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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
(4 November - 6 December 1985)

CLASSIFICATION OF THE IRREDUCIBLE REPRESENTATIONS
(CARTAN-WEYL THEOREM OF HIGHEST WEIGHT)

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These are preliminary lecture notes, intended only for distribution to participants.

𝔤 real semi-simple (or reductive) Lie algebra with Cartan subalgebra $\mathfrak{h}$.

Root system R. For $\alpha \in R$ the hyperplane orthogonal to α is

$$(\alpha, 0) = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \; ; \; \langle\lambda, \alpha\rangle = 0\}$$

the union $\bigcup_{\alpha \in R} (\alpha, 0)$ is the Cartan-Stiefel diagram of R, $D(R)$.

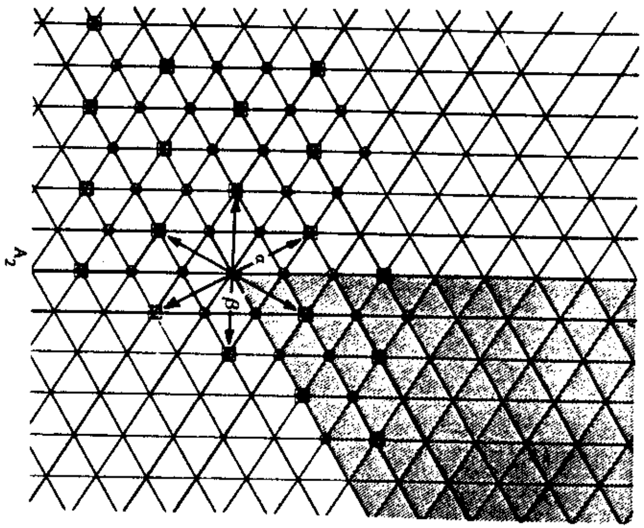
the complement $\mathfrak{h}^*_{\mathbb{R}} - D(R)$ is an open set with connected components which are open cones; the closures of these are the Weyl chambers of R. The Weyl group W acts simply transitively on the Weyl chambers. Let F be a fundamental system; the set $\{\lambda \in \mathfrak{h}^*_{\mathbb{R}} ; \langle\lambda, \alpha_i\rangle \geq 0 \;\; i = 1, \ldots, l\}$ is then a Weyl chamber called the fundamental one

We pick out two lattices in $\mathfrak{h}^*_{\mathbb{R}}$ i.e. the root lattice $\mathcal{R}$ i.e. the subgroup generated by R and the lattice of integral forms $\mathfrak{I}$ i.e. the subgroup of all λ s.t. $2\langle\lambda, \alpha\rangle / \langle\alpha, \alpha\rangle$ is an integer, each α in R. Write τ_i for τ_{α_i}, then the dual basis to $\{\tau_i\}$ is $\{\lambda_i\}$ $(i = 1, \ldots, l)$ the set of fundamental weights.

Write $\rho = \lambda_1 + \ldots + \lambda_l$ the sum of the fundamental weights

$$= \frac{1}{2} \sum_{\alpha \in R^+} \alpha \qquad \text{half the sum of the positive roots.}$$

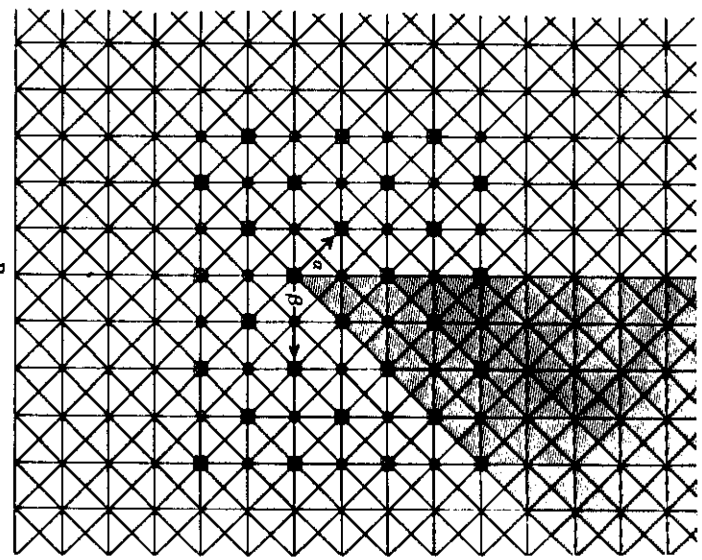
The position of λ_i on the i^{th} edge of the fundamental chamber is (as $\langle\lambda_i, \alpha_j\rangle = 0$, $i \neq j$ and $\alpha_i(\tau_i) = 2$) given by the intersection with the plane orthogonal to α_i through the point $\frac{1}{2}\alpha_i$.



