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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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SYMPLECTIC ACTIONS OF LIE GROUPS AND GEOMETRIC QUANTIZATION.

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These are preliminary lecture notes, intended only for distribution to participants.

1. Introduction.

Geometric Quantization is an attempt to find a consistent way to pass from Classical Mechanics to Quantum Mechanics. It began with the work of Dirac [D] who contrasted the Poisson bracket of Classical Mechanics with the commutator bracket of Quantum Mechanics. van Hove [vH] studied the problem of finding a representation of the group of canonical transformations by unitary operators in a Hilbert space so that the infinitesimal representations would preserve the two brackets. He found that if physically reasonable requirements of irreducibility were imposed then it was in fact impossible to quantize all classical observables simultaneously. The modern geometrical methods of Kostant [K] and Souriau [S] ~~to~~ and others have been developed to examine in as great a generality as possible what can be achieved. In these lectures I intend to describe the theory as formulated by Kostant and some of its applications to representation theory. The Borel-Weil Theorem as described by Professor Narasimhan, and the Kirillov method of orbits described by Professor Raghunathan [redacted] can both be interpreted as applications of Geometric Quantization to Representation Theory.

In five lectures the picture I can give of this subject which has been evolving for more than 20 years is somewhat incomplete, so I have added a bibliography at the end which should be consulted for further details. I shall emphasize the group theoretical aspects as befits the

topics of this College. I shall begin with a study of groups acting on symplectic manifolds and relate this to the Lie algebra cohomology groups introduced in Professor Gutt's lectures. Then I shall describe the general quantization proceedure for symplectic manifolds and apply this to the group theoretical setting to produce representati
In the last part I shall describe some of the results obtained by this method such as the Borel-Weil theorem and Kirillov method mentioned above.

There is no standard sign convention in symplectic geometry, and the signs I have chosen are of course self-consistent, but differ from those of Professor Gutt's last lecture and from some of the standard references. With this warning the reader should be able to translate the formulas I give here into his own preferred conventions.

2. Symplectic manifolds

A symplectic manifold is a pair (M, ω) consisting of a smooth manifold M of even dimension and a 2-form ω which is

(i) closed : $d\omega = 0$

and

(ii) non-degenerate.

(ii) [redacted] can be stated several ways. In coordinates $x^1, \dots, x^{2n}$

$$\omega = \frac{1}{2} \sum_{i,j} \omega_{ij}\, dx^i \wedge dx^j$$

and non-degenerate means the matrix (ω_{ij}) is non-singular. This is equivalent to requiring the $2n$-form ω^n be without zeros, so defines a volume

$$\mu = \frac{(-1)^{[n/2]}}{n!}\, \omega^n$$

on M called the Liouville volume. A third definition of non-degenerate is that the map $TM \longrightarrow T^*M$

$$X \longmapsto i_X \omega \quad \text{or} \quad X^i \longmapsto X^i \omega_{ij}$$

is an isomorphism from the tangent to cotangent spaces. Thus ω can be used to raise and lower indices just like a metric tensor. There is an important difference however, namely that ω is skew-symmetric, and (i) says, essentially, that ω is flat.

Example 1. The basic example is $\mathbb{R}^{2n}$ with coordinates divided into two groups $(p_1, \dots, p_n, q^1, \dots, q^n)$ and

$$\omega = \sum_i dp_i \wedge dq^i .$$

In this case

$$\mu = dp_1 \wedge \dots \wedge dp_n \wedge dq^1 \wedge \dots \wedge dq^n$$

is just the Lebesgue volume in phase space.

Example 2. Another important example is the cotangent bundle $M = T^*C$ of a configuration space C. In this case let $\pi : M \longrightarrow C$ be the cotangent projection and $x^1, \dots, x^n$ be coordinates on some chart U on C. Define

$$q^i = x^i \circ \pi$$

on $\pi^{-1}(U)$, an open set in M. If [redacted] $\beta \in \pi^{-1}(U)$, it is a cotangent vector at some point $x = \pi(\beta)$, so can be expanded

$$\beta = \sum_i p_i(\beta)\, dx^i,$$

and as β varies this defines smooth functions p_i on $\pi^{-1}(U)$. It is easy to check that

$$\Theta = \sum_i p_i\, dq^i$$

is a 1-form on M independent of the coordinates chosen on C and

$$\omega = d\Theta = \sum_i dp_i \wedge dq^i$$

is then a symplectic form. Θ is called the canonical 1-form and ω the canonical 2-form on the cotangent bundle T^*C. Θ can be defined intrinsically by

$$\Theta_\beta(X) = \beta(\pi_* X) \quad , \ \forall\, X \in T_\beta M .$$

In the above example we saw that the symplectic form looked just the same in a special coordinate system as the basic example $\mathbb{R}^{2n}$. This is true in fact for all symplectic manifolds:

