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COLLEGE ON

REPRESENTATION THEORY OF LIE GROUPS

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A SURVEY OF SOME APPLICATIONS OF REPRESENTATIONS OF COMPACT LIE GROUPS IN GEOMETRY

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These are preliminary lecture notes, intended only for distribution to participants.

INTRODUCTION

The aim of these lectures is to give a rapid introduction to several situations in geometry and topology where the theory of roots and weights is of practical value. The first concerns characteristic classes of homogeneous vector bundles, which dates from work of A. Borel and F. Hirzebruch in the 1950's, and leads to a connection between the Weyl Character Formula and the Atiyah-Singer Index Theorem first described by R. Bott. The second concerns minimal immersions of homogeneous spaces, a subject that has become topical again in the last few years in the form of harmonic map theory. The third is the study of the Yang-Mills equations, which are currently of interest to mathematical physicists.

The first two lectures sketch the relevant theory. This is well documented in textbooks, so we shall emphasize only certain practical aspects, giving short shrift to other results however fundamental they may be. For example, as we shall always deal with compact Lie groups, we assume without loss of generality that finite dimensional real (or complex) representations are orthogonal (or unitary), and we omit any mention of the Haar measure and its properties. The Peter-Weyl Theorem will appear in connection with induced representations and the Laplacian, but we omit any discussion of important consequences such as the finite dimensionality of irreducible representations and the existence of faithful representations. Thus we do not claim to give a balanced summary of the theory of compact Lie groups and their representations, but rather will concentrate on the

definition and properties of roots and weights (in I) and homogeneous spaces (in II). We assume the reader has some basic acquaintance with the idea of a Lie group, its Lie algebra, and the exponential map, and with the (compact) matrix groups O_n, U_n and Sp_n (consisting of $n \times n$ real orthogonal, unitary, and "quaternionic unitary" matrices respectively). Certain obvious conventions, such as all manifolds and maps being smooth, and all subgroups of Lie groups being closed, will be maintained throughout without comment.

The word "homogeneous" figures prominently in the list of section headings for each lecture. This indicates what is really the central theme of III, IV, and V, i.e. the consideration of a general theory in the special case where objects are homogeneous with respect to the action of some compact Lie group. There are two benefits of doing this, the most obvious one being that the homogeneous examples are particularly susceptible to explicit computation and thus contribute to one's understanding of the general theory. The other is that, in practice, homogeneous objects often turn out to be more fundamental than a priori might seem likely. An example of this which we do not discuss is the description of (essentially) all vector bundles over a homogeneous space in terms of homogeneous bundles; another example (in differential geometry) is given in IV.

CONTENTS

I. Representations of Compact Lie Groups

(1.1) Real Representations. Let G be a compact connected Lie group, and let $\mathfrak{g}$ be its Lie algebra, which we identify with the tangent space T_eG at the identity of G or with the algebra of left-invariant vector fields on G. We take on $\mathfrak{g}$ the inner product $\langle\ ,\ \rangle$ given by minus the Killing form. The adjoint representation of G, denoted $Ad(G)$ or just Ad, is the homomorphism

$$Ad : G \longrightarrow SO(\mathfrak{g})$$

defined by $Ad(g)Y = (d/dt)(\exp tX)Y(\exp tX)^{-1}|_{t=0}$, for $g = \exp X \in G$, $X,Y \in \mathfrak{g}$. The adjoint representation of $\mathfrak{g}$, denoted $ad(\mathfrak{g})$ or just ad, is the homomorphism

$$ad : \mathfrak{g} \longrightarrow Sk(\mathfrak{g})$$

defined by $ad = d(Ad)_e$. ($Sk(\mathfrak{g})$ denotes the skew symmetric endomorphisms of $\mathfrak{g}$; $Sk(\mathfrak{g})$ is the Lie algebra of $SO(\mathfrak{g})$.)

More generally, a (real, finite dimensional) representation of G on V is by definition a homomorphism

$$\theta : G \longrightarrow SO(V)$$

where V is a (real, finite dimensional) inner product space. Thus, V may be regarded as a G-module, by defining $g \cdot X = \theta(g)X$ for $g \in G$, $X \in V$. We refer to V as the representation space of θ. There is an associated

representation of $\mathfrak{g}$ defined by $\theta = d\theta_e$, and the diagram below commutes, where the vertical right hand map is $X \longmapsto \sum_{n=0}^{\infty} X^n/n!$.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\theta} & Sk(V) \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\theta} & SO(V) \end{array}$$

The representation θ (or θ) is said to be reducible if it factors through some $SO(W)$ for a subspace W of V, in which case it factors through $S(O(W) \times O(W^{\perp}))$ and the G-module V is the direct sum of the G-modules W and $W^{\perp}$. Otherwise, the representation is said to be irreducible. For example, Ad is irreducible if and only if G is simple.

(1.2) Complex Representations. One may consider complex or quaternionic representations by replacing $SO(V)$ by $U(V)$ (when V is complex with a Hermitian inner product) or by $Sp(V)$ (similarly). If V is an F-vector space for some field F, and if F', F'' are fields with $F' \subseteq F \subseteq F''$, one may form the F'-vector space $V^{F'}$ (by regarding V as an F'-vector space) and the F''-vector space $V \otimes F''$. Using the fields $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H}$ gives various possibilities (see Chapter III of [Ad]). However, the significant fact is that the maps

$$V \longrightarrow V \otimes \mathbb{C} \quad (V \text{ real})$$
$$V \longrightarrow V^{\mathbb{C}} \quad (V \text{ quaternionic})$$

are injective (3.28 of [Ad]), so that one only need consider complex representations. (We shall use the notation $\theta \otimes \mathbb{C}$, $\theta^{\mathbb{C}}$ for representations in what follows.) Moreover, it is often useful to convert to complex vector spaces in order to use the special properties of $\mathbb{C}$. For example, if V and W are real, $T \in \mathrm{Hom}(V,W)$ defines a complex linear transformation $T \in \mathrm{Hom}(V \otimes \mathbb{C}, W \otimes \mathbb{C})$, by "extending T by complex linearity". More generally, any real tensor may be extended to a corresponding complex tensor; we shall do this without comment and without introducing new notation for the extended tensor. A familiar example in differential geometry is the embedding of the space of real k-forms $\Lambda^k V$ in the space of complex forms $\Lambda^k(V \otimes \mathbb{C})$; an example of particular importance in these lectures is the extension of a Lie algebra representation θ of $\mathfrak{g}$ to one (also called θ) of $\mathfrak{g} \otimes \mathbb{C}$. As further evidence of the value of complexifying, we mention the adjoint representations of the groups SO_n, U_n, Sp_n. These have natural representations on $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$ respectively, which we shall denote by Λ. Then $Ad(SO_n) \cong \Lambda^2\Lambda$, but $Ad(U_n)$ and $Ad(Sp_n)$ are best described in terms of the corresponding complex modules. Indeed, $Ad(U_n) \otimes \mathbb{C} \cong \bar{\Lambda} \otimes \Lambda$ and $Ad(Sp_n) \otimes \mathbb{C} \cong S^2\Lambda^{\mathbb{C}}$. These descriptions follow from the usual identifications of the Lie algebras of SO_n, U_n and Sp_n with (respectively) the skew symmetric, skew Hermitian and skew quaternionic-Hermitian matrices.

The <u>character</u> of a representation $\theta : G \longrightarrow U(V)$ is the function $\chi_\theta : G \longrightarrow \mathbb{C}$ defined by $\chi_\theta(g) = \mathrm{tr}\, \theta(g)$. It is a well known fact (see [Ad], Chapter III) which we quote without proof that $\chi_\theta = \chi_\phi$ if and only if $\theta \cong \phi$.

(1.3) <u>Tori</u>. A <u>torus</u> in G is a subgroup T which is isomorphic to a product $S^1 \times \dots \times S^1$ of circle groups $(S^1 = \{z \in \mathbb{C} \mid |z| = 1\})$; T is a <u>maximal torus</u> if it is not properly contained in any other torus in G. It is

elementary that any torus is contained in a maximal torus, that a maximal torus is precisely a maximal connected abelian subgroup of G, and that any element of G is contained in some maximal torus. However, the following result is the fundamental reason for the importance of maximal tori (see [Ad], Chapter IV, for a topological proof, or [Bol] for a Morse theoretic proof).

<u>Theorem</u>: Let T be a fixed maximal torus of G. Then $G = \bigcup_{g \in G} gTg^{-1}$, and if T' is any maximal torus of G, there exists some $g \in G$ with $T' = gTg^{-1}$. □

As a corollary, one sees that the dimension of a maximal torus is independent of the choice of torus; this number is called the <u>rank</u> of G. A more significant fact is that any character χ_θ, being a class function ($\chi_\theta(ghg^{-1}) = \chi_\theta(h)$ for all $g,h \in G$), is determined by its restriction to a fixed maximal torus. Since χ_θ determines θ, this means that the representation theory of a Lie group may be analysed completely in terms of a maximal torus. This is the basis for the theory of roots and weights, which we shall come to in section 1.4. First, we need some simple results on the real and complex representations of a torus T. The circle group S^1 as defined above is naturally isomorphic with U_1. We shall also identify it with SO_2, by assigning to $z = e^{\sqrt{-1}\,t}$ the matrix

$$M(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

We identify the Lie algebra of S^1 with $\mathbb{R}$ so that the exponential map is $t \longmapsto e^{\sqrt{-1}\,t}$.

Proposition: Any representation $\theta : T \longrightarrow U_n$ is equivalent to a sum $\alpha_1 \oplus \dots \oplus \alpha_n$ of one dimensional representations. If $T \cong S^1 \times \dots \times S^1$ (k factors), we may write $\alpha_i(e^{\sqrt{-1}\,t_1}, \dots, e^{\sqrt{-1}\,t_k}) = e^{\sqrt{-1}\,\alpha_i(t_1,\dots,t_k)}$ where $\alpha_i \in t^*$ is the Lie algebra representation associated to α_i. □

This is elementary since the unitary matrices $\theta(z)$ may be simultaneously diagonalized for all $z \in T$, as is the following consequence.

Corollary: Any representation $\theta : T \longrightarrow SO_n$ is equivalent to a sum $\alpha_1 \oplus \dots \oplus \alpha_m \oplus (n - 2m)$ where $m \leq 2n$, $(n - 2m)$ denotes a trivial representation of dimension $n - 2m$, and $\alpha_i : T \longrightarrow SO_2$ is a two dimensional representation. We may write

$$\alpha_i(e^{\sqrt{-1}\,t_1}, \dots, e^{\sqrt{-1}\,t_k}) = M(\alpha_i(t_1,\dots,t_k))$$

where α_i is the Lie algebra representation associated to α_i. □

Finally we remark that as maximal tori for the groups U_n and Sp_n one may take the subgroups consisting of complex diagonal matrices, so these groups have rank n. For the group SO_{2n} one may take the subgroup consisting of matrices of the form

$$\begin{pmatrix} M(x_1) & & \\ & \ddots & \\ & & M(x_n) \end{pmatrix},$$

while for SO_{2n+1} one may take matrices of the form

$$\begin{pmatrix} M(x_1) & & & \\ & \ddots & & \\ & & M(x_n) & \\ & & & 1 \end{pmatrix}.$$

Hence SO_{2n} and SO_{2n+1} both have rank n. For further details we refer to [Ad].

(1.4.) Roots and Weights for Lie Groups. Let T be a fixed maximal torus of G. If $\theta : G \longrightarrow U_n$ is a representation of G on $\mathbb{C}^n$, $\theta \cong \Gamma_1 \oplus \dots \oplus \Gamma_n$ (by the proposition), where Γ_i is one dimensional with associated Lie algebra representation $\gamma_i \in t^*$.

Definition: The weights of θ (with respect to T) are the linear forms $\gamma_1/2\pi, \dots, \gamma_n/2\pi$ (repetitions are possible; the number of times $\lambda/2\pi$ is repeated is called the multiplicity of $\lambda/2\pi$).

(The factor 2π is introduced so as to force the weights to take integer values on the integer lattice $\exp^{-1}(e)$.) Without loss of generality, we may therefore assume

$$\theta(z) = \begin{pmatrix} \Gamma_1(z) & & \\ & \ddots & \\ & & \Gamma_n(z) \end{pmatrix}, \quad z \in T.$$

The weights obviously determine the character of θ, and hence θ itself (up to equivalence).

Definition: The roots of G (with respect to T) are the non-zero weights of $Ad \otimes \mathbb{C}$.

Let the roots be $\Delta = \{\pm\alpha_1, \ldots, \pm\alpha_\ell\}$. As T is a maximal connected abelian subgroup, $\mathfrak{t}$ is a maximal trivial submodule of $\mathfrak{g}$ for the representation Ad, so we may write

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{i=1}^{\ell} V_i ,$$

where the matrix of $Ad(\exp(x))$ on the subspace V_i is $M(2\pi\alpha_i(x))$ for any $x \in \mathfrak{t}$. It turns out that $\alpha_i \neq \alpha_j$ for $i \neq j$ (see [Ad]).

We now state the fundamental classification theorems for compact Lie groups and their complex representations, which form the main goal of this lecture.

Theorem: The "root system" Δ of a compact connected simple Lie group determines the group up to local isomorphism. □

Moreover, "root systems" may be characterized in a purely combinatorial fashion, from which may be obtained the well known list of groups of this type: SU_n $(n > 1)$, SO_{2n} $(n > 3)$, Sp_n $(n > 2)$, SO_{2n+1} $(n > 3)$, G_2, F_4, E_6, E_7, E_8. This is proved in [He].

To explain the classification of representations, some further notation is needed; for results quoted without proof we refer to [Ad]. The Weyl group of G (with respect to T) is $W = \{Ad(g) \in SO(\mathfrak{g}) \mid Ad(g)T \subseteq T\}/T$. This is a finite group which acts naturally on $\mathfrak{t}$ and hence on $\mathfrak{t}^*$. This action will be denoted $\lambda \longmapsto w\cdot\lambda$. It follows easily from the definition of weights that W permutes the weights of any representation, and in particular it permutes the roots of G. The hyperplanes $\operatorname{Ker} \alpha_i \subseteq \mathfrak{t}$ divide up $\mathfrak{t}$ into convex subsets called the Weyl chambers, which are themselves permuted under the action of W, each of which is determined by a set of inequalities of the form $\epsilon_1\alpha_1 > 0, \ldots, \epsilon_k\alpha_k > 0$ (ϵ_i is 1 or -1). (Not every such set of inequalities defines a Weyl chamber, in general, as the forms $\alpha_1, \ldots, \alpha_k$ need not be independent.) Let us choose (arbitrarily) $\epsilon_1\alpha_1, \ldots, \epsilon_k\alpha_k$ of the above form, and call the corresponding chamber D the fundamental Weyl chamber. We refer to $\Delta^+ = \{\epsilon_1\alpha_1, \ldots, \epsilon_k\alpha_k\}$ as the positive roots for D. Using the identification $\mathfrak{t} \cong \mathfrak{t}^*$ provided by the metric $\langle\, ,\, \rangle$ on $\mathfrak{g}$ restricted to $\mathfrak{t}$, we associate to D the so called fundamental dual Weyl chamber $D^* \subseteq \mathfrak{t}^*$.

Now let G be simply connected. (There is no loss of generality here as the universal covering space of a compact Lie group is also a compact Lie group, and the representations of the group may be considered as a subring of the representations of the universal cover.)

