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SMR/161 - 27

COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
(4 November - 6 December 1985)

INDUCED REPRESENTATIONS (cont.d)

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These are preliminary lecture notes, intended only for distribution to participants.

Induced Representations (Contd)

As before let G be a locally compact group and H a closed subgroup. Let π be a unitary representation of H on a Hilbert space $\mathbb{H}$. The induced representation I_π of G was a realised on a Hilbert space $\mathbb{H}_\pi$ whose elements can be identified with suitable measurable functions $f: G \longrightarrow \mathbb{H}$ satisfying the condition

$$f(gh) = \rho(h)^{-1}_{\cdot}\,\pi(h)^{-1} f(g) \qquad (*)$$

(Recall that $\rho = \Delta_G^{-1}\Delta_H$). Now if $\varphi \in C_c(G/H)$ multiplication by φ (considered as an H-invariant function on G) defines a bounded operator

$$M_\varphi : \mathbb{H}_\pi \longrightarrow \mathbb{H}_\pi$$

(via the identification of $\mathbb{H}_\pi$ as a subspace $\mathbb{H}$-valued functions on G satisfying $*$). The map $\varphi \longmapsto M_\varphi$ $(=M(\varphi))$ is a $*$-algebra homomorphism of $C_c(G/H)$ in $B(\mathbb{H})$, the algebra of bounded operators on $\mathbb{H}$: we have for $\varphi, \psi \in C_c(G/H)$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}
&(i) && M(\varphi+\psi) = M(\varphi) + M(\psi)\\
&(ii) && M(\varphi\psi) = M(\varphi)M(\psi)\\
&(iii) && M(\lambda\varphi) = \lambda M(\varphi) \qquad\qquad (**)\\
&(iv) && M(\bar{\varphi}) = M(\varphi)^*
\end{aligned}$$

Also one has – as is easy to check – for $g \in G$,

$$M(I_\pi(g)\,\varphi\, I_\pi(g)^{-1}) = M({}^g\varphi) \qquad (***)$$

where for $g \in G$, ${}^g\varphi \in C_c(G/H)$ is given by ${}^g\varphi(x) = \varphi(g^{-1}x)$.

<u>Definition</u> An algebra homomorphism $M: C_c(X) \to B(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space, $B(\mathcal{H})$ = algebra of bounded operators on it) where X is a locally compact space is called a projection valued measure if ($\forall \varphi \in C_c(G/H)$) $\|M(\varphi)\| \le \operatorname{Sup}\{|\varphi(x)| \mid x \in X\}$

(Standard measure theory techniques enable one to deduce that one can extend M to a $*$-algebra homomorphism of the algebra of <u>bounded</u> <u>measurable</u> functions on G/H onto $B(\mathcal{H})$. The main point is that if ($\varphi \in C_c(G/H)$) $\varphi \ge 0$, $M(\varphi)$ is self adjoint and non-negative

~~and~~ so that $\varphi_n \in C_c(G)$ is a monotone increasing sequence converging to a measurable function φ, $\lim_{n\to\infty} M(\varphi_n)$ is either infinite or yields non-negative self adjoint operator which we can define as $M(\varphi)$. Since $M(1_E) = M(1_E \cdot 1_E) = M(1_E)^2$, $P_E = M(1_E)$ is a self adjoint projection for any (Borel) measurable set E, 1_E being its characteristic function. Hence the name projection valued measure).

In the sequel we will also assume that $M(1_{G/H})$ is the identity. This last operator being a projection onto a subspace on whose orthogonal complement all of $C_c(G/H)$ operates trivially, one can always replace $\mathcal{H}$ by its image under $M(1_{G/H})$ to secure this condition.

With this back-ground we can now state Mackey's theorem which enables one to decide when a representation of G is induced from that of a subgroup H.