<u>Theorem 1</u>. (Darboux) Each point of a symplectic manifold (M, ω) has a neighbourhood on which are defined coordinates $(p_1, \dots, p_n, q^1, \dots, q^n)$ (to be called <u>Darboux coordinates</u>) such that on this neighbourhood

$$\omega = \sum_i dp_i \wedge dq^i .$$

Theorem 1 says that all symplectic manifolds look the same locally and only differ in their global properties. We shall see later examples which differ considerably from the non-compact examples above in their global properties.

Let $C^\infty(M)$ denote the algebra of smooth functions on M. If $f \in C^\infty(M)$ then df is a 1-form on M and corresponds with a vector field X_f on M under the isomorphism of TM and T^*M defined by ω:

$$i_{X_f}\omega = df.$$

X_f is called the <u>Hamiltonian vector field</u> determined by the <u>Hamiltonian function</u> f. In Darboux coordinates

$$X_f = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} .$$

If we have two functions $f, g \in C^\infty(M)$ we form a third

$$\{f, g\} = X_f(g) = dg(X_f) = i_{X_g}\omega\,(X_f) = \omega(X_g, X_f)$$

called the Poisson bracket of f and g. It clearly agrees with the standard Poisson bracket on phase space $\mathbb{R}^{2n}$ in a Darboux coordinate system:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} .$$

$\{f, g\}$ is obviously skew-symmetric:

$$\{f, g\} = -\{g, f\}$$

and bilinear. Further

$$\begin{aligned} \mathcal{L}_{X_f}\omega &= d(i_{X_f}\omega) + i_{X_f}d\omega \\ &= d(df) \\ &= 0, \end{aligned}$$

so that

$$\begin{aligned} 0 &= (\mathcal{L}_{X_f}\omega)(X_g, X_h) \\ &= X_f(\omega(X_g, X_h)) - \omega([X_f, X_g], X_h) - \omega(X_g, [X_f, X_h]) \\ &= \{f, \{h, g\}\} + [X_f, X_g](h) - [X_f, X_h](g) \\ &= -\{f, \{g, h\}\} + \{f, \{g, h\}\} - \{g, \{f, h\}\} - \{f, \{h, g\}\} + \{h, \{f, g\}\} \\ &= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \end{aligned}$$

which is the Jacobi identity. Hence we have shown:

<u>Theorem 2</u>. The Poisson bracket makes $C^\infty(M)$ into a Lie algebra.

A 1-parameter group of diffeomorphisms of M is a curve σ_t of diffeomorphisms such that

$$\sigma_0 = id_M, \qquad \sigma_{t_1} \circ \sigma_{t_2} = \sigma_{t_1 + t_2}.$$

If we have such a group σ_t then

$$X_x = \left.\frac{d}{dt}\right|_0 \sigma_{-t}(x)$$

defines a vector-field on M which we call the *infinitesimal generator* of σ_t. Not every vector-field is an infinitesimal generator on all of M but the existence theorem for ordinary differential equations says that any vector-field is locally an infinitesimal generator. If X is globally an infinitesimal generator it is called complete and σ_t is said to be the flow generated by X.

