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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
(4 November - 6 December 1985)

THE PLANCHEREL FORMULA FOR SEMISIMPLE LIE GROUPS.

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These are preliminary lecture notes, intended only for distribution to participants.

Lectures - Trieste - November '85
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Introduction

Let $f: S \to \mathbb{C}$ be a complex valued function on the circle group. View f as function $\mathbb{R} \to \mathbb{C}$ of period 2π. If $f \in L^1(S) \cap L^2(S)$,

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-inx}$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{inx} dx$$

and the formula is understood to mean

$$f = \sum_{-\infty}^{\infty} \hat{f}(n) \chi_{-n} \quad \text{in } L^2(S)$$

where

$$\chi_k(x) = e^{ikx} \quad (k \in \mathbb{Z})$$

This Fourier inversion formula expresses f in terms of the unitary characters on the circle group S.

If $\chi \in \hat{S}$, the unitary characters on S, and $f \in L^1(S)$, set

$$\chi(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\chi(x)dx.$$

Denote the right translates of f by

$$[r_x(f)](y) = f(y+x).$$

Then

$$\chi_n(r_x f) = \hat{f}(n)e^{-inx}$$

so we can write the Fourier inversion formula as

$$f(x) = \sum_{\chi\in\hat{S}} \chi(r_x f).$$

These formulas are true pointwise — not just in the L^2 sense — if $f \in C^\infty(S)$. In that case $p(n)\hat{f}(n)$ is bounded for every polynomial p.

Roughly the same thing holds for functions $f: \mathbb{R} \to \mathbb{C}$. If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-i\xi x}d\xi$$

where

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\xi x}dx$$

and the formula is understood to mean

$$f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\xi)\chi_{-\xi} \quad \text{in } L^2(\mathbb{R})$$

where

$$\chi_\eta(x) = e^{i\eta x} \qquad (\eta\in\mathbb{R}).$$

Here the L^2 sum expresses $L^2(\mathbb{R})$ as a direct integral of one dimensional spaces $\langle \chi_\xi \rangle$ and the Fourier inversion formula expresses f in terms of unitary characters on the additive group $\mathbb{R}$.

If $\chi \in \hat{\mathbb{R}}$ and $f \in L^1(\mathbb{R})$ then, as before, $\chi(f) = \int_{-\infty}^{\infty} f(x)\chi(x)dx$. Essentially as before,

$$\chi_\xi(r_x f) = \sqrt{2\pi}\,\hat{f}(\xi)e^{-i\xi x},$$

so the Fourier inversion formula becomes

$$f(x) = \frac{1}{2\pi}\int_{\hat{\mathbb{R}}} \chi_\xi(r_x f)\,d\xi.$$

These formulas hold pointwise if f belongs to the Schwartz space

$$\mathcal{C}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \text{if } n \geq 0 \text{ integer, } p \text{ poly then } p(x)\frac{d^n f}{dx^n} \text{ bounded}\}.$$

In fact, $f \in \mathcal{C}(\mathbb{R}) \Leftrightarrow \hat{f} \in \mathcal{C}(\mathbb{R})$.

In this sequence of lectures I hope to explain how the classical results, just described, go over to semisimple Lie groups. The plan is

Lecture 1: Introduction
Compact Lie Groups

Lecture 2: Structure of Real
Semisimple Groups

Lecture 3: Smooth Vectors
Characters
Harish-Chandra Modules

Lecture 4: Standard Representations
Plancherel Formula

Lecture 5: Universal Cover of
the Conformal Group

§1. Generalities.

Let G be a locally compact topological group, dx the left invariant Haar measure

$$\int_G f(x)\,dx = \int_G f(yx)\,dx .$$

A <u>representation</u> of G on a Banach space $\mathcal{B}$ is a homomorphism

$$\pi : G \to GL(\mathcal{B})$$

into the group of bounded linear transformations (with bounded inverse) of $\mathcal{B}$, which is continuous in the sense

if $b \in \mathcal{B}$, $\beta \in \mathcal{B}^*$ then $\varphi_{b,\beta}$ is continuous

where $\varphi_{b,\beta}$ is the "matrix coefficient"

$$\varphi_{b,\beta}(x) = \beta\big(\pi(x)b\big).$$

If there is an $M > 0$ such that each $\|\pi(x)\| \le M$ then π is <u>uniformly bounded</u>.

If $f \in C_c(G)$, or if π is uniformly bounded and $f \in L^1(G)$, then we have a bounded linear operator on $\mathcal{B}$ given by

$$\pi(f) = \int_G f(x)\pi(x)\,dx .$$

In other words,

$$\beta(\pi(f)b) = \int_G f(x)\beta(\pi(x)b)\,dx$$

for $b \in \mathcal{B}$, $\beta \in \mathcal{B}^*$.

We are really interested in the case where π is <u>unitary</u>: $\mathcal{B}$ is a Hilbert space and every $\pi(x)$ is a unitary operator. But we still have to know about Banach representations.

In general write $\mathcal{B}_\pi$ for the representation space of π. If π is unitary, write $\mathcal{H}_\pi$.

As usual, representations π and π' are <u>equivalent</u> if there is a Banach space isomorphism $\mathcal{B}_\pi \xrightarrow{\Psi} \mathcal{B}_{\pi'}$ such that

$$\Psi(\pi(x)b) = \pi'(x)\Psi(b),$$

<u>unitarily equivalent</u> if they are unitary and Ψ is an isometry. We use the standard notions of <u>subrepresentation</u>, <u>quotient representation</u>, <u>direct sum</u>, <u>irreducible representation</u>, $\otimes$, Hom, etc. Denote

$\hat{G}$: unitary equivalence classes of irreducible unitary representations of G.

Under certain circumstances

(unimodular, separable, type I) $\hat{G}$ has a Borel structure and a unique measure μ such that

<u>IF</u> $f \in L^1(G) \cap L^2(G)$

<u>THEN</u> $\pi(f)$ is HS, i.e. $[\pi] \in \hat{G}$, $[\pi] \mapsto \|\pi(f)\|^2_{HS}$ is measurable, and

$$\|f\|^2_{L^2(G)} = \int_{\hat{G}} \|\pi(f)\|^2_{HS} \, d\mu(\pi)$$

(Theorem of Mautner and Segal)

Sometimes (vagueness intentional) this can be refined to

$$f(x) = \int_{\hat{G}} \operatorname{trace} \pi(r_x f) \, d\mu(\pi).$$

We are going to do just that, for real semisimple Lie groups.

§2. Compact Groups

Let K be a compact group. The basic facts are the Frobenius-Schur orthogonality relations and complete continuity of the left regular representation. These lead to the Peter-Weyl theorem. When K is a Lie group, this all is quite explicit.

The <u>left regular</u> representation of K is the unitary representation L on $L^2(K)$ given by

$$[L(x)f](y) = f(x^{-1}y).$$

If $\phi \in C_c(G)$ then

$$[L(\phi)f](y) = \int_K \phi(x) f(x^{-1}y)\,dx$$

$$= \int_K \phi(yx^{-1}) f(x)\,dx$$

shows $L(\phi)f = \phi * f$ (convolution) and shows that $L(\phi)$ is given by a continuous kernel. That proves that $L(\phi)$ is a completely continuous operator on $L^2(K)$, i.e. that L is completely continuous. First consequence: $L = \sum_{\widehat{K}} m_\pi \cdot \pi$ with $m_\pi < \infty$. Second consequence: if $[\pi] \in \widehat{K}$ then $\dim \pi < \infty$ and $m_\pi > 0$.

For the orthogonality, first normalize $\int_K dx = 1$. If u, v are in $\mathcal{H}_\pi$ set

$$f_{u,v} : x \mapsto \langle u, \pi(x) v \rangle,$$

matrix coefficient of π^*. Write $E(\pi)$: subspace of $L^2(K)$ spanned by the $f_{u,v}$, $u, v \in \mathcal{H}_\pi$. The orthogonality relations are

1. If $[\pi] \neq [\pi']$ in $\widehat{K}$, if $f \in E(\pi)$ and $f' \in E(\pi')$, then $f * f' = 0$ and $f \perp f'$ in $L^2(K)$.

2. If $[\pi] \in \widehat{K}$ and $u, v, u', v' \in \mathcal{H}_\pi$ then

$$f_{u,v} * f_{u',v'} = \frac{1}{\deg \pi} \langle u', v \rangle f_{u,v'}$$

and

$$\langle f_{u,v}, f_{u',v'} \rangle = \frac{\langle u, u' \rangle \overline{\langle v, v' \rangle}}{\deg \pi}$$

Combining these:

<u>Peter-Weyl Theorem</u>. If K is compact then

$$L^2(K) = \sum_{\widehat{K}} E(\pi)$$

where $E(\pi) \cong \mathcal{H}_\pi \otimes \mathcal{H}_{\pi^*}$, with the left/right action of K on the left/right factor. If $f \in L^2(K)$ then $f(x) = \sum_{\widehat{K}} \deg(\pi) \operatorname{trace} \pi^*(r_x f)$, pointwise if $f \in C_c(K)$.

Now let K be connected, $\mathfrak{k}$ its Lie algebra, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{k}$, and

$$T = \{k \in K : \mathrm{Ad}(k)|_{\mathfrak{t}} = 1\}$$

the corresponding Cartan subgroup of K.

$K = U(n)$:

$$\mathfrak{k} = \{\xi \in \mathbb{C}^{n \times n} : \xi^* = -\xi\}$$
$$\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n) : \theta_i \text{ real}\}$$
$$T = \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) : \theta_i \text{ real}\}$$

$\mathfrak{t}$ is diagonable/$\mathbb{C}$ in the adjoint action $\mathrm{ad}(\xi) : \eta \mapsto [\xi, \eta]$ on $\mathfrak{k}$, and is its own centralizer, so

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha \in \Phi} \mathfrak{k}_\alpha$$

where $\Phi \subset \mathfrak{t}_{\mathbb{C}}^*$ consists of the nonzero joint eigenvalues, i.e. the <u>roots</u>. Φ is the <u>root system</u>, $\mathfrak{k}_\alpha$ is the

α-root space, $x_\alpha \in \mathfrak{k}_\alpha$ is an α-root vector. In fact here $\Phi \subset i\mathfrak{t}^*$. If we decompose

$$\mathfrak{k} = \mathfrak{z} \oplus [\mathfrak{k},\mathfrak{k}], \quad \mathfrak{t} = \mathfrak{z} + (\mathfrak{t} \cap [\mathfrak{k},\mathfrak{k}])$$

where $\mathfrak{z}$ is the center of $\mathfrak{k}$, then Φ spans $i(\mathfrak{t} \cap [\mathfrak{k},\mathfrak{k}])^*$.

The finite set of hyperplanes $\alpha^\perp = \{\xi \in i\mathfrak{t} : \alpha(\xi) = 0\}$, $\alpha \in \Phi$, cuts out a set of convex open cones in $i\mathfrak{t}$, the Weyl chambers. If $\mathcal{D}$ is a Weyl chamber it determines

$$\Phi^+ = \{\alpha \in \Phi ; \alpha > 0 \text{ on } \mathcal{D}\},$$

which is a positive root system in the sense

$$\Phi = \Phi^+ \cup -\Phi^+ \qquad (\text{disjoint})$$

$$\alpha, \beta \in \Phi^+, \ \alpha+\beta \in \Phi \Rightarrow \alpha+\beta \in \Phi^+.$$

That in turn determines a simple root system $\Sigma = \{\alpha \in \Phi^+ : \alpha \notin \Phi^+ + \Phi^+\}$.

Facts: $\Sigma = \{\alpha \in \Phi^+ : \alpha^\perp \cap cl(\mathcal{D})$ is open in $\alpha^\perp\}$ ie $\alpha^\perp$ is a wall of $\mathcal{D}$; Σ is a basis of $i(\mathfrak{t} \cap [\mathfrak{k},\mathfrak{k}])^*$.

Case $U(n)$:

$\varepsilon_i \in \mathfrak{t}_{\mathbb{C}}^*$ by $\varepsilon_i \cdot \mathrm{diag}(a_1, \dots, a_n) = a_i$

$\Phi = \{\varepsilon_i - \varepsilon_j : i \neq j\}$.

$\mathfrak{u}(n)_{\varepsilon_i - \varepsilon_j}$ spanned by matrix E_{ij}: 1 in (i,j) place, else 0.

$\Phi^+ = \{\varepsilon_i - \varepsilon_j : i < j\}$ so

$$\sum_{\alpha \in \Phi^+} \mathfrak{u}(n)_\alpha = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}.$$

$\Sigma = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{n-1} - \varepsilon_n\}$

The Killing form

$$\langle \xi, \eta \rangle = \mathrm{trace}(\mathrm{ad}(\xi)\,\mathrm{ad}(\eta))$$

is a symmetric bilinear form on $\mathfrak{k}$, negative definite on $[\mathfrak{k},\mathfrak{k}]$, kernel $\mathfrak{z}$. So its $\mathbb{C}$-bilinear extension is positive definite on $i(\mathfrak{t}\cap[\mathfrak{k},\mathfrak{k}])$. If $\alpha\in\Phi$ define $t_\alpha\in i(\mathfrak{t}\cap[\mathfrak{k},\mathfrak{k}])$ by $\alpha(\xi)=\langle t_\alpha,\xi\rangle$. If $\alpha,\beta\in\Phi$ define $\langle\alpha,\beta\rangle=\langle t_\alpha,t_\beta\rangle$. Then

$$\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}=0 \text{ or } \pm1 \text{ or } \pm2 \text{ or } \pm3$$

and

$$\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\cdot\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}=0 \text{ or } 1 \text{ or } 2 \text{ or } 3.$$

The <u>Dynkin diagram</u> of $[\mathfrak{k},\mathfrak{k}]$ is the graph with vertices Σ where

i) α,β joined by $\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\cdot\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ lines

ii) if $\|\alpha\|<\|\beta\|$ then arrow points to α

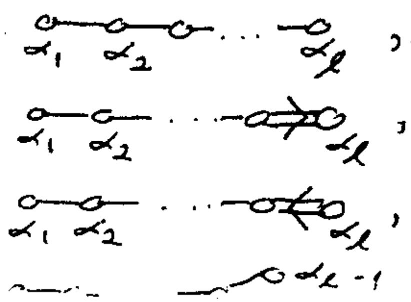
<u>examples</u>

A_ℓ , $\ell\geq1$

B_ℓ , $\ell\geq2$

C_ℓ , $\ell\geq3$

D_ℓ , $\ell\geq4$



Case $U(n)$:

$\Sigma=\{\alpha_1,\dots,\alpha_{n-1}\}$, $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$

diagram A_{n-1}

The <u>weight lattice</u> for $(K,\mathfrak{t})$ is

$$\Lambda=\{\lambda\in i\mathfrak{t}^*:\ e^\lambda \text{ defined on } T^0\}$$

where

$$e^\lambda:\ \exp(\xi)\mapsto e^{\lambda(\xi)} \text{ for } \xi\in\mathfrak{t}.$$

Case $U(n)$:

$$\Lambda=\{\textstyle\sum n_j\varepsilon_j:\ n_j \text{ integers}\}.$$

If $\pi\in\hat{K}$ we get representation $d\pi$ of $\mathfrak{k}$ by $d\pi(\xi)=\frac{d}{dt}\Big|_{t=0}\pi(\exp t\xi)$. The joint eigenvalues of $d\pi|_{\mathfrak{t}}$ form a finite subset $M_\pi\subset\Lambda$, the <u>weight system</u> of π (or $d\pi$). There is a unique <u>highest weight</u> $\lambda\in M_\pi$, characterized by

$$\alpha\in\Sigma\Rightarrow\lambda+\alpha\notin M_\pi$$

The highest weight λ is

dominant: $\langle\lambda,\alpha\rangle \geqq 0$ for $\alpha\in\Sigma$

integral: $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$ for $\alpha\in\Sigma$

so it can be encoded by putting the integer $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ over the vertex α in the diagram (omit zeroes):

vector representation ψ of $U(n)$:

$\overset{1}{\circ}-\circ\cdots-\circ$, $\lambda=\varepsilon_1$

$\Lambda^k(\psi)$, $1\leqq k\leqq n-1$:

$\underset{\alpha_1}{\circ}-\underset{\alpha_2}{\circ}\cdots-\overset{1}{\underset{\alpha_k}{\circ}}-\cdots-\underset{\alpha_{n-1}}{\circ}$, $\lambda=\varepsilon_1+\dots+\varepsilon_k$

Fact: $\pi,\pi'\in\hat{K}$ are equivalent if and only if they have the same highest weights.