Theorem (Weyl Character Formula): There is a one to one correspondence between the (equivalence classes of) irreducible complex representations of a simply connected compact connected Lie group G, and those linear forms $\lambda \in D^*$ $(\subseteq \mathfrak{t}^*)$ which satisfy $\lambda(\exp^{-1}(e)) \subseteq \mathbb{Z}$. Explicitly, the character χ

of the representation associated to such a λ is given by the formula

$$\chi \circ \exp|_{t} = \sum_{w\in W} \sigma(w)e^{2\pi\sqrt{-1}\, w\cdot(\lambda+\delta)} / \sum_{w\in W} \sigma(w)e^{2\pi\sqrt{-1}\, w\cdot\delta}$$

where $\delta = (1/2) \sum_{\alpha\in\Delta^+} \alpha$ and $\sigma(w)$ $(= \pm 1)$ denotes the determinant of w on t . □

A basic fact about irreducible representations is that such a representation θ has a distinguished weight, the so called maximal weight relative to a given choice of positive roots of G. The linear form λ associated to θ in the theorem is precisely this maximal weight. (The appearance of the factor 2π in the definition of weights is designed to ensure that λ satisfies the condition $\lambda(\exp^{-1}(e)) \subseteq \mathbb{Z}$.) Hence, the theorem shows that an irreducible representation is determined not just by its weights, but in fact by its maximal weight alone.

As an example, consider the group $G = SU_2$, which consists of all complex matrices of the form $\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$. The Lie algebra may be identified with the set of skew-Hermitian matrices of the form $\begin{pmatrix} \sqrt{-1}\,t & z \\ -\bar z & -\sqrt{-1}\,t \end{pmatrix}$ with $t \in \mathbb{R}$, $z \in \mathbb{C}$, $t^2 + |z|^2 = 1$. As a maximal torus T one may take the circle subgroup consisting of matrices with $\beta = 0$; its Lie algebra t then consists of matrices with $z = 0$. The exponential map $t \to T$ is given by $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$, i.e.

$$\begin{pmatrix} \sqrt{-1}\,t & 0 \\ 0 & -\sqrt{-1}\,t \end{pmatrix} \longrightarrow \begin{pmatrix} e^{\sqrt{-1}\,t} & 0 \\ 0 & e^{-\sqrt{-1}\,t} \end{pmatrix} .$$

We identify t with $\mathbb{R}$ by assigning to $\begin{pmatrix} \sqrt{-1}\,t & 0 \\ 0 & -\sqrt{-1}\,t \end{pmatrix}$ the number $2\pi t$. Using the fact that $Ad(X)Y = XYX^{-1}$, one sees that there are just two roots $\pm\alpha$, with $\alpha(t) = 2t$. Those $\lambda \in t^*$ which satisfy $\lambda(\exp^{-1}(e)) \subseteq \mathbb{Z}$ are of the form $\lambda = \lambda_n$, $\lambda_n(t) = nt$, for $n \in \mathbb{Z}$. If we choose $\Delta^+ = \{\alpha\}$, D^* is $\{t \in \mathbb{R} \mid t > 0\}$, and so the forms λ_n which parametrize irreducible representations of SU_2 are those with $n > 0$. Let the representation corresponding to λ_n be θ_n. The Weyl group has just two elements, which act on $\mathbb{R}$ by $t \longmapsto \pm t$, so the Weyl character formula gives

$$\chi_n \begin{pmatrix} \alpha & 0 \\ 0 & \bar\alpha \end{pmatrix} = (\alpha^{n+1} - \alpha^{-(n+1)})/(\alpha - \alpha^{-1})$$

for the restriction to T of the character of θ_n. Since this expression is just $\alpha^n + \alpha^{n-2} + \ldots + \alpha^{-n}$, we deduce that $\dim \theta_n = n + 1$ and that the weights of θ_n are λ_n , λ_{n-2}, ..., λ_{-n}. In fact, a concrete realization of θ_n is provided by the action of SU_2 on the space of homogeneous polynomials in two variables e_0 , e_1 which is specified by

$$\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \cdot e_0 = \alpha e_0 - \bar\beta e_1$$

$$\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \cdot e_1 = \beta e_0 + \bar\alpha e_1 .$$

This is equivalent to the n-th symmetric power $S^n A$ of the standard representation A of SU_2. For an example where rank $G = 2$, see [Bo1].

(1.5) Roots and Weights for Lie Algebras. Because of the basic correspondence between compact (semisimple) Lie groups G and complex (semisimple) Lie algebras $\otimes \mathbb{C}$ (see Chapters II and III of [He]), the discussion of 1.4 may be carried out for a representation θ of $\mathfrak{g} \otimes \mathbb{C}$ on $\mathbb{C}^n$ entirely in the framework of such Lie algebras. We sketch the details as this provides some useful notation (and illustrates the philosophy of 1.2 concerning the value of complexifying). First, maximal tori in G are replaced by Cartan subalgebras in $\mathfrak{g} \otimes \mathbb{C}$, and we shall fix one, $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$ (where $\mathfrak{t}$ is the Lie algebra of a fixed maximal torus T of G). Equivalence of maximal tori under conjugation (the theorem of 1.3) is replaced by equivalence of Cartan subalgebras under $\mathrm{Ad} \otimes \mathbb{C}$.

Definition: If $\hat{\lambda} \in \mathfrak{h}^*$, let $W_{\hat{\lambda}} = \{X \in \mathbb{C}^n \mid \theta(H)X = \hat{\lambda}(H)X \text{ for all } H \in \mathfrak{h}\}$. If $W_{\hat{\lambda}} \neq \{0\}$, $\hat{\lambda}$ is called a weight of θ (with respect to $\mathfrak{h}$), with weight space $W_{\hat{\lambda}}$.

If $\hat{\lambda}$ is a weight of θ, $\theta(H)$ is equal to the scalar transformation $\hat{\lambda}(H)$ on $W_{\hat{\lambda}}$; hence $\theta(\exp H) = e^{\hat{\lambda}(H)}$ for all $H \in \mathfrak{t}$ $(\subseteq \mathfrak{h})$. This means that $\hat{\lambda}/2\pi\sqrt{-1}$ is a weight λ of θ in the sense of 1.4. (Recall that we use the same notation for a linear form $\lambda \in \mathfrak{t}^*$ and its $\mathbb{C}$-linear extension $\lambda \in \mathfrak{h}^*$.) If the decomposition of $\mathbb{C}^n$ corresponding to $\theta \cong \Gamma_1 \oplus \dots \oplus \Gamma_n$ is written $\mathbb{C}^n = W_{\lambda_1} \oplus \dots \oplus W_{\lambda_n}$, then the weights of θ are $\hat{\lambda}_i = 2\pi\sqrt{-1}\,\lambda_i$ $(i = 1,\dots,n)$ and $W_{\lambda_i} = W_{\hat{\lambda}_i}$. The multiplicity of λ_i is just $\dim W_{\lambda_i}$.

Definition: The roots of $\mathfrak{g} \otimes \mathbb{C}$ (with respect to $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$) are the non-zero weights of ad (considered as a representation of $\mathfrak{g} \otimes \mathbb{C}$ on $\mathfrak{g} \otimes \mathbb{C}$).

The weights of ad are 0 (with multiplicity the rank of G), and $\{\pm 2\pi\sqrt{-1}\,\alpha_i \mid 1 \leq i \leq \ell\}$ where $\Delta = \{\pm\alpha_i \mid 1 \leq i \leq \ell\}$ are the roots of G, (each with multiplicity one). The Jacobi identity shows that if $X \in V_\alpha$, $Y \in V_\beta$ $(\alpha, \beta \in \Delta)$, then $\mathrm{ad}(X)Y = [X,Y] \in V_{\alpha+\beta}$ $(= \{0\}$ if $\alpha + \beta \notin \Delta)$. From this it is easily shown (a) that $V_\alpha \perp V_\beta$ if $\alpha + \beta \neq 0$, and (b) that $\langle\ ,\ \rangle$ is non-degenerate on $\mathfrak{h}$. By (b) we may choose $H_\alpha \in \mathfrak{h}$ such that $\langle H, H_\alpha\rangle = -\hat{\alpha}(H)$ for all $H \in \mathfrak{h}$. For any $E_\alpha \in V_\alpha$, $E_{-\alpha} \in V_{-\alpha}$ $(\alpha \in \Delta^+)$, invariance of $\langle\ ,\ \rangle$ under ad gives $\langle [E_\alpha, E_{-\alpha}], H\rangle = \langle E_\alpha, [E_{-\alpha}, H]\rangle$ $= \langle E_\alpha, \hat{\alpha}(H)E_{-\alpha}\rangle = -\langle H, H_\alpha\rangle\langle E_\alpha, E_{-\alpha}\rangle$ for all $H \in \mathfrak{h}$. Hence $[E_\alpha, E_{-\alpha}] = -\langle E_\alpha, E_{-\alpha}\rangle H_\alpha$. If $E_\alpha, E_{-\alpha} \neq 0$, then $\langle E_\alpha, E_{-\alpha}\rangle \neq 0$ (otherwise, using (a) above, this would contradict the non-degeneracy of $\langle\ ,\ \rangle$ on $\mathfrak{g} \otimes \mathbb{C}$, i.e. the semisimplicity of $\mathfrak{g} \otimes \mathbb{C}$). Hence, on choosing any non-zero E_α, we may choose $E_{-\alpha}$ so that $\langle E_\alpha, E_{-\alpha}\rangle = -1$. To summarize, we have chosen $E_\alpha, E_{-\alpha}$ so that the "root space decomposition"

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{t} \otimes \mathbb{C} \oplus \sum_{\alpha \in \Delta^+} V_\alpha \oplus \sum_{\alpha \in \Delta^+} V_{-\alpha}$$

is

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{t} \otimes \mathbb{C} \oplus \sum_{\alpha \in \Delta^+} \mathbb{C}E_\alpha \oplus \sum_{\alpha \in \Delta^+} \mathbb{C}E_{-\alpha}$$

and so that

(1) $\langle E_\alpha, E_{-\alpha}\rangle = -1$, $[E_\alpha, E_{-\alpha}] = H_\alpha$

(2) $[\mathbb{C}E_\alpha, \mathbb{C}E_\beta] = \mathbb{C}E_{\alpha+\beta}$ if $0 \neq \alpha + \beta \in \Delta$.

The real decomposition $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\pm\alpha_i \in \Delta} V_i$ is included in the root space decomposition as follows:

$$\mathfrak{t} = \sum_{\alpha\in\Delta} \sqrt{-1}\,\mathbb{R} H_\alpha$$

$$V_i = \mathbb{R}(E_{\alpha_i} - E_{-\alpha_i}) \oplus \sqrt{-1}\,\mathbb{R}(E_{\alpha_i} + E_{-\alpha_i}).$$

It should be noted that our choice of positive roots $\Delta^+ \subseteq \Delta$ gives rise to a complex structure J on each V_i, by setting $JE_\alpha = \sqrt{-1}\,E_\alpha$, $JE_{-\alpha} = -\sqrt{-1}\,E_{-\alpha}$ ($\alpha = \epsilon_i\alpha_i \in \Delta^+$). With respect to this, $V_{-\alpha} = \overline{V}_\alpha$, so $V_i \otimes \mathbb{C} = V_{\alpha_i} \oplus V_{-\alpha_i} = V_\alpha \oplus \overline{V}_\alpha$. Further details of the discussion in this paragraph may be found in Chapter III of [He].

We shall point out some simple properties of the "weight space decomposition" $\mathbb{C}^n = \sum_\lambda W_\lambda$ which will be useful in IV. First, it follows from the definition that if $\lambda \neq \mu$, then W_λ and W_μ are orthogonal with respect to the given Hermitian inner product on $\mathbb{C}^{n+1}$. (For the case $\theta = \mathrm{ad}$, this is defined on $\otimes\,\mathbb{C}$ by $\langle\langle X,Y\rangle\rangle = \langle X,\overline{Y}\rangle$, complex conjugation being specified by $\overline{E}_\alpha = -E_{-\alpha}$ for all $\alpha \in \Delta$.) Second, if $X \in W_\lambda$, then $\theta(E_\alpha)X \in W_{\lambda+\alpha}$ if $\lambda + \alpha$ is a weight of θ (and zero otherwise). To prove this, just observe that $\theta(H)\theta(E_\alpha)X = \theta[H,E_\alpha]X + \theta(E_\alpha)\theta(H)X = \hat{\alpha}(H)\theta(E_\alpha)X + \hat{\lambda}(H)\theta(E_\alpha)X = (\hat{\alpha} + \hat{\lambda})(H)\theta(E_\alpha)X$ for all $H \in \mathfrak{h}$.

(1.6) Notes. The basic texts referred to above are [Ad] and [He]. The latter contains historical notes which demonstrate the importance of the work of E. Cartan, F. Killing, S. Lie, and H. Weyl. Particularly lucid expositions are the survey article [Bol], and the articles of E. Stiefel referred to therein, in which the "Stiefel diagram" is developed. An informal and practical treatment of representation theory appears in [Ch].

II. Homogeneous Spaces

(2.1) Homogeneous Spaces. Throughout this lecture G will be a compact connected Lie group and H a subgroup of G. The space of cosets $G/H = \{gH \mid g \in G\}$ is called a homogeneous space. The orthogonal complement of the subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$ (minus the Killing form) will be denoted $\mathfrak{m}$. If $X \in \mathfrak{g}$, we define the vector field X^* on G/H by $X^*_{gH} = (d/dt)(\exp tX \cdot gH)|_{t=0}$. This should be contrasted with the identification between elements of $\mathfrak{g}$ and left-invariant vector fields on G: the map $X \longmapsto X^*$ is not a Lie algebra homomorphism (indeed, $[X^*,Y^*] = -[X,Y]^*$). There is no suggestion that G/H may be parallelized by vector fields of the form X^* (such vector fields have zeros, in general). Since $\langle\ ,\ \rangle$ is invariant under the adjoint representation, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is an H-module decomposition for this representation. We write $\mathrm{Ad} = \mathrm{Ad}(G) = \mathrm{Ad}(H) \oplus \mathrm{Ad}(G/H)$.

Two special cases are particularly important. First, G/H is said to be symmetric if there is an automorphism σ of G such that $\sigma^2 = 1$ and $G_\sigma = \{g \in G \mid \sigma(g) = g\}$ satisfies $(G_\sigma)_0 \subseteq H \subseteq G_\sigma$, where $(G_\sigma)_0$ denotes the identity component of G_σ. The induced automorphism of $\mathfrak{g}$, also denoted σ, has the property $\sigma^2 = 1$; the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the eigenspace decomposition for σ. From this it is clear that $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}$. This is the basic algebraic property of a symmetric space, for if $\mathfrak{h}$ is any subalgebra of $\mathfrak{g}$ and $\mathfrak{m} = \mathfrak{h}^\perp$ satisfies $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{m}$, then one may define a suitable automorphism σ of $\mathfrak{g}$ by $\sigma|_\mathfrak{h} = 1$, $\sigma|_\mathfrak{m} = -1$. If G is simply connected, σ gives an automorphism σ of G which makes G/H into a symmetric space. The theory of maximal tori and roots for a Lie group carries over to a symmetric space, from which it is possible to obtain a classification of compact symmetric spaces. Note that the group G itself can be considered as the symmetric space $G \times G/\Delta$, $\Delta = \{(g,g) \mid g \in G\}$, where σ is induced by the map $(x,y) \longrightarrow (y,x)$ of $G \times G$. For this material, see [He].