A_2

Figure 1.

Figs 1,2,3 are the Cartan-Stiefel diagram for A_2, B_2, G_2. In $\mathfrak{h}^*_{\mathbb{R}}$ the points marked □ form the lattice $\mathcal{R}$, ○ the lattice $\mathfrak{I}$; the vectors α, β form a fundamental system.



B_2

Figure 2.

The fundamental Weyl chamber is shaded.

structure developed, Cartan subalgebra h etc.

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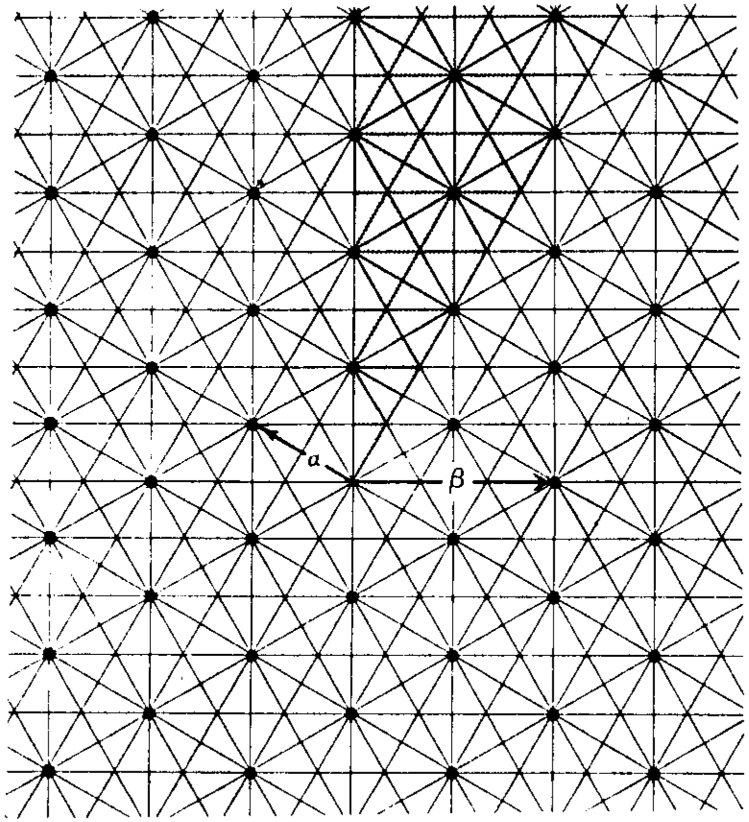
sis of weight vectors, say $v_1, v_2, \ldots, v_n$, with associated weights $\rho_1, \rho_2, \ldots, \rho_n$. For any H in h the operator $\exp(\phi(H))$ is diago-



G_2

Figure 3.

nal, with diagonal elements $\exp(\rho_r(H))$. We modify this by a factor $2\pi i$, and call the trace of $\exp(2\pi i\phi(H))$, as function of H,

the *character* χ of ϕ; so that $\chi(H) = \Sigma_r \exp(2\pi i\rho_r(H))$. (It depends, of course, only on the equivalence class of ϕ.) As a matter of fact, we only consider H in $h_{\mathbb{R}}$ (this can also be interpreted as considering $\exp(2\pi\phi(iH))$ with iH in the subalgebra $ih_{\mathbb{R}}$ of the compact form of $\mathfrak{g}$; Ch. II, no. 10).