Theorem. Let G, H be as above. Let $\sigma : G \to B(\mathcal{H})$ be a unitary irreducible representation of G on $\mathcal{H}$. Suppose we are also given a projection valued measure $M : C_c(G/H) \longrightarrow B(\mathcal{H})$ such that $\sigma(g)\, M(\varphi)\, \sigma(g)^{-1} = M(\partial\varphi)$ for all $\varphi \in C_c(G/H)$. Then if $\mathcal{H}$ is separable, there is a unitary representation π of H on a Hilbert space $\mathbb{H}$ and an isometry $\mathcal{H} \xrightarrow{\Phi} \mathbb{H}_\pi$ such that for all $g \in G$

$$\Phi^{-1} I_\pi(g)\, \Phi = \sigma(g).$$

The rest of the lecture is devoted to the proof of this theorem. We need to go back a little and talk about measures on G/H. We have seen that the Haar measure $\mu : C_c(G) \longrightarrow \mathbb{C}$ defines by passage to the quotient a linear functional

$$\bar{\mu} : C_c(G, H; \rho^2) \longrightarrow \mathbb{C}.$$

Now $\bar{\mu}$ extends to a larger class of measurable functions for G satisfying $f(gh) = \bar{\rho}^2(h) f(g)$ for all $h \in H$ viz the functions of the form $I_{\rho^2}(u)(x) = \int_H u(xh) \rho^2(h)\, dh$, $u \in L^1(G)$ ~~(By Fubini's theorem~~ (By a Fubini-like theorem one proves the integral exist for almost all x). We denote this class of functions by $L^1(G, H, \rho^2)$. If $f : G \to \mathbb{C}$ is a measurable function with $f(gh) = \bar{\rho}^2(h) f(g)$ for all $h \in H$, we will say that $f \in L^1_{loc}(G, H, \rho^2)$ if $\varphi f \in L^1(G, H; \bar{\rho}^2)$ for all $\varphi \in C_c(G/H)$. Suppose now that $f_0 \in L^1_{loc}(G, H; \bar{\rho}^2)$ is such that $f_0 > 0$ — such a continuous f_0 exist as can be seen using a partition of unity on G/H. Then f_0 defines a Borel measure ν_0 on G/H if we set

$$\nu_0(\varphi) = \bar{\mu}(f_0 \varphi), \quad \varphi \in C_c(G).$$

Evidently if $f \in L^1_{loc}(G, H; \rho^2)$, $\varphi \longmapsto \bar{\mu}(f \cdot \varphi)$, $\varphi \in C_c(G)$

defines a (complex) measure ν_f on G/H and since

$$\nu_f(\varphi) = \bar{\mu}(f \cdot \varphi) = \bar{\mu}(f/f_0 \cdot f_0 \varphi) = \nu_0(f/f_0 \cdot \varphi),$$

$\nu_f << \nu_0$. ~~[struck out]~~

The translate ${}^g\nu_0$ of ν_0 ($g \in G$) is a measure on G/H which is evidently of the form $\nu_{{}^g f_0}$ and is thus absolutely continuous with respect to ν_0 : $\nu_{{}^g f_0}$ $(= \nu_{f_0})$ $<< \nu_0$. ~~Now it is well known~~ Now one has a converse for this

If ν is a measure on G/H such that ${}^g\nu << \nu$ for all $g \in G$, then $\nu = \nu_f$ for some $f > 0$ in $L^1_{loc}(G, H, \bar{\rho}^2)$

We will assume this result. We will say a measure ν on G/H is absolutely continuous with respect to $\bar{\mu}$ iff there is an $f_\nu \in L^1_{loc}(G, H, \bar{\rho}^2)$ such that

$$\nu(\varphi) = \bar{\mu}(f_\nu \varphi).$$

~~The discussion above shows~~ and we will call f_ν the Radon Nikodym derivative of ν w.r.t $\bar{\mu}$ and also denote $\frac{d\nu}{d\bar{\mu}}$. Observe that if $f_0 > 0$

is in $L^1_{loc}(G,H,\rho^2)$, then a measure ν on G/H is absolutely continuous with respect to $\tilde{\mu}$ iff it is (in the usual sense) absolutely continuous with respect to ν_{f_0}.

Suppose now that $\{e_n \mid 1 \le n < \infty\}$ is an orthonormal basis of $\mathcal{H}$. Let ν_0 be the measure $C_c(G/H)$ defined by $\nu_0(\varphi) = \sum_{1\le n<\infty} 2^{-i}\langle M(\varphi)e_i, e_i\rangle$
Then if $E \subset G/H$ is a Borel set

$$\nu_0(E) = \sum_{1\le n<\infty} 2^{-i}\langle M(1_E)e_i, e_i\rangle = \sum_{1\le i<\infty} 2^{-i}\langle P_E e_i, e_i\rangle$$