The second class of homogeneous spaces we shall consider is that of generalized flag manifolds, i.e. where $H = C(S)$ is the centralizer of some (not necessarily maximal) torus of G. The name is taken from the example $G = U_n$, $H = U_{n_1} \times U_{n_2} \times \dots \times U_{n_k}$, where G/H may be identified with the manifold $F(m_1,m_2,\dots,m_k)$ where $m_i = n_1 + \dots + n_i$: by definition, $F(m_1,m_2,\dots,m_k)$ consists of the "flags" $\{0\} \subseteq E_{m_1} \subseteq E_{m_2} \subseteq \dots \subseteq E_{m_k} = \mathbb{C}^n$ where E_i is a subspace of $\mathbb{C}^n$ of dimension i. If $S = T$ is a maximal torus, $C(S) = T$, and G/T is called simply a flag manifold. These two classes of homogeneous spaces are not mutually exclusive, for example the Grassmannian $Gr_k(\mathbb{C}^n) = F(k,n)$ is both a symmetric space and a generalized flag manifold. For a detailed discussion of generalized flag manifolds, see [BH].

(2.2) Homogeneous Bundles. We shall be interested in bundles on a homogeneous space which are themselves "homogeneous". Recall that if $\pi : P \longrightarrow M$ is a principal H-bundle, and H acts on a space F, one may form the associated bundle $E \to M$ with fibre F as follows. First, one sets $E = P \times_H F$ where H acts on $P \times F$ by $h(p,f) = (ph, h^{-1}f)$, then one defines a projection $E \to M$ as the map induced by $(p,f) \longmapsto \pi(p)$. If F is a vector space, and if H acts linearly, the associated bundle is a vector bundle.

Definition: A homogeneous vector bundle over G/H is a bundle associated to the principal bundle $G \to G/H$ by a representation θ of H. Such a bundle will be denoted $\alpha(\theta)$.

For example, the tangent bundle of G/H is a homogeneous vector bundle:

Proposition: $T(G/H) \cong \alpha(Ad(G/H))$.

Proof. If $\pi : P \to M$ is any principal bundle with group G, $TP \cong \pi^*TM \oplus T_F$ where T_F is the "bundle of tangents to the fibres". Note that π^*TM is naturally trivial, and that a trivialization $T_F \cong P \times \mathfrak{g}$ may be defined by associating to a vector $V \in T_F$ over $p \in P$ the element $\gamma'(0) \in \mathfrak{g}$, where γ is a one-parameter subgroup of G such that $t \longrightarrow p\gamma(t)$ has tangent vector V at 0. The action of G on TP, induced by right translation $R_g : P \to P$ $(g \in G)$, acts via the trivialization on $P \times \mathfrak{g}$ by the formula $g \cdot (p,V) = (pg, Ad(g)^{-1}V)$. This is because the one-parameter subgroup corresponding to $dR_g(V)$ is $g^{-1}\gamma g$. Thus T_F/G is the bundle associated to $P \to M$ by the representation $Ad(G)$. Applying this to $\pi : G \longrightarrow G/H$, we have $T_F/H \cong \alpha(Ad(H))$. But $(TG)/H \cong \alpha(Ad(G)|_H)$, so π^*TM/H $(\cong TM) \cong \alpha(Ad(G/H))$, as required. □

The "α-construction" is functorial, so one has, for example, $\Lambda^i T(G/H) \cong \alpha(\Lambda^i Ad(G/H))$. We shall use the notation α_P for the construction as applied to a general principal bundle $\pi : P \to M$.

(2.3) Homogeneous Geometrical Structures. In this section we shall write T for $T(G/H)$, to simplify notation. By a geometrical structure on G/H (or rather, on T), we mean a distinguished tensor or differential operator such as an almost complex structure, a metric, or a connection. Since T is homogeneous, it is natural to concentrate on homogeneous, or invariant structures, namely those which are determined by their behaviour at $o \in G/H$ (and then transported around by the action of G).

An almost complex structure on G/H is an (endomorphism) $J \in \Gamma(T^* \otimes T)$ such that $J^2 = -1$; J is homogeneous if it is specified by an H-invariant (endomorphism) $J \in \mathfrak{m}^* \otimes \mathfrak{m}$ with $J^2 = -1$. We have noted in 1.5 of I that a choice of positive roots Δ^+ for G with respect to a maximal torus S determines a complex structure on the vector space $T_o(G/S) \cong \sum_{\pm\alpha_i \in \Delta} V_i$. This is clearly S-invariant, hence it defines a homogeneous almost complex structure on G/S. It turns out (see [BH]) that the almost complex structure is integrable, and that any such homogeneous complex structure is obtained by some choice of positive roots. The most general homogeneous almost complex structure is obtained by changing J to -J on an arbitrary selection of subspaces V_i.

A Riemannian metric on G/H is a (symmetric) form $g \in \Gamma(S^2T^*)$ which gives an inner product on each fibre of T ; a homogeneous Riemannian metric is given by an H-invariant form $g \in S^2\mathfrak{m}^*$. For example, minus the Killing form restricts to such a form on $\mathfrak{m}$, which we shall continue to call < , >. Note that if $\mathfrak{m}$ is an irreducible H-module, any two such forms must differ by a constant multiple. (This is an application of Schur's Lemma.) For example, this is the case for the Grassmannian $Gr_k(\mathbb{C}^n)$. For the generalized flag manifold G/S , $\mathfrak{m}$ decomposes as $\sum_{\pm\alpha_i \in \Delta} V_i$ as an S-module, so there are many

homogeneous metrics. However, each module V_i is certainly irreducible (being two dimensional and non-trivial), so any homogeneous metric differs from a fixed metric (such as < , >) only by a constant multiple on each V_i.

An affine connection on G/H is a "covariant derivative" operator $\nabla : \Gamma(T) \longrightarrow \Gamma(T^* \otimes T)$ which is characterized by the properties:

a) $\nabla_X Y$ (= $\nabla Y(X)$) is $C^\infty(G/H)$-linear in X.

b) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$ for all $f \in C^\infty(G/H)$.

We say that ∇ is homogeneous (or invariant) if it is preserved by the left translation maps $L_g : G \longrightarrow G$, $g \in G$. This means $dL_g(\nabla_X Y) = \nabla_X^{L_g^*TG} dL_g(Y)$ (see 4.1 of IV for a formula for $\nabla^{L_g^*TG}$). Such a connection is determined by a linear transformation of H-modules

$$\Lambda : \mathfrak{m} \longrightarrow \mathfrak{m}^* \otimes \mathfrak{m}$$

by means of $(\nabla_{X^*}Y^*)_o = \Lambda(X)(Y) + [X^*,Y^*]_o$ $(X,Y \in \mathfrak{m})$. This is proved in [KN] , volume II, Chapter X. Two examples are of particular interest. If $\Lambda = 0$, ∇ is called the canonical connection (geometrically, it has the property that one-parameter subgroups in G give geodesics in G/H , and parallel translation along geodesics is given by left multiplication). If $\Lambda(X)(Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}}$ (the component in $\mathfrak{m}$ of $\frac{1}{2}[X,Y] \in \mathfrak{g}$) , ∇ is the Levi-Cività connection for the homogeneous Riemannian metric given by < , >. It should be noted that these coincide if G/H is symmetric.

(2.4) The Hodge-Laplacian. The vector space $C^\infty(G)$ consisting of (smooth) real-valued functions on G may be made into an (infinite dimensional) G-module by defining $(g\cdot f)(h) = f(hg)$ for $g,h \in G$, $f \in C^\infty(G)$. Any (finite dimensional) irreducible real representation $\theta : G \longrightarrow SO_n$ is contained in $C^\infty(G)$ in the sense that if $V_1, \ldots, V_n$ is an orthonormal basis for $\mathbb{R}^n$, the functions $\theta_{ij} : g \longmapsto \langle\langle \theta(g)V_j, V_i \rangle\rangle$, $1 \le j \le n$, span a subspace A_i of $C^\infty(G)$ which is a G-module isomorphic to $\mathbb{R}^n$. (The inner product on $\mathbb{R}^n$ is denoted $\langle\langle\ ,\ \rangle\rangle$.) The isomorphism is that which assigns to V_j the function $\theta_{ij} \in A_i$; this holds for each fixed i. Similar remarks apply to complex representations.

Theorem (Peter-Weyl): As θ varies through the equivalence classes of irreducible complex representations of G, the functions θ_{ij} form a complete orthogonal subset in the L^2 sense of $C^\infty(G) \otimes \mathbb{C}$. □

We refer to [Ze] for a proof and for a discussion of various important consequences. This theorem is the basis for harmonic analysis on Lie groups.

A real class 1 representation of (G,H) is an irreducible representation $\theta : G \longrightarrow SO_n$ for which there exists some non-zero vector in $\mathbb{R}^n$ whose isotropy subgroup contains H. If $V_1, \ldots, V_k$ is an orthonormal basis for the subspace of such vectors, which is extended to an orthonormal basis $V_1, \ldots, V_n$ of $\mathbb{R}^n$, we obtain functions $\theta_{ij} \in C^\infty(G/H)$ for $1 \le i \le n$, $1 \le j \le k$, by the same formula used earlier. We obtain a copy of the G-module $\mathbb{R}^n$ in the infinite dimensional G-module $C^\infty(G/H)$ for each j, providing we make $C^\infty(G/H)$ into a G-module by defining $(g\cdot f)(xH) = f(g^{-1}xH)$. Similar remarks apply to complex class 1 representations, and the

Peter-Weyl theorem generalizes to this situation (see [Ze]), which permits harmonic analysis on homogeneous spaces.

The Hodge-Laplacian on a Riemannian manifold M is defined to be the differential operator $\Delta = dd^* + d^*d$ where d is the usual exterior derivative on forms and d^* is its adjoint with respect to the metric. It is an operator on forms (which preserves degree); we shall simply be interested in the case of 0-forms, i.e. functions, in which case we have $\Delta f = d^*df$. If $X_1, \ldots, X_m$ form an orthonormal basis for T_xM, $x \in M$, then one obtains the formula $(\Delta f)_x = -\sum_{i=1}^m (X_i^2 f)_x$ (where X_i is regarded as a differential operator on functions). Now let $M = G$ with the Riemannian metric given by $\langle\ ,\ \rangle$. If $X_1, \ldots, X_m$ is an orthonormal basis for $\mathfrak{g}$, the corresponding left-invariant vector fields on G form an orthonormal basis of T_gG at each $g \in G$, hence we may write $\Delta = -\sum_{i=1}^m X_i^2$.

Proposition: (a) If A is a finite dimensional G-submodule of $C^\infty(G) \otimes \mathbb{C}$, then $\Delta A \subseteq A$.

(b) If A is irreducible, $\Delta|_A$ is a real scalar operator.

(c) if $\theta : G \longrightarrow U_n$ is an irreducible representation of G, consider the identification $\mathbb{C}^n \cong A_i$ $(\subseteq C^\infty(G) \otimes \mathbb{C})$ defined above. Then the operator Δ on A_i corresponds to the operator $-\sum_{j=1}^m \theta(X_j)^2$ on $\mathbb{C}^n$.

Proof. (a) If suffices to show $X_jA \subseteq A$. This is clear, as by definition, $(X_jf)_x = (d/dt)f(x \exp tX_j)|_{t=0}$.

(b) This follows from Schur's Lemma, since Δ is a symmetric G-module transformation.

(c) It suffices to show that the operator X_j corresponds to the operator $\theta(X_j)$. The function f corresponding to $V \in \mathbb{C}^n$ is defined by $f(g) = \langle\langle\theta(g)V, V_i\rangle\rangle$ (see earlier). So $(X_j f)(g) = (d/dt)\langle\langle\theta(g\ \exp(tX_j))V, V_i\rangle\rangle|_{t=0} = \langle\langle\theta(g)\theta(X_j)V, V_i\rangle\rangle$. □

In general, the Casimir operator (of θ) with respect to the basis $X_1, \ldots, X_m$ is defined to be the operator $-\sum_{i=1}^{m} \theta(X_i)^2$. This holds for all complex representations, irreducible or not. The key fact is that it is G-module transformation. (This is usually expressed more abstractly, by saying that $\sum_{i=1}^{m} X_i^2$ is in the centre of the universal enveloping algebra of $\mathfrak{g}$.) For use in IV we note that if the basis $X_1, \ldots, X_m$ is chosen as the elements $(1/\sqrt{2})(E_\alpha - E_{-\alpha})$, $\sqrt{(-1/2)}(E_\alpha + E_{-\alpha})$ for $\alpha \in \Delta^+$, together with any orthonormal basis $H_1, \ldots, H_k$ of $\mathfrak{t}$, then the Casimir operator is $-\sum_{i=1}^{k} \theta(H_i)^2 + \sum_{\alpha\in\Delta} \theta(E_\alpha)\theta(E_{-\alpha})$. Finally, when θ is irreducible, it is easy to calculate the "Casimir constant" c for which $\sum_{i=1}^{m} \theta(X_i)^2 V = cV$ for all vectors V. To do this, take V to be a weight vector for the maximal weight $\hat{\lambda}$, i.e. $\theta(H)V = \hat{\lambda}(H)V$ for all $H \in \mathfrak{h}$. Using the basis for $\mathfrak{g}$ just described, we have (a) if $-\alpha \in \Delta^+$, $\theta(E_\alpha)\theta(E_{-\alpha})V = 0$, since $\theta(E_{-\alpha})V \in W_{\lambda-\alpha} = \{0\}$ (it follows from the maximality of $\hat{\lambda}$ that $\hat{\lambda} - \hat{\alpha}$ is not a weight), (b) if $\alpha \in \Delta^+$, $\theta(E_\alpha)\theta(E_{-\alpha})V = \theta([E_\alpha, E_{-\alpha}])V + \theta(E_{-\alpha})\theta(E_\alpha)V = \theta(H)_\alpha V = \hat{\lambda}(H_\alpha)V$, and (c) $\theta(H_i)^2V = \hat{\lambda}(H_i)^2V$. Thus $c = \sum_{i=1}^{k} \hat{\lambda}(H_i)^2 - \sum_{\alpha\in\Delta^+} \hat{\lambda}(H_\alpha)$. If $\langle\ ,\ \rangle$ on $\mathfrak{h}$ is extended to $\mathfrak{h}^*$ in the usual way, we get $c = \langle\hat{\lambda},\hat{\lambda}\rangle + \langle\hat{\lambda},2\hat{\delta}\rangle = \langle\hat{\lambda}, \hat{\lambda}+2\hat{\delta}\rangle = -4\pi^2\langle\lambda, \lambda+2\delta\rangle$ (recall that $\delta = (1/2)\ \sum_{\alpha\in\Delta^+} \alpha)$, which is a real number. In particular, if the representation is non-trivial, c is strictly negative. Finally, if θ is an irreducible real representation, we claim that $\sum_{i=1}^{m} \theta(X_i)^2$ is still a scalar operator. For $\theta \otimes \mathbb{C}$ either is irreducible, in which case the assertion follows, or is a sum $\psi \oplus \overline{\psi}$ where ψ is irreducible. In the latter case, the formula $c = \langle\hat{\lambda}, \hat{\lambda}+2\hat{\delta}\rangle$ shows that $\sum_{i=1}^{m} \psi(X_i)^2 = \sum_{i=1}^{m} \overline{\psi}(X_i)^2$, since if $\hat{\lambda}$ is the maximal weight of ψ with respect to the choice of positive roots Δ^+, $-\hat{\lambda}$ is the maximal weight of $\overline{\psi}$ with respect to the choice of positive roots $-\Delta^+$. Hence the assertion is true in this case.