The character, in fact every term of the sum, takes the same value at two H's whose difference lies in the lattice $\mathcal{T}$, since the ρ_r are integral forms. In other words, χ is periodic, with the elements of $\mathcal{T}$ as periods. Even more, χ is a finite Fourier series on $h_{\mathbb{R}}$ with respect to the bases $\{r_i\}$. To put this somewhat differently: The quotient group $h_{\mathbb{R}}/\mathcal{T}$ is isomorphic to the ℓ-dimensional torus ($\mathbb{R}^\ell$ modulo the lattice of integral vectors, direct sum of ℓ copies of $\mathbb{R}/\mathbb{Z}$). Each $\exp \circ 2\pi i\rho$, for ρ in $\mathcal{I}$, is a homomorphism of $h_{\mathbb{R}}$ into the unit circle $U = \{z : |z| = 1\}$ of $\mathbb{C}$, with $\mathcal{T}$ in the kernel; thus it can be considered as a homomorphism of $h_{\mathbb{R}}/\mathcal{T}$ into U, or, in the usual terminology of Abelian groups, a character of $h_{\mathbb{R}}/\mathcal{T}$. (For some purposes it might be better to "divide" $h_{\mathbb{R}}$ by the lattice $2\pi\mathcal{T}$.)

We use the symbol e_ρ for $\exp \circ 2\pi i\rho$; these are then functions on $h_{\mathbb{R}}$ (even for ρ any element of $h_{\mathbb{R}}^*$) and on $h_{\mathbb{R}}/\mathcal{T}$. We have $e_\rho \cdot e_\sigma = e_{\rho+\sigma}$ (pointwise product, as functions, on the left), and $e_\rho = (e_{\lambda_1})^{n_1} \cdot (e_{\lambda_2})^{n_2} \cdot \ldots \cdot (e_{\lambda_\ell})^{n_\ell}$, if $\rho = \Sigma\, n_i\lambda_i$, where the λ_i are the fundamental weights. The e_ρ, with ρ in $\mathcal{I}$, are *all* the characters of $h_{\mathbb{R}}/\mathcal{T}$ (we take this as well known), and the correspondence $\rho \longrightarrow e_\rho$ is an isomorphism of $\mathcal{I}$ with the character group of $h_{\mathbb{R}}/\mathcal{T}$. The finite integral linear combinations $\Sigma\, a_\rho e_\rho$ of the e_ρ (such as the character χ above) form the *integral group ring* $\mathcal{G}$ of the character group of $h_{\mathbb{R}}/\mathcal{T}$ (or the isomorphic integral group ring $\mathbb{Z}\mathcal{I}$ of $\mathcal{I}$. (Actually, to make this precise, Let ϕ be a finite dimensional representation of $\mathfrak{g}$ on a complex vector space V (if $\mathfrak{g}$ is reductive but not s.s. we consider ϕ irreducible). For $\lambda \in \mathfrak{h}^*$ the weight space (possibly zero)

$$V_\lambda' = \{ v \in V ; \phi(\mathfrak{z}) v = \lambda(\mathfrak{z}) v , \forall \mathfrak{z} \in \mathfrak{h} \}$$

Any non-zero v in V_λ is called a weight vector with weight λ. Write $m_\lambda = \dim V_\lambda$ and call m_λ the multiplicity of λ as a weight of ϕ.

Propn

(i) V is spanned by weight vectors.

(ii) The weights are in $\mathfrak{I}$.

(iii) The (finite) set of weights is invariant under the Weyl group i.e. if λ is a weight so is $w_\alpha(\lambda) = \lambda - \lambda(T_\alpha)\alpha$ for each α in R; the α-string of λ i.e. the weights $\lambda + t\alpha$, $t \in \mathbb{Z}$, consists of the integral t between $-t'$ and t'' and $\lambda(T_\alpha) = t' - t''$.

(iv) $m_\lambda = m_{w\lambda}$ for λ a weight and each w in W.

Proof

i) Restrict ϕ to $\mathfrak{g}^{(\alpha)} \cong sl(2,\mathbb{C})$. By A_1 reps theory each $\phi(T_\alpha)$ is semi-simple and the $\phi(\mathfrak{z})$ $\mathfrak{z} \in \mathfrak{h}$ commute; thus there is a simultaneous diagonalization.

ii) The eigenvalues of $\phi(T_\alpha)$ are integers running in steps of two from a max $+r$ to a min $-r$; but these are $\lambda(T_\alpha)$ for λ a weight.

iii) Let v lie in V_λ, then $E_\alpha v$ lies in $V'_{\lambda+\alpha}$. With $r = \lambda(T_\alpha)$,

the vectors v, $E_{-\alpha}v$, $E_{-\alpha}^2 v$, …, $E_{-\alpha}^r v$ are non-zero with weights λ, $\lambda-\alpha$, $\lambda-2\alpha$, …, $\lambda - r\alpha$.