Fact: every $\lambda\in\Lambda$ is integral, and the highest weights of representations $\pi\in\hat{K}$ are just the dominant elements of Λ.

These two facts constitute Cartan's "highest weight" theory. Write π_λ for the representation of highest weight λ.

Case $U(n)$:

$\lambda=\sum n_k(\varepsilon_1+\dots+\varepsilon_k)$: $\overset{n_1}{\circ}-\overset{n_2}{\circ}\cdots-\overset{n_{n-1}}{\circ}$

π_λ is the component of

$$\bigotimes^{n_1}(\Lambda^1\psi)\otimes\dots\otimes\bigotimes^{n_{n-1}}(\Lambda^{n-1}\psi)$$

that contains the highest weight vector

$v_1^{n_1}\otimes\dots\otimes v_{n-1}^{n_{n-1}}$

where $e_1,\dots,e_n$ usual basis of $\mathbb{C}^n$

$v_k=e_1\wedge\dots\wedge e_k$

$v^m=v\otimes\dots\otimes v$

§3. Maximal Compact Subgroups

Let G be a connected semisimple Lie group. In other words its Lie algebra $\mathfrak{g}$ has Killing form $\langle \xi, \eta \rangle = tr(ad(\xi) ad(\eta))$ that is nondegenerate! Then G has

- Cartan involution θ:

$\theta \in Aut(G)$, $\theta^2 = 1$, $Z_G \subset G^\theta$,

G^θ / Z_G max compact subgroup of G/Z_G

examples:

G	θ	$K = G^\theta$
$SL(n;F)$, $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$	$x \mapsto {x^*}^{-1}$	$SU(n;F)$
$SU(p,q;F)$	$x \mapsto {x^*}^{-1}$	$S(U(p;F) \times U(q;F))$

[$x^* = {}^t x^{-1}$; $SU(n;\mathbb{R}) = SO(n)$ special orthogonal group
$SU(n;\mathbb{C}) = SU(n)$ (complex) unitary group
$SU(n;\mathbb{H}) = Sp(n)$ quaternion unitary group
$SU(p,q;F)$: for hermitian form, matrix $\begin{pmatrix} -I_p & \\ & I_q \end{pmatrix}$]

$K = G^\theta$ is a maximal compactly embedded subgroup of G. It is compact $\Leftrightarrow$ Z_G finite.

If $G = SU(p,q)$ complex indefinite-unitary group then universal cover $\pi: \tilde{G} \to G$ has infinite cyclic kernel, $Z_{\tilde{G}} = \pi^{-1}(\{e^{2\pi i k/(p+q)} I\})$ infinite, $\tilde{K}$ not compact but is compact modulo $Z_{\tilde{G}}$

It is the same if G has only finitely many components, if G is reductive rather than semisimple, ($\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}]$, $\mathfrak{z}$ = center, $[\mathfrak{g},\mathfrak{g}]$ semisimple) or both:

if $G = O(p,q)$: $K = O(p) \times O(q)$

if $G = GL(n;\mathbb{C})$: $K = \mathbb{C}^\times \cdot U(n)$, $\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$.

In any case: K meets every component of G

$K \cap G^0$ is connected

But here's the catch; there are two ways to go

algebraic: $Z_{[G^0, G^0]}$ always finite; G linear

$K = G^\theta$ always compact

this way: $G = GL(n;\mathbb{C}) \Rightarrow K = U(n)$

analytic: $Z_{G^0} \subset K$, K compact only mod Z_{G^0}

this way: $G = GL(n;\mathbb{C}) \Rightarrow K = \mathbb{C}^\times \cdot U(n)$

There is no difference for (finite covers of) linear semisimple groups

e.g. flavors $SO(2,4)$ or $SU(2,2)$ of the conformal group

but there is a difference for the case where G^0 may have infinite center. WE WILL USE THE ANALYTIC APPROACH - these talks are analytically oriented, and I want to consider the universal cover of the conformal group.

some decompositions:

1. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ $\quad \pm 1$ eigenspaces of Θ

   $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$

   $G = K\cdot\exp(\mathfrak{p})$

   $(k,\xi)\mapsto k\cdot\exp_G(\xi)$ diffeo of $K\times\mathfrak{p}$ onto G

   ex $G = GL(n;\mathbb{R})$, has 2 components
   $K = O(n)$, $\mathfrak{p}$ = symmetric matrices
   polar decomp: $g\in G \Rightarrow g = k\cdot\exp(\xi)$, $\xi\in\mathfrak{p}$, and this is unique

2. $\mathfrak{a}$: max abelian subspace of $\mathfrak{p}$

   $A = \exp_G(\mathfrak{a})$

   $\Rightarrow G = KAK$. Cartan decomposition

   ex $G = GL(n;\mathbb{C})$: $\mathfrak{a} = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} : a_i\in\mathbb{R} \right\}$

   $\therefore g\in G \Rightarrow g = kak'$ with $k,k'\in U(n)$,
   a diagonal with positive real entries

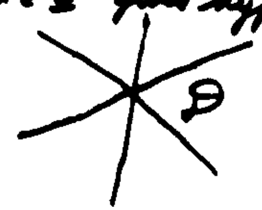
   let $M = Z_K(\mathfrak{a}) = Z_K(A)$: then $G = KAK$ is unique up to $(km)ak' = k\cdot(mam^{-1})\cdot mk'$ when $m\in M'$, $a\in A$, $k,k'\in K$

   $' = N_K(\mathfrak{a})$

3. let Φ denote the nonzero joint eigenvalues of $\mathfrak{a}$ on $\mathfrak{g}$ (action $\mathrm{ad}(\xi)$: $y\mapsto[\xi,y]$),

   $$\mathfrak{g} = \underbrace{(\mathfrak{m}+\mathfrak{a})}_{\mathfrak{z}_\mathfrak{g}(\mathfrak{a})} + \sum_{\substack{\alpha\in\Phi \\ \text{roots}}} \underbrace{\mathfrak{g}_\alpha}_{\substack{\text{root space: all } y \ni \\ [\xi,y]=\alpha(\xi)y\ \forall\xi\in\mathfrak{a}}}$$

Each $\alpha\in\Phi$ gives hyperplane $\alpha = 0$ in $\mathfrak{a}$



These cut out some convex open cones, WEYL CHAMBERS

fact: $\alpha\in\Phi \Rightarrow -\alpha\in\Phi$. So a Weyl chamber $\mathcal{D}$ defines a positive root system Φ^+

$\Phi = \Phi^+ \cup -\Phi^+$ (disj); if $\alpha,\beta\in\Phi^+$, $\alpha+\beta\in\Phi$ then $\alpha+\beta\in\Phi^+$

by: $\Phi^+ = \{\alpha\in\Phi : \alpha$ takes positive values on $\mathcal{D}\}$

let $\mathfrak{n} = \sum_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$ $\qquad \mathfrak{n}^- = \Theta\mathfrak{n} = \sum_{\alpha\in\Phi^+}\mathfrak{g}_{-\alpha}$

$N = \exp_G\mathfrak{n}$ $\qquad N^- = \exp_G\mathfrak{n}^-$

Then

$G = KAN$ Iwasawa decomposition

$K\times\mathfrak{a}\times\mathfrak{n}\to G$, by $(k,\xi,y)\mapsto k\cdot\exp_G(\xi)\cdot\exp_G(y)$ is a diffeomorphism

ex $G = GL(n;\mathbb{C})$,
$\Downarrow$
$g = kan$ with $k\in U(n)$
a diag, pos real entries
$n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

§3½. Cartan Subgroups & Subalgebras

def A *Cartan subalgebra* (CSA) in a reductive Lie algebra $\mathfrak{g}$ is a maximal $\{$abelian subalg $\mathfrak{h}$, $\mathrm{ad}_{\mathfrak{g}}\mathfrak{h}$ diagonalizable over $\mathbb{C}\}$.

ex $\mathfrak{g} = \mathfrak{gl}(4;\mathbb{R})$

1. fundamental (maximally compact) CSA $\mathfrak{h}$
$$\mathfrak{t}_H = \left\{\begin{pmatrix} 0 & a & & \\ -a & 0 & & \\ & & 0 & b \\ & & -b & 0\end{pmatrix}\right\},\quad \mathfrak{a}_H = \left\{\begin{pmatrix} c & & & \\ & c & & \\ & & d & \\ & & & d\end{pmatrix}\right\}$$
$$\mathfrak{h} = \mathfrak{t}_H + \mathfrak{a}_H,\quad \mathfrak{t}_H = \mathfrak{h}\cap\mathfrak{k},\quad \mathfrak{a}_H = \mathfrak{h}\cap\mathfrak{p}$$

2. intermediate CSA $\mathfrak{j}$:
$$\mathfrak{t}_J = \left\{\begin{pmatrix} 0 & a & & \\ -a & 0 & & \\ & & 0 & 0 \\ & & 0 & 0\end{pmatrix}\right\},\quad \mathfrak{a}_J = \left\{\begin{pmatrix} c & & & \\ & c & & \\ & & b_1 & \\ & & & b_2\end{pmatrix}\right\}$$
$$\mathfrak{j} = \mathfrak{t}_J + \mathfrak{a}_J,\quad \mathfrak{t}_J = \mathfrak{j}\cap\mathfrak{k},\quad \mathfrak{a}_J = \mathfrak{j}\cap\mathfrak{p}$$

3. maximally split CSA $\mathfrak{v}$:
$$\mathfrak{t}_V = \left\{\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0\end{pmatrix}\right\},\quad \mathfrak{a}_V = \left\{\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4\end{pmatrix}\right\}$$
$$\mathfrak{v} = \mathfrak{t}_V + \mathfrak{a}_V = \mathfrak{a}_V \subset \mathfrak{p}$$

Construction of all (conj classes of) CSA in $\mathfrak{g}$:

start: $\mathfrak{a}\subset\mathfrak{p}$ max abelian subspace

$\mathfrak{t}$ CSA in $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$

$\mathfrak{h} = \mathfrak{t}+\mathfrak{a}$ maximally split CSA in $\mathfrak{g}$

S: set of strongly orthogonal ($\alpha,\beta\in S \Rightarrow \alpha\pm\beta\notin\Phi$) roots in $\Phi = \Phi(\mathfrak{g},\mathfrak{h})$ such that $\alpha\in S \Rightarrow \alpha(\mathfrak{t}) = 0$

i.e. $\alpha(\mathfrak{h})\subset\mathbb{R}$ is a *real* root

each $\alpha\in S$ gives
$$\mathfrak{g}[\alpha] = (\mathfrak{g}_\alpha\cap\mathfrak{g}) + (\mathfrak{g}_{-\alpha}\cap\mathfrak{g}) + ([\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]\cap\mathfrak{g})$$
$$\ni x_\alpha \qquad \ni y_\alpha \qquad \ni h_\alpha = [x_\alpha, y_\alpha]$$
such that $[h_\alpha, x_\alpha] = 2x_\alpha$, $[h_\alpha, y_\alpha] = -2y_\alpha$

$\cong \mathfrak{sl}(2;\mathbb{R})$ by $h_\alpha\to\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $x_\alpha\to\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $y_\alpha\to\begin{pmatrix}0&0\\1&0\end{pmatrix}$

h_α spans $\mathfrak{h}\cap\mathfrak{g}[\alpha]\subset\mathfrak{a}$

"replace" h_α by $x_\alpha - y_\alpha$:
$$\mathfrak{t}\mapsto\mathfrak{t}_S = \mathfrak{t} + \sum_{\alpha\in S}(x_\alpha - y_\alpha)\mathbb{R}$$
$$\mathfrak{a}\mapsto\mathfrak{a}_S = \bigcap_{\alpha\in S}\{\xi\in\mathfrak{a} : \alpha(\xi) = 0\}$$
$$\mathfrak{h}\mapsto\mathfrak{h}_S = \mathfrak{t}_S + \mathfrak{a}_S$$

Fact: 1) if $\mathfrak{h}'$ is any CSA in $\mathfrak{g}$ then $\exists\, x\in G$, $\mathrm{Ad}(x)\mathfrak{h}'$ is one of the $\mathfrak{h}_S$

2) if $\exists\, x\in G$, $\mathrm{Ad}(x)\mathfrak{h}_S = \mathfrak{h}_{S'}$ then $\exists\, x\in N_G(\mathfrak{h})$, $\mathrm{Ad}(x)^* S' = S$

Alternative construction

Start with $\mathfrak{t}$ CSA in $\mathfrak{k}$, $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$ is "maximally compact" or "fundamental" CSA in $\mathfrak{g}$.

As before: S set of strongly orthogonal imaginary roots of $(\mathfrak{g},\mathfrak{h})$: $\mathfrak{h} = \mathfrak{t}+\mathfrak{a}$ under $\theta \Rightarrow \alpha(\mathfrak{a}) = 0$ for $\alpha\in S$

This time, replace

$$x_\alpha - y_\alpha \in \mathfrak{h} \cap \underbrace{\mathfrak{g}[\alpha]}_{\cong \mathfrak{su}(2)} \quad \text{by } h_\alpha = [x_\alpha, y_\alpha]$$

$$Z \mapsto Z_{(S)} = \bigcap_{\alpha \in S} \{\zeta \in Z : \alpha(\zeta) = 0\}$$

$$\mathfrak{a} \mapsto \mathfrak{a}_{(S)} = \mathfrak{a} + \sum_{\alpha \in S} h_\alpha \mathbb{R}$$

<u>definition</u> The <u>Cartan subgroup</u> of G corresponding to a CSA $\mathfrak{h} \subset \mathfrak{g}$ is

$$H = \{x \in G : \mathrm{Ad}(x)\zeta = \zeta \quad \forall \zeta \in \mathfrak{h}\}.$$

If $\mathfrak{h}$ is θ-stable, so is H, and

$H = T \times A$, $\quad T = H \cap K$ not necessarily connected even if G is conn

$A = \exp(\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p})$

<u>topological components of</u> CSG $H \subset G$:

for each real root $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ set

$h_\alpha \in \mathfrak{a}_H$: element dual to $\check{\alpha} = 2\alpha/\|\alpha\|^2$

$x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha = \theta x_\alpha \in \mathfrak{g}_{-\alpha}$ so $[x_\alpha, y_\alpha] = h_\alpha$

$\gamma_\alpha = \exp_G(\pi(x_\alpha - y_\alpha))$

Then Z_{G^0} and the γ_α generate a group $Z_{\mathfrak{a}_H}$ such that $\quad H \cap G^0 = Z_{\mathfrak{a}_H} H^0$

§4. Smooth Vectors

Let π: Banach rep of G

π is T.I. if $\mathcal{B}_\pi$ has no proper invariant subspace

π is T.C.I. if $\quad T : \mathcal{B}_\pi \to \mathcal{B}_\pi$ bounded, $n \geq 1$,

$\{v_1, \ldots v_n\} \subset \mathcal{B}_\pi$, $\epsilon > 0$

$\Rightarrow \exists f \in C_c(G)$, $\|\pi(f) - T)v_i\| < \epsilon \;\forall i$

Note TCI $\Rightarrow$ TI.