In the symplectic case, if we have a Hamiltonian vector field X_H generating σ_t then

$$x(t) = \sigma_t(x)$$

satisfies Hamilton's equations. In Darboux coordinates

$$\dot{p}_i = \frac{d}{dt} p_i \circ \sigma_t = -X_H(p_i) = \{p_i, H\} = -\frac{\partial H}{\partial q^i},$$

$$\dot{q}^i = \frac{d}{dt} q^i \circ \sigma_t = -X_H(q^i) = \{q^i, H\} = \frac{\partial H}{\partial p_i},$$

which are the usual Hamilton's equations of motion. If f is a function on $C^\infty(M)$ and we define $f_t = f\circ\sigma_t$ then

$$\frac{d}{dt} f_t = -X_H(f_t) = \{f_t, H\}.$$

Example. $M = \mathbb{R}^6$, $\omega = dp_1\wedge dq^1 + dp_2\wedge dq^2 + dp_3 \wedge dq^3$. If $H = \frac{1}{2m}(p_1^2+p_2^2+p_3^2) + V(q^1,q^2,q^3)$ then the equations of motion are

$$\dot{q}^i = p_i/m, \qquad \dot{p}_i = -\frac{\partial V}{\partial q^i}$$

or

$$m\,\ddot{q}^i = -\frac{\partial V}{\partial q^i}.$$

Thus symplectic manifolds and the Poisson bracket represent a coordinate-free way of formulating Classical Mechanics and allow ~~substa~~ substantial generalisation, as well as making available geometrical methods for studying qualitative properties of such systems.

Let σ be a diffeomorphism of M. σ is called a *canonical transformation* or *symplectomorphism* if it preserves ω:

$$\sigma^*\omega = \omega.$$

This is easily seen to be the same as preserving Darboux coordinate systems, or the Poisson brackets

$$\{f,g\}\circ\sigma = \{f\circ\sigma, g\circ\sigma\}$$

of functions. If σ_t is a 1-parameter group of symplectomorphisms its infinitesimal generator X will satisfy

$$\mathcal{L}_X\omega = \left.\frac{d}{dt}\right|_0 \sigma_{-t}^*\omega = \left.\frac{d}{dt}\right|_0 \omega = 0.$$

Thus

$$d(i_X\omega) + i_X d\omega = 0$$

or

$$d(i_X\omega) = 0.$$

This argument is reversible, so we see a vector-field X generates a 1-parameter group of symplectomorphisms if and only if $i_X\omega$ is a closed 1-form. The Poincaré Lemma tells us such 1-forms locally have the form df for some function f and so X has the form X_f locally. For this reason we call X locally Hamiltonian if $i_X\omega$ is closed:

$$\mathrm{Ham}^\circ = \{X : d\, i_X\omega = 0\}.$$

We have a subspace Ham of Ham° of vector fields X_f with f globally defined and
$$\mathrm{Ham} = \{ X : i_X\omega \text{ exact} \}.$$
Thus $\mathrm{Ham}^\circ \supset \mathrm{Ham}$ and
$$\mathrm{Ham}^\circ/\mathrm{Ham} \cong \text{Closed 1-forms}/\text{Exact 1-forms} \cong H^1(M;\mathbb{R})$$
the first de Rham cohomology group of M.

Suppose $X, Y \in \mathrm{Ham}^\circ$ and Z is any vector field, then
$$\begin{aligned}(i_{[X,Y]}\omega)(Z) &= \omega([X,Y], Z) \\ &= X(\omega(Y,Z)) - (\mathcal{L}_X\omega)(Y,Z) - \omega(Y,[X,Z]) \\ &= X(i_Y\omega)(Z) - (i_Y\omega)([X,Z]) \\ &= (d(i_Y\omega))(X,Z) + Z(i_Y\omega)(X) \\ &= (\mathcal{L}_Y\omega)(X,Z) + Z(\omega(Y,X))\end{aligned}$$
so
$$\circledast \qquad i_{[X,Y]}\omega = d(\omega(Y,X)).$$
Thus $[X,Y] \in \mathrm{Ham}$ and hence we have shown

<u>Theorem 3.</u>
$$0 \to \mathrm{Ham} \hookrightarrow \mathrm{Ham}^\circ \to H^1(M;\mathbb{R}) \to 0$$
is an exact sequence of Lie algebras if $H^1(M;\mathbb{R})$ is given the structure of an abelian Lie algebra.