(iv) From (iii) as $\phi(E_{-\alpha})^r$ has no kernel on V_λ, we have $m_\lambda \leq m_{w_\alpha \lambda}$ each α in R, thus $m_\lambda \leq m_{w\lambda}$ or $m_\lambda = m_{w\lambda}$ each w in W. □

A weight λ is called extreme if $\lambda + \alpha$ is not a weight for each α in R^+. Extreme weights exist eg can take a maximal weight in the given order or one of maximal norm and transform into the fundamental chamber by an element of W. Let v be an extreme weight vector ie one whose weight λ is extreme. Define V_v to be the subspace of V spanned by the vectors $E_{-i_k} \dots E_{-i_2} E_{-i_1} v$ with $k = 0, 1, 2, \dots$ and $1 \leq i_j \leq l$; these have weights $\lambda - \alpha_{i_1} - \dots - \alpha_{i_k}$.

Propn

V_v is an invariant subspace of V.

Proof

Sufficient to show invariance under E_i, E_{-i} $i = 1, \dots, l$ as these generate $\mathfrak{g}_{\mathbb{C}}$. Invariance under E_{-i} is clear; also $E_i E_{-i_k} \dots E_{-i_1} v$ $= E_{-i_k} E_i E_{-i_{k-1}} \dots E_{-i_1} v + [E_i, E_{-i_k}] E_{-i_{k-1}} \dots E_{-i_1} v$ so by induction on k also get invariance under E_i. □

Corollary

If V is irreducible under ϕ, then there is exactly one extreme weight (called the highest weight); it is dominant, (belongs to $\mathfrak{g}^d$) maximal in norm and in given order, and of multiplicity 1; all other weights are of the form $\mu = \lambda - \sum_i n_i \alpha_i$ with non-negative integers n_i.

Proof
$V = V_r$ as above. Any weight of maximal order or norm is extreme. □

Construction: Any complex irreducible $\mathfrak{g}$-module is equivalent to the quotient of a Verma module by a maximal proper $\mathfrak{g}$-submodule.
Then

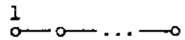
Theorem
There is a 1-1 correspondence between the irreducible complex finite dim representations (up to equivalence) of $\mathfrak{g}$ reductive semi-simple and $\mathfrak{g}^d$, which assigns the extreme weight.

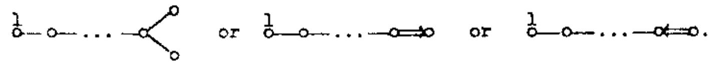
Example The adjoint reps of $\mathfrak{g}_{\mathbb{C}}$: The non-zero weights w.r.t. $\mathfrak{h}$ are the roots R, each occurs with multiplicity 1; the highest weight is the maximal root, for $\mathfrak{g}$ simple. The adjoint reps of $\mathfrak{g}$ is irreducible iff $\mathfrak{g}$ is simple.

Write ϕ_λ for the irreducible reps of $\mathfrak{g}$ with highest weight λ. Can describe this on the Dynkin diagram: if $\lambda(T_i)$ is positive, write it above the vertex corresponding to the simple root α_i.

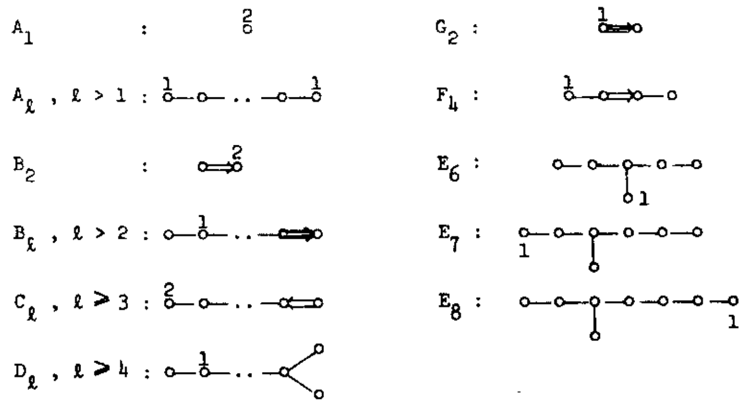
c.f. J. Wolf

Now suppose that V is a finite dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Then V has lowest weight $w_0(\lambda)$, and V^* is the finite dimensional irreducible $\mathfrak{g}$-module with highest weight $-w_0(\lambda)$; $w_0 R^+ = R^-$. $w_0 \in W$