The Hodge-Laplacian on G/H (with its Riemannian metric defined by $\langle\ ,\ \rangle$) may be treated in a similar fashion. If $X_1, \ldots, X_m$ is an orthonormal basis for $\mathfrak{m}$, then we have $(\Delta f)_o = -(\sum_{i=1}^{m} (X_i^*)^2 f)_o$.

Corollary: The proposition goes through for $C^\infty(G/H) \otimes \mathbb{C}$ in place of $C^\infty(G) \otimes \mathbb{C}$ (with i and j interchanged). □

(2.5) Induced Representations. If θ is a representation of G and $i : H \to G$ is the inclusion of the subgroup H, the restriction of the representation to H will sometimes be denoted $i^*\theta$. Conversely, given a representation θ of H, one may define the induced representation $i_*\theta$ of G to be the infinite dimensional space $\Gamma(\alpha(\theta))$ of (smooth) sections of the homogeneous bundle $\alpha(\theta)$ (see 2.2), together with the action of G defined by means of the formula $(g\cdot s)(xH) = s(g^{-1}xH)$. If V is the representation space of θ, we write $i_*V = \Gamma(\alpha(\theta))$. For example, if $\theta : H \longrightarrow SO_1$ is the trivial representation of H on $\mathbb{R}$, $i_*\mathbb{R}$ is the G-module $C^\infty(G/H)$.

In the following basic observation, which describes the structure of $i_*\theta$, $\mathrm{Hom}_G(\phi_1,\phi_2)$ denotes the vector space of G-module transformations from the representation space of ϕ_1 to that of ϕ_2.

Theorem (Frobenius Reciprocity): Let ϕ be a representation of G. Then $\mathrm{Hom}_G(\phi, i_*\theta)$ is naturally isomorphic (as a vector space) to $\mathrm{Hom}_H(i^*\phi, \theta)$.

Proof. Let V be the representation space for θ, and W that for ϕ. From the definition of $\alpha(\theta)$, sections s of $\alpha(\theta)$ correspond to elements $f \in C^\infty(G) \otimes V$ such that $f(gh) = \theta(h)^{-1}f(g)$ for all $h \in H$, $g \in G$. (Define $f(g) = X$, where $s(gH) \in G \times_H V$ is the equivalence class of $(g,X) \in G \times V$.) The action of G on $\Gamma(\alpha(\theta))$ corresponds to the action of G on $C^\infty(G) \otimes V$ given by $(g\cdot f)(x) = f(g^{-1}x)$. (Note that this "left action" is not the natural "right action" considered at the beginning of this section.) Thus we have identified $\Gamma(\alpha(\theta))$ with a G-submodule of $C^\infty(G) \otimes V$. If now $\phi : W \longrightarrow \Gamma(\alpha(\theta))$ is a G-module transformation, we obtain $\psi : W \longrightarrow V$ by regarding $\phi(w)$ as an element of $C^\infty(G) \otimes V$ and defining $\psi(w) = \phi(w)(e)$ for all $w \in W$. If $h \in H$, $\psi(\phi(h)w) = \phi(\phi(h)w)(e) = (h\cdot\phi(w))(e) = \phi(w)(h^{-1}) = \theta(h)\phi(w)(e) = \theta(h)\psi(w)$, so ψ is an H-module transformation. Conversely, if $\psi : W \longrightarrow V$ is an H-module transformation, define $\phi : W \longrightarrow C^\infty(G) \otimes V$ by $\phi(w)(g) = \psi(\phi(g)^{-1}w)$. If $x \in G$, $\phi(\phi(x)w)(g) = \psi(\phi(g^{-1}x)w) = \phi(w)(x^{-1}g) = (x\cdot\phi(w))(g)$, so ϕ is a G-module transformation. The maps $\phi \longmapsto \psi$ and $\psi \longmapsto \phi$ are clearly linear inverses of each. □

If the representation space of ϕ is W (as in the proof), denote by A_ϕ the image of the evaluation map $W \otimes \mathrm{Hom}_G(\phi, i_*\theta) \longrightarrow \Gamma(\alpha(\theta))$. The Peter-Weyl theorem may now be extended to show that the spaces A_ϕ, as ϕ varies over

equivalence classes of irreducible complex representations, form a complete orthogonal system of subspaces for the space of L^2-sections of $\alpha(\theta)$. (By taking $\theta : H \longrightarrow U_1$ to be the trivial representation on $\mathbb{C}$, we obtain the version of the Peter-Weyl theorem for $C^\infty(G/H) \otimes \mathbb{C}$ mentioned earlier.) See [Bo2] for comments on this. The proposition of 2.4 may also be extended, provided Δ is replaced by the appropriate operator on $\Gamma(\alpha(\theta))$ (c.f. IV). Finally, we note that if θ and ϕ are complex irreducible, then it makes sense because of the extended Peter-Weyl theorem to say that the number of subrepresentations of $i_*\theta$ which are equivalent to ϕ is precisely $\dim \mathrm{Hom}_G(\phi, i_*\theta) = \dim \mathrm{Hom}_H(i^*\phi, \theta)$. By elementary representation theory, the latter number is the number of subrepresentations of $i^*\phi$ equivalent to θ (in any decomposition of $i^*\phi$ into irreducible summands). For example, the number of times ϕ appears in $C^\infty(G/H) \otimes \mathbb{C}$ is $\dim\{X \mid \phi(H)X = X\}$.

(2.6) Notes. The conventions regarding vector fields on G and G/H are unavoidably rather troublesome. If one wishes to consider $\mathfrak{g}$ as the algebra of left-invariant vector fields, and to deal with the space G/H of left cosets, one has to accept that results for G/H do not agree exactly with those for G when $H = \{e\}$, e.g. in the Peter-Weyl theorem for G the group acts by right multiplication, whereas for G/H it acts by left multiplication. For the theory of symmetric spaces, see [He], [Lo] or [KN]; for results on generalized flag manifolds see [BH]. An elementary discussion of the Laplacian (Laplace-Beltrami operator) is given in [Wr]. The Peter-Weyl theorem is proved in [Ze] using the Stone-Weierstrass theorem; an approach using the fact that the Laplacian is an elliptic operator is given in [Wr]. Our discussion of induced representations is taken from [Bo2].

III. Characteristic Classes of Homogeneous Bundles

(3.1) Cohomology of Homogeneous Spaces. There are various ways of defining characteristic classes, but since our aim is to discuss the connection with representations we shall just give one convenient definition in this section. Moreover, we shall only use cohomology groups with real coefficients, and consider only Chern and Pontrjagin classes. References for unproved assertions will be given in the last section.

If G is a compact Lie group, one may construct a principal G-bundle $EG \to BG$ which is "universal" in the sense that any other principal G-bundle $P \to M$ is the pull-back f^*EG for some map $f : M \to BG$. For example, if $G = U_n$, one may take $BU_n = \lim_{N\to\infty} Gr_n(\mathbb{C}^{N+n})$, $EU_n = \lim_{N\to\infty} W_{N+n,n}$ (the Stiefel manifold $W_{N+n,n}$ is the space of ordered n-tuples of vectors in $\mathbb{C}^{N+n}$ which are orthonormal with respect to the standard Hermitian inner product). The projection $EU_n \to BU_n$ comes from the natural maps $W_{N+n,n} \to Gr_n(\mathbb{C}^{N+n})$. If $G = SO_n$ one may take $BSO_n = \lim_{N\to\infty} Gr_n(\mathbb{R}^{N+n})$, $ESO_n = \lim_{N\to\infty} V_{N+n,n}$, where the Stiefel manifold $V_{N+n,n}$ is the real version of $W_{N+n,n}$. An important consequence is that the tautologically defined vector bundles $W_n \to BU_n$, $V_n \to BSO_n$ are then universal for vector bundles of rank n with structural groups U_n, SO_n respectively.

It is a well known fact (see later) that the real cohomology rings of BU_n, BSO_n are as follows:

(a) $H^*BU_n \cong \mathbb{R}[c_1,\dots,c_n]$ i.e. the polynomial ring on $c_1,\dots,c_n$, where $c_i \in H^{2i}BU_n$.

(b) $H^*BSO_{2n+1} \cong \mathbb{R}[p_1,\dots,p_n]$ where $p_i \in H^{4i}BSO_{2n+1}$, and $H^*BSO_{2n} \cong \mathbb{R}[p_1,\dots,p_n,q]$ where $p_i \in H^{4i}BSO_{2n}$, $q \in H^{2n}BSO_n$, and $q^2 = p_n$.

Definition: (i) If $P \to M$ is a principal U_n-bundle, and $P = f^*EU_n$, then the ith Chern class of P is $c_i(P) = f^*c_i$.

(ii) $P \to M$ is a principal SO_n-bundle, and $P = f^*ESO_n$, the ith Pontrjagin class of P is $p_i(P) = f^*p_i$. The Euler class of P is $e(P) = f^*q$.

(iii) The Chern, Pontrjagin and Euler classes of a vector bundle with structural group U_n or SO_n are defined similarly.

Note that although f is not unique, it is unique up to homotopy, so the definition makes sense. Note also that the characteristic classes of a vector bundle are the same as those of the associated principal bundle.

To go further, we need some results on the real cohomology rings of homogeneous spaces (of which (a) and (b) above are special cases). First consider the case of a torus $T = S^1 \times \dots \times S^1$ (n factors). Then $H^*T \cong \otimes^n H^*S^1 \cong \Lambda(x_1,\dots,x_n)$, i.e. the exterior algebra on generators $x_1,\dots,x_n \in H^1T$. On the other hand, $BT \cong BS^1 \times \dots \times BS^1$ and $BS^1 \cong BU_1 = \lim_{N\to\infty} \mathbb{C}P^N = \mathbb{C}P^\infty$. Since $H^*\mathbb{C}P^N \cong \mathbb{R}[y]/(y^{N+1})$ for some $y \in H^2\mathbb{C}P^N$, we have $H^*\mathbb{C}P^\infty \cong \mathbb{R}[y]$. Hence $H^*BT \cong \mathbb{R}[y_1,\dots,y_n]$, i.e. the polynomial algebra on generators $y_1,\dots,y_n \in H^2BT$. There is a natural relation between H^*T and H^*BT which is best seen by considering the spectral sequence of the fibre bundle $ET \to BT$: the transgression τ gives an isomorphism $H^1T \to H^2BT$ in this situation. Now, H^1T is naturally isomorphic to the vector space $\mathfrak{t}^*$, hence so is H^2BT. The point of all this is that we may identify H^*BT in a natural way with the symmetric algebra $S^*\mathfrak{t}^*$ (or the algebra of all homogeneous polynomials on $\mathfrak{t}$).

More generally, consideration of the spectral sequence of $EG \to BG$ leads to a relation between H^*G and H^*BG. It was proved by H. Hopf [Ho] that H^*G must be an exterior algebra $\Lambda(x_1,\dots,x_k)$ on odd dimensional generators, from which it follows using the spectral sequence that H^*BG is a polynomial algebra on generators $y_1,\dots,y_k$, where $\tau(x_i) = y_i$ (and so $\deg y_i = \deg x_i + 1$). Now let H be a subgroup of G. The space BH may be identified with EG/H, so one has a map $BH \to BG$ which is a bundle with fibre G/H. The spectral sequence of this gives a relation between H^*BH, H^*BG and H^*G/H. The case when H is a maximal torus T is particularly interesting:

Theorem 1: The image of the map $H^*BG \to H^*BT$ is the subring $(H^*BT)^W$ consisting of invariants under the action of the Weyl group W of G; W acts on H^*BT via the identification $H^*BT \cong S^*\mathfrak{t}^*$. The map is injective, so $H^*BG \cong (H^*BT)^W$. □

This gives the result quoted above for $G = U_n$, for W is the symmetric group S_n acting in the standard way on $H^*BT \cong \mathbb{R}[x_1,\dots,x_n]$, so $H^*BU_n \cong \mathbb{R}[c_1,\dots,c_n]$ where c_i is the i-th elementary symmetric function of $x_1,\dots,x_n$. The case $G = SO_n$ is similar.

If H is a subgroup of maximal rank, i.e. $H \supseteq T$ for some maximal torus T, then the theorem can be used to express H^*G/H in terms of the Weyl groups W_H and W_G of H and G.

Theorem 2: The kernel of the map $(S^*\mathfrak{t}^*)^{W_H} \cong H^*BH \to H^*G/H$ (induced by the inclusion of the fibre G/H for the bundle $BH \to BG$) is the ideal I generated by those elements of $(S^*\mathfrak{t}^*)^{W_G}$ $(\subseteq (S^*\mathfrak{t}^*)^{W_H})$ which are of positive degree. The map is surjective, so $H^*G/H \cong (S^*\mathfrak{t}^*)^{W_H}/I$. □

For example, let $G = U_n$, $H = U_p \times U_q$, $p+q = n$, so that $G/H \cong Gr_p(\mathbb{C}^n) \cong Gr_q(\mathbb{C}^n)$. Then $W_G \cong S_n$, $W_H \cong S_p \times S_q$, and $H^*BH \cong \mathbb{R}[\sigma_1,\dots,\sigma_p, \tau_1,\dots,\tau_q]$ where σ_i is the i-th symmetric polynomial in $x_1,\dots,x_p$ and τ_i is that in $x_{p+1},\dots,x_n$. The ideal I is generated by the symmetric polynomials in $x_1,\dots,x_n$ of positive degree. Dividing $(S^*\mathfrak{t}^*)^{W_H} \cong \mathbb{R}\,\sigma_1,\dots,\sigma_p, \tau_1,\dots,\tau_q]$ by I is equivalent to imposing all relations of the form $\Sigma\, \sigma_i\tau_{k-i} = 0$, $k > 0$. For more information on theorems 1 and 2 see [BH] and [Br].

(3.2) Characteristic Classes of Homogeneous Bundles. We can now give the basic relation between the characteristic classes of a homogeneous bundle and the weights of the corresponding representation. We begin with the universal situation. Recall that if θ is a representation of G, $\alpha_{EG}(\theta)$ denotes the vector bundle associated to $EG \to BG$ by θ.

Proposition 3: Let $\theta : G \to U_n$ be a complex representation with weights $\lambda_1,\ldots,\lambda_n$ (with respect to a fixed maximal torus T). Then $c_i\alpha_{EG}(\theta) = \sigma_i(\lambda_1,\ldots,\lambda_n)$, i.e. the ith elementary symmetric function of $\lambda_1,\ldots,\lambda_n$ (considered as an element of H^*BG via the identification $H^*BG \cong S^*t^*)^W$).

Proof. It is easy to see that $\alpha_{EG}(\theta) = f^*W_n$ where $f : BG \to BU_n$ is the map induced by $\theta : G \to U_n$. The result now follows from theorem 1 and the succeeding remarks. □

If $\theta : H \to U_n$ is a representation of H, a similar argument applies to the bundle $\alpha(\theta)$ associated to $G \to G/H$, using theorem 2.