Schur's Lemma If π is TCI, $T : \mathcal{B}_\pi \to \mathcal{B}_\pi$ is bounded, $\pi(x)T = T\pi(x) \;\forall x \in G$, ~~then~~ T is <u>scalar</u>

$\boxed{\text{Treat also}}$ Cor. Then $\pi|_{Z_G}$ gives a ~~number~~ <u>quasicharacter</u> $\zeta_\pi : Z_G \to \mathbb{C}^\times$

Fact: unitary + TI $\Rightarrow$ TCI, so Schur's Lemma is good also for TI unitary π, i.e. for $\pi \in \widehat{G}$. Then $\zeta_\pi \in \widehat{Z}_G$ unitary character.

<u>Def</u> $v \in \mathcal{B}_\pi$ is C^∞ if: whenever $v^* \in \mathcal{B}_\pi^*$,

$x \mapsto v^*(\pi(x)v)$ is C^∞ fn on G

i.e. $x \mapsto \pi(x)v$ is weakly C^∞

Write $\mathcal{B}$ for $\mathcal{B}_\pi$, $\mathcal{B}_\infty$ for space of C^∞ vectors, π_∞ for rep of G on $\mathcal{B}_\infty$

$U(\mathfrak{g})$ acts on $\mathcal{B}_\infty$

$$\zeta \in \mathfrak{g},\ v \in \mathcal{B}_\infty \Rightarrow d\pi(\zeta)v \overset{\text{def}}{=} \frac{d}{dt}\Big|_{t=0} \pi(\exp_G(t\zeta))v$$

— exists because $v \in \mathcal{B}_\infty$

$\pi(C_c^\infty(G))\mathcal{B} \subset \mathcal{B}_\infty$: Let $\Xi \in \mathcal{U}(\mathfrak{g})$, $x \in G$, $f \in C_c^\infty(G)$

$$\Xi_x \cdot \pi(x)\pi(f)v = \Xi_x \int_G \pi(x)f(y)\pi(y)v\,dy$$

$$= \Xi_x \int_G f(x^{-1}y)\pi(y)v\,dy \text{ ; move } \Xi_x \text{ past } \int : f \in C_c^\infty, \text{ etc}$$

$\therefore \mathcal{B}_\infty$ dense in $\mathcal{B}$.

Theorem If $\mathrm{Ad}(x)$ inner on $\mathfrak{g}_c$ $\forall x \in G$

need here: $\mathrm{Ad}(x)$ trivial on center $\mathfrak{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g}_c)$

Then: if π is TCI, it represents $\mathfrak{Z}(\mathfrak{g})$ by scalars on $\mathcal{B}_\infty$

These scalars define a homomorphism of associative algebras

$$\chi_\pi : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$$ INFINITESIMAL CHARACTER OF THE TCI REP. π

In other words, for each Casimir $D \in \mathfrak{Z}(\mathfrak{g})$,

if $v \in \mathcal{B}_\infty$ then $d\pi(D)\cdot v = \chi_\pi(D)v$

and then of course this extends by cont. to

if $v \in \mathcal{B} = \mathcal{B}_\pi$ then $d\pi(D)\cdot v = \chi_\pi(D)v$

FIRST MAJOR INGREDIENT IN SEMI-SIMPLE REPRESENTATION THEORY

Cor If $\mathrm{Ad}(x)$ inner on $\mathfrak{g}_c$ $\forall x \in G$, $[\pi] \in \hat{G}$

Then $\exists$ homomorphism $\chi_\pi : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ infinitesimal character of π such that, $\forall D \in \mathfrak{Z}(\mathfrak{g})$, $d\pi(D)v = \chi_\pi(D)v$ directly if $v \in \mathcal{B}_\infty$; by continuity from $\mathcal{B}_\infty$ in general

Brief indication of why these things are true

<u>Schur's Lemma</u>: π TCI $\Rightarrow$ every bounded $T : \mathcal{B}_\pi \circlearrowleft$ that commutes with π is scalar

Let $0 \neq v \in \mathcal{B}$. If v, Tv linearly indep:

$\{f_\alpha\} \subset C_c(G) \ni \{\pi(f_\alpha)v\} \to v$ AND $\{\pi(f_\alpha)Tv\} \to 0$

$\therefore v = \lim \pi(f_\alpha)Tv = T \lim \pi(f_\alpha)v = Tv$ contra

$\therefore v \in \mathcal{B} \Rightarrow Tv = c(v)v$, $c(v) \in \mathbb{C}$

$0 \neq u, v \Rightarrow$ take $\{f_\alpha\} \subset C_c(G)$, $\pi(f_\alpha)u \to v$, apply T, see $c(u) = c(v)$

<u>unitary TI $\Rightarrow$ TCI</u> $\mathcal{A}$ = W*-alg of all bounded T on $\mathcal{H} = \mathcal{H}_\pi$ s.t.

if $\{v_i\}$ finite subset of $\mathcal{H}$, $\varepsilon > 0$

then $\exists f \in M_c(G)$ comp supp Radon meas: each $\|(\pi(f) - T)v_i\| < \varepsilon$

$M_c(G)$: $\pi(\mu)$ def by $(\pi(\mu)v, v') = \int (\pi(x)v, v')\,d\mu$

$\therefore \pi(G) \subset \pi(M_c(G)) \subset \mathcal{A}$, π is TI so $\mathcal{A}' = \mathbb{C}$,

$\Rightarrow \mathcal{A} = (\mathcal{A}')'$ all bounded operators

$\therefore \pi$ is TCI

<u>$\exists$ of infinitesimal character</u>

$\mathcal{U}(\mathfrak{g})$: view as distrib on G supported at 1

$\therefore \mathcal{U}(\mathfrak{g}) \subset$ (comp supported distrib, $\mathcal{D}_c'(G)$)

with $\mathfrak{Z}(\mathfrak{g}) \subset$ center of convol. alg. $\mathcal{D}_c'(G)$

π_∞ gives rep of $\mathcal{D}_c'(G)$ on $\mathcal{B}_\infty = \pi(C_c^\infty(G))\cdot\mathcal{B}$

$\therefore \pi_\infty$ gives rep of $\mathcal{D}_c'(G)$ on $\mathcal{B}_\infty^* = \pi^*(C_c^\infty(G))\mathcal{B}^*$

latter weakly dense in $\mathcal{B}^*$

As in Schur's Lemma: Z central in $\mathcal{D}_c'(G)$

$\Rightarrow$ acts on $\mathcal{B}_\infty$ by a scalar $\chi_\pi(Z)$. Weak cont argument: same for $Z \in \mathfrak{Z}(\mathfrak{g})$

Suppose Z_{G^0} finite, so K is compact. If $\chi \in \hat{K}$ then the normalized character

$$\tau_\chi(k) = \dim\chi \cdot \operatorname{trace}\chi(k)$$

defines orthogonal projection

$$L^2(K) = \sum_{\chi' \in \hat{K}} V_{\chi'} \otimes V_{\chi'}^* \longrightarrow V_\chi \otimes V_\chi^*$$

by $f \mapsto \bar{\tau}_\chi *_K f = f *_K \bar{\tau}_\chi$, and we get a continuous projection of $\mathcal{B}$ to its K-isotypic component

$$\mathcal{B}(\chi) = \{v \in \mathcal{B} : \pi|_K \text{ acts on (span } \pi(K)v) \text{ by } \chi\}$$

by the same trick:

$$\pi(\bar{\tau}_\chi) : \mathcal{B} \to \mathcal{B}(\chi) \text{ by } v \mapsto \int_K \bar{\tau}_\chi(k)\pi(k)v\,dk$$

The situation is similar if $Z = Z_{G^0}$ is infinite. Let $\zeta \in \hat{Z}$, $\hat{K}_\zeta = \{\chi \in \hat{K} : \chi|_Z \text{ contains } \zeta\}$. Then $\mathcal{B}(\chi) \neq 0 \Rightarrow \chi \in \hat{K}_{\zeta_\pi}$. $L^2(K/Z, \zeta_\pi) = \sum_{\hat{K}_{\zeta_\pi}} V_{\chi'} \otimes V_{\chi'}^*$, same projections – but this time integrals on K/Z:

$$\pi(\bar{\tau}_\chi) : \mathcal{B} \to \mathcal{B}(\chi) \text{ by } v \mapsto \int_{K/Z} \bar{\tau}_\chi(k)\pi(k)v\,d(kZ)$$

The first application of $\mathcal{Z}(\mathfrak{g})$ is reconstruction of v from its K-components:

__Theorem__ If $v \in \mathcal{B}_\infty$ then $\sum \pi(\bar{\tau}_\chi)v$ is absolutely convergent ($\sum \|\pi(\bar{\tau}_\chi)v\| < \infty$) to v.

model: abs. convergence of Fourier series for $f \in C^\infty$(circle)

Idea of proof:

$\langle , \rangle$ pos def $\mathrm{Ad}(K)$-invariant bilinear form on $\mathfrak{k}$, $\xi_1, \dots, \xi_n$ ON basis, $\Omega_K = -\sum \xi_i^2 \in \mathcal{Z}(\mathfrak{k})$, [illegible] and skew-adj of $d\chi(\xi_i)$ gives $\chi(1+\Omega_K) = c_\chi \geq 1$ and $(1+\Omega_K)\tau_\chi = c_\chi \tau_\chi$. K/Z compact $\Rightarrow \pi|_K$ uniformly bounded, $\|\pi(k)\| \leq a$, so

$$\|\pi(\bar{\tau}_\chi)v\| \leq (\dim\chi)^2 a\|v\|$$

$\therefore m$ integer $\geq 0 \Rightarrow$ [use $\pi(\bar{\tau}_\chi)v = c_\chi^{-m}\pi(\bar{\tau}_\chi)\pi_\infty(1+\Omega_K)^m v$]

(†) $\|\pi(\bar{\tau}_\chi)v\| \leq c_\chi^{-m}(\dim\chi)^2 a\|\pi_\infty(1+\Omega_K)^m v\|$

Suppose for the moment that we know

(*) $m \gg 0 \Rightarrow \sum_{\chi \in \hat{K}_\zeta} (\dim\chi)^2 c_\chi^{-m} < \infty \quad \forall \zeta \in \hat{Z}$

Then combine (†) and (*):

$$\sum_{\chi \in \hat{K}} \|\pi(\bar{\tau}_\chi)v\| = \sum_{\chi \in \hat{K}_{\zeta_\pi}} \|\pi(\bar{\tau}_\chi)v\|$$

$$\leq \Big\{\sum_{\chi \in \hat{K}_{\zeta_\pi}} (\dim\chi)^2 c_\chi^{-m}\Big\} a\|\pi_\infty(1+\Omega_K)^m v\|$$

which gives the absolute convergence of $\sum \pi(\bar{\tau}_\chi)v$, for $v \in \mathcal{B}_\infty$.

$\sum \pi(\bar{\tau}_\chi)v$ converges to v: let $v_0 = \sum \pi(\bar{\tau}_\chi)v$, so $\pi(\bar{\tau}_\chi)(v - v_0) = 0 \ \forall \chi \in \hat{K}$. Let (f_n) be an approximate identity in $L^1(K/Z, \zeta_\pi)$. Then for each n have $F_n \subset \hat{K}_{\zeta_\pi}$ finite, $h_n \in \sum_{\chi \in F_n} V(\chi) \otimes V(\chi)^*$ such that

$$\sup|h_n - f_n| < 2^{-n}$$

$$\therefore v - v_0 = \lim_{n\to\infty} \pi(f_n)(v - v_0) = \lim_{n\to\infty} \pi(h_n)(v - v_0)$$

$$= \lim_{n\to\infty} \pi\Big(h_n *_{K/Z} \sum_{F_n} \bar{\tau}_\chi\Big) = \lim_n \sum_{F_n} \pi(h_n)\pi(\bar{\tau}_\chi)(v - v_0) = 0$$

Finally, prove (*). Frobenius reciprocity: can assume K connected. Let T be a Cartan subgp s.t. T/Z max torus of K/Z. If $\chi_\lambda \in \hat{K}$: highest weight λ, then

$$\dim \chi_\lambda = \prod_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{t})} \frac{\langle \lambda+\rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$$ WEYL DEGREE FORMULA

where Σ^+ is a positive root system, $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$.

So $\dim \chi_\lambda = p(\lambda)$, poly in λ. Fact:

$$c_{\chi_\lambda} = 1 + (\|\lambda+\rho\|^2 - \|\rho\|^2)$$

$\hat{K}_\zeta$ has finite set $F \ni \chi_\lambda \in \hat{K}_\zeta \setminus F \Rightarrow \|\lambda+\rho\|^2 - \|\rho\|^2 > \frac{1}{2}\|\lambda\|^2$.

Now $\sum_{\hat{K}_\zeta \setminus F} (\dim \chi)^2 c_\chi^{-m} \leq 2^m \sum_{\lambda\ \text{h.wt.}} (1+\|\lambda\|^2)^{-m} |p(\lambda)|^2$

which proves (*)

Theorem $\sum_\chi \mathcal{B}_\omega \cap \mathcal{B}(\chi)$ is dense in $\mathcal{B}$

Essentially a corollary

Corollary $\mathcal{B}_\omega \cap \mathcal{B}(\chi)$ is dense in $\mathcal{B}(\chi)$

§5. Analytic and K-finite vectors

A vector $v \in \mathcal{B}$ is analytic if $x \mapsto \pi(x)v$ is weakly analytic on G, i.e. if $v' \in \mathcal{B}' \Rightarrow x \mapsto v'(\pi(x)v)$ is analytic. $\mathcal{B}_\omega$ = space of analytic vectors

Note $\mathcal{B}_\omega \subset \mathcal{B}_\infty$; $x \in G \Rightarrow \pi(x)\mathcal{B}_\omega \subset \mathcal{B}_\omega$; $u \in \mathcal{U}(\mathfrak{g}) \Rightarrow d\pi(u)\mathcal{B}_\omega \subset \mathcal{B}_\omega$. Taylor series:

Prop. If $v \in \mathcal{B}_\omega$ then $\exists$ nbhd $\mathcal{O}$ of $0 \in \mathfrak{g}$ s.t.

$$\sum_{m=0}^{\infty} \frac{1}{m!} \underbrace{d\pi_\omega}_{\text{action of } G \text{ on } \mathcal{B}_\omega}(\xi)^m v \text{ converges to } d\pi(\exp \xi)v \quad \forall \xi \in \mathcal{O}$$

... why one tackles analytic vectors

Cor If $\mathcal{B}_0$ is a $d\pi_\omega(\mathcal{U}(\mathfrak{g}))$-stable subspace of $\mathcal{B}_\omega$ THEN closure ($\mathcal{B}_0$ in $\mathcal{B}$) is a $\pi(G)$-stable subspace of $\mathcal{B}$

Nelson's Theorem $\mathcal{B}_\omega$ is dense in $\mathcal{B}$.