In the formula $\circledast$ above we may take $X = X_f$, $Y = X_g$ then
$$\begin{aligned}i_{[X_f,X_g]}\omega &= d\,\omega(X_g,X_f) \\ &= d\{f,g\} \\ &= i_{X_{\{f,g\}}}\omega\end{aligned}$$

which tells us
$$[X_f, X_g] = X_{\{f,g\}}$$
or that the map $j : C^\infty(M) \to \mathrm{Ham}$ which assigns to a function f the corresponding Hamiltonian vector field X_f is a homomorphism of Lie algebras. By definition it is onto, whilst its kernel consists of all functions f with $X_f = 0$ or equivalently $df = 0$. If M is connected f must then be constant. If we regard $\mathbb{R}$ as a subspace of $C^\infty(M)$ consisting of the constant functions we have
$$\{c, f\} = X_c(f) = 0 \quad , \quad c \in \mathbb{R} \ , \ f \in C^\infty(M),$$
so $\mathbb{R}$ sits as the centre of $C^\infty(M)$. We thus have

<u>Theorem 4.</u> $C^\infty(M)$ is a central extension by $\mathbb{R}$ of the Lie algebra of Hamiltonian vector fields
$$0 \to \mathbb{R} \xrightarrow{i} C^\infty(M) \xrightarrow{j} \mathrm{Ham} \to 0.$$

3. Lie groups of symplectomorphisms.

Let (M,ω) be a symplectic manifold (we assume from now on that M is connected) and G a Lie group acting smoothly on M. We denote by $\mathfrak{g}$ the Lie algebra of G and for each $\xi \in \mathfrak{g}$ we have a 1-parameter group σ_t^ξ of diffeomorphisms

$$\sigma_t^\xi(x) = \exp t\xi \cdot x \quad , \quad x \in M.$$

Denote by $\tilde{\xi}$ the vector-field on M which is the infinitesimal generator of σ_t^ξ, so

$$\tilde{\xi}_x = \frac{d}{dt}\Big|_0 \exp -t\xi \cdot x .$$

If $\mathfrak{X}(M)$ denotes the Lie algebra of vector fields on M we have:

<u>Theorem 1.</u> $\sigma : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ given by $\sigma(\xi) = \tilde{\xi}$ is a homomorphism of Lie algebras.

<u>Proof.</u>

$$\begin{aligned}([\tilde{\xi},\tilde{\eta}]f)(x) &= \tilde{\xi}_x(\tilde{\eta}f) - \tilde{\eta}_x(\tilde{\xi}f) \\ &= \frac{d}{dt}\Big|_0 \tilde{\eta}_{\exp -t\xi\cdot x}(f) - \frac{d}{ds}\Big|_0 \tilde{\xi}_{\exp -s\eta\cdot x}(f) \\ &= \frac{\partial^2}{\partial s \partial t}\Big|_{0,0} f(\exp -s\eta \exp -t\xi \cdot x) + f(\exp +t\xi \exp -s\eta\cdot x) \\ &= \frac{\partial^2}{\partial s\partial t}\Big|_{0,0} f(\exp t\xi \exp -s\eta \exp -t\xi \cdot x) \\ &= \frac{\partial^2}{\partial s \partial t}\Big|_{0,0} f(\exp -s(\mathrm{Ad}\exp t\xi(\eta)) \cdot x) \\ &= \frac{d}{dt}\Big|_0 \widetilde{\mathrm{Ad}\exp t\xi(\eta)}_x f \\ &= \widetilde{[\xi,\eta]}_x f . \quad \blacksquare \end{aligned}$$

Now suppose G acts by symplectomorphisms (or symplectically):

$$g^* \omega = \omega \quad \forall g \in G.$$

Such groups often arise naturally in physics from kinematical considerations. If the physical predictions of a theory are to be independent of observations where the observers are related by a group G, then G must act symplectically since we can write the dynamics using the Poisson bracket and hence the form ω. For instance in Newtonian theory inertial frames are related by the Galilean group and in special relativity by the Poincaré group. These groups thus must act symplectically in their respective phase spaces. We thus are interested in studying the symplectic manifolds on which a given Lie group G acts.