Proposition 4: Let H be a subgroup of G of maximal rank. Let $\theta : H \to U_n$ be a complex representation with weights $\lambda_1,\ldots,\lambda_n$. Then $c_i\alpha(\theta) = [\sigma_i(\lambda_1,\ldots,\lambda_n)]$, i.e. the equivalence class of $\sigma_i(\lambda_1,\ldots,\lambda_n)$ in $(S^*t^*)^{W_H}/I$. □

To deal with Pontrjagin classes, we need to examine the description of H^*BSO_n more closely. If T is the standard maximal torus (see I) we have $H^*BT \cong \mathbb{R}[x_1,\ldots,x_m] \cong S^*t^*$, where $n = 2m+1$ or $n = 2m$. For $n = 2m+1$, the Weyl group (as a transformation group of t^*) is generated by S_m together with the transformations which change the sign of any of $x_1,\ldots,x_m$. Hence $p_i \in H^{4i}BSO_{2m+1}$ may be identified with $\sigma_i(x_1^2,\ldots,x_m^2)$. For $n = 2m$, the Weyl group is generated by S_m together with transformations which change the sign of any even number of the $x_1,\ldots,x_m$. Hence

$p_i \in H^{4i}BSO_{2m}$ is again identified with $\sigma_i(x_1^2,\ldots,x_m^2)$, and q with $x_1\ldots x_m$. Note that this shows the definition of p_i is consistent with respect to the inclusion $SO_{2m} \to SO_{2m+1}$, $A \longmapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$.

Proposition 5: Let $\phi : G \to SO_n$ be a real representation such that $\phi \otimes \mathbb{C}$ has non-zero weights $\pm\mu_1,\ldots,\pm\mu_k$. Then $p_i\alpha_{EG}(\phi) = [\sigma_i(\mu_1^2,\ldots,\mu_k^2)]$. If n is even, then $e\alpha_{EG}(\phi) = [\mu_1\ldots\mu_k]$ if $k = n$, and zero otherwise.

Proof. From the results on real and complex representations of tori given in I, it is clear that the non-zero weights of $\phi \otimes \mathbb{C}$ occur in pairs of the form $\pm\mu$. The method of proof of proposition 3 extends to this situation. □

Proposition 6: Let H be a subgroup of G of maximal rank. Let $\phi : H \to SO_n$ be a real representation such that $\phi \otimes \mathbb{C}$ has non-zero weights $\pm\mu_1,\ldots,\pm\mu_k$. Then $p_i\alpha(\phi) = [\sigma_i(\mu_1^2,\ldots,\mu_k^2)]$. If n is even, then $e\alpha(\phi) = [\mu_1\ldots\mu_k]$ if $k = n$, and zero otherwise.

Proof. As for proposition 4, using the remarks in the proof of proposition 5. □

It is convenient to introduce the total Chern class $c(P) = \sum_{i\geq 0} c_i(P)$ and the total Pontrjagin class $p(P) = \sum_{i\geq 0} p_i(P)$ (where $c_0 = 1$, $p_0 = 1$). Using propositions 4 and 6 we then have

$$c(\alpha(\theta)) = [\prod_{j=1}^{n} (1 + \lambda_j)]$$

$$p(\alpha(\phi)) = [\prod_{j=1}^{k} (1 + \mu_j^2)] \ .$$

If $\phi \otimes \mathbb{C} = \theta$,

$$c(\alpha(\theta)) = [\prod_{j=1}^{k} (1 + \mu_j)(1 - \mu_j)]$$

from which one may deduce the well known relations between Chern and Pontrjagin classes which exist when a real bundle is complexified or vice versa. The formulae for the total Chern and Pontrjagin classes in terms of weights exhibit the "splitting principle", which says that as far as these characteristic classes are concerned, any complex vector bundle behaves like a direct sum of line bundles.

As an application, consider the tangent bundle $T(G/H) \equiv \alpha\ (Ad(G/H))$ of a generalized flag manifold. If $H = T$ (a maximal torus), the representation space of $Ad(G/T)$ is $\mathfrak{m} = \sum_{\pm\alpha_i \in \Delta} V_i$ (see 1.4 of I), so the weights of $Ad(G/T)$ are the roots of G and one has $p(T(G/T)) = [\prod_{\pm\alpha_i \in \Delta} (1 + \alpha_i^2)]$. Since W_H consists of the identity element here, theorem 2 of 3.1 shows that $p(T(G/T)) = 0$. More generally (see [BH]), $\mathfrak{h}$ is always of the form $\mathfrak{t} \oplus \sum_{i \in I} V_i$ for some subset I of $\{1,\dots,\ell\}$, so $\mathfrak{m} = \sum_{i \notin I} V_i$, and the weights of $Ad(G/H) \otimes \mathbb{C}$ are $\{\pm \alpha_i \mid i \notin I\}$ (this subset of Δ is usually called the set of <u>complementary roots</u> for G/H). Then $p(T(G/H)) = [\prod_{i \notin I} (1 + \alpha_i^2)]$ (which is in general non-zero). If a set Δ^+ of positive roots is chosen, G/H acquires an almost complex structure (as explained in 2.3 of II), hence $Ad(G/H) = \theta^{\mathbb{R}}$ for some complex representation θ. The Chern classes of the "complex" tangent bundle $T_{1,0}\ G/H \equiv \alpha(\theta)$ are then given by $c(T_{1,0}\ G/H) = [\prod_{\alpha \in \Delta^+} (1 + \alpha)]$.

(3.3) <u>Homogeneous Differential Operators</u>. In this section and the next we shall follow the article [Bo2] of R. Bott, in which the relation between the Weyl Character Formula (see 1.4 of I) and the Atiyah-Singer Index Theorem is sketched. Let $D : \Gamma(E) \to \Gamma(F)$ be a differential operator, where E and F are complex vector bundles over a compact manifold M. If D is elliptic, which we shall always assume, both $\mathrm{Ker}\ D$ and $\mathrm{Coker}\ D$ are finite dimensional vector spaces, and the <u>index</u> of D is defined to be the integer $\dim(\mathrm{Ker}\ D) - \dim(\mathrm{Coker}\ D)$. For example, if $E = F = \Lambda^i T^*M$ and M is Riemannian, the Hodge-Laplacian $dd^* + d^*d$ is an elliptic operator. (However, because it is also self-adjoint, $\mathrm{Coker}\ D = \mathrm{Ker}\ D^* = \mathrm{Ker}\ D$, and so the index is zero.)

Let $M = G/H$ and assume $E = \alpha(\theta)$, $F = \alpha(\phi)$ for some complex representations θ, ϕ of H on vector spaces V, W. An (elliptic) differential operator $D : \Gamma(E) \to \Gamma(F)$ is said to be <u>homogeneous</u> if it is a G-module transformation, where the G-module structures on $\Gamma(E)$ and $\Gamma(F)$ are defined by the formula $(g\cdot s)(xH) = s(g^{-1}xH)$, as in 2.5 of II. For such an operator, $\mathrm{Ker}\ D$ and $\mathrm{Coker}\ D$ are themselves (finite dimensional) G-modules. The virtual G-module $\mathrm{Ker}\ D - \mathrm{Coker}\ D$ will be denoted $\chi(D)$. The "dimension function" extends to the ring $R(G)$ of virtual G-modules in an obvious way, and we have $\text{index}(D) = \dim \chi(D)$.

<u>Theorem</u>: $\chi(D) = i_*(V - W)$.

<u>Proof</u>. Let α be any (finite dimensional) irreducible complex representation of G on a vector space U. Let $\Gamma_\alpha(E)$ and $\Gamma_\alpha(F)$ denote the (finite dimensional) subspaces of $\Gamma(E)$ and $\Gamma(F)$ respectively consisting of sums of G-modules equivalent to U (c.f. 2.5 of II). Since D is homogeneous, it restricts to a G-module transformation $D_\alpha : \Gamma_\alpha(E) \to \Gamma_\alpha(F)$, and $\chi(D) = \sum_\alpha \chi(D_\alpha)$, where the sum is over the equivalence classes of irreducible representations. Now, if $C : S \to T$ is any transformation of (finite dimensional) G-modules, the exact sequence

$$0 \longrightarrow \text{Ker } C \longrightarrow S \longrightarrow T \longrightarrow \text{Coker } C \longrightarrow 0$$

shows that Ker C - Coker T is equivalent to $S - T$. Hence $\chi(D) \cong \sum_\alpha (\Gamma_\alpha(E) - \Gamma_\alpha(F))$ (since D is elliptic, all but a finite number of terms here are zero). By Frobenius Reciprocity (see 2.5 of II), $\Gamma_\alpha(E) \cong \dim \text{Hom}_H(i^*U,V)U$ and $\Gamma_\alpha(F) \cong \dim \text{Hom}_H(i^*U,W)U$, so $\chi(D) \cong \sum_\alpha \dim \text{Hom}_H(i^*U,V-W)U \cong i_*(V-W)$. □

This is a considerable simplification of the problem of finding index(D). First, the result says that index(D) depends on the bundles E and F and not on D itself. Second, we have shown implicitly that $i_*(V-W)$, which a priori is a formal infinite sum, is actually finite.

(3.4) <u>The Index Theorem</u>. For an elliptic operator $D : \Gamma(E) \to \Gamma(F)$, the Atiyah-Singer Index Theorem expresses index(D) in terms of the characteristic classes of TM and the symbol $\sigma(D)$, the latter being a bundle constructed algebraically from E, F and the highest order part of D. The formula may be written

$$\text{index}(D) = (-1)^{\dim M}(\text{ch } \sigma(D)\text{td}(TM \otimes \mathbb{C}))[TM]$$

although we do not propose to explain this, except to say that ch and td represent combinations of Chern classes which are then evaluated on the homology class $[TM]$. A detailed introduction together with several examples is given in [Sh]. In the homogeneous situation described in 3.3, the role of $\sigma(D)$ is played by $V - W$; the Index Theorem should give a formula for $\text{index}(D) = \dim i_*(V-W)$ in terms of the characteristic classes of $\alpha(\theta)$, $\alpha(\phi)$ and $TG/H \cong \alpha(\text{Ad}(G/H))$, and hence (because of 3.2) in terms of the weights of θ and ϕ and the roots of G. We shall now explain how the Weyl Character Formula performs this function.

The basic point is the interpretation of the Weyl Character Formula in terms of induced representations, due to Bott. Referring back to 1.4 of I, we see that if G is a simply connected group with a fixed inclusion $i : T \to G$ of a maximal torus T, and if Ψ is an irreducible complex representation whose maximal weight $\lambda \in \underline{t}^*$ corresponds to the one dimensional representation Λ of T, then the formula may be written

$$(i^*\Psi)\Omega = \sum_{w\in W} \sigma(w)\ w\cdot(\Lambda + \Delta)$$

where Δ is the representation of T corresponding to δ, and $\Omega = \sum_{w\in W} \sigma(w)\ w\cdot\Delta$. This must be interpreted as a formula in the ring $R(T)$ of virtual representations of T. We need to quote the following additional facts, all of which may be found in [Ad].

(1) $i^* : R(G) \to R(T)$ is injective, the image being the subring $I(T) = R(T)^W$ consisting of invariants of W.

(2) The subgroup $A(T) = \{\Psi \in R(T) \mid w\cdot\Psi = \sigma(w)\Psi\}$ of "alternating elements" is a free module over $I(T)$, generated by Ω.

(3) If α_1, α_2 are representations of G on complex vector spaces U_1, U_2, the form $\langle\ ,\ \rangle_G$ defined by $\langle\alpha_1,\alpha_2\rangle_G = \dim \mathrm{Hom}_G(U_1,U_2)$ extends to a non-degenerate symmetric bilinear form on $R(G)$, relative to which a set of representatives of the equivalence classes of irreducible representations is an orthonormal basis.

(4) The "Weyl Integration Formula" holds: $\langle\alpha_1,\alpha_2\rangle_G = (1/|W|)\langle i^*\alpha_1, \Omega\bar{\Omega}\ i^*\alpha_2\rangle_T$.

<u>Theorem</u>: If $\Gamma \in R(T)$, $\Omega(i^*i_*(\bar{\Omega}\Gamma)) = \sum_{w\in W} \sigma(w)w\cdot\Gamma$.

<u>Proof</u>. For any $\Gamma' \in R(T)$, $\langle\Gamma', i_*(\bar{\Omega}\Gamma)\rangle_G = \langle i^*\Gamma' \in \bar{\Omega}\Gamma\rangle_T$ (by Frobenius Reciprocity). Since $i^*\Gamma'$ and $\langle\ ,\ \rangle_T$ are W-invariant, one may write this

as $(1/|W|) \sum_{w\in W} \langle i^*\Gamma', w\cdot(\bar{\Omega}\Gamma)\rangle_T$. As $\bar{\Omega} \in A(T)$, $w\cdot\bar{\Omega} = \sigma(w)\bar{\Omega}$. So $\sum_{w\in W} w\cdot(\bar{\Omega}\Gamma) = \sum_{w\in W} \sigma(w)(w\cdot\Gamma) = \Omega\ i^*\Gamma''$ for some $\Gamma'' \in R(G)$ (by (1) and (2)). Hence $\langle\Gamma', i_*(\bar{\Omega}\Gamma)\rangle_G = (1/|W|)\langle i^*\Gamma', \bar{\Omega}\Omega\Gamma''\rangle_G = \langle\Gamma',\Gamma''\rangle_T$ by (4)). So by (3), $i_*(\bar{\Omega}\Gamma) = \Gamma''$. Applying i^* gives $i^*i_*(\bar{\Omega}\Gamma) = i^*\Gamma''$, and multiplying by Ω gives $\Omega(i^*i_*(\Omega\Gamma)) = \Omega i^*\Gamma'' = \sum_{w\in W} \sigma(w)(w\cdot\Gamma)$, as required. □

Thus, if λ is the maximal weight of Ψ, this theorem says that

$$i_*\Lambda = \Psi,$$

i.e. the procedure of passing from the maximal weight to the irreducible representation is precisely that of taking the induced representation.

The theorem extends to the situation where T is replaced by a subgroup $H \supseteq T$. Let the inclusions be denoted $a : H \to G$, $b : T \to H$, and let $\Omega(H)$, $\Omega(G)$ be $\sum_{w\in W_H} \sigma(w)\ w\cdot\Delta$, $\sum_{w\in W_G} \sigma(w)\ w\cdot\Delta$ respectively. Define $\Omega = \Omega(G/H) \in R(H)$ by $\Omega(H)\ b^*\Omega = \Omega(G)$. If $\Gamma \in R(H)$, one has

$$\Omega(G)(i^*a_*(\bar{\Omega}\Gamma)) = \sum_{\sigma\in W_G/W_H} \sigma(w)\ w\cdot(\Omega(H)b^*\Gamma)$$

(see [Bo2] for the proof). In particular, if Γ is an irreducible representation of H, $a_*(\bar{\Omega}\Gamma)$ is an irreducible representation of G. The final step in obtaining the index formula is to interpret this formula (or at least, the result of taking dimensions) in terms of characteristic classes. We shall simply quote the result,

$$\dim a_*(\overline{\Omega}\Gamma) = T(\xi)ch(\alpha(\Gamma))[G/H],$$

in which $T(\xi)$ is a characteristic class defined in terms of a certain homogeneous differential operator L which has the property $\chi(L) = \Omega$.

