ and consequently) $T = M_\psi$, $\psi \in L^\infty(\hat{G})$ where $M_\psi(f) = \psi \cdot f$, $f \in L^2(\hat{G})$. Now in the realisation $\hat{\sigma}$, the group $\hat{G}$ acts on $L^2(\hat{G})$ by translations. It follows that M_ψ commutes with all translations on $\hat{G}$. But this means that ψ is invariant under translations in $\hat{G}$ i.e. ψ is a constant almost every where. Thus T is necessarily a scalar so that $\hat{\sigma}$ (and hence σ as well) is an irreducible representation of $H(G)$.

The representation σ constructed above is in fact an example of an induced representation. To see this consider the <u>abelian</u> subgroup $1 \times \hat{G} \times \mathbb{T}$ of $\mathbb{H}(G)$. Let $\pi : \hat{G} \times \mathbb{T} \longrightarrow \mathbb{T}$ be the character $(\chi, z) \longrightarrow z$. Consider the representation I_π. The representation space of I_π consists of functions

$$f : G \times \hat{G} \times \mathbb{T} \longrightarrow \mathbb{C}$$

such that

$$f((g,\chi,t)(1,\chi',t')) = f(g,\chi,t) \cdot t'$$

In other words $f(g,\chi,t)$ is determined by $f(g,1,1)$. One sees that as a function on G, $f \in L^2(G)$. Thus the representation I_π is on $L^2(G)$. It is also obvious that the subgroup G ($= G \times 1 \times 1$) acts via left translations. The action of $1 \times \hat{G} \times 1$ on the left is given as follows :

$$(1,\chi,1)(g',1,1) = (g',\chi\chi',\chi(g')t)$$

so that

$$\begin{aligned} f((1,\chi^{-1},1)(g',1,1)) &= f(g',\chi^{-1},\chi^{-1}(g')) \\ &= f((g',1,1)(1,\chi^{-1},\chi^{-1}(g')) \\ &= \chi(g') \, f(g',1,1) \end{aligned}$$

Thus proves our contention that the representation on $L^2(G)$ is equivalent to I_π. We will later see that all infinite dimensional irreducible representations of $\mathbb{H}(G)$ are obtained by analogous constructions.

4) <u>The group SL(2, $\mathbb{R}$)</u>. Any matrix $g \in G = SL(2,\mathbb{R})$ can be written as a product

$$g = k.a.n$$

where $k \in SO(2)$ (also denoted K), $a \in A$ the group of diagonal matrices $\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} | \lambda \in \mathbb{R}^+\}$ and $n \in N = \{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} | u \in \mathbb{R}\}$. Let H denote the "Borel" subgroup

$$\{\begin{pmatrix}a & u\\ 0 & a^{-1}\end{pmatrix} | a \in R^*, \; u \in \mathbb{R}\}.$$

The modular character on H is given by $\begin{pmatrix}a & u\\ 0 & a^{-1}\end{pmatrix} \longrightarrow a^2$ suppose now that $g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ then a simple calculation shows the following :

$$g = k.a.n$$

where $k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ $a = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ with $re^{i\zeta} = \alpha - i\gamma$. Consider now the 1-dimensional unitary representation π of H on $\mathbb{C}$ given by

$$\pi\begin{pmatrix} a & \mu \\ 0 & a^{-1} \end{pmatrix} = (\mathrm{Sgn}\ a)^m \ \exp it \log |a|$$

on $\mathbb{C}$ with $t \in \mathbb{R}$. Since the map

$$(k,a,n) \longrightarrow k.a.n$$

of $K \times A \times N \longrightarrow G$ is a diffeomorphism. The modular character on H is given by the formula $\Delta_H\begin{pmatrix} a & \mu \\ 0 & a^{-1} \end{pmatrix} = |a|^2$. It follows that the representation space for I_π consists of measurable functions f on G satisfying

$$f(k\ a\ n) = e^{(it-1)\log\lambda}\ f_0(k) \qquad (*)$$

where $f_0 \in L^2(K)$ and $f_0(-k) = (-1)^m f_0(k)$ for all $k \in SO(2)$, $k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ $a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $n = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$. We may thus identify the representation space with a subspace of $L^2(K)$. It is also clear that under this identification K acts on this subspace of $L^2(K)$ by left translations. We will now take a look at the action of $\tilde{\mu} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ on this subspace of $L^2(K)$. For this we have for $f_0 \in L^2(K)$, taking f to be the corresponding function on G (as in $*$ above), we have

$$(L_{\tilde{u}}f)_0(k) = f_0(k_u)\ \exp(it-1)\ \log\lambda_u$$

where $z_u = \cos\theta - \sin\theta + i\sin\theta = \lambda_u e^{i\theta_u}$ and $k_u = \begin{pmatrix} \cos\theta_u & \sin\theta_u \\ -\sin\theta_u & \cos\theta_u \end{pmatrix}$. Suppose now that $V \subset L^2(K)$ is a G-invariant subspace of our representation space; then V decomposes into a direct sum of eigen spaces under the compact group so that V contains a nonzero eigen vector for K. We conclude that the

function e_n given by

$$e_n\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \exp in\theta$$

belongs to V for some $n \in Z$ (n is odd or even depending on the parity of m). Taking f to be such that $f_0 = e_n$ we see that

$$(L_{\tilde{u}}f)_0 = (z_u/|z_u|)^n \exp\ (it-1)\ \log|z_u|$$
$$= [z_u^n/|z_u|^n]\ |z_u|^{it-1}$$
$$= z_u^n\ |z_u|^{it-1-n}$$

(with $z_u = \cos\theta - u\sin\theta + i\sin\theta$) belongs to V. It is now easy to see that $\frac{d}{du}(L_uf)_0|_{u=0} \in V$ — $\{(L_{u+h}f)_0 - (L_uf)_0\}/h$ tends to $\frac{d}{du}(L_uf)_0$ in $L^2(K)$.

$$\frac{d}{du}(L_{\tilde{u}}f)_0(k) = \frac{d}{du}\ z_u^n\ (|z_u|^2)^{(it-1-n)/2}$$
$$= nz_u^{n-1}(-\sin\theta)\ |z_u|^{(it-1-n)}$$
$$+ z_u^n(it-1-n)/2|z_u|^{it-1-n-2}\ 2(\cos\theta - u\sin\theta)(-\sin\theta)$$

so that

$$\frac{d}{du}(L_uf)_0(k)|_{u=0} = -n\ e^{i(n-1)\theta}\ \sin\theta - e^{in\theta}(it-1-n)/2\cdot\sin 2\theta$$
$$= i/2\{(it-1-n)/2\ e^{i(n+2)\theta} + ne^{in\theta} - (it-1+n)/2\ e^{i(n-2)\theta}\}$$

Now if $t \neq 0$ one concludes (using the fact that V is K-stable) that e_{n+2} and $e_{n-2} \in V$; and a simple induction argument shows then that $e_r \in V$ for all r with the same parity as n. When $t = 0$ and m is odd it is not difficult to see that the closure of the subspaces

$$\coprod_{\substack{n \equiv m (\mathrm{mod}\ 2) \\ n > 0}} \mathbb{C}\, e^{in\theta}\ ,\quad \coprod_{\substack{n \equiv m (\mathrm{mod}\ 2) \\ n}} \mathbb{C}\, e^{in\theta}$$

are G-stable. When $t = 0$ and m is even once again I_π is irreducible. The family of representations constructed above is called the principal series of representations of $SL(2,\mathbb{R})$.