If $\chi \in \hat{K}$: $\mathcal{B}_\omega(\chi) = \mathcal{B}_\omega \cap \mathcal{B}(\chi)$

$\mathcal{B}_K = \sum \mathcal{B}_\omega(\chi) = \mathcal{B}_\omega \cap \sum \mathcal{B}(\chi)$

note $\mathcal{B}_K \subset \mathcal{B}_\omega \subset \mathcal{B}_\infty$

Theorem $\mathcal{B}_K$ is dense in $\mathcal{B}$

This is a corollary of Nelson's Theorem. Let $u \in \mathcal{B}$, $\varepsilon > 0$. $\mathcal{B}_\omega$ dense $\Rightarrow \exists v \in \mathcal{B}_\omega$, $\|u-v\| < \frac{1}{2}\varepsilon$. $v \in \mathcal{B}_\infty \Rightarrow \sum \pi(\Xi_\chi)v \to v$ uniformly $\Rightarrow \exists$ partial sum $w = \sum_F \pi(\Xi_\chi)v$, $F \subset \hat{K}$ finite, $\|w-v\| < \frac{1}{2}\varepsilon$. If $f: G \to \mathcal{B}$ is analytic, $h: K \to \mathbb{C}$ is analytic, then $G \to \mathcal{B}$ by $x \mapsto \int_K f(xk)h(k)\,dk$ analytic. $\therefore w \in \mathcal{B}_\omega$, $\|u-w\| < \varepsilon$.

Cor $\chi \in \hat{K}$, $\dim \mathcal{B}(\chi) < \infty \Rightarrow \mathcal{B}(\chi) \subset \mathcal{B}_\omega$

Cor $\mathcal{B}_K$ is $\pi_\omega(\mathcal{U}(\mathfrak{g}))$-invariant

For the first: $\pi(\Xi_\chi)\mathcal{B}_\omega$ is dense in $\mathcal{B}(\chi)$, contained in $\mathcal{B}_\omega$

For the second:

$D \in \mathcal{U}(\mathfrak{g})$, $k \in K \Rightarrow \pi(k)\pi_\omega(D)v = \pi_\omega(\mathrm{Ad}(k)D) \cdot \pi(k)v$ for $v \in \mathcal{B}_K$. $\{\pi(k)D : k \in K\}$ spans a finite dim subspace of $\mathcal{U}(\mathfrak{g})$, the $\pi(k)v$ span a finite dim subspace of $\mathcal{B}_K$, so the $\pi(k)\pi_\omega(D)v$ span f. dim. subspace of $\mathcal{B}$. $\therefore \pi_\omega(D)v \in \mathcal{B}_K$

... $\mathcal{B}_K$ ... a ... K) module. This is the real key point in Harish-Chandra theory

Theorem Let G: connected semisimple Lie group.
π: top. irred Banach rep, $\mathcal{B} = \mathcal{B}_\pi$
Then π has infinitesimal character $\Leftrightarrow$ TCI
And in that case, π is K-finite (each $\mathcal{B}(\kappa)$ finite dim), each $\mathcal{B}(\kappa) \subset \mathcal{B}_\omega$, $\mathcal{B}_K = \sum \mathcal{B}(\kappa)$

proof Saw: TCI $\Rightarrow$ has inf. character
Let π have infinitesimal character
$\chi = \chi_\pi : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$. $\forall\, 0 \neq v \in \mathcal{B}_K$: $\pi_\omega(\mathcal{U}(\mathfrak{g}))v$ has $\pi(G)$-stable closure, so dense in $\mathcal{B}$. Contained in $\mathcal{B}_K$. So $\pi_\omega(\mathcal{U}(\mathfrak{g}))v = \sum_\kappa \pi_\omega(\mathcal{U}(\mathfrak{g}))v \cap \mathcal{B}_\omega(\kappa)$.
Let $\mathfrak{J}$ = annihilator of v in $\mathcal{U}(\mathfrak{k})$, left ideal of finite codim.

note: connectedness not really used; just need
i) G has normal abelian subgroup S such that S centralizes G^0 and G/G^0S finite
ii) $x \in G \Rightarrow \mathrm{Ad}(x)$ inner on $\mathfrak{g}_{\mathbb{C}}$

Corollary G conn semisimple (as above), $\pi \in \hat{G}$, $f \in L^1(G)$
$\Rightarrow \pi(f)$ completely continuous. In other words G is CCR ('liminaire'), hence type I.

pf. π is TCI, hence K-finite, so the $\pi(\bar{\xi}_\kappa)\pi(f) = \pi(\bar{\xi}_\kappa * f)$ have finite rank. But $\pi(\sum \bar{\xi}_\kappa * f)$ converges strongly to $\pi(f)$ because $\sum \bar{\xi}_\kappa * f$ converges L^1 to f.

a rep. ψ of an associative algebra $\mathcal{A}$ (such as $\mathcal{U}(\mathfrak{g})$) on a vector space V is
__algebraically irreducible__ if $\nexists$ nontrivial $\psi(\mathcal{A})$-invariant subspace
__algebraically completely irreducible__ if
$\{u_1, \dots, u_n\} \subset V$ linearly independent
$\{v_1, \dots, v_n\} \subset V$ arbitrary
$\Rightarrow \exists\, a \in \mathcal{A}$ with $\psi(a)u_i = v_i \ \forall i$

Theorem G connected unimodular Lie group
K compact subgroup
π K-finite Banach rep of G, $\mathcal{B} = \mathcal{B}_\pi$
$\Rightarrow$ equivalent: (i) π is TI (ii) π is TCI
(iii) π_K (of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{B}_K$) is AI, (iv) π_K is ACI

Combine these last 2 theorems:

Theorem G connected semisimple (as above notes on p.31), π TCI Banach rep of G on $\mathcal{B}$ (e.g. $\pi \in \hat{G}$). Then $d\pi_K$ is an alg completely irred rep of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{B}_K = \sum \mathcal{B}(\kappa)$, where K acts by finite mult rep compatible with the subalgebra $\mathcal{U}(\mathfrak{k})$ of $\mathcal{U}(\mathfrak{g})$.

def $(\mathcal{B}_K ; \mathcal{U}(\mathfrak{g}), K)$ is the HARISH-CHANDRA module associated to $(\mathcal{B}, \pi)$.

§6 K-multiplicities

Theorem G: connected semisimple Lie group
(or reductive group of the larger class, p. 31)

K-multiplicity theorem

K: max compactly embedded subgroup
π: TCI rep of G in Banach space B
χ: irreducible rep of K

Then multiplicity $m(\chi, \pi|_K) \leq \dim \chi$.
(we'll only actually prove it when G is a linear group)

Step 1 A associative alg, n integer > 0
R: set of rep. of A such that
R is complete ($0 \neq a \in A \Rightarrow \exists \psi \in R,\ \psi(a) \neq 0$)
$\psi \in R \Rightarrow \dim \psi \leq n$
$\Rightarrow$ every TCI Banach rep of A has dimension $\leq n$

Step 2 Suppose that G has a complete (for $C_c(G)$) set of Banach reps such that $\exists \chi \in \hat{K}$ and $n > 0$ for which every $\psi \in R$ has $m(\chi, \psi|_K) \leq n$.
Then if π is TCI Banach rep of G then $m(\chi, \pi|_K) \leq n$

Step 3 Every finite dim irreducible rep of G is equivalent to a subrep of some
$$\pi_\nu = \mathrm{Ind}_{AN}^{G}(e^\nu), \quad \nu \in \mathfrak{a}_{\mathbb{C}}^*$$
where $G = KAN$ is the Iwasawa decomp.

Before continuing, let me indicate how these steps contribute to the proof of the Theorem.

Suppose step 1 carried out. Given $\chi \in \hat{K}$ define $C_{c,\chi}(G) = \bar{\tau}_\chi * C_c(G) * \bar{\tau}_\chi$. It is an associative algebra — convolution — and $\pi(C_{c,\chi}(G))B(\chi) \subset B(\chi)$. Since π is TCI, and
$$\pi_\chi: \text{rep of } C_{c,\chi}(G) \text{ on } B(\chi)$$
is TCI: given T bounded linear on $B(\chi)$, set $\tilde{T} = T \circ \pi(\bar{\tau}_\chi)$ bounded linear on B. If $\{f_\alpha\}$ in $C_c(G)$ with $|\pi(f_\alpha)| \to \tilde{T}$ then
$$\pi_\chi(\bar{\tau}_\chi * f_\alpha * \bar{\tau}_\chi) \to T$$
Now step 2 follows from step 1: the ψ_χ, $\psi \in R$, form a complete set of Banach reps of $C_{c,\chi}$ of dimension $\leq n \cdot \dim \chi$, so $m(\chi, \pi|_K) \leq n$.

Now what we need is:
1) to verify step 1
2) to find a complete set R of Banach reps of G s.t. $\psi \in R \Rightarrow m(\chi, \psi|_K) \leq \dim \psi$ for all $\chi \in K$.

Then the result of step 2 will give the Theorem.

note use of non-unitary Banach representations

If G is a linear group then the finite dim. reps of G form a complete set of Banach reps. Once we have step 3, for π finite dim irreducible, $\pi \subset \pi_\nu$, we have
$$m(\chi, \pi|_K) \leq m(\chi, \pi_\nu|_K) \underset{\text{using FR}}{=} m(e^\nu|_M, \chi|_M) \leq \dim \chi$$
so we are done

If G is not linear: Then it is a lot more difficult to find a complete set of Banach reps with the required properties, but one can use the same idea. Every $\pi \in \hat{G}$ is equivalent to a sub-rep of some

$$\pi_{\mu,\nu} = \mathrm{Ind}_{MAN}^{G} (\mu \otimes e^{\nu}), \quad \mu \in \hat{M}, \ \nu \in \mathfrak{a}_{\mathbb{C}}^{*}.$$

A slightly weaker result, which is good enough for us, is Harish-Chandra's

subquotient theorem: every $\pi \in \hat{G}$ is equiv. to a quotient of a subrep of some $\pi_{\mu,\nu}$

Then as before

$$m(K, \pi|_K) \leq m(K, \pi_{\mu,\nu}) = m(\mu, K|_M) \leq \dim K.$$

The subquotient theorem, and the slightly stronger statement due to Casselman and Jacquet, are beyond the scope of these lectures.

<u>proof for step 1</u> Let $\Lambda(n)$ = least integer $\hat{}$ such that $\xi_1, \dots \xi_n \in \mathfrak{gl}(n;\mathbb{R}) \Rightarrow \sum (\operatorname{sign} \sigma) \xi_{\sigma(1)} \cdots \xi_{\sigma(n)} = 0$. Since Λ^n of $\mathfrak{gl}(n;\mathbb{R}) = 0$ for $n > n^2$ have $\Lambda(n) \leq n^2+1$. Induction: $\Lambda(n) \geq \Lambda(n-1)+2$. Now let ϕ, TCI Banach rep of A on B with $\dim B > n$. Let $B_0 \subset B$ subspace of $\dim = n+1$. As $\Lambda(n+1) > \Lambda(n)$, $\exists$ $T_1, \dots T_{\Lambda(n)} \in E(B_0)$ such that

$$[T_1, \dots T_{\Lambda(n)}] \overset{\text{def}}{=} \sum (\operatorname{sign} \sigma) \xi_{\sigma(1)} \cdots \xi_{\sigma(\Lambda(n))} \neq 0$$

Extend T_i to bounded linear $\tilde{T}_i$ on B. As ϕ is TCI $\exists$ net $\{a_\alpha\} \subset A$, $\phi(a_\alpha) \to \tilde{T}_1$ in the strong topology. $\therefore \lim [\phi(a_\alpha), \tilde{T}_2, \dots, \tilde{T}_{\Lambda(n)}] = [\tilde{T}_1, \dots \tilde{T}_{\Lambda(n)}] \neq 0$. $\therefore \exists\, y_1 \in A$ with $[\phi(y_1), \tilde{T}_2, \dots$ Iterate: get y_2 then y_3 ... such that $0 \neq [\phi y_1, \phi y_2, \dots \phi y_{\Lambda(n)}] = \phi[y_1, \dots, y_{\Lambda(n)}]$. Thus $[y_1 \cdots y_{\Lambda(n)}] \neq 0$. $\therefore R$ complete $\Rightarrow \exists\, \psi \in$ with $\psi[y_1, \dots, y_{\Lambda(n)}] \neq 0$. Contradicts hypothesis that $\dim \psi \leq n$.

<u>proof for step 3</u>

ψ finite dim irred rep of G on V. Lie's theorem for ψ^* and the solvable group AN: $\exists\ 0 \neq v^* \in V^*$, $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, such that $\psi^*(an) v^* = e^{-(\nu+\rho)}(a) v^*$

If $v \in V$, $\phi_v : x \mapsto \langle \psi(x)^{-1} v, v^* \rangle$ satisfies

$$\phi_v(xan) = e^{-(\nu+\rho)}(a)\, \phi_v(x)$$

$\therefore v \mapsto \phi_v$ intertwines V with subrep of π_ν subrep equiv V $\because$ V irred and $\phi_v(1) \neq 0$ when $\langle v, v^* \rangle \neq 0$.