(3.5) Notes. Useful references for characteristic classes, from various points of view, are [BT], [Hi], [MS], [Sw]. The results on the cohomology of Lie groups are basically due to A. Borel - see [Br] for a survey and further references. The connection with representation theory is developed in the series of papers [BH] of A. Borel and F. Hirzebruch. For further results on homogeneous bundles, and in particular the Borel-Weil-Bott theorem, see [Bo3].

IV. Minimal Immersions and Harmonic Maps

(4.1) The Second Fundamental Form. Let E be a real vector bundle over a manifold M with connection $\nabla^E : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$. We write $\nabla^E s(X) = \nabla^E_X s$ for $s \in \Gamma(E)$, $X \in \Gamma(TM)$. If F is another bundle with connection ∇^F, a connection ∇ in $E^* \otimes F$ may be defined in a standard way by the formula $(\nabla_X \phi)s = \nabla^F_X \phi(s) - \phi(\nabla^E_X s)$.

Now suppose M is Riemannian, with associated (Levi-Cività) connection ∇^{TM}. Replacing $E^* \otimes F$ by $T^*M \otimes E$ in the last paragraph we obtain $\nabla : \Gamma(T^*M \otimes E) \longrightarrow \Gamma(\otimes^2 T^*M \otimes E)$. Repeating the process we obtain $\nabla : \Gamma(\otimes^i T^*M \otimes E) \longrightarrow \Gamma(\otimes^{i+1} T^*M \otimes E)$ for any $i > 0$; it is usual to call all these operators ∇^E (regarding them as natural extensions of the original ∇^E). The contraction $\Delta^E = tr(\nabla^E)^2 : \Gamma(E) \longrightarrow \Gamma(E)$ is called the Laplacian on sections on E (with respect to a local frame field $X_1, \ldots, X_m$ for M, $\Delta^E(s) = \sum_{i=1}^{m} (\nabla^E)^2(X_i, X_i)(s)$). For example, if $E = M \times \mathbb{R}$ is the trivial bundle, with connection defined by $\nabla^E_X f = Xf$, Δ^E is minus the Hodge-Laplacian on functions introduced in II.

If $f : M \longrightarrow N$ is a map of (oriented) Riemannian manifolds whose connections are ∇^{TM}, ∇^{TN}, a related construction is possible using the fact that $df : TM \to TN$ defines an element $df \in \Gamma(T^*M \otimes f^*TN)$. For f^*TN we take the connection ∇^{f^*TN} induced by ∇^{TN}, which is characterized by the formula $\nabla^{f^*TN}_X(Y \circ f) = (\nabla^{TN}_{df(X)} Y) \circ f$ (see 1.7 of [EL2]). Using the formula of the first paragraph, we obtain $\nabla(df) \in \Gamma(\otimes^2 T^*M \otimes f^*TN)$.

Definition: The second fundamental form of $f : M \to N$ is the tensor field $\nabla(df)$; by definition $\nabla(df)(X,Y) = \nabla_X^{f^*TN} df(Y) - df(\nabla_X^{TM} Y)$ for $X, Y \in \Gamma(TM)$.

Note that $\operatorname{tr} \nabla(df)$ is just the Laplacian if $N = \mathbb{R}$. In general, it is an f^*TN-valued (symmetric) two form on M, and is called the tension field (see 2.5 of [EL2] for an expression in terms of local coordinates).

Definition: A map $f : M \to N$ is said to be harmonic if $\operatorname{tr} \nabla(df) = 0$.

It turns out (2.4 of [ES2]) that f is harmonic precisely when it is an extremum for the energy functional

$$E(f) = \frac{1}{2} \int_M \|df\|^2 .$$

In other words the equation $\operatorname{tr} \nabla(df) = 0$ is the Euler-Lagrange equation for E.

An important special case is the classical concept of a minimal immersion. Suppose $f : M \to N$ is an immersion which is also isometric, i.e. $f^*h = g$ where g, h are the metrics on M, N respectively. Then it is easy to show that $\nabla(df)$ takes values in the normal bundle $\nu \subseteq f^*TN$, and the expression $(1/m)\operatorname{tr} \nabla(df)$ is known as the mean curvature normal field.

Definition: An isometric immersion is said to be minimal if its mean curvature normal field vanishes.

It is well known (see §2 of [ES]) that an isometric immersion is minimal if and only if it is an extremum for the volume functional

$$V(f) = \int_M (\det f^*h)^{1/2} .$$

Of course, an isometric immersion is minimal if and only if it is harmonic. However, it is important to realise that the concepts of harmonic and minimal arise from different variational problems: in the former case, one fixes metrics g and h and considers (the energy of) all maps, whereas in the latter case one fixes only h and considers (the volume of) immersions, taking the induced metric f^*h on M.

(4.2) Homogeneous Maps. Now let $M = G/H$, $N = G'/H'$ be homogeneous spaces, with invariant Riemannian metrics whose associated Levi-Cività connections are given by linear maps $\Lambda : \otimes^2 \mathfrak{m} \to \mathfrak{m}$, $\Lambda' : \otimes^2 \mathfrak{m}' \to \mathfrak{m}'$ as explained in II. Let $\theta : G \to G'$ be a homomorphism such that $\theta(H) \subseteq H'$, and let $f_\theta : G/H \to G'/H'$ be the induced map. We shall refer to such maps as homogeneous maps. The second fundamental form is determined by its behaviour at the identity coset $o \in G/H$:

Lemma: For $X, Y \in \mathfrak{m}$, $\nabla(df_\theta)(X,Y) = [\theta(X)_{\mathfrak{h}'}, \theta(Y)_{\mathfrak{m}'}] + \Lambda'(\theta(X)_{\mathfrak{m}'}, \theta(Y)_{\mathfrak{m}'}) - \theta(\Lambda(X,Y))_{\mathfrak{m}'}$, where the suffices denote the appropriate components of the vectors.

Proof. We shall write f instead of f_θ to simplify notation. By definition, $\nabla(df)(X,Y) = \nabla(df)(X^*,Y^*)_o$, and $\nabla(df)(X^*,Y^*) = \nabla_{X^*}^{f^*TN} df(Y^*) - df(\nabla_{X^*}^{TM} Y^*)$. Since $df(Y^*) = \theta(Y)^* \circ f$, $\nabla_{X^*}^{f^*TN} df(Y^*) = (\nabla_{df(X)}^{TN} \theta(Y)^*) \circ f = (\nabla_{\theta(X)^* \circ f}^{TN} \theta(Y)^*) \circ f$. Hence $\nabla(df)(X,Y) = (\nabla_{\theta(X)^*}^{TN} \theta(Y)^*)_o - df(\nabla_{X^*}^{TM} Y^*)_o$. The second term is evaluated by noting that $df(X) = \theta(X)_{\mathfrak{m}'}$ and

$(\nabla^{TM}_{X^*} Y^*)_o = \Lambda(X,Y) - [X,Y]_{\mathfrak{m}}$. To evaluate the first term we need an expression for $(\nabla^{TN}_{U^*} V^*)_o$ when $U,V \in \mathfrak{g}'$. Writing $U = U_{\mathfrak{h}'} + U_{\mathfrak{m}'}$ we may replace U^* by $(U_{\mathfrak{m}'})^*$ since $U^*_o = (U_{\mathfrak{m}'})^*_o$ and $\nabla_A B$ is tensorial in A. Writing $V = V_{\mathfrak{h}'} + V_{\mathfrak{m}'}$ we obtain $(\nabla^{TN}_{U^*} V^*)_o = (\nabla^{TN}_{(U_{\mathfrak{m}'})^*} (V_{\mathfrak{h}'})^*)_o - [U_{\mathfrak{m}'}, V_{\mathfrak{m}'}]_{\mathfrak{m}'} + \Lambda'(U_{\mathfrak{m}'}, V_{\mathfrak{m}'})$. Since $\nabla_A B - [A,B]$ is tensorial in B, and $(V_{\mathfrak{h}'})^*_o = 0$, we have $(\nabla^{TN}_{(U_{\mathfrak{m}'})^*} (V_{\mathfrak{h}'})^*)_o = [(U_{\mathfrak{m}'})^*, (V_{\mathfrak{h}'})^*]_o = -[U_{\mathfrak{m}'}, V_{\mathfrak{h}'}]$. Thus $(\nabla^{TN}_{U^*} V^*)_o = -[U_{\mathfrak{m}'}, V]_{\mathfrak{m}'} + \Lambda'(U_{\mathfrak{m}'}, V_{\mathfrak{m}'})$. The required formula follows. □

We shall only use this formula here when g and h are the metrics $\langle\, ,\,\rangle$ given by minus the Killing forms. In this case $\Lambda(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}}$ and similarly for Λ', so we obtain:

$$\nabla(df_\theta)(X,Y) = \frac{1}{2}\left([\theta(X)_{\mathfrak{h}'}, \theta(Y)_{\mathfrak{m}'}] - [\theta(X)_{\mathfrak{m}'}, \theta(Y)_{\mathfrak{h}'}]\right)$$

This gives the following simple criterion for harmonicity.

<u>Proposition</u>: Let $X_1, \ldots, X_m$ be a basis for $\mathfrak{m}$, orthonormal with respect to $\langle\, ,\,\rangle$. The map $f_\theta : G/H \to G'/H'$ is harmonic with respect to the metrics $\langle\, ,\,\rangle$ if and only if $\sum_{i=1}^{m} [\theta(X_i)_{\mathfrak{h}'}, \theta(X_i)_{\mathfrak{m}'}] = 0$. □

<u>Example 1</u>: Let $\theta : G \longrightarrow U_{n+1}$ be a representation of G on $\mathbb{C}^{n+1}$, let $V \in \mathbb{C}^{n+1}$ $(V \neq 0)$, let H be contained in the isotropy subgroup of $[V] \in \mathbb{C}P^n$ for the induced action of G on $\mathbb{C}P^n$, and let $H' \cong U_1 \times U_n$ be the isotropy subgroup of $[V]$ for the action of U_{n+1} on $\mathbb{C}P^n$. Thus one has

$f_V = f_\theta : G/H \longrightarrow \mathbb{C}P^n \cong (U_{n+1}/(U_1 \times U_n))$, whose image is the orbit $G\cdot[V]$. We shall express the condition that f_V be harmonic in terms of the vector V. Let $L = [V]$, so that $\mathbb{C}^{n+1} = L^\perp \oplus L$. The description of $Ad(U_{n+1}) \otimes \mathbb{C}$ at the end of 1.2 of I gives identifications:

$$\mathfrak{h}' \otimes \mathbb{C} \cong Hom(L,L) \oplus Hom(L^\perp,L^\perp)$$
$$\mathfrak{m}' \otimes \mathbb{C} \cong Hom(L,L^\perp) \oplus Hom(L^\perp,L)\,,$$

with respect to which we write $X = X_1 + X_2$ for $X \in \mathfrak{h}' \otimes \mathbb{C}$, $Y = Y_1 + Y_2$ for $Y \in \mathfrak{m}' \otimes \mathbb{C}$. In terms of matrices:

$$X = \begin{pmatrix} X_2 & 0 \\ 0 & X_1 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & Y_1 \\ Y_2 & 0 \end{pmatrix}.$$

From this we obtain $[X,Y] = (X_2 \circ Y_1 - Y_1 \circ X_1) + (X_1 \circ Y_2 - Y_2 \circ X_2)$. With $X = \theta(A)_{\mathfrak{h}'}$, $Y = \theta(B)_{\mathfrak{m}'}$, the element $X_2 \circ Y_1 - Y_1 \circ X_1 \in Hom(L,L^\perp)$ is the operator $p^\perp\theta(A)p\theta(B)p - p^\perp\theta(B)p^\perp\theta(A)p$, where p, $p^\perp$ are the orthogonal projections on L, $L^\perp$ respectively. Hence f_V is harmonic if and only if $0 = \sum_{i=1}^{m} p^\perp\theta(X_i)(p - p^\perp)\theta(X_i)p \in Hom(L,L^\perp)$ and $0 = \sum_{i=1}^{m} p^\perp\theta(X_i)(p^\perp - p)\theta(X_i)p \in Hom(L^\perp,L)$, where $X_1, \ldots, X_n$ is an orthonormal basis for $\mathfrak{m}$. (These two conditions are the same - the duplication arises because we complexified the real second fundamental form to facilitate computation.) In other words:

(*) $f_V : G/H \longrightarrow \mathbb{C}P^n$ is harmonic if and only if V is an eigenvector of the operator $\sum_{i=1}^{m} \theta(X_i)(p^\perp - p)\theta(X_i)$.

If V is a weight vector for θ, i.e. $V \in W_\lambda$ for some weight λ with respect to a maximal torus T, we may take $H = T$. Choosing $\{X_i \mid 1 \le i \le m\}$ to be $\{(1/\sqrt{2})(E_\alpha - E_{-\alpha}), \sqrt{(-1/2)}(E_\alpha + E_{-\alpha}) \mid \alpha \in \Delta^+\}$, the condition becomes that V should be an eigenvector of the operator $\sum_{\alpha\in\Delta} \theta(E_\alpha)(p^\perp - p)\theta(E_{-\alpha})$. Since $\theta(E_{-\alpha})V$ is either zero or a weight vector with weight $\lambda - \alpha$, it is orthogonal to V, so the operator reduces essentially to $\sum_{\alpha\in\Delta} \theta(E_\alpha)\theta(E_{-\alpha})$, which has the same eigenvectors as the Casimir operator. Thus, f_V is harmonic if and only if V is an eigenvector of the Casimir operator (for θ). For example, if θ is irreducible, f_V is harmonic for any weight vector V.

Many examples of harmonic maps from homogeneous spaces to Grassmannians and generalized flag manifolds $F(m_1,\dots,m_k)$ may be produced using weight vectors of representations as in the last paragraph (see [Gu]).

Example 2: Let θ be a representation of G either (a) on $\mathbb{R}^{n+1}$ or (b) on $\mathbb{C}^{n+1}$. Let V be a vector in the unit sphere $S^n \cong SO_{n+1}/SO_n$ or $S^{2n+1} \cong U_{n+1}/U_n$, and H be the isotropy subgroup of V for the action of G on the sphere. The condition for the embedding f_V of the orbit $G\cdot V$ to be harmonic may be written down as in example 1. One obtains:

(*) f_V is harmonic if and only if V is an eigenvector of the operator $\sum_{i=1}^{m} \theta(X_i)p^\perp\theta(X_i)$.

If θ is irreducible this represents a non-trivial condition in case (b), but no condition at all in case (a) since $\theta(X_i)$ is skew-symmetric and so $\theta(X_i)V$ is always orthogonal to V (the operator then has the same eigenvectors as the Casimir operator, which we are assuming (see 2.4 of II) is a scalar operator). Thus, all orbits in the unit sphere of an irreducible real representation are harmonic.

This is not true for a complex representation (see below for an example). Note that if θ is a complex representation on $\mathbb{C}^{n+1}$, its orbits in S^{2n+1} are precisely the same as those of the underlying real representation, and the Casimir operators of θ and $\theta^{\mathbb{R}}$ are identical. Hence if θ is irreducible, the orbits are all harmonic with respect to the metric $\langle\ ,\ \rangle$ on S^{2n+1} coming from the identification $S^{2n+1} \cong SO_{2n+2}/SO_{2n+1}$, but are not in general harmonic with respect to the metric coming from the identification $S^{2n+1} \cong U_{n+1}/U_n$. For example, consider the representation $\theta = S^n\lambda : SU_2 \to U(P_n)$ of SU_2 on the space P_n of homogeneous polynomials in e_0, e_1 of degree n, where explicit computations are possible. All orbits are locally isomorphic to SU_2, except for that of the zero weight vectors (when n is even), which is locally isomorphic to S^2. It turns out that the orbits which are harmonic with respect to the metric defined by $S^{2n+1} \cong U_{n+1}/U_n$ are precisely the orbits of the weight vectors.