$P_E = M(1_E)$ a self adjoint projection. Thus if $\nu_0(E) = 0$, $\langle P_E e_i, e_i\rangle = \|P_E e_i\|^2 = 0$ $\forall\, i$ so that $P_E = 0$. Since $P_{gE} = \sigma(g)P_E\sigma(g)^{-1}$, we see that $\nu_0(gE) = 0$ iff $\nu_0(E) = 0$. Thus $\nu_0 = \nu_{f_0}$ for some $f_0 > 0$, $f_0 \in L^1_{loc}(G,H,\rho^2)$. Consider now for vectors $v, w \in \mathcal{H}$, the linear functional

$$\varphi \longmapsto \langle M(\varphi)v, w\rangle$$

(φ bounded measurable on G/H) This is a complex measure $\nu_{v,w}$ on G/H which is absolutely continuous with respect to ν_0: this is easily seen by usual polarisation argument, reducing the problem to the case when $v = w$. We can therefore define $\frac{d\nu_{v,w}}{d\tilde{\mu}} = \lambda(v,w)$ as a measurable complex valued function G satisfying

$$\lambda(v,w)(gh) = \rho(h)^{-2}\lambda(v,w)(g), \quad g \in G,\ h \in H.$$

(It is easy to see that $\lambda(v,w) \in L^1(G, H\rho^2)$)
(the equality is to be treated as equality outside a set of measure zero in G which will depend on $h \in H$

Claim Let $\mathcal{H}_0 \subset \mathcal{H}$ be the $\mathbb{C}$-linear span of the set $\{\sigma(\varphi)v = \int_G \varphi(g)\sigma(g)v\,dg \mid \varphi \in C_c(G), v \in \mathcal{H}\}$
Then for $v, w \in \mathcal{H}_0$, $\lambda(v,w)$ is (almost everywhere equal to) a continuous function on G.

Assume the claim. Then on $\mathcal{H}_0$ we can define a antilinear pairing as follows:

$$\langle\langle v, w\rangle\rangle = \lambda(v,w)(1)$$

($\lambda(v,w)$ being continuous this makes sense).

We observe that $\lambda_{v,v} \geq 0$ as $\nu_{v,v}(\varphi) \geq 0$ if $\varphi \geq 0$. (Note that if $v = M(\varphi)w$, $\varphi \in C_c(G/H)$

$$\varphi\,\lambda(v,w) = \lambda(M(\varphi)v, w)$$

so that for $v \in \mathcal{H}_0$, $\langle\langle M(\varphi)v, M(\varphi)v\rangle\rangle = 0$ if $\varphi(1) = 0$). $\langle\langle\ ,\ \rangle\rangle$ defines the structure of a Prehilbert space on $\mathcal{H}_0/\mathcal{H}_0^\perp$ where $\mathcal{H}_0^\perp$ is the subspace $\{v \in \mathcal{H}_0 \mid \langle\langle v, v\rangle\rangle = 0\}$. Now $\mathcal{H}$ is G-stable as is easily seen; and $\mathcal{H}_0^\perp$ is H-stable because we have $\lambda(\sigma(g)v, \sigma(g)w) = {}^g\lambda(v,w)$ and if $g \in H$, ${}^g\lambda(v,w)(1) = \lambda(v,w)(g^{-1}) = \rho^2(g)\,\lambda(v,w)(1)$. Thus one sees that H acts on $\mathcal{H}_0/\mathcal{H}_0^\perp$. Let τ denote this action on H. If $(\ ,\ )$ denotes the scalar product on $\mathcal{H}_0/\mathcal{H}_0^\perp$, we see that $\langle\tau(h)\bar{v}, \tau(h)\bar{w}\rangle = \rho(h)^2(\bar{v}, \bar{w})$ where

$\bar{v}, \bar{w} \in \mathcal{H}_0/\mathcal{H}_0^\perp$. Let $\mathbb{H}$ be the completion of $\mathcal{H}_0/\mathcal{H}_0^\perp$ w.r.t $(\ ,\ )$. We then get a unitary representation of H on $\mathbb{H}$. (The continuity of this action is not difficult to check). Suppose now that $f \in \mathcal{H}_0$. We can then define a measurable function $\hat{f}$ on G with values in $\mathbb{H}$ by setting

$$\hat{f}(g) = p[\sigma(g)f]$$

where $p : \mathcal{H}_0 \longrightarrow \mathbb{H}$ is the natural map. Observe that $(\hat{f}(g), \hat{f}(g)) = \lambda(\sigma(g)f, \sigma(g)f)(1)$ $= \lambda(f,f)(g)$. and $\bar{\mu}(\lambda(f,f)) = \nu_{f,f}(1) = \langle f, f\rangle$. Thus $f \mapsto \hat{f}$ preserves norms so that it extends to an isometry of $\mathcal{H}$ on $\mathbb{H}_\pi$. (We have left some details to be checked). This proves the theorem modulo the claim.