This completes the proof of the K-multiplicity theorem when G is a linear group

§7. Global Character

Theorem Let G be a connected semisimple (or larger class mentioned earlier) Lie group, π a TCI Banach rep on a HILBERT space $\mathcal{H}$. If $f \in C_c^\infty(G)$ then $\pi(f)$ is a trace class operator on $\mathcal{H}$. Also,

$\Theta_\pi : C_c^\infty(G) \to \mathbb{C}$ by $\Theta_\pi(f) = \operatorname{trace} \pi(f)$ is a distribution on G and Θ_π satisfies the system of differential equations

$$D(\Theta_\pi) = \chi_\pi(D)\Theta_\pi \qquad \forall D \in \mathfrak{Z}(\mathfrak{g})$$

Let π, π' be K-finite (e.g. TI) unitary rep. of G. Then they are unitarily equiv. $\Longleftrightarrow \Theta_\pi = \Theta_{\pi'}$

Θ_π is the "global" or "distribution" character of π. If $\dim \pi < \infty$ then notice that, formally, Θ_π is just given by

$$\Theta_\pi(f) = \int_G f(x) \operatorname{trace} \pi(x)\, dx$$

step 1 (valid for G locally compact unimod. K compact emb. subgroup π Banach rep on $\mathcal{H}$ s.t. $\dim \mathcal{H}(\chi) \leq m_\pi \dim \chi \quad \forall \chi \in \hat{K}$)

Theorem
$f \in L^2(G)$, f compactly supported $\Rightarrow \pi(f)$ Hilbert-Schmidt on $\mathcal{H}$

Pf. If T bounded with bounded inverse

$$\|\pi(f)\|_{HS} = \|T^{-1}\cdot \pi(f) \cdot T\|_{HS} \leq \|T^{-1}\| \cdot \|\pi(f)\|_{HS} \cdot \|T\|$$

Use this with a T such that $k \mapsto T\cdot \pi(k) \cdot T^{-1}$ is unitary. $\therefore$ can assume $\pi|_K$ unitary and thus different $\mathcal{H}(\chi)$ orthogonal

2. Let $F \subset G$ compact, $\supset \operatorname{supp}(f)$
$\varphi \in C_c^+(G)$, $\varphi \equiv 1$ on KF
If $\{f_n\}$ continuous, supp in F, $\{f_n\} \to f$ in $L^2(G)$ then $\{f_n\} \to f$ in $L^1(G)$ so $\|\pi(f_n) - \pi(f)\| \to 0$. Let us check that $\{\pi(f_n)\}$ is Cauchy HS.

first: if $h \in C_c(G)$ supported on F
then $\exists\ N = N(F,\pi)$ such that
$$\|\pi(h)\|_{HS} \le N \cdot \|h\|_{L^2(G)}$$

because
$$\|\pi(h)\|_{HS} \le \int_G \Big\| \int_K h(kx)\pi(kx)\,dk \Big\|_{HS} dx$$
$$\le \underbrace{\int_G \Big\| \int_K h(kx)\pi(k)\,dk \Big\|_{HS}}\ \underbrace{\|\pi(x)\|}_{\le M \text{ on } KF} dx$$
$$\le m_\pi^{1/2} \Big\| \int_K h(kx) L_K(k)\,dk \Big\|_{HS}$$
because $\pi|_K$ unitary, $\chi \in \hat{K}$ occurs $\le m_\pi \dim\chi$ times in π

$$\le M m_\pi^{1/2} \int_G \underbrace{\Big\| \int_K h(kx) L_K(k)\,dk \Big\|_{HS}} dx$$
$$= \Big(\int_K |h(kx)|^2 dk\Big)^{1/2} \text{ by Peter-Weyl}$$
$$\le M m_\pi^{1/2} \int_G \Big(\int_K |h(kx)|^2 dk \Big)^{1/2} dx$$
$$\le M m_\pi^{1/2} \|\phi\|_{L^2(G)} \|h\|_{L^2(G)}$$

Now: $\{\pi(f_n)\}$ Cauchy $L^2(G) \Rightarrow$ Cauchy HS
$\therefore \to T$ Hilbert Schmidt: $\|\pi(f_n) - T\|_{HS} \to 0$

But: $\|\pi_n(f) - T\| \le \|\pi_n(f) - T\|_{HS}$
$\therefore\ \pi(f) = T$ in HS

step 2:

Theorem $f \in C_c^\infty(G) \Rightarrow \pi(f)$ is trace class

Pf Can assume $\pi|_K$ unitary

A = closure $\{\pi(f) : f \in C_c^\infty(G)\}$ in the Banach space of bounded linear operators on $\mathcal{H}$

$\pi(x)\pi(f) = \pi(L_G(x)f)$, $\pi(f)\pi(x) = \pi(R_G(x)f)$

$\therefore$ have Banach reps
$$l(x) : A \to \pi(x)A, \quad r(x) : A \to A\pi(x)^{-1}$$
of G on A. If $f \in C_c^\infty(G)$ then $\pi(f)$ is a C^∞ vector for l and r. Let $\psi = l \otimes r$, rep of $G \times G$ on A.
$$\psi(x,y) : A \to \pi(x) A \pi(y)^{-1}$$
Can assume K conn so $\exists\ D_0 \in \mathfrak{Z}([\mathfrak{k},\mathfrak{k}])$,
$$\chi(D_0) = (\dim\chi)^2 \chi(1) \quad \forall \chi \in \hat{K}$$
Now $f \in C_c^\infty(G) \Rightarrow f$ is C^∞ vector for ψ
$\Rightarrow \psi(L_G(D_0) R_G(D_0) f)$ is C^∞ vector for ψ

$$\therefore \sum_{\tau_1,\tau_2\in\hat K} \underbrace{\|\psi(\bar\tau_{\chi_1\otimes\chi_2})\pi(L_G(D_0)R_G(D_0)f)\|}_{} < \infty$$

$$= \ell(\bar\tau_{\chi_1})r(\bar\tau_{\chi_2})\ell_\infty(D_0)r_\infty(D_0)\pi(f)$$

$$= \ell_\infty(D_0)\ell(\bar\tau_{\chi_1})r_\infty(D_0)r(\bar\tau_{\chi_2})\pi(f)$$

$$= (\dim\chi_1)^2(\dim\chi_2)^2\ell(\bar\tau_{\chi_1})r(\bar\tau_{\chi_2})\pi(f)$$

$$\therefore \sum_{\tau_1,\tau_2\in\hat K} (\dim\chi_1)^2(\dim\chi_2)^2\|\pi(\bar\tau_{\chi_1})\pi(f)\pi(\bar\tau_{\chi_2})\| < \infty$$

But if $(v_i)_{i\in b_\chi}$ on basis of $\mathcal{H}(\chi)$:

$$\sum_{i,j\in b=\cup b_\chi} |\langle\pi(f)v_i,v_j\rangle| = \sum_{\substack{\chi_1,\chi_2\in\hat K\\ i,j\in b_{\chi_1},b_{\chi_2}}} |\langle\pi(f)v_i,v_j\rangle|$$

$$\leq \sum_{\chi_1,\chi_2\in\hat K} (\dim\mathcal{H}(\chi_1))(\dim\mathcal{H}(\chi_2))\|\pi(\bar\tau_{\chi_1})\pi(f)\pi(\bar\tau_{\chi_2})\|$$

$$\leq m_\pi^2 \sum_{\chi_1,\chi_2} (\dim\chi_1)^2(\dim\chi_2)^2\|\pi(\bar\tau_{\chi_1})\pi(f)\pi(\bar\tau_{\chi_2})\|$$

which we just saw finite

$\therefore \sum |\langle\pi(f)v_i,v_j\rangle| < \infty$, i.e. f is trace class

on basis of $\mathcal{H}$

Step 3: $f\mapsto \operatorname{trace}\pi(f)$ is continuous from $C_c^\infty(G)$ to $\mathbb{C}$

proof As before we may assume $\pi|_K$ unitary, $D_0\in\mathfrak{z}([\mathfrak{k},\mathfrak{k}])$ such that $d\chi(D_0)=(\dim\chi)^2\cdot\chi(1)$

Compute

$$|\operatorname{trace}\pi(f)| \leq \sum_{\chi\in\hat K} |\operatorname{trace}(\pi(\bar\tau_\chi)\pi(f)\pi(\bar\tau_\chi))|$$

$$\leq \sum_{\chi\in\hat K} (\dim\mathcal{H}(\chi))\|\pi(\bar\tau_\chi)\pi(f)\|$$

$$\leq m_\pi \sum_{\chi\in\hat K} (\dim\chi)^2\|\pi(\bar\tau_\chi)\pi(f)\|$$

$$= m_\pi \sum_{\chi\in\hat K} \|\pi(\bar\tau_\chi)\pi(L_G(D_0)f)\|$$

As $\pi(L_G(D)f)$ is C^∞ vector for ℓ (as in last theorem) we have (as in section on C^∞ vectors) $D\in\mathcal{U}(\mathfrak{k})$ independent of f s.t.

$$\sum_{\chi\in\hat K} \|\pi(\bar\tau_\chi)\pi(L_G(D_0)f)\| = \sum_{\chi\in\hat K} \|\ell(\bar\tau_\chi)\pi(L_G(D_0)f)\|$$

$$\leq \|\ell(D)\pi(L_G(D_0)f)\| = \|\pi(L_G(DD_0f))\|$$

i.e.

$$|\operatorname{trace}\pi(f)| \leq \|\pi(L_G(DD_0f))\| \quad \text{for all } f\in C_c^\infty(G)$$

Now let $F\subset G$ compact, $\{f_n\}\subset C_c^\infty(G)$ all supported in F, $L_G(D)f_n\to 0$ uniformly on F. Then

$$\|\pi(L_G(D')f_n)\| \leq \int_G |(L_G(D')f_n)(x)|\,\|\pi(x)\|\,dx \to 0$$

for all $D'\in\mathcal{U}(\mathfrak{g})$

$$\therefore |\operatorname{trace}\pi(f_n)| \leq m_\pi\|\pi(L_G(DD_0f_n))\| \to 0$$

now $f\mapsto\operatorname{trace}\pi(f)$ is continuous q.e.d.

$\Theta_\pi(f) = \operatorname{trace}\pi(f)$, global character, now is shown to exist, Θ_π distribution

Step 4 π, π' (K-finite) unitarily equivalent

$$\Longleftrightarrow \Theta_\pi = \Theta_{\pi'}$$

($\Rightarrow$) is clear

($\Leftarrow$) is tedious and I won't do it here.

See Garth Warner's book, volume 1. This uses the notions of infinitesimal equivalence, Naimark equivalence, Banach equivalence, ...

Let π TCI rep of G on $\mathcal{H}$. Then Θ_π is invariant (under conjugation):

$$\Theta_\pi(f\circ \mathrm{Ad}(x)) = \mathrm{trace} \int_G f(xgx^{-1})\,\pi(g)\,dg \qquad \text{(definition)}$$

$$= \mathrm{trace} \int_G f(g)\,\pi(x^{-1}gx)\,dg \qquad \text{(unimodularity)}$$

$$= \mathrm{trace}\left(\pi(x^{-1})\,\pi(f)\,\pi(x)\right)$$

$$= \mathrm{trace}\ \pi(f) = \Theta_\pi(f)$$

Θ_π is an eigendistribution of $\mathfrak{Z}(\mathfrak{g})$, joint eigenvalues given by the infinitesimal character χ_π:

$$D\in \mathfrak{Z}(\mathfrak{g}) \Rightarrow (D\Theta_\pi)(f) = \Theta_\pi(Df) = \mathrm{trace}\ \pi(Df)$$

$$= \mathrm{trace}(\pi_\infty(D)\pi(f)) = \mathrm{trace}(\chi_\pi(D)\pi(f))$$

$$= \chi_\pi(D)\,\mathrm{trace}\ \pi(f) = \chi_\pi(D)\Theta_\pi(f)$$

$$\boxed{D\Theta_\pi = \chi_\pi(D)\Theta_\pi \quad \forall\, D\in \mathfrak{Z}(\mathfrak{g})}$$

This system of PDE shows that there is an analytic function T_π on regular set $G' = \{x\in G : \mathfrak{g}^{\mathrm{Ad}(x)} \text{ is Cartan subalg}\}$ such that $\Theta_\pi|_{G'} = T_\pi$, i.e.

if $f\in C_c^\infty(G')$

then $\Theta_\pi(f) = \int_{G'} T_\pi(x) f(x)\,dx$

In effect, if $x\in G'$ one can find $D\in \mathfrak{Z}(\mathfrak{g})$ that is elliptic in a neighborhood of x, so Θ_π satisfies an elliptic equation in neighborhood of x, hence given there by $T_{\pi,x}$ analytic function. The $T_{\pi,x}$ match up because Θ_π is well defined.

One can prove that T_π has only finite jump discontinuities across the singular set $G\setminus G'$. This is a deep theorem of Harish Chandra, simplified about 10 years ago by Atiyah + Schmid. So – $T_\pi \in L^1_{loc}(G)$ and

$$f\in C_c^\infty(G) \Rightarrow \Theta_\pi(f) = \int_G T_\pi(x) f(x)\,dx$$

PLANCHEREL THEOREM FOR THE UNIVERSAL COVER OF THE CONFORMAL GROUP

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§0. INTRODUCTION

Ever since the discovery of conformal invariance of Maxwell's equations, conformal groups and conformal structures have had an important role in mathematical physics. For the most part, this role has been confined to consequences of the geometric action of the conformal group and to the use of certain unitary representations. Now there is a fairly explicit theory of harmonic analysis on the conformal group. It seems likely that this theory will be of some physical importance, e.g. in partial wave analysis.

When a group G acts by geometric symmetries on a space X, it also acts on various spaces of functions on X. These functions are better understood by taking the symmetries into account. That is, of course, the basic idea in Fourier analysis. It has also been exploited in the use of spherical harmonics, where $G = SO(3)$ and $X = S^2$, and in the application of other sorts of special functions.

Now the machinery is available for the case where G is the conformal group or one of its coverings, and X is either the space G itself or is a symmetric homogeneous space of G.

In this article we describe some of those developments, first sketching the general theory and then describing the case of the simply connected covering group $\widetilde{SU}(2,2)$ of the conformal group.

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We assume that the reader is well acquainted with Mackey theory but not so well acquainted with Harish-Chandra theory.

The results in §§1 and 2 have been known for some time. The Plancherel formula in §3 was worked out by Harish-Chandra in the 1960's (published somewhat later in [1], [2] and [3]) for semisimple groups with finite center. We recently developed another approach ([4],[5]) which allows infinite center, as in the group $\widetilde{SU}(2,2)$. In §4 we work out the constants to obtain an explicit formula for $\widetilde{SU}(2,2)$.

§1. TEMPERED REPRESENTATIONS OF SEMISIMPLE GROUPS

We describe the representations involved in the Plancherel formula for a semisimple group. To do this for a class of semisimple groups, one must ensure that certain subgroups belong to the same class. Our class consists of the reductive Lie groups G that have a closed normal abelian subgroup Z such that

(1.1) Z centralizes the identity component G^0 and G/ZG^0 is finite

and

(1.2) If $x \in G$ then $\mathrm{Ad}(x)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$.

Here "reductive" means that the Lie algebra $\mathfrak{g}$ of G is (commutative)$\oplus$(semisimple).

If $\pi \in \hat{G}$, the set of equivalence classes of irreducible unitary representations of G, then π has three types of characters. The *central character* ζ_π is the scalar valued function on the center Z_G that is given by $\pi(z) = \zeta_\pi(z)\cdot I$ where I is the identity on the representation space $\mathcal{H}(\pi)$. The *infinitesimal character* χ_π is the map on the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ given by $d\pi(D) = \chi_\pi(D)\cdot I$ for every Casimir operator D. We view it as a homomorphism $\mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ of associative algebras. The *character* or *distribution character* $\Theta(\pi)$ is the Schwartz distribution on G given by

$$(1.3) \qquad \Theta(\pi:f) = \mathrm{trace}\ \pi(f) \quad \text{for} \quad f \in C_c^\infty(G)$$

where $\pi(f) = \int_G f(x)\pi(x)dx$. The equivalence class of π is specified by $\Theta(\pi)$. The differential equations

$$(1.4) \qquad D\Theta(\pi) = \chi_\pi(D)\cdot\Theta(\pi) \quad \text{for} \quad D \in \mathcal{Z}(\mathfrak{g})$$

lead to the information that $\Theta(\pi)$ is integration against a locally L^1 function $T(\pi)$,

(1.5) $$\Theta(\pi:f) = \int_G T(\pi:x)f(x)dx \quad ,$$

which is real analytic on a certain dense open subset ("regular set") G' in G.

We may suppose $Z \cap G^0 = Z_{G^0}$. Then $\pi \in \hat{G}$ belongs to the *relative discrete series* if its coefficients

(1.6) $$\phi_{u,v}(x) = \langle u, \pi(x)v\rangle_{\mathcal{H}(\pi)} \quad , \qquad u,v \in \mathcal{H}(\pi) \ ,$$

are L^2 modulo Z. The representations we will use will be constructed from relative discrete series representations.

Choose a Cartan involution θ of G. In other words, θ is an automorphism of G, $\theta^2 = 1$, θ is the identity on $Z_G(G^0)$, and the fixed point set K of θ satisfies: $K/Z_G(G^0)$ is a maximal compact subgroup of $G/Z_G(G^0)$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, i.e. a maximal diagonalizable (over $\mathbb{C}$) subalgebra. Then $H = \{x \in G: Ad(x)\xi = \xi \text{ for all } \xi \in \mathfrak{h}\}$ is the corresponding Cartan subgroup. One can find $x \in G^0$ such that $Ad(x)\mathfrak{h}$ and xHx^{-1} are θ-stable.