We conclude this section with some remarks on the relation between energy and volume for orbits of representations. If $\theta : G \to SO_{n+1}$ is an irreducible representation, one may consider for each $V \in S^n$ the composition

$$G \xrightarrow{\pi} G/H \xrightarrow{f_V} S^n$$

where π is the natural projection. As in example 2, all the maps $f_V\circ\pi$ are harmonic, hence they must have the same energy as they form a connected set of critical points of the energy functional. (In contrast, the orbits of θ in S^n do not in general have the same volume.) This may be seen directly by calculating the energy $E(f_V) = (1/2)\int_G \|df_V\|^2 = (1/2)\int_G \operatorname{tr} f_V^* h$ where h is the metric $\langle\ ,\ \rangle$ on S^n. Relative to the metric $\langle\ ,\ \rangle$ on $\mathfrak{g}$, the pull-back $f_V^* h$ is given by the matrix $(\langle\langle\theta(X_i)V,\ \theta(X_j)V\rangle\rangle)$ where $X_1, \dots, X_m$ is an orthonormal basis for $\mathfrak{g}$ and $\langle\langle\ ,\ \rangle\rangle$ is the inner product on $\mathbb{R}^{n+1}$. Hence $\operatorname{tr} f_V^* h = \sum_{i=1}^{m} \langle\langle\theta(X_i)V, \theta(X_i)V\rangle\rangle = \sum_{i=1}^{m} -\langle\langle\theta(X_i)^2 V,\ V\rangle\rangle = \langle\langle CV, V\rangle\rangle$ where C is the Casimir operator for θ. Thus the energy is given essentially by the Casimir constant. (Note that the formula $E(f_V) = (1/2)\int_G \langle\langle CV, V\rangle\rangle = (1/2)\langle CV, V\rangle \mathrm{Vol}(G)$ holds for any real θ, irreducible or not. The critical points of the function $S^n \longrightarrow \mathbb{R}$, $V \longmapsto \langle\langle CV, V\rangle\rangle$ are just the eigenvectors of C, as predicted by example 2.) As a specific example, consider the representation $\theta = \mathrm{Ad}$, which is irreducible for a simple group G. The orbits in the unit sphere $S(\mathfrak{g}) \subseteq \mathfrak{g}$ are parametrized by the points of $A = D \cap S(\mathfrak{g})$, where D is the fundamental Weyl chamber (see 1.4 of I). Let $M \cong G/H$ be the orbit of any point $x \in A$. Then $\mathfrak{h} = \mathfrak{t} \oplus \sum_{i\in I} V_i$ where $I = \{i \mid \pm\alpha_i(x) = 0\}$. If we take the orthonormal basis $\{(1/\sqrt{2})(E_{\alpha_i} - E_{\alpha_i})$, $\sqrt{(-1/2)}(E_{\alpha_i} + E_{\alpha_i}) \mid i \notin I\}$ for $\mathfrak{m}$, the induced metric on $\mathfrak{m}$ is given by the diagonal matrix

$$\begin{pmatrix} \ddots & & & \\ & \alpha_i(x)^2 & & \\ & & \alpha_i(x)^2 & \\ & & & \ddots \end{pmatrix} \quad (i \notin I).$$

Hence $E(f_V) = \mathrm{Vol}(G) \sum_{i\notin I} \alpha_i(x)^2 = \mathrm{Vol}(G) \sum_{\alpha\in\Delta^+} \alpha(x)^2$. This is easily seen to be constant, for $\sum_{\alpha\in\Delta} \alpha(x)^2 = (-1/4\pi^2) \sum_{\alpha\in\Delta} \hat{\alpha}(x)^2 = (-1/4\pi^2)\operatorname{tr}(\operatorname{ad}x\ \operatorname{ad}x) = (1/4\pi^2)\langle x,x\rangle = 1/4\pi^2$. On the other hand, $V(f_V) = \int_M (\det f^*h)^{1/2} = \mathrm{Vol}(G) \prod_{i\notin I} \alpha_i(x)^2$. For general results on the volume functional restricted to orbits of representations see [HL].

(4.3) <u>Minimal Immersions of Homogeneous Spaces in Spheres</u>. In the 1960's, M. do Carmo and N. R. Wallach (see [Wa]) studied minimal isometric immersions of homogeneous spaces (in particular, spheres) into the sphere $S^n \cong SO_{n+1}/SO_n$ with its metric $\langle\ ,\ \rangle$. They made use of the following criterion for minimality due to T. Takahashi (see [Ta]).

<u>Proposition</u>: Let M be a Riemannian manifold of dimension m, and let rS^n be the sphere of radius r in $\mathbb{R}^{n+1}$ with metric induced from the Euclidean metric $\langle\langle\ ,\ \rangle\rangle$ of $\mathbb{R}^{n+1}$. Let $f : M \longrightarrow rS^n$ be an isometric immersion. The map f is minimal if and only if $i\circ f$ is an eigenfunction of Δ, where $i : rS^n \longrightarrow \mathbb{R}^{n+1}$ is the inclusion and Δ is the Laplacian on M (on $\mathbb{R}^{n+1}$ - valued functions). Moreover, the eigenvalue concerned is necessarily $-m/r^2$.

<u>Proof</u>. First, by the remarks in 4.1, $\Delta(i\circ f) = \operatorname{tr} \nabla d(i\circ f)$. Next, $\nabla(i\circ f)(X,Y) = di(\nabla df(X,Y)) + \nabla di(df(X),df(Y))$ for $X,Y \in \Gamma(TM)$ (2.20 of [EL2]). Finally, $\nabla di(U,V)_x = -\langle\langle U,V\rangle\rangle x$, by direct calculation. If f is minimal, $i\circ f$ is clearly an eigenfunction of Δ. The converse is true because $x \perp \mathrm{Im}(di)_x$. The eigenvalue is $-\langle\langle df, df\rangle\rangle = -m/r^2$. We are using the fact that the metric on rS^n is $(1/r^2)\langle\ ,\ \rangle$. □

The next result is proposition 8.1 of [Wa].

Proposition: Let $\theta : G \longrightarrow SO_{n+1}$ be a class 1 representation of (G,H), with $\theta(H)V = V$ for some $V \in \mathbb{R}^{n+1}$, such that the induced map $f_V : G/H \longrightarrow S^n$ is an immersion. Let the isotropy action of H (i.e. the representation $Ad(G/H)$) be irreducible. Then f_V is a minimal isometric immersion with respect to a multiple of the metric $\langle\ ,\ \rangle$ on G/H.

Proof. The induced metric on G/H is homogeneous, hence is a multiple of $\langle\ ,\ \rangle$ since $Ad(G/H)$ is irreducible. By the corollary to the proposition of 2.4 of II, $i \circ f_V$ is an eigenfunction of Δ. Hence the result follows from the previous proposition. □

This and example 2 of 4.2 amount to the same statement, of course. The significance of the second approach is that a converse result may be proved, which leads to a classification of certain types of minimal immersions. We shall give a brief indication of this, referring to [Wa] for more details.

Let G/H be symmetric. It is a theorem of E. Cartan (see theorem 9.1 of [Wa]) that each finite dimensional irreducible submodule V_α of $C^\infty(G/H)$ occurs exactly once. Let X be any point in the unit sphere $S(V_\alpha)$ of V_α. The map f_α defined by $f_\alpha(gH) = g \cdot X$ is then an immersion of G/H into $S(V_\alpha)$, called the standard minimal immersion associated to V_α.

Theorem (Theorem 9.1 of [Wa]): Let $f : G/H \longrightarrow S^n$ be a minimal isometric immersion of the symmetric space G/H (with metric a multiple of $\langle\ ,\ \rangle$) into

the sphere $S^n \cong SO_{n+1}/SO_n$ (with its metric $\langle\ ,\ \rangle$). Assume that $f(G/H)$ is not contained in a great sphere of S^n. Then for some α there is a linear isometric injection $A : \mathbb{R}^{n+1} \longrightarrow V_\alpha$, and a linear map $B : V_\alpha \longrightarrow V_\alpha$, such that $A \circ f = B \circ f_\alpha$.

Proof. Let $i : S^n \longrightarrow \mathbb{R}^{n+1}$ be the inclusion. As f is minimal, $\Delta(i \circ f) = \lambda(i \circ f)$ for some $\lambda < 0$. The linear map $\mathbb{R}^{n+1} \longrightarrow C^\infty(G/H)$ defined by $i \circ f$ is injective (since $f(G/H)$ is not contained in a great sphere), and its image is contained in the λ-eigenspace of Δ. This is an irreducible G-module whose Casimir constant is λ, which without loss of generality we may take to be V_α. The result follows. □

(4.4) Non-homogeneous maps. The theorem of the last section shows that the standard minimal immersions $f_\alpha : G/H \longrightarrow SO(V_\alpha)$, which are examples of what in 4.2 we have called homogeneous maps, play a fundamental role in the description of arbitrary isometric minimal immersions $G/H \longrightarrow S^n$. Indeed, do Carmo and Wallach use the theorem to show that isometric minimal immersions between certain spheres may be parametrized by compact convex subspaces of Euclidean space. The homogeneous maps $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^n$, induced by the irreducible representations $\{S^m\Lambda \mid m > 0\}$ of SU_2, play an analogous role in the classification of arbitrary harmonic maps $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^n$. For details of the classification we refer to [EW] and for the relation with homogeneous maps to [Gu].

(4.5) Notes. For connections and differential geometry in general we refer to [KN]. The study of harmonic maps began with [ES]; we recommend [EL1] and [EL2] for a modern introduction together with basic facts concerning connections. Many references on the much older subject of minimal immersions can be found in [KN]. A recent survey is [La]. The results of do Carmo and Wallach on minimal immersions of spheres generalized work of E. Calabi (see [Ca]) on minimal immersions $S^2 \longrightarrow S^n$. It was also [Ca] which, several years later, provided motivation for the methods used in [EW] to study harmonic maps $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^n$. The subject of harmonic maps from $\mathbb{C}P^1$ to homogeneous spaces was initially revived and is currently being developed, partly because it represents the "σ-model" of mathematical physics, which exhibits some analogies with Yang-Mills theory (see V). For information on this, see [Ha], especially the articles of J. P. Bourguignon and D. Burns.

V. The Yang-Mills Equations for Homogeneous Bundles

(5.1) The Curvature Tensor. Let E be a real vector bundle with a connection ∇^E, over a Riemannian manifold with its Levi-Cività connection. In 4.1 of IV we defined the extended operator $\nabla^E : \Gamma(\otimes^i T^*M \otimes E) \to \Gamma(\otimes^{i+1} T^*M \otimes E)$. By anti-symmetrizing, one obtains a new operator

$$D^E : \Gamma(\Lambda^i T^*M \otimes E) \to \Gamma(\Lambda^{i+1} T^*M \otimes E).$$

This is in fact independent of the connection on M: it is given explicitly by the formula $D^E(\omega \otimes s) = \omega \wedge \nabla^E s + d\omega \otimes s$. The composition $D^E \nabla^E : \Gamma(E) \to \Gamma(\Lambda^2 T^*M \otimes E)$ is $C^\infty(M)$-linear, hence it determines an element $F(\nabla^E) \in \Gamma(\Lambda^2 T^*M \otimes E^* \otimes E)$, which is called the curvature tensor of the connection ∇^E. If $E = TM$ and ∇^E is the Levi-Cività connection, we have $E^* \cong E$ and $F(\nabla^E)$ lies in $\Gamma(\Lambda^2 T^*M \otimes \Lambda^2 TM)$; this is the classical curvature tensor of the Riemannian Manifold M. Similar definitions apply to a connection $\nabla^E : \Gamma(E) \to \Gamma((T^*M \otimes \mathbb{C}) \otimes E)$ in a complex vector bundle E. The curvature tensor is now an element of $\Gamma(\Lambda^2(T^*M \otimes \mathbb{C}) \otimes E^* \otimes E)$.

If V is any (real or complex) vector bundle on M, we shall use the notation

$$\Omega^i V = (\Lambda^i T^*M \otimes V) \quad \text{or} \quad \Gamma(\Lambda^i (T^*M \otimes \mathbb{C}) \otimes V)$$

for the space of (real or complex) "V-valued i-forms on M". Thus $F(\nabla^E) \in \Omega^2 E^* \otimes E$. If V has a connection ∇^V, then one has as usual the extended operator $D^V : \Omega^i V \to \Omega^{i+1} V$. Taking $V = E^* \otimes E$, the Bianchi identity says that $D^{E^*\otimes E} F(\nabla^E) = 0$. This is a formal consequence of the definitions. Now suppose V has a Riemannian structure, and that M is compact and oriented. Then $\Lambda^i(T^*M(\otimes \mathbb{C})) \otimes V$ acquires a Riemannian structure $\langle\langle\ ,\ \rangle\rangle$ and we may define an inner product $(\ ,\ )$ on $\Omega^i V$ by $(\alpha,\beta) = \int_M \langle\langle \alpha,\beta \rangle\rangle$. Using this we obtain the adjoint of D^V, $(D^V)^* : \Omega^i V \to \Omega^{i-1} V$. The (generalized) Hodge-Laplacian is defined to be $(D^V)(D^V)^* + (D^V)^*(D^V)$, and a form in $\Omega^i V$ is said to be harmonic if it is annihilated by this operator. When V is the trivial bundle with its standard connection, we obtain the Hodge-Laplacian on forms defined in II. Finally, if $* : \Omega^i V \to \Omega^{\dim M - i} V$ is the Hodge star operator (characterized by the property that $\theta \wedge (* \theta)$ is (θ,θ) times the volume form of M), we have the usual formula $(D^V)^* = (-1)^{1 + (1 + i)\dim M} * D^V *$ on $\Omega^i V$.

(5.2) Connections in Principal Bundles. We shall discuss the Yang-Mills equations in the context of principal bundles. In this section we shall merely introduce the relevant notation and point out the relation with 5.1, referring to [KN], volume I, Chapter II and [AB], §3, for further explanation. A connection form in a principal G-bundle $\pi : P \to M$ is a $\mathfrak{g}$-valued 1-form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ which satisfies two conditions. First, for all $X \in \mathfrak{g}$, $\omega(X') = X$, where $X' \in \Gamma(T^*P)$ in the vector field defined by $X'_p = (d/dt)(p \exp tX)|_{t=0}$. Second, $\omega(dR_g(X)) = Ad(g)^{-1}\ \omega(X)$ for all $g \in G$, $X \in \Gamma(TP)$, i.e. ω is "pseudo-tensorial". Given such an ω, one has the "horizontal subbundle" $\operatorname{Ker} \omega \subseteq TP$, and to any vector $X \in T_m M$ and $p \in \pi^{-1}(m)$ one may associate the "horizontal lift" $\tilde{X} \in T_p P$. The curvature form of ω is the 2-form $\Omega \in \Gamma(\Lambda^2 T^*P \otimes \mathfrak{g})$ defined by $\Omega = d\omega + \frac{1}{2}[\omega,\omega]$. This is "tensorial", i.e. it descends to a section F of $\Lambda^2 T^*M \otimes Ad(P)$, where $Ad(P)$ is the bundle associated to P by the representation Ad(G) of G. Note that ω itself does not in general descend to a section of $T^*M \otimes Ad(P)$, although it may be represented locally as a $\mathfrak{g}$-valued 1-form on M.