To prove the claim it suffices to show

that $\lambda(\sigma(\varphi)v, \sigma(\psi)w)$ is continuous for $\varphi, \psi \in C_c(G)$.

We have now

$$\langle f \int_G \varphi(g)\sigma(g)v\,dg, \int_G \psi(h)\sigma(h)v\,dh \rangle^* =$$

$$\int_G \int_G \varphi(g)\psi(h)\langle f\,\sigma(g)v, \sigma(h)w\rangle\,dg\,dh =$$

$$\int_G \int_G \varphi(g)\psi(h)\langle {}^{g^{-1}}f\,v, \sigma(g^{-1}h)w\rangle\,dg\,dh =$$

$$\int_G \int_G \varphi(g)\psi(h)\;\bar\mu\left[\lambda(v, \sigma(g^{-1}h)w)\,{}^{g^{-1}}f\right]dg\,dh =$$

$$\int_G \int_G \varphi(g)\psi(h)\;\bar\mu\left[{}^{g}\lambda(v, \sigma(g^{-1}h)w)\cdot f\right]dg\,dh =$$

$$\bar\mu\left[f \int_G \int_G \varphi(g)\psi(h)\,{}^{g}\lambda(v, \sigma(g^{-1}h)w)\right]dg\,dh.$$

Thus we find

$$\lambda(\sigma(\varphi)v, \sigma(\varphi)w)(x) = \int_G \int_G \varphi(g)\psi(h)\,\lambda(v, \sigma(g^{-1}h)w)(g^{-1}x)\,dg\,dh$$

Setting $x^{-1}g = t$, $x^{-1}h = u$, we get

$$\lambda(\sigma(\varphi)v, \sigma(\varphi)w)(x) = \int_G \int_G \varphi(xt)\psi(xu)\,\lambda(v, \sigma(t^{-1}u)w)(t^{-1})\,dt\,du$$

$$= \int_G \int_G \varphi(xt^{-1})\Delta_G(t)\psi(xu)\,\lambda(v, \sigma(tu)w)(t)\,dt\,du.$$

Now if x, x' are sufficiently near each other we have $|\varphi(xt) - \varphi(x't)|\Delta_G < \epsilon$, $|\psi(xu) - \psi(x'u)| < \epsilon$ for all $t, u \in G$ — uniform continuity guarantees this as $\varphi, \psi \in C_c(G)$. We conclude thus (setting $F = \lambda(\sigma(\varphi)v, \sigma(\psi)w)$)

$$|F(x) - F(x')| \le \epsilon \int_K dt \int_K |\lambda(v, \sigma(tu)w)(t)|\,du$$

$$= \int_K dt \int_K |\lambda(v, \sigma(u)w)(t)|\,du$$

where K is a suitably chosen compact set.

Let θ in $C_c(G)$ be such that $\theta(K) = 1$.

Then $|F(x) - F(x')| \le \epsilon \int_K du \int_G dt\, |\lambda(v, \sigma(u)w)(t)|\,|\theta|$

$$= \epsilon \int_K du\; \bar\mu\{I_{\rho 2}(\theta \cdot |\lambda(v, \sigma(u)w)|$$

$$= \epsilon \int_K du\; \bar\mu\{I_1(\theta)\cdot|\lambda(v, \sigma(u)w)|\}$$

$$\le \epsilon \int_K du\, \langle I_1(\theta)v, v\rangle^{1/2} \langle I_1(\theta)w, w\rangle^{1/2}$$

$$\le \mu(K)\cdot\epsilon\cdot M\,\|v\|\,\|w\|$$

The compact set K can be chosen to be ~~fixed~~ ~~for~~ independent of $\epsilon > 0$ for all sufficiently small ϵ. This shows $\lambda(\sigma(\varphi)v, \sigma(\psi)w)$ is continuous, proving the claim