If G has relative discrete series representations, then K contains a Cartan subgroup of G. Conversely, let $T \subset K$ be a Cartan subgroup of G, $\mathfrak{t}$ its Lie algebra, and $\Phi^+ = \Phi^+(\mathfrak{g},\mathfrak{t})$ a system of positive roots. Let $\rho = \frac{1}{2}\Sigma_{\Phi^+}\alpha$ and set

(1.7) $$\Lambda' = \lambda \in i\mathfrak{t}^*: \begin{cases} \langle\lambda,\alpha\rangle \neq 0 \text{ for all } \alpha \in \Phi^+ \text{ and} \\ e^{\lambda-\rho} \text{ is well defined on } T^0 . \end{cases}$$

If $\lambda \in \Lambda'$ then there is a unique relative discrete series representation π_λ^0 of G^0 such that

(1.8) $$T(\pi_\lambda^0 : x) = (\text{constant})\Delta(x)^{-1} \sum_{w \in W^0} \det(w)\, e^{w\lambda}$$

where $x \in T^0 \cap G'$, $\Delta(x) = \prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})$, and W^0 is the Weyl group of (G^0,T^0). The central character of π_λ^0 is $e^{\lambda-\rho}|_{Z_{G^0}}$. If $\chi \in Z_G(G^0)\hat{}$ agrees with π_λ^0 on Z_{G^0}, then

(1.9) $$\chi \otimes \pi_\lambda^0 \in (G^\dagger)\hat{} \ , \qquad G^\dagger = Z_G(G^0)G^0$$

is well defined and is a relative discrete series representation of $G^\dagger$. Finally, the relative discrete series of G consists of the

(1.10) $$\pi_{\chi,\lambda} = \operatorname{Ind}_{G^\dagger}^{G}(\chi \otimes \pi_\lambda^0) \ , \qquad \lambda \in \Lambda' \ , \quad \chi \in Z_G(G^0)\hat{}$$

where π_λ^0 and χ agree on Z_{G^0}.

Now let H be any θ-stable Cartan subgroup of G. Then $\mathfrak{h} = \mathfrak{t}_H \oplus \mathfrak{a}_H$, ± 1 eigenspaces of $\theta|_{\mathfrak{h}}$, and $H = T_H \times A_H$ where $T_H = H \cap K$ and $A_H = \exp_G(\mathfrak{a}_H)$. Then the centralizer $Z_G(A_H) = M_H \times A_H$ where $\theta M_H = M_H$ and T_H is a Cartan subgroup of M_H. Let $\Phi^+_{\mathfrak{a}_H} = \Phi^+(\mathfrak{g},\mathfrak{a}_H)$ be a system of positive $\mathfrak{a}_H$-roots, $\mathfrak{n}_H = \sum_{\alpha \in \Phi^+_{\mathfrak{a}}} \mathfrak{g}_\alpha$, N_H the corresponding analytic subgroup of G, and $P_H = M_H A_H N_H$ the associated "cuspidal parabolic" subgroup of G. If $\nu \in \Lambda'_{M_H}$, η_ν^0 is the corresponding relative discrete series representation of M_H, and for $\chi \in Z_{M_H}(M_H^0)\hat{}$ consistent with η_ν^0 then $\eta_{\chi,\nu} = \operatorname{Ind}_{M_H^\dagger}^{M_H}(\chi \otimes \eta_\nu^0)$ is the corresponding relative discrete series representation of M_H. If $\sigma \in \mathfrak{a}_H^*$ now $\eta_{\chi,\nu} \otimes e^{i\sigma} \in (M_H A_H)\hat{}$ extends to P_H, trivial on N_H, and we have the unitary representation

(1.11) $$\pi_{\chi,\nu,\sigma} = \operatorname{Ind}_{P_H}^{G}(\eta_{\chi,\nu} \otimes e^{i\sigma}) \quad .$$

It does not depend on choice of $\Phi^+_{\mathfrak{a}_H}$. The representations (1.11) of G constitute the *H-series*. If $A_H = \{1\}$ that is the *relative discrete series*. If A_H is maximal it is the *principal series*. Given ν and χ, $\pi_{\nu,\chi,\sigma}$ is irreducible for almost all σ. The irreducible constituents of representations $\pi_{\nu,\chi,\sigma}$, H variable, are the *tempered* representations of G.

When we are dealing with several Cartan subgroups we will write

(1.12a) $\pi(H:\chi:\nu:\sigma)$ for the H-series representation $\pi_{\chi,\nu,\sigma}$,

(1.12b) $\Theta(H:\chi:\nu:\sigma)$ for the distribution character of $\pi_{\chi,\nu,\sigma}$, and

(1.12c) $\Theta(H:\chi:\nu:\sigma:f)$ for the trace of $\pi_{\chi,\nu,\sigma}(f)$, $f \in C_c^\infty(G)$.

§2. TEMPERED REPRESENTATIONS OF $\widetilde{SU}(2,2)$

The conformal group is usually realized as the identity component $SO(2,4)$ of the orthogonal group of the real bilinear form $-x_1y_1 - x_2y_2 + x_3y_3 + \ldots + x_6y_6$ on $\mathbb{R}^{2,4}$. The space of light-like lines in $\mathbb{R}^{2,4}$ is the conformal completion of Minkowsky space $\mathbb{R}^{1,3}$, and that gives the action of $SO(2,4)$ there. We will find it much more convenient to use the complex form

(2.1) $$SU(2,2) = \{x \in GL(4;\mathbb{C}): xJx^* = J \text{ and } \det x = 1\}$$

of the double cover of $SO(2,4)$, $J = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, for the linear group. The two to one correspondence is

(2.2) $SU(2,2) \to SO(2,4)$ by Λ^2 (vector representation) ,

antisymmetrization of the usual representation of $SU(2,2)$ on $\mathbb{C}^4$.

Write G for the universal covering group $\widetilde{SU}(2,2)$ of $SU(2,2)$, $\psi: G \to SU(2,2)$ for the covering, and $Z_1 = \mathrm{Ker}(\psi)$. Since $SU(2,2)$ has center of order 4, Z_1 has index 4 in $Z = Z_G$.

The maximal compactly embedded subgroup $K = [K,K] \times Z_K^0$ where $[K,K] \cong SU(2) \times SU(2)$ maps one-to-one to $SU(2,2)$, and where $Z_K^0 \cong \mathbb{R}$. Let $\mathfrak{t}$ denote the compactly embedded Cartan subalgebra of $\mathfrak{g}$ given by

(2.3) $$\mathfrak{t} = \{\mathrm{diag}(i\theta_1, i\theta_2, i\theta_3, i\theta_4) = \theta_j \text{ real}, \Sigma\theta_j = 0\} .$$

Then $\mathfrak{t} = (\mathfrak{t} \cap [\mathfrak{k},\mathfrak{k}]) \oplus \mathfrak{z}_K$ with

(2.4) $$\mathfrak{t} \cap [\mathfrak{k},\mathfrak{k}] = \{\mathrm{diag}(i\theta_1, -i\theta_1, i\theta_2, -i\theta_2) : \theta_j \text{ real}\}$$
$$\mathfrak{z}_K = \{\mathrm{diag}(i\theta, i\theta, -i\theta, -i\theta) : \theta \text{ real}\} .$$

This gives us a parametrization of the corresponding Cartan subgroup T of G:

(2.5a) $$T = \{t(\theta_1,\theta_2) z_u : 0 \le \theta_1 < 2\pi,\ 0 \le \theta_2 < 2\pi,\ -\infty < u < \infty\}$$

where

(2.5b) $$t(\theta_1,\theta_2) = \exp_G \mathrm{diag}(i\theta_1, -i\theta_1, i\theta_2, -i\theta_2) \quad \text{and}$$

(2.5c) $$z_u = \exp_G \mathrm{diag}(iu, iu, -iu, -iu) .$$

Notice that ψ sends $t(\theta_1,\theta_2)z_u$ to $\mathrm{diag}\left(e^{i(\theta_1+u)}, e^{i(-\theta_1+u)}, e^{i(\theta_2-u)}, e^{i(-\theta_2-u)}\right)$, so $Z = Z_G = \psi^{-1}(\{\pm I, \pm iI\})$ is given by

(2.6) $$\{t(0,0)z_{k\pi},\ t(\pi,\pi)z_{k\pi},\ t(0,\pi)z_{(k+\frac12)\pi},\ t(\pi,0)z_{(k+\frac12)\pi} : k \in \mathbb{Z}\} .$$

Let $\varepsilon_j : \mathrm{diag}(a_1,a_2,a_3,a_4) \mapsto a_j$ as usual. Then $\Phi^+(\mathfrak{g},\mathfrak{t}) = \{\varepsilon_i - \varepsilon_j : 1 \le i < j \le 4\}$. The compact roots are $\{\varepsilon_1-\varepsilon_2,\ \varepsilon_3-\varepsilon_4\}$. The only sets of strongly orthogonal (sums and differences are not roots) noncompact positive roots are, up to K-conjugacy

(2.7) $$\emptyset , \quad \{\varepsilon_1-\varepsilon_3\} \quad \text{and} \quad \{\varepsilon_1-\varepsilon_3,\ \varepsilon_2-\varepsilon_4\} .$$

It follows from general theory that $\mathfrak{g}$ has exactly three conjugacy classes of Cartan subalgebras: $\mathfrak{t}$, $\mathfrak{j}$ and $\mathfrak{h}$, where $\mathfrak{t}$ is given by (2.3) and

(2.8) $$\mathfrak{j} = \left\{ \begin{pmatrix} i\theta_1 & 0 & u & 0 \\ 0 & i\theta_2 & 0 & 0 \\ u & 0 & i\theta_1 & 0 \\ 0 & 0 & 0 & i\theta_4 \end{pmatrix} : u, \theta_j \text{ real},\ 2\theta_1 + \theta_2 + \theta_4 = 0 \right\},$$

(2.9) $$\mathfrak{h} = \left\{ \begin{pmatrix} i\theta_1 & 0 & u_1 & 0 \\ 0 & i\theta_2 & 0 & u \\ u & 0 & i\theta_1 & 0 \\ 0 & u & 0 & i\theta_4 \end{pmatrix} : u_j, \theta_j \text{ real},\ \theta_1 + \theta_2 = 0 \right\} .$$

Consider the "intermediate" Cartan subgroup J corresponding to $\mathfrak{j}$. First,

(2.10a) $$\mathfrak{a}_J = \left\{ \begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{t}_J = (\mathfrak{t}_J \cap [\mathfrak{k},\mathfrak{k}]) + (\mathfrak{t}_J \cap [\mathfrak{m}_J,\mathfrak{m}_J])$$

where

(2.10b) $$\mathfrak{t}_J \cap [\mathfrak{k},\mathfrak{k}] = \left\{ \begin{pmatrix} i\theta & & & \\ & -i\theta & & \\ & & i\theta & \\ & & & -i\theta \end{pmatrix} \right\}, \quad \mathfrak{t}_J \cap [\mathfrak{m}_J,\mathfrak{m}_J] = \left\{ \begin{pmatrix} 0 & & & \\ & i\theta & & \\ & & 0 & \\ & & & -i\theta \end{pmatrix} \right\}$$

Here $T_J = \{t(\theta-u, \theta+u)z_u\}$ contains Z_G. As $T_J = Z(\mathfrak{a}_J)T_J^0$ where $Z(\mathfrak{a}_J)$ is generated by Z_G and $\gamma_{\varepsilon_1-\varepsilon_3} = \exp_G \mathrm{diag}(i\pi, 0, -i\pi, 0) = t(\frac{\pi}{2}, -\frac{\pi}{2})z_{\pi/2} \in T_J^0$ we have

(2.11a) $$T_J = \{t(\theta-u, \theta+u)z_u\}, \text{ connected} .$$

Similarly $M_J = Z(\mathfrak{a})M_J^0$ where $Z(\mathfrak{a})$ is generated by Z_G, $\gamma_{\varepsilon_1-\varepsilon_3}$, and $\gamma_{\varepsilon_2-\varepsilon_4} = t(-\frac{\pi}{2}, \frac{\pi}{2})z_{\pi/2} \in T_J^0$, so

(2.11b) $$M_J \cong \widetilde{SU}(1,1) \times S^1 , \text{ connected} .$$

Now consider the maximally split Cartan subgroup H corresponding to $\mathfrak{h}$. Here

(2.12a) $$\mathfrak{a}_H = \left\{ \begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 0 & 0 & v \\ u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{t}_H = \left\{ \begin{pmatrix} i\theta & & & \\ & -i\theta & & \\ & & i\theta & \\ & & & -i\theta \end{pmatrix} \right\} \subset [\mathfrak{k},\mathfrak{k}]$$

$T_H = Z(\mathfrak{a})T_H^0$ with $Z(\mathfrak{a})$ generated by $\gamma_{\varepsilon_1-\varepsilon_3}$ and $\gamma_{\varepsilon_2-\varepsilon_4}$ as above, and $T_H^0 = \{t(\theta,\theta) : 0 \le \theta \le 2\pi\}$. Compute

$$\gamma_{\varepsilon_2-\varepsilon_4} \in \gamma_{\varepsilon_1-\varepsilon_3} \cdot T_H^0$$

$$t(0,0)z_{k\pi},\ t(\pi,\pi)z_{k\pi} \in (\gamma_{\varepsilon_1-\varepsilon_3})^{2k} \cdot T_H^0$$

$$t(0,\pi)z_{(k+\frac{1}{2})\pi},\ t(\pi,0)z_{(k+\frac{1}{2})\pi} \in (\gamma_{\varepsilon_1-\varepsilon_3})^{2k+1} \cdot T_H^0$$

to see that

(2.12b) $$T_H = \bigcup_{n=-\infty}^{\infty} (\gamma_{\varepsilon_1-\varepsilon_3})^n T_H^0 = M_H \cong \mathbb{Z} \times T_H^0 .$$

Now we have the Cartan subgroups and the associated cuspidal parabolic subgroups. So we can parametrize the tempered series.