Let us write ϕ_λ for the finite dimensional irreducible representation of highest weight $\lambda \in \Lambda^*_{\mathfrak{g}^d}$. If $\mathfrak{g}$ is semisimple we can "describe" ϕ_λ on the Dynkin diagram as follows. Whenever the non-negative integer $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is positive, we write it next to the vertex that corresponds to the simple root α. Thus, for the classical groups, the ordinary ("vector") representation is "described" as 

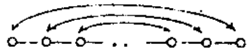
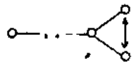
or 

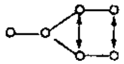
The adjoint representations are



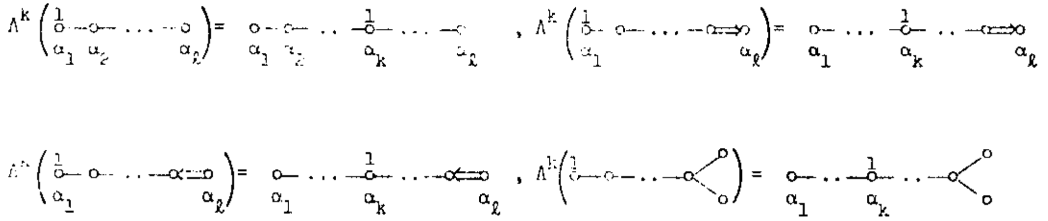
cf J. Wolf

If $\mathfrak{g}$ is simple, then ϕ_λ is equivalent to its dual $\phi_{-w_0(\lambda)}$ if and only if its Dynkin diagram is stable under

 for A_ℓ ($\ell \geq 2$),  for D_{2n+1},

 for E_6.

One also has some handy computational tricks, such as



(V, ϕ) finite-dim representation of $\mathfrak{g}$. $\tilde{G}$ simply connected with Lie algebra $\mathfrak{g}$. ϕ is the differential of $\tilde{\pi}$

$$\begin{tikzcd} \tilde{G} \arrow[r, "\tilde{\pi}"] & GL(V) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\phi"'] & \mathfrak{gl}(V) \arrow[u, "\exp"'] \end{tikzcd}$$

Suppose G connected has Lie algebra $\mathfrak{g}$. $\tilde{G} \xrightarrow{p} G$, $\operatorname{Ker} p = \pi_1(G)$

$p \circ \exp_{\tilde{G}} = \exp_G \circ p_{*,e}$.

$$\begin{tikzcd} \tilde{G} \arrow[r, "\tilde{\pi}"] \arrow[d, "p"'] & GL(V) \\ G \arrow[ur, dashed] & \end{tikzcd}$$

If $\tilde{\pi}(\operatorname{Ker} p) = \{I\}$ ie $\operatorname{Ker} p \subseteq \operatorname{Ker} \tilde{\pi}$ then define $\pi(g) = \tilde{\pi}(\tilde{g})$ where $p(\tilde{g}) = g$

($p(\tilde{g}_1) = g$ implies that $\tilde{g}\tilde{g}_1 \in \operatorname{Ker} p$ so $\tilde{\pi}(\tilde{g}) = \tilde{\pi}(\tilde{g}_1)$)

The irreducible finite-dim representations of G are given by those ϕ irreducible of $\mathfrak{g}$ with $\operatorname{Ker} p \subseteq \operatorname{Ker} \tilde{\pi}$.

G compact, connected Lie group with Lie algebra $\mathfrak{g}$. $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1$, $\mathfrak{z}$ the centre, $\mathfrak{g}_1 = [\mathfrak{g}\mathfrak{g}]$ the derived algebra which is semi-simple. The Killing form $B(\,,\,)$ is -ve definite on $\mathfrak{g}_1$. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$, then

$\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}_1$ with $\mathfrak{h}_{1\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let H be the associated maximal torus to $\mathfrak{h}$. The unitary character group $\hat{H}$ is identified with a lattice

$$\Lambda \subseteq \sqrt{-1}\,\mathfrak{h}^* \quad (* \text{ the real dual}) \text{ by}$$

$$\chi \longmapsto \lambda, \quad \chi(\exp \xi) = e^{\lambda(\xi)}, \quad \xi \in \mathfrak{h}.$$

Let Γ be the unit lattice of H (of G) i.e.

$$\Gamma_H = \{\xi \in \mathfrak{h};\ \exp \xi = e\} \quad e \text{ the identity element of } G$$