If $G = SO_n$, the vector bundle E associated to P (by the standard representation $\Lambda : SO_n \to SO_n$) has a Riemannian metric; P is recovered from E as the bundle of orthogonal frames. There is a bijective correspondence between connection forms ω in P and connections ∇ in E which are compatible with the metric (i.e. $X\langle\langle s,t\rangle\rangle = \langle\langle \nabla_X s,t\rangle\rangle + \langle\langle s,\nabla_X t\rangle\rangle$), which is given explicitly in terms of a local orthonormal frame field $s_1,\ldots,s_n$ of E by $\nabla_X s_i = \sum_{j=1}^{n} \omega_{ij}(X)s_j$ where (ω_{ij}) is a local representation of ω on M. The conditions of being compatible with the metric shows that the curvature tensor $F(\nabla)$ of 5.1 actually lies in $\Omega^2\Lambda^2 E$ (note that $E^* \cong E$ here), and in fact this coincides with the form $F \in \Omega^2 Ad(P)$ defined above, using $Ad(P) \cong \Lambda^2 E$ (see 1.1 of I). If $G = U_n$, similar remarks apply to the Hermitian vector bundle E associated to P. Here, $Ad(P) \otimes \mathbb{C} \cong \overline{E} \otimes E$, and the compatibility of a connection ∇ in E with the Hermitian metric says that the complex curvature form $F(\nabla) \in \Omega^2\ E^* \otimes E \cong \Omega^2\ \overline{E} \otimes E$ is the $\mathbb{C}$-linear extension of the form $F \in \Omega^2 Ad(P)$. Following our usual convention regarding $\mathbb{C}$-linear extension of tensors, we shall just write $F = F(\nabla)$.

More generally, if θ is a representation of G, one obtains from a connection form in P a connection in the associated vector bundle E which preserves a "G-structure", and P is recoverable from E as a subbundle of the frame bundle of E (at least, if θ is injective). For example, if $\theta = Ad(G)$, a connection form ω in P gives rise to a connection D in Ad(G). The Bianchi identity now says that $DF = 0$ (it should be noted that both D and F depend on ω here), which reduces to the formula given above when $G = SO_n$ or U_n.

We shall be interested in the space of all connection forms $\mathcal{A}(P)$ in a fixed principal bundle P. Although a connection form is not tensorial, the difference of two such is, and hence induces an element of $\Omega^1 Ad(P)$. Thus, if we fix a connection form $\omega_0 \in \Gamma(T^*P \otimes \mathfrak{g})$, we obtain an identification of $\mathcal{A}$ with the vector space $\Omega^1 Ad(P)$. (In terms of connections ∇, ∇' in a vector bundle, although $\nabla_X Y$ is not $C^\infty(M)$-linear in Y, $\nabla_X Y - \nabla'_X Y$ is.)

(5.3) <u>Homogeneous Connections</u>. Let $M = G/H$, with its Riemannian metric (and Levi-Civitá connection) coming from $\langle\ ,\ \rangle$. If $\theta : H \to U$ is a homomorphism of H into another compact Lie group U, H acts on U by the formula $h \cdot u = \theta(h)u$, so one may form the bundle P_θ associated to the principal bundle $G \to G/H$ with fibre U. This is a principal U-bundle; we refer to a bundle of the form P_θ as a <u>homogeneous principal bundle</u>. This should be compared with the definition of homogeneous vector bundle in 2.2 of II. Note that if $U = SO_n$ or U_n, i.e. if θ is a representation of H, then P_θ is just (orthonormal or unitary) frame bundle of $\alpha(\theta)$. A connection form ω in P_θ is said to be homogeneous (or invariant) if it satisfies $R_g^*\omega = \omega$ for $g \in H$. Such connection forms are in one to one correspondence with H-module transformations

$$\Lambda : \mathfrak{m} \longrightarrow \mathfrak{u}$$

(where H acts on $\mathfrak{u}$ by $h \cdot X = (Ad(u)\theta(h))X$), since the horizontal subbundle is in this case determined by the horizontal subspace $\Lambda(\mathfrak{m})$ in a single fibre $\mathfrak{u}$. For the details of this, see [KN] volume I, Chapter II and volume II, Chapter X. For example, if $\theta = Ad(G/H)$ (so $U = SO(\mathfrak{m})$), homogeneous connections in T(G/H) which are compatible with $\langle\ ,\ \rangle$ correspond to H-module transformations $\Lambda : \mathfrak{m} \to \Lambda^2\mathfrak{m}$; this is in agreement with the definition of homogeneous connections given in 2.3 of II.

The connection form ω_0 corresponding to $\Lambda = 0$ is called the canonical connection form in P_θ. Relative to this choice, we identify the space of all connection forms $\mathcal{A}(P_\theta)$ with $\Omega^1 Ad(P_\theta)$ as explained in 5.2. Since $Ad(P_\theta) = \alpha(Ad(U) \circ \theta)$ and $TM \cong \alpha(Ad(G/H))$, we may further identify $\Omega^1 Ad(P_\theta)$ with the H-module $i_*(\mathfrak{m}^* \otimes \mathfrak{u})$ (see 2.5 of II). The subspace $\mathcal{H}(P_\theta)$ of homogeneous connection forms may be identified according to the last paragraph by the trivial submodule of $\mathfrak{m}^* \otimes \mathfrak{u}$, which we denote by $(\mathfrak{m}^* \otimes \mathfrak{u})^H$. There is a natural inclusion $(\mathfrak{m}^* \otimes \mathfrak{u})^H \to i_*(\mathfrak{m}^* \otimes \mathfrak{u})$, whereby a fixed point of H in $\mathfrak{m}^* \otimes \mathfrak{u}$ defines a "constant" section of the bundle, and by Frobenius Reciprocity this is in fact an isomorphism onto the subspace of such sections. To summarize, then, we have the following diagram:

$$\begin{array}{ccc} \mathcal{H}(P_\theta) & \cong & (\mathfrak{m}^* \otimes \mathfrak{u})^H \\ \cap & & \cap \\ \mathcal{A}(P_\theta) & \cong & i_*(\mathfrak{m}^* \otimes \mathfrak{u}) . \end{array}$$

This was utilised by H. T. Laquer in [Lq], whose work we shall describe in 5.5.

(5.4) The Yang-Mills Equations. Let M be a compact oriented Riemannian manifold, and let $P \to M$ be a principal G-bundle. The Yang-Mills functional is the map $L : \mathcal{A}(P) \to \mathbb{R}$ defined by

$$L(\omega) = \int_M \langle\langle F_\omega, F_\omega \rangle\rangle$$

where F_ω is the curvature form of ω. Since $\mathcal{A}(P)$ is an affine space (which we identify with the vector space $\Omega^1 \operatorname{Ad}(P)$ on choosing some $\omega_0 \in \mathcal{A}(P)$), to study the critical points of L it suffices to consider variations of ω of the form $\omega_t = \omega + t\eta$ ($t \in \mathbb{R}$, $\eta \in \Omega^1 \operatorname{Ad}(P)$). It is relatively straightforward to calculate $L(\omega_t)$, and one obtains (see [AB], §4):

$$L(\omega_t) = L(\omega) + 2t(D_\omega \eta, F_\omega) + t^2((D_\omega \eta, D_\omega \eta) + (F,[\eta,\eta])) + o(t^2)$$

where $D_\omega : \Omega^1 \operatorname{Ad}(P) \to \Omega^2 \operatorname{Ad}(P)$ is the operator defined using the connection in $\operatorname{Ad}(P)$ given by ω (as in 5.2). The next two results follow from this.

Proposition 1: The connection form ω is an extremum for L if and only if $D^*_\omega F_\omega = 0$. □

Since $D_\omega F_\omega = 0$ (the Bianchi identity), the critical points of L are those forms ω whose curvature F_ω is harmonic. The Yang-Mills equations are the equations

$$D^*_\omega F_\omega = 0, \quad D_\omega F_\omega = 0,$$

and such an ω is said to be a Yang-Mills connection form.

Proposition 2: If ω is an extremum of L, the Hessian Q of L at ω is given by $Q(\eta) = (D^*_\omega D_\omega \eta + *[* F_\omega, \eta], \eta)$ for $\eta \in \Omega^1\operatorname{Ad}(P)$. □

The Hessian is a quadratic form defined on the tangent space to $\mathcal{A}(P)$ at ω, which we identify with $\mathcal{A}(P)$ itself and hence with $\Omega^1 \operatorname{Ad}(P)$. If $f : X \to \mathbb{R}$ is a function on a compact manifold X, the Hessian at a critical point $x \in X$ is the quadratic form on $T_x X$ whose matrix in terms of local coordinates $x_1, \dots, x_n$ is $(\partial^2 f/\partial x_i \partial x_j)$. If the Hessian is non-degenerate at every critical point, f is said to be a Morse function, and there is then a well known relation between the cohomology of X and the indices of (the Hessians of) the critical points (see [AB], §1). This relation often persists in some form even when the Hessian at a critical point is degenerate (i.e. has positive nullity). To generalize this theory to manifolds which are not finite dimensional has proved to be a very difficult task, however, and perhaps the only significant example where it has been done directly is that of the energy functional on paths in a Riemannian manifold, i.e. the original work of M. Morse in the 1930's. In [AB], M. F. Atiyah and R. Bott show that it may effectively be done for the Yang-Mills functional L, in the case where M is a Riemann surface, by transferring the problem to one concerning holomorphic structures on vector bundles. It is therefore of some interest to identify the critical points of L, and to compute the index and nullity of their Hessians.

The problem may be simplified by making the following observations (see [AB], §4). First, since L is invariant under the (infinite dimensional) group $\mathcal{G}$ of automorphisms of P ("gauge transformations"), the critical points come in $\mathcal{G}$-orbits, hence if ω is critical then the whole of the tangent space to $\mathcal{G}\cdot\omega$ will be annihilated by Q. If $T_\omega\mathcal{A}(P)$ is identified with $\Omega^1 Ad(P)$ as usual, the tangent space to $\mathcal{G}\cdot\omega$ at ω becomes identified with the image of $D_\omega : \Omega^0 Ad(P) \to \Omega^1 Ad(P)$. (The space $\Omega^0 Ad(P)$ may be identified with the Lie algebra of $\mathcal{G}$.) Hence it suffices to find the index and nullity of Q on $(\text{Im } D_\omega)^\perp = \text{Ker } D_\omega^*$.

Proposition 3: The index and nullity of a Yang-Mills connection form ω on Ker D_ω^* are finite, and equal to the index and nullity respectively of the quadratic form $\hat{Q}$ on Ω^1 Ad(P), where $\hat{Q}(\eta) = ((D_\omega D_\omega^* + D_\omega^* D_\omega)\eta + *[* F_\omega, \eta], \eta)$.

Proof. The form $\hat{Q}$ is positive definite on Im D_ω, and agrees with Q on $(\text{Im } D_\omega)^\perp = \text{Ker } D_\omega^*$. Since the Hodge-Laplacian $D_\omega D_\omega^* + D_\omega^* D_\omega$ is elliptic with non-negative eigenvalues, it follows that $\hat{Q}$ has finite index and nullity. □

(5.5) Homogeneous Yang-Mills Connections. The results in this section are due to H. T. Laquer. First, it is a straightforward matter to show that the canonical connection form ω_0 in any homogeneous bundle P_θ satisfies the Yang-Mills equations. The problem then is to use proposition 3 of 5.4 to calculate the index and nullity at ω_0 in various situations. We have identified $\Omega^1 Ad(P)$ with $i_*(\mathfrak{m}^* \otimes \mathfrak{u})$, and this may be identified with the subspace of

$C^\infty(G) \otimes (\mathfrak{m}^* \otimes \mathfrak{u})$ consisting of functions which are H-invariant (c.f. the proof of Frobenius Reciprocity in 2.5 of II). Let Δ be the Laplacian on $C^\infty(G) \otimes (\mathfrak{m}^* \otimes \mathfrak{u})$. The following is a special case of proposition 3.2 of [Lq].

Proposition: Let $\mathfrak{m}'$ and $\mathfrak{u}'$ be irreducible submodules of $\mathfrak{m}$ and $\mathfrak{u}$ respectively. If G/H is symmetric, $\hat{Q}(\eta) = ((-\Delta + c(\mathfrak{u}')I)\eta, \eta)$ for all $\eta \in i_*(\mathfrak{m}'^* \otimes \mathfrak{u}')$, where $c(\mathfrak{u}')$ is the Casimir constant (< 0) associated to $\mathfrak{u}'$ (as in 2.4 of II). □

Thus, on $i_*(\mathfrak{m}'^* \otimes \mathfrak{u}')$, the nullity of $\hat{Q}$ is the dimension of the $c(\mathfrak{u}')$-eigenspace of Δ, and the index is the sum of the dimensions of the λ-eigenspaces for which $\lambda > c(\mathfrak{u}')$. The λ-eigenspace of Δ is a sum of irreducible G-modules whose Casimir constants are λ; the number of times such a G-module appears in $i_*(\mathfrak{m}'^* \otimes \mathfrak{u}')$ may be determined by Frobenius Reciprocity.

Extensive calculations show that when P_θ is the bundle $G \to G/H$, and G/H is irreducible, the index I and nullity N of ω_0 are zero except in the following cases:

(a) G/H a simple Lie group (I = 1)

(b) $G/H = SO_{n+1}/SO_n \cong S^n$, $n \geq 5$ (I = n+1)

$G/H = Sp_{n+1}/Sp_1 \times Sp_n \cong HP^n$, $n \geq 2$ (N = n(2n + 3))

(c) $G/H = SO_5/SO_4 \cong S^4$ (N = 10), E_6/F_4 (I = 54), $F_4/Spin_9$ (I = 26).

One may also restrict to variations of ω_0 which are entirely within the space of homogeneous connection forms. If G/H is a simple Lie group K, then $(\mathfrak{m}^* \otimes \mathfrak{u})^H \cong (\mathfrak{k}^* \otimes \mathfrak{k})^K$ is clearly one dimensional, and it turns out that $\hat{Q}(n) = -(1/2)(n,n)$ for all $n \in (\mathfrak{m}^* \otimes \mathfrak{u})^H$. Thus, the negative direction in (a) above is accounted for precisely by the homogeneous connections forms. If G/H is an irreducible symmetric space which is not a Lie group, then $(\mathfrak{m}^* \otimes \mathfrak{u})^H = \{0\}$, i.e. ω_0 is the only homogeneous connection form.

For any homogeneous bundle P_θ over a symmetric space G/H with G semisimple it follows from the proposition that $\hat{Q}(n) = (-1/2)(n,n)$ for $n \in (\mathfrak{m}^* \otimes \mathfrak{u})^H$, hence ω_0 gives a local maximum for L restricted to (P_θ). For further results we refer to [Lq].

(5.6) Notes. It is not known whether there exists any Yang-Mills connection in a complex vector bundle on S^4 which has positive index (a case of importance to mathematical physicists). For other results on homogeneous connections, see [It].

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