If G acts symplectically on (M,ω), then the infinitesimal generators $\tilde{\xi}$, $\xi \in \mathfrak{g}$, will be locally Hamiltonian, so we have a homomorphism

$$\sigma : \mathfrak{g} \longrightarrow \mathrm{Ham}^\circ .$$

It is natural to ask if the image actually lies in the Hamiltonian vector fields. The reason this is important is the following: If a Hamiltonian H generates a flow σ_t and $f_t = f \circ \sigma_t$ is the time evolution of f, then we saw in section 2 that

$$\frac{df_t}{dt} = \{f, H\}_t$$

so that f will be a constant of the motion: $f = f_t \ \forall t$ if and only if $\{f,H\} = 0$. But the latter is symmetric in f and H, so H must be invariant under the flow of f. Thus if we have a group G of symmetries of H:

$$H\circ g = H \quad \forall g \in G$$

and if each $\tilde{\xi}$ is Hamiltonian: $\tilde{\xi} = X_{f^{\xi}}$ then each f^{ξ} will be a constant of the motion.

There are two simple conditions which will guarantee that $\tilde{\xi} \in \mathrm{Ham}$ for all $\xi \in \mathfrak{g}$. From Theorem 3 of §2 we have $\mathrm{Ham} = \mathrm{Ham}^{\circ}$ when $H^1(M;\mathbb{R}) = 0$. Thus, for example, if M is simply-connected every locally Hamiltonian vector field is Hamiltonian. Alternatively we can use the fact that $[\mathrm{Ham}^{\circ}, \mathrm{Ham}^{\circ}] \subset \mathrm{Ham}$ to deduce that if $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ then $\tilde{\xi}$ will be Hamiltonian for each ξ. Thus we have

<u>Theorem 2.</u> If G acts symplectically on (M,ω) and either

(a) $H^1(M;\mathbb{R}) = 0$

or

(b) $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ (i.e. $H^1(\mathfrak{g};\mathbb{R}) = 0$)

then

$$\sigma(\mathfrak{g}) \subset \mathrm{Ham}.$$

If both these conditions fail we can replace M by its simply-connected covering manifold $\tilde{M}$ which has an induced symplectic form $\tilde{\omega}$. G may no longer act, but its simply-connected covering group $\tilde{G}$ does (assume G is connected) and $\tilde{G}$ preserves $\tilde{\omega}$. Condition (a) now holds for $\tilde{M}$ and so on $\tilde{M}$ each vectorfield $\tilde{\xi}$ will be Hamiltonian.

Let us say that an action of a Lie group G on a symplectic manifold (M,ω) is <u>almost Hamiltonian</u> if each $\tilde{\xi}$ is a Hamiltonian vector field. Then the curves $\exp t\xi\cdot x$ in M will be the solutions of the Hamiltonian equations of motion for the corresponding Hamiltonian function. If G has an almost Hamiltonian action on (M,ω) we thus have the following picture

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \xrightarrow{i} & C^{\infty}(M) & \xrightarrow{j} & \mathrm{Ham} & \longrightarrow & 0 \\ & & & & & & \uparrow{\scriptstyle\sigma} & & \\ & & & & & & \mathfrak{g} & & \end{array}$$

where σ is a homomorphism of Lie algebras. Since $\mathfrak{g}$ is finite dimensional and j is surjective we can always find a linear map $\lambda : \mathfrak{g} \to C^{\infty}(M)$ with $j\circ\lambda = \sigma$. If we can do so with λ a homomorphism of Lie algebras

$$\{\lambda(\xi), \lambda(\eta)\} = \lambda([\xi,\eta])$$

then we shall say that we have a <u>Hamiltonian action</u> of G on (M,ω). $\lambda(\xi)$ will not be unique since we can add a constant without changing $j(\lambda(\xi)) = \tilde{\xi}$. So we have in general to specify λ in order to give the Hamiltonian action completely. We call (M,ω,λ) a <u>Hamiltonian G-space</u> if λ is a homomorphism with $j\circ\lambda = \sigma$. In the next section we examine the question of the existence and uniqueness of λ.