The space Λ' of (1.7) for the Cartan subgroup T of G is

(2.13a) $$\Lambda_T' = \{\lambda_{n,m,h} = \tfrac{n}{2}(\varepsilon_1-\varepsilon_2) + \tfrac{m}{2}(\varepsilon_3-\varepsilon_4) + \tfrac{h}{4}(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4),\ n,m \text{ integers} \neq 0,\ h \text{ real},\ n \pm m \pm h \neq 0\} .$$

Here notice that

(2.13b) $$\exp \lambda_{n,m,h}: t(\theta_1,\theta_2)z_u \mapsto e^{in\theta_1} e^{im\theta_2} e^{ihu} .$$

Since G is connected, its relative discrete series consists of the

(2.14a) $$\pi(T: n: m: h) = \pi_{\lambda_{n,m,h}} \quad \text{where} \begin{cases} n,m \text{ integers} \neq 0 \\ h \text{ real} \\ n \pm m \pm h \neq 0 . \end{cases}$$

The Weyl group $W(G,T)$ is generated by reflections in compact roots $\varepsilon_1-\varepsilon_2$ and $\varepsilon_3-\varepsilon_4$. Thus

(2.14b) $$\pi(T: n: m: h) = \pi(T: n': m': h') \Leftrightarrow n = \pm n',\ m = \pm m',\ h = h' .$$

The space Λ' of (1.7) for the Cartan subgroup T_J of M_J is

(2.15a) $$\Lambda_J' = \{\lambda_{n,h} = \tfrac{n}{4}(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4) + \tfrac{h}{2}(\varepsilon_2-\varepsilon_4):\ n \text{ integer},\ h \neq 0\} .$$

Here

(2.15b) $$\exp \lambda_{n,h}: t(\theta-u,\ \theta+u)z_u \longmapsto e^{in\theta} e^{2ihu} .$$

Since M_J is connected, its relative discrete series consists of the

(2.16a) $$\eta(T_J: n: h) = \eta_{\lambda_{n,h}} \quad \text{where } n \text{ integer},\ h \neq 0 .$$

Parameterize $\mathfrak{a}_J^*$ by $\sigma_s\begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = su$; so $i\sigma_s$ comes by Cayley transform from $\frac{s}{2}(\varepsilon_1-\varepsilon_3) \in i\mathfrak{t}^*$. Now the J-series of G consists of the

(2.16b) $$\pi(J: n: h: s) = \operatorname{Ind}_{M_J A_J N_J}^{G}(\eta(T_J: n: h) \otimes e^{i\sigma_s})$$

for $n \in \mathbb{Z}$, h and s real, $h \neq 0$. The Weyl group $W(G,J)$ is generated by reflection in the real root, which is 1 on $\mathfrak{t}_J$ and -1 on $\mathfrak{a}_J$, so

(2.16c) $$\pi(J: n: h: s) = \pi(J: n': h': s') \Leftrightarrow n = n',\ h = h',\ s = \pm s' .$$

The space Λ' of (1.7) for the Cartan subgroup T_H of M_H is

(2.17a) $$\Lambda_H' = \{\lambda_n = \tfrac{n}{4}(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4):\ n \text{ integer}\} .$$

Here

(2.17b) $$\exp \lambda_n: t(\theta,\theta) \longrightarrow e^{in\theta} .$$

Now the relative discrete series of $M_H = \langle\gamma_{\varepsilon_1-\varepsilon_3}\rangle \times T_H^0$ consists of the unitary characters

(2.18a) $$\eta(T_H: n: h): (\gamma_{\varepsilon_1-\varepsilon_3})^m\, t(\theta,\theta) \mapsto e^{\pi imh} e^{in\theta} ,$$

n integer and $0 \leq h < 2$. Parameterize $\mathfrak{a}_H^*$ by $\sigma_{s,t}\begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 0 & 0 & v \\ u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \end{pmatrix} = su + tv$. Then the H-series of G (which is the principal series) consists of the

(2.18b) $$\pi(H: n: h: s: t) = \operatorname{Ind}_{M_H A_H N_H}^{G}(\eta(T_H: n: h) \otimes e^{i\sigma_{s,t}})$$

for n integer, $0 \leq h < 2$, s and t real. The Weyl group $W(G,H)$ is generated by conjugation by

$$\begin{pmatrix} i & & & \\ & 1 & & \\ & & -i & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & i & & \\ & & 1 & \\ & & & -i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} .$$

The first two are trivial on $\mathfrak{t}$, hence on $T_H \subset T$; the first sends $\sigma_{s,t} \to \sigma_{-s,t}$ and the second sends $\sigma_{s,t} \to \sigma_{s,-t}$. The third, call it w, is -1 on $\mathfrak{t}_H$, interchanges $\gamma_{\varepsilon_1-\varepsilon_3}$ and $\gamma_{\varepsilon_2-\varepsilon_4}$, and sends $\sigma_{s,t} \to \sigma_{t,s}$. Compute $w: \gamma_{\varepsilon_1-\varepsilon_3}^m t(\theta,\theta) \to \gamma_{\varepsilon_2-\varepsilon_4}^m t(-\theta,-\theta) = \gamma_{\varepsilon_1-\varepsilon_3}^m t(m\pi-\theta,\ m\pi-\theta)$. We conclude that

(2.18c) $$\pi(H:n:h:s:t) = \pi(H:n':h':s':t') \Leftrightarrow \text{either } (n',h',s',t') = (n,h,\pm s,\pm t) \text{ or } (n',h',s',t') = (-n,h+n,\pm t,\pm s) .$$

§3. PLANCHEREL THEOREM FOR SEMISIMPLE GROUPS

We describe the Plancherel formula for the class of reductive Lie groups specified in (1.1) and (1.2). Here we enlarge Z if necessary so that $Z \cap G^0 = Z_{G^0}$ -- just replace Z by ZZ_{G^0}. Let $\mathrm{Car}(G)$ denote a set of representatives of the conjugacy classes of Cartan subgroups of G, chosen so that $\theta H = H$ for all $H \in \mathrm{Car}(G)$. The Plancherel formula says that, if $f \in C_c^\infty(G)$, then

$$\begin{aligned} f(x) = c_G \sum_{H \in \mathrm{Car}(G)} c_{H \cap G^0}^{-1} \int_{\chi \in Z_{M_H}(M_H)^\wedge} \deg(\chi) \\ \times \sum_{\nu \in \Lambda_H'} \int_{\sigma \in \mathfrak{a}_H^*} \Theta(H: \chi: \nu: \sigma: r_x f) \\ \times \left| \prod_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{h})} \langle \nu + i\sigma, \alpha \rangle \prod_{\beta \in \Phi_{\mathbb{R}}^+(\mathfrak{g},\mathfrak{h})} \bar{p}_\beta(\chi:\sigma) \right| d\sigma d\chi \end{aligned} \tag{3.1}$$

where $r_x f$ is the right translate of f by x, $(r_x f)(y) = f(yx)$. In this section we explain the ingredients of (3.1).

First, for the formula to make any sense at all, we must normalize Haar measures on the groups over which we integrate.

Let $G_1 = ZG^0/Z$, let θ_1 denote the Cartan involution derived from θ, let K_1 denote the fixed point set of θ_1, and let B_1 be a fundamental (as compact as possible) Cartan subgroup of G_1. Warning: this notation differs slightly from that of [4], and the following normalizations of Haar measures are streamlined over the ones in [4], because we do not need certain auxiliary groups for the final formula (3.1).

Write $\langle\ ,\ \rangle$ for the Killing form on $\mathfrak{g}_1$ and $(\ ,\ )$ for the associated positive definite form, $(\xi,\eta) = -\langle \xi, \theta_1 \eta \rangle$. Split $B_1 = T_1 \times A_1$ as in §1. Then T_1 is a torus; give it Haar measure of total mass $1/|\pi_1(G_{1\mathbb{C}})|$, where $G_{1\mathbb{C}}$ is the complexification of G_1 and $\pi_1(G_{1\mathbb{C}})$ is its fundamental group. The exponential map is a diffeomorphism from $\mathfrak{a}_1$ to A_1; give A_1 the Haar measure corresponding to the $(\ ,\ )$-euclidean structure of $\mathfrak{a}_1$. Now B_1 carries the product Haar measure.

Let $\mathfrak{g}_1'$ denote the regular subset of $\mathfrak{g}_1$. It consists of all elements of $\mathfrak{g}_1$ whose centralizers are Cartan subalgebras. The subset e, all elements of $\mathfrak{g}_1$ whose centralizers are conjugate to $\mathfrak{b}_1$, is open in $\mathfrak{g}_1$ and inherits a measure from the $(\ ,\ )$-euclidean structure there. Define a G_1-invariant measure on G_1/B_1 by

$$\int_e f(\xi)d\xi = \int_{G_1/B_1} \left\{ \int_{\mathfrak{b}} \left| \prod_{\alpha \in \Phi^+} \alpha(\xi) \right|^2 \cdot f(\mathrm{Ad}(x)\xi)d\xi \right\} d(xB_1) \tag{3.2a}$$

where $\Phi^+ = \Phi^+(\mathfrak{g}_1,\mathfrak{b}_1)$ is a positive root system and where $f \in C_c^\infty(e)$. Normalize Haar measure on G_1 by

$$\int_{G_1} f(x)dx = \int_{G_1/B_1} \left\{ \int_{B_1} f(xb)db \right\} d(xB_1) \tag{3.2b}$$

for $f \in C_c^\infty(G_1)$. Now we have normalized Haar measure $d(xZ)$ on $ZG^0/Z = G_1$.

Fix a Haar measure on $Z_G(G^0)$. If it is compact use the invariant measure of total mass 1. If it is discrete use counting measure. Then

$$\int_{Z_G(G^0)} f(z)dz = \sum_{z_0 \in Z_G(G^0)/Z} \int_Z f(z_0 z)dz \tag{3.3a}$$

specifies Haar measure on Z, and

$$\int_{ZG^0} f(x)dx = \int_{ZG^0/Z} \left\{ \int_Z f(xz)dz \right\} d(xZ) \tag{3.3b}$$

defines Haar measure on ZG^0. At last, we have Haar measure on G defined by

$$\int_G f(x)dx = \sum_{y \in G/ZG^0} \int_{ZG^0} f(yx)dx \ . \tag{3.3c}$$

It is independent of choice of Z.

Now that Haar measure on G is normalized, the operators

$$\pi(H: \chi: \nu: \sigma: r_x f) = \int_G f(yx)\pi(H: \chi: \nu: \sigma: x)dy \tag{3.4a}$$

are specified for $f \in C_c^\infty(G)$, and it makes sense to talk about their traces

$$\Theta(H: \chi: \nu: \sigma: r_x f) = \operatorname{trace} \pi(H: \chi: \nu: \sigma: r_x f) \ . \tag{3.4b}$$

Those traces are the basic ingredient in the Plancherel formula (3.1).

Next, we look at the measures $d\chi$ on the $Z_{M_H}(M_H^0)^\wedge$ that occur in (3.1). Given our choice of Haar measure on $Z_G(G^0)$, we fixed Haar measure on Z by (3.3a), and that normalizes Haar measure on $\hat{Z}$ by

$$f(x) = |Z_G(G^0)/Z| \int_{\hat{Z}} f_\zeta(x)d\zeta \tag{3.5a}$$

where, for $f \in C_c^\infty(G)$ and $\zeta \in \hat{Z}$ we denote

$$f_\zeta(x) = \int_Z f(xz)\zeta(z)dz \ . \tag{3.5b}$$

We now normalize $d\chi$ by

(3.6) $$\int_{Z_{M_H}(M_H^0)^\wedge} \phi(X)\deg(X)dX = \int_Z \sum_{X\in Z_{M_H}(M_H)^\wedge_\zeta} \phi(X)\deg(X)d\zeta$$

where $Z_{M_H}(M_H^0)^\wedge_\zeta = \{X \in Z_{M_H}(M_H^0)^\wedge : X|_Z$ has ζ as summand$\}$. This is equivalent to the normalization in [4, Lemma 6.12].

The measure $d\sigma$ on $\mathfrak{a}_H^*$ in (3.1) is also normalized through the abelian Fourier transform. If $f \in C_c^\infty(\mathfrak{a}_H)$ then the (,)-euclidean structure on $\mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}] \supset \mathfrak{a}_H$ specifies $\hat{f}: \mathfrak{a}_H^* \to \mathbb{C}$ by

(3.7a) $$\hat{f}(\sigma) = \int_{\mathfrak{a}_H} f(\xi)\, e^{i\sigma(\xi)}\, d\xi \quad ,$$

and we normalize $d\sigma$ by

(3.7b) $$f(\xi) = (2\pi)^{-\dim \mathfrak{a}_H} \int_{\mathfrak{a}_H^*} \hat{f}(\sigma)\, e^{-i\sigma(\xi)}\, d\sigma \quad .$$

The constant c_G in (3.1) is given by

(3.8) $$c_G = |\pi_1(G_{1\mathbb{C}})| \cdot \frac{W(G^0, B\cap G^0)}{|G/Z_G(G^0)G^0| \cdot (2\pi)^{r+p}}$$

where B is a fundamental Cartan subgroup of G (e.g. the inverse image of $B_1 \subset G_1 = ZG^0/Z$), where $W(G^0, B\cap G^0)$ is the Weyl group

$$\{x \in G^0 : Ad(x)\mathfrak{b} = \mathfrak{b}\}/(B \cap G^0) \quad ,$$

where $r = |\Phi^+(\mathfrak{g},\mathfrak{b})|$, and where $p = \operatorname{rank} G - \operatorname{rank} K = \operatorname{rank} A_B$.

Given $H \in Car(G)$, $\theta H = H$, let $\Phi_{\mathbb{R}}(\mathfrak{g},\mathfrak{h})$ denote the set of real roots in $\Phi(\mathfrak{g},\mathfrak{h})$. So $\Phi_{\mathbb{R}}(\mathfrak{g},\mathfrak{h}) = \{\alpha \in \Phi(\mathfrak{g},\mathfrak{h}) : \alpha(\mathfrak{j}) \subset \mathbb{R}$, i.e. $\alpha(\mathfrak{t}_H) = 0\}$ and is a root system.

$\Phi_{\mathbb{R}}(\mathfrak{m}_B + \mathfrak{a}_B, \mathfrak{h})$ is spanned by strongly orthogonal roots, hence is a direct sum of simple root systems with that property. For each simple summand there is a number that comes out of the theory of two-structures and evaluates to

(3.9)

summand	A_1	B_{2n}	B_{2n+1}	C_ℓ	D_{2n}	G_2	F_4	E_7	E_8
number	1	2^{n-1}	2^n	1	2^{n-1}	2	2	8	16

and $Q(\mathfrak{g},\mathfrak{h})$ is the product (over the simple summands) of those numbers. Let $R(\mathfrak{g},\mathfrak{h})$ denote the set of strongly orthogonal roots of $(\mathfrak{g},\mathfrak{b})$ used to define $\mathfrak{h}$ by the Cayley transform procedure. Then

(3.10) $$c_{H\cap G^0} = |W(G^0, H\cap G^0)| \cdot |H\cap K^0/H\cap K^0 \cap M_B^\dagger| \cdot Q(\mathfrak{g},\mathfrak{h}) \cdot \prod_{\alpha\in R(\mathfrak{g},\mathfrak{h})} \|\alpha\|$$

Given $\alpha \in \Phi_{\mathbb{R}}(\mathfrak{g},\mathfrak{h})$ we denote

$$h_\alpha^* \in \mathfrak{a}_H: \text{ element dual to } \alpha^\vee = 2\alpha/\|\alpha\|^2 \quad ,$$
$$x_\alpha \in \mathfrak{g}_\alpha: \text{ normalized by } [x_\alpha, \theta x_\alpha] = h_\alpha^* \quad ,$$
$$z_\alpha = x_\alpha - \theta x_\alpha \text{ and } \gamma_\alpha = \exp_G(\pi z_\alpha) \quad .$$

Z_{G^0} and the γ_α generate a group $Z(\mathfrak{a}_H)$ such that $H \cap G^0 = Z(\mathfrak{a}_H)H^0$. If $\sigma \in \mathfrak{a}_H^*$ and $X \in Z_{M_H}(M_H^0)$ then

(3.11a) $$p_\alpha(X:\sigma) = \sinh\left(\frac{2\pi\langle\sigma,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right) \left\{\cosh\left(\frac{2\pi\langle\sigma,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right) 1_k - \tfrac{1}{2} e^{\rho_\alpha}(\gamma_\alpha)\,[X(\gamma_\alpha) + X(\gamma_\alpha)^{-1}]\right\}^{-1}$$

is a scalar matrix, where $k = \deg(X)$ and ρ_α is half the sum of $\{\beta \in \Phi^+(\mathfrak{g},\mathfrak{h}) : \beta|_{\mathfrak{a}_H}$ is a multiple of $\alpha\}$. The factor $\bar{p}_\alpha(X:\sigma)$ in (3.1) is the value of this scalar,

(3.11b) $$\bar{p}_\alpha(X:\sigma) = \deg(X)^{-1} \cdot \operatorname{trace} p_\alpha(X:\sigma) \quad .$$

This completes the description of the terms involved in the Plancherel formula (3.1).