$\lambda \in \sqrt{-1}\,\mathfrak{h}^*$ is the differential of $\chi \in \hat{H}$

iff $\lambda(\Gamma_H) \subseteq 2\pi\sqrt{-1}\,\mathbb{Z}$

Also if $\Gamma_{\cdot} = \{\xi \in \mathfrak{h};\ \exp \xi \in Z\}$ Z the centre of G

$$\pi_1(G) = \Gamma_{\cdot} / $$

Let R be the root system of (G, H) i.e. of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. The root spaces $\mathfrak{g}^\alpha$ have dimension one and

$\mathrm{Ad}(h)\,\mathfrak{g}^\alpha = \mathfrak{g}^\alpha$ each $h \in H$ $\quad (\mathrm{Ad}(\exp \xi) = e^{\mathrm{ad}\,\xi},\ \xi \in \mathfrak{h})$

therefore $R \subseteq \Lambda$. In particular the roots take pure imaginary values on $\mathfrak{h}$. Also

$$\mathfrak{h}_{1,\mathbb{R}} = \sqrt{-1}\,\mathfrak{h}_1$$

$\Lambda \subseteq I$ for if $\alpha \in R$ then $\{\tau_\alpha, E_\alpha, E^\alpha\}$ spans a subalgebra $\cong sl(2,\mathbb{C})$; here we can choose root vectors E_α so that $\bar{E}_\alpha = E^\alpha$ (where $\bar{\ }$ is conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$) Then with

$$\xi_\alpha = (E_\alpha - E^\alpha) + \sqrt{-1}\,(E_\alpha + E^\alpha), \text{ we have that}$$

$\{\sqrt{-1}\,\tau_\alpha, \xi_\alpha, \xi^\alpha\}$ spans a subalgebra of $\mathfrak{g}_1$ isomorphic to $su(2)$;

to each $\alpha \in R$, G_1 has a connected subgroup isomorphic to $SU(2)$. Now $\sqrt{-1}\,\tau_\alpha$ lies in $\mathfrak{h}$ and $\tau_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so as $2\pi\sqrt{-1}\,\tau_\alpha$ lies in the unit lattice of $SU(2)$ we have $\lambda(\tau_\alpha) \in \mathbb{Z}$ for each $\alpha \in R$.

Let (V, π) be a complex finite-dim representation of G. π is completely reducible $V = V_1 \oplus \dots\dots \oplus V_m$. Say that the weights of π w.r.t H are those of $d\pi$ w.r.t $\mathfrak{h}$. Each irreducible repn of H is one-dimensional, therefore determined by a character; thus can choose a basis $\{v_1, \dots\dots, v_d\}$ of V such that on v_i, $\pi|_H$ is a character χ_i with weight λ_i. Thus each weight of π lies in Λ.

One can also show that a complex finite-dim representation of a semi-simple Lie group G (connected) is completely reducible and that the differential ϕ has all weights integral (i.e. in I) in the following way: take a compact real form $\mathfrak{k}$ of $\mathfrak{g}_{\mathbb{C}}$ and a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ which is the complexification of a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Let $\tilde{K}$ be the simply connected group with Lie algebra $\mathfrak{k}$. $\tilde{K}$ is compact and has a maximal torus $\tilde{H}$ with Lie algebra $\mathfrak{h}$. ϕ restricted to $\mathfrak{k}$ is the differential of $\tilde{\pi}$ on $\tilde{K}$, therefore ϕ is completely reducible and any weight lies in the lattice $\tilde{\Lambda}$ of $\tilde{\tilde{H}}$.

Recall the definition of a regular element of $\mathfrak{g}$. $\mathfrak{h} \subseteq \mathfrak{g}$, $\zeta \in \mathfrak{h}$ is regular iff $\alpha(\zeta) \neq 0$ for each $\alpha \in R$.

There is the Weyl group $W(G, H) = N_G(H)/H$ and also the Weyl group $W(R)$ of the root system R.

<u>Proposition</u>

$$W(G, H) = W(R)$$

<u>Proof</u>

Let $\zeta = 2^{-1/2}(E_\alpha - E^{\alpha})$ then the reflection $w_\alpha = \mathrm{Ad}(\exp(\pi\langle\alpha,\alpha\rangle)^{1/2}\zeta)\big|_{\mathfrak{h}}$, $\alpha \in R$.