§4. EXPLICIT PLANCHEREL FORMULA FOR $\widetilde{SU}(2,2)$

The first step is to normalize Haar measure as in §3 for $G = \widetilde{SU}(2,2)$. This comes down to the following.

(a) Note that the Killing form $\langle\xi,\eta\rangle = \operatorname{trace}(ad(\xi)ad(\eta))$ on $\mathfrak{g} = \mathfrak{su}(2,2)$ is given by

(4.1) $$\langle\xi,\eta\rangle = 8 \cdot \operatorname{trace}(\xi\eta) \quad .$$

This defines the euclidean structure on $\mathfrak{g}$ by

(4.2) $$(\xi,\eta) = -\langle\xi, \theta\eta\rangle = 8 \cdot \operatorname{trace}(\xi\eta^*) \quad .$$

It gives a volume element on the open subset $\mathfrak{e} = \bigcup_{x\in G} Ad(x)\mathfrak{t}'$, and on $\mathfrak{t}$ itself.

(b) $\int_{\mathfrak{e}} f(\xi)d\xi = \int_{G/T}\left\{\int_{\mathfrak{t}} |\prod_{\alpha\in\Phi^+} \alpha(\xi)|^2 f(Ad(x)\xi)d\xi\right\} d(xT)$ defines the G-invariant measure on G/T, where $\Phi^+ = \Phi^+(\mathfrak{g},\mathfrak{t})$ and $f \in C_c^\infty(\mathfrak{e})$.

(c) Normalize Haar measure on the compact Cartan subgroup T/Z of $G/Z = SU(2,2)/\{\pm I, \pm iI\}$ to have total volume $\tfrac{1}{4}$. Then Haar measure on G is given by

$$(4.3)\qquad \int_G f(x)dx = \int_{G/T}\int_{T/Z}\sum_{z\in Z} f(xtz)d(tZ)\ d(xT) ,$$

where $f \in C_c^\infty(G)$.

Our Haar measure on $Z = Z_G = Z_G(G^0)$ is counting measure, and $|Z_G(G^c)/Z| = 1$ in (3.5a). So the Haar measure on $\hat{Z} = Z_G(G^0)^\wedge$ is normalized by (3.5b) to total mass 1.

In (2.13), $\exp(\lambda_{n,m,h})|_Z = 1$ precisely when $h/2$ is an integer and

$$(-1)^n = (-1)^m = (-1)^{h/2} ,$$

so we may view

$$\hat{Z} = \{\exp(\lambda_{0,0,h})|_Z : 0 \leq h < 4\} \cup \{\exp(\lambda_{1,0,h})|_Z : 0 \leq h < 4\} .$$

Now

$$\int_{\hat{Z}} \phi(\zeta)d\zeta = \tfrac{1}{8}\sum_{n=0,1}\int_0^4 \phi(\exp(\lambda_{n,0,h})|_Z)dh .$$

That gives us (remember: $G = M_T$),

$$(4.4)\qquad \int_{Z_G(G^0)^\wedge}\sum_{\substack{\nu\in\Lambda_T\\ \nu,X \text{ agree}}}\phi(\nu)dX = \tfrac{1}{8}\sum_{\substack{m,n\\ \text{integers}}}\int_{-\infty}^{\infty}\phi(\lambda_{n,m,h})dh .$$

In (2.15), $\exp(\lambda_{n,h})|_Z = 1$ precisely when $n/2$ is an integer and $(-1)^h = (-1)^{n/2}$, so $\hat{Z} = \{\exp(\lambda_{0,h}) : 0 \leq h < 2\} \cup \{\exp(\lambda_{1,h}) : 0 \leq h < 2\}$. Now

$$\int_{\hat{Z}} \phi(\zeta)d\zeta = \tfrac{1}{4}\sum_{n=0,1}\int_0^2 \phi(\exp(\lambda_{n,h})|_Z)dh .$$

That gives us

$$(4.5)\qquad \int_{Z_{M_J}(M_J^0)^\wedge}\sum_{\substack{\nu\in\Lambda_J\\ \nu,X \text{ agree}}}\phi(\nu)dX = \sum_{n \text{ integer}}\int_{-\infty}^{\infty}\phi(\lambda_{n,h})dh .$$

In (2.18a) express $\eta(T_H : n : h) = X_h \otimes \exp(\lambda_n)$. Then $(X_h \otimes \exp\lambda_n)|_Z = 1$ exactly when h and $n/2$ are integers with $(-1)^h = (-1)^{n/2}$, so

$$\hat{Z} = \{X_h \otimes \exp\lambda_0 : 0 \leq h < 2\} \cup \{X_h \otimes \exp\lambda_1 : 0 \leq h < 2\} .$$

Using (2.12b) and (2.17), now

$$(4.6)\qquad \int_{Z_H(M_H^0)^\wedge}\sum_{\substack{\nu\in\Lambda_H\\ \nu,X \text{ agree}}}\phi(\nu)dX = \sum_{n \text{ integer}}\int_0^2 \phi(X_h \otimes \lambda_n)dh .$$

Haar measure on $\mathfrak{a}_J$ is given by $\int_{\mathfrak{a}_J} f(\xi)d\xi = \int_{-\infty}^{\infty} f(r\xi_1)dr$ where $\|\xi_1\| = 1$, say $\xi_1 = \tfrac{1}{4}\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$ using (4.1). Since $\sigma_s(\xi_1) = s/4$, the normalization (3.7) becomes

$$(4.7)\qquad \int_{\mathfrak{a}_J^*}\phi(\sigma)d\sigma = \tfrac{1}{4}\int_{-\infty}^{\infty}\phi(\sigma_s)ds .$$

Similarly, Haar measure on $\mathfrak{a}_H$ is given by $\int_{\mathfrak{a}_H} f(\xi)d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r_1\xi_1 + r_2\xi_2)dr_1dr_2$ where $\xi_1 = \tfrac{1}{4}\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$ and $\xi_2 = \tfrac{1}{4}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix}$, so $\sigma_{s,t}(\xi_1) = s/4$ and $\sigma_{s,t}(\xi_2) = t/4$ give us

$$(4.8)\qquad \int_{\mathfrak{a}_H^*}\phi(\sigma)d\sigma = \tfrac{1}{16}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\sigma_{s,t})dsdt .$$

As seen just before (2.14), $W(G^0, B\cap G^0) = W(G,T)$ has order 4 in (3.8). Since G is connected, $|G/Z_G(G^c)G^c| = 1$. Evidently $r = G$ and $p = 0$. Finally $G_{1\mathbb{C}} = (G/Z)_{\mathbb{C}} = SL(4;\mathbb{C})/\{\pm I, \pm iI\}$ has fundamental group of order 4. So

$$(4.9)\qquad c_G = 16/(2\pi)^6 .$$

Note $\mathfrak{m}_T = \mathfrak{g}$. Since $\Phi_{\mathbb{R}}(\mathfrak{g},\mathfrak{t}) = \emptyset$, $\Phi_{\mathbb{R}}(\mathfrak{g},\mathfrak{j})$ is of type A_1, and $\Phi_{\mathbb{R}}(\mathfrak{g},\mathfrak{h})$ is of type $A_1 \times A_1$, in each case (3.9) gives $Q(\mathfrak{g},\cdot) = 1$. From (4.1), all roots $\alpha = \varepsilon_i - \varepsilon_j$ have $\|\alpha\|^2 = \|\varepsilon_i\|^2 + \|\varepsilon_j\|^2 = \tfrac{1}{4}$, so $\prod_{\alpha\in R(\mathfrak{g},\cdot)}\|\alpha\|$ is 1, $\tfrac{1}{2}$, $\tfrac{1}{4}$ for $\mathfrak{t}$, $\mathfrak{j}$, $\mathfrak{h}$. $W(G,T)$, $W(G,J)$ and $W(G,H)$ were seen in §2 to have respective orders 4, 2 and 8. As $G = M_T^\dagger = M_B^\dagger$, $|T\cap K^0/T\cap K^0\cap M_T^\dagger| = |J\cap K^0/J\cap K^0\cap M_T^\dagger| = |H\cap K^0/H\cap K^0\cap M_T^\dagger| = 1$. Now

$$(4.10)\qquad c_T = 4 , \quad c_J = 1 , \quad \text{and} \quad c_H = 2 .$$

$\Phi_{\mathbb{R}}(\mathfrak{g},\mathfrak{t}) = \emptyset$ so there are no $\tilde{p}_\beta(X:\sigma)$-terms for T.

$\Phi_{\mathbb{R}}^+(\mathfrak{g},\mathfrak{j}) = \{\beta\}$, the Cayley transform of $\varepsilon_1 - \varepsilon_3$.

Compute $\dfrac{2\pi\langle\sigma_s,\beta\rangle}{\langle\beta,\beta\rangle} = \pi s$, $\rho_\beta(\mathrm{diag}(\pi i, 0, -\pi i, 0))3\pi i$, so $e^{\rho_\beta}(\gamma_\beta) = -1$, and $(\exp\lambda_{n,h})(\gamma_\beta) = (\exp\lambda_{n,h})(t(\tfrac{\pi}{2}, -\tfrac{\pi}{2})z_{\pi/2}) = e^{i\pi n}e^{i\pi h}$. Thus

(4.11) $$\tilde{p}_\beta(\exp \lambda_{n,h} : \sigma_s) = \frac{\sinh(\pi s)}{\cosh(\pi s) + (-1)^n \cos(\pi h)}$$

$\Phi_{\mathbb{R}}(\mathfrak{g},\mathfrak{h}) = \{\beta_1, \beta_2\}$, respective Cayley transforms of $\varepsilon_1-\varepsilon_3$ and $\varepsilon_2-\varepsilon_4$. Compute $\frac{\langle 2\pi\sigma_{s,t}, \beta_j\rangle}{\langle \beta_j, \beta_j\rangle} = \pi s$ for $j=1$, πt for $j=2$. As above, $e^{\rho_\beta}(\gamma_\beta) = -1$ for $\beta = \beta_1, \beta_2$. Also

$$\eta(T_H : n : h)(\gamma_{\beta_1}) = e^{i\pi h}$$

and

$$\eta(T_H : n : h)(\gamma_{\beta_2}) = e^{i\pi n} e^{i\pi h} .$$

So

(4.12) $$\prod_{\beta \in \Phi^+_{\mathbb{R}}(\mathfrak{g},\mathfrak{h})} \tilde{p}_\beta(\eta(T_H:n:h) : \sigma_{s,t}) = \frac{\sinh(\pi s)}{\cosh(\pi s) + \cos(\pi h)} \; \frac{\sinh(\pi t)}{\cosh(\pi t) + (-1)^n \cos(\pi h)} .$$

Finally, using $\|\varepsilon_i\|^2 = 1/8$, we glance back at (2.13a) to check

$$\prod_{1 \leq i < j \leq 4} \left\langle \frac{n}{2}(\varepsilon_1-\varepsilon_2) + \frac{m}{2}(\varepsilon_3-\varepsilon_4) + \frac{h}{4}(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4), \varepsilon_i - \varepsilon_j \right\rangle$$

$$= n\tfrac{1}{2}(n-m+h)\tfrac{1}{2}(n+m+h)\tfrac{1}{2}(-n+m+h)\tfrac{1}{2}(-n-m+h)m\|\varepsilon_i\|^6 ,$$

that is,

(4.13) $$\prod_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t})} \langle \lambda_{n,m,h}, \alpha\rangle = 2^{-22} nm(n+m+h)(n+m-h)(n-m+h)(n-m-h) .$$

Similarly, using (2.15a) and the fact that σ_s comes from $\frac{s}{2}(\varepsilon_1-\varepsilon_3)$ by Cayley transform,

(4.14) $$\prod_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{j})} \langle \lambda_{n,h} + i\sigma_s, \alpha\rangle = -2^{-22} ihs|(n+h)+is|^2|(n-h)+is|^2$$

and, using (2.17a) and the fact that $\sigma_{s,t}$ comes from $\tfrac{1}{2}(s(\varepsilon_1-\varepsilon_3)+t(\varepsilon_2-\varepsilon_4))$ by Cayley transform,

(4.15) $$\prod_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{h})} \langle \lambda_n + i\sigma_{s,t}, \alpha\rangle = 2^{-22} st|n+i(s+t)|^2|n+i(s-t)|^2 .$$

Now we are ready to put specific values into (3.1). Break the sum over $\mathrm{Car}(G) = \{T, J, H\}$ into

(4.16) $$f(x) = f_T(x) + f_J(x) + f_H(x) \quad \text{for} \quad f \in C_c^\infty(G) .$$

Then, from (3.1) and the preceding results of this section,

(4.17) $$f_T(x) = 2^{-29}\pi^{-6} \sum_{m,n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Theta(T:n:m:h:r_x f) \times |nm(n+m+h)(n+m-h(n-m+h)(n-m-h)|\,dh ,$$

(4.18) $$f_J(x) = 2^{-28}\pi^{-6} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Theta(J:n:h:s:r_x f) \times \left| hs|n+h+is|^2|n-h+is|^2 \frac{\sinh(\pi s)}{\cosh(\pi s)+(-1)^n\cos(\pi h)} \right| dhds ,$$

and

(4.19) $$f_H(x) = 2^{-31}\pi^{-6} \sum_{n=-\infty}^{\infty} \int_0^2 \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Theta(H:n:h:s:t:r_x f) \times \left| st|n+i(s+t)|^2|n+i(s-t)|^2 \frac{\sinh(\pi s)}{\cosh(\pi s)+\cos(\pi h)} \; \frac{\sinh(\pi t)}{\cosh(\pi t)+(-1)^n\cos(\pi h)} \right| dsdt \right\} dh .$$

Combining these and using (2.14b), (2.16c) and (2.18c), we finally arrive at

4.20 THEOREM. *Let G be the universal covering of the conformal group. In the normalizations and notation described above, if $f \in C_c^\infty(G)$ and $x \in G$ then*

$$2^7\pi^6 f(x) = \sum_{m,n=1}^{\infty} \int_{-\infty}^{\infty} \Theta(T:n:m:h:r_x f)\,|nm(n+m+h)(n+m-h)(n-m+h)(n-m-h)|\,dh$$

$$+ \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} \Theta(J:n:h:s:r_x f) \left| hs|n+h+is|^2|n-h+is|^2 \frac{\sinh(\ s)}{\cosh(\pi s)+(-1)^n\cos(\pi h)} \right| ds \right\} dh$$

$$+ \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \int_0^2 \left\{ \int_0^{\infty}\int_0^{s} \Theta(H:n:h:s:t:r_x f) \left| st|n+i(s+t)|^2 \times |n+i(s-t)|^2 \frac{\sinh(\pi s)}{\cosh(\pi s)+\cos(\pi h)} \; \frac{\sinh(\pi t)}{\cosh(\pi t)+(-1)^n\cos(\pi h)} \right| dtds \right\} dh$$

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