$W(G,H)$ acts simply on the Weyl chambers: for suppose $wC = C$, some chamber C. Let n be the order of w and $\zeta \in C$, then with

$$\zeta_0 = (1/n)(\zeta + w\zeta + \dots + w^{n-1}\zeta)$$

we have $w\zeta_0 = \zeta_0$ and $\zeta_0 \in C$. ζ_0 is a regular element,

and if $w = \mathrm{Ad}\, g$, $g \in G$ we have $g \exp \zeta_0 g^{-1} = \exp \zeta_0$. Now there is a maximal torus H_1 containing g and $\exp \zeta_0$, but as ζ_0 is regular $\mathfrak{h}_1 = \mathfrak{g}_{\zeta_0} = \mathfrak{h}$ and $H_1 = H$. Hence $w = \mathrm{Ad}\, g|_{\mathfrak{h}}$ is the identity. □

Also $W(R)$ acts transitively on the Weyl chambers.

<u>Theorem</u> (Cartan Weyl)

$\hat{G}$ is in 1-1 correspondence with $\Lambda \cap \mathfrak{g}^*$.

<u>Proof</u> (cf N. Wallach)

Let G_1 be the connected subgroup of G with Lie algebra $\mathfrak{g}_1$. G_1 is compact. Let $\lambda \in \Lambda \cap \mathfrak{g}^*$ and $(V^\lambda, \tilde{\pi}) \in \hat{\tilde{G}}_1$ correspond to λ. The highest weight space V^λ_λ is one dimensional and each weight differs from λ by a sum of positive roots. Then since the kernel of the covering homomorphism $\rho: \tilde{G}_1 \longrightarrow G_1$ is central and $\rho \circ \exp_{\tilde{G}_1} = \exp_{G_1} \circ \rho_{*,e}$, we have that

$\tilde{\pi}(\mathrm{Ker}\,\rho) = \{I\}$ (as $\lambda \in \Lambda$), thus we obtain $(V^\lambda, \pi) \in \hat{G}_1$.

Let Z_0 be the connected subgroup of G corresponding to $\mathfrak{z}$ (the identity component of the center), then

$$G = (Z_0 \times G_1)/F \quad \text{where } F = \{(g^{-1}, g);\ g \in Z_0 \cap G_1\}$$

a finite subgroup. Now $Z_0 \subseteq H$ and $\lambda_0 = \lambda|_{\mathfrak{z}}$ is the differential of a character χ_0 of Z_0. Define for $z \in Z_0$, $g \in G_1$

$$\pi(z, g) = \chi_0(z)\,\pi(g)$$

then $\Pi|_F$ is the identity, hence $(V^\lambda, \Pi) \in \hat{G}$.
λ is called the highest weight. $\square$

<u>Corollary</u>
Let (V, ϕ) be a finite-dimensional complex representation of $\mathfrak{g}$.
Then ϕ is the differential of a representation Π of G
iff each weight of ϕ is the differential of a character of H
(ie lies in Λ)

<u>Proof</u>
$\phi|_{\mathfrak{g}_1}$ is completely reducible $V = V_1 \oplus \dots \oplus V_m$ with ϕ_j
irreducible of highest weight λ_j. Then $\phi = d\Pi$ iff $\lambda_j \in \Lambda$ each j. $\square$

For $\mu \in \Lambda$, let e^μ denote the corresponding character. Then in the integral group ring $\mathbb{Z}[\Lambda]$, define

$$A(\mu) = \sum_{w \in W(G,H)} (\det w)\, e^{w\mu}$$

We suppose that half the sum of the positive roots ρ lies in Λ.

Weyl's character formula:
Let χ_λ be the character of $(V^\lambda, \Pi) \in \hat{G}$, then

$$A(\rho)\, \chi_\lambda|_H = A(\lambda + \rho)$$

Weyl's degree formula:

$$\dim V^\lambda = \prod_{\alpha \in R^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$$

In writing these lecture notes, the following literature was referred to

(1) Atiyah M.F et al Representation Theory of Lie Groups (Conference proceedings LMS 34)

(2) Mackey G.W Unitary group representations in physics, probability and number theory.

(3) Price J. Lie groups and compact groups

(4) Samelson H. Notes on Lie algebras

(5) Wallach N. Harmonic analysis on homogeneous spaces.

(6) Wolf J.A Foundations of representation theory for semi-simple Lie groups (Lectures at the Nato advanced study institute Liege, Belgium 1977)