<u>Examples</u> 1. The basic example is $M = \mathbb{R}^{2n}$ with $\omega = \sum dp_i \wedge dq^i$. Clearly the group $\mathbb{R}^{2n}$ of translations preserves ω since

ω has constant coefficients. The infinitesimal action is given by the directional derivative

$$\tilde{\xi}_{(p,q)} = \frac{d}{dt}\Big|_0 (p,q) - t\xi \quad , \quad \xi \in \mathbb{R}^{2n}$$

$$= \xi_1 \frac{\partial}{\partial p_1} + \cdots + \xi_n \frac{\partial}{\partial p_n} + \xi_{n+1}\frac{\partial}{\partial q^1} + \cdots + \xi_{2n}\frac{\partial}{\partial q^n},$$

so

$$i_{\tilde{\xi}}\,\omega = \xi_1 dq^1 + \cdots + \xi_n dq^n - \xi_{n+1} dp_1 - \cdots - \xi_{2n} dp_n$$

$$= d\,(\xi_1 q^1 + \cdots + \xi_n q^n - \xi_{n+1} p_1 - \cdots - \xi_{2n} p_n).$$

Thus we see explicitly this action is locally Hamiltonian. We shall in the next section that in fact this action is not Hamiltonian.

2. Consider again $(\mathbb{R}^{2n}, \omega = \sum dp_i \wedge dq^i)$, but with the group $Sp(2n;\mathbb{R})$ of linear transformations preserving ω. If

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{matrix} \}\,n \\ \}\,n \end{matrix}$$

then $g \in Sp(2n,\mathbb{R})$ if ${}^t g J g = J$. $Sp(2n,\mathbb{R})$ acts by $\binom{p}{q} \longrightarrow g\binom{p}{q}$.
The Lie algebra $sp(2n,\mathbb{R})$ consists of ξ linear with

$${}^t\xi J + J\xi = 0,$$

and if $x = \binom{p}{q}$ then

$$\tilde{\xi}_x = \frac{d}{dt}\Big|_0 \exp -t\xi\, x = \sum_i (-\xi x)^i \partial_i = (-\xi x)^t \partial_x,$$

$$= \sum_i -(Ap+Bq)_i \frac{\partial}{\partial p_i} - (Cp - {}^tAq)^i \frac{\partial}{\partial q^i}$$

where

$$\xi = \begin{bmatrix} A & B \\ C & -{}^tA \end{bmatrix} \quad , \quad {}^tB = B, \quad {}^tC = C.$$

Then

$$\tilde{\xi} \lrcorner\, \omega = -\sum_i (Ap+Bq)_i dq^i - (Cp - {}^tAq)^i dp_i$$

$$= \frac{1}{2} d\,\left({}^tp Cp - 2{}^tp{}^tAq - {}^tqBq\right)$$

$$= -\frac{1}{2} d\,({}^tp, {}^tq) J\xi \binom{p}{q} \quad = \quad d\left(-\frac{1}{2}\,{}^tx J\xi x\right)$$

Thus $\lambda(\xi)(p,q) = -\frac{1}{2}$ is a quadratic Hamiltonian for ξ. We thus see the action is almost Hamiltonian. In fact

$$\{\lambda(\xi), \lambda(\eta)\} = \tilde{\xi}(\lambda(\eta)) = (-\xi x)^t \partial_x \left(-\tfrac{1}{2}\,{}^tx J\eta x\right)$$
$$= {}^tx J\eta\xi x.$$

But

$$\lambda([\xi,\eta]) = -\frac{1}{2}\,{}^tx J(\xi\eta - \eta\xi) x$$

and

$${}^tx J\xi\eta x = {}^t(J\xi\eta x)\, x = -\,{}^tx\, {}^t\eta\, {}^t\xi J x$$
$$= -{}^tx J\eta J J \xi J J x$$
$$= -{}^tx J\eta\xi x$$

so

$$\{\lambda(\xi), \lambda(\eta)\} = \lambda([\xi,\eta])$$

nd hence this action is Hamiltonian.

A special case is $n = 3$, $\xi_1 = \begin{bmatrix} L_1 & 0 \\ 0 & L_1 \end{bmatrix}$, $L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

enerating rotations around the e_1 axis in $\mathbb{R}^3$, then

$$\lambda(\xi_1) = -{}^tp L_1 q = q^2 p_3 - q^3 p_2$$

the component of angular momentum in the e_1 direction. nilarly for ξ_2, ξ_3 obtained by replacing L_1 by L_2, L_3. en $\lambda(\xi_2) = q^3 p_1 - q^1 p_3$, $\lambda(\xi_3) = q^1 p_2 - q^2 p_1$. These nctions satisfy $\{q^2p_3 - q^3p_2,\ q^3p_1 - q^1p_3\} = q^1p_2 - q^2p_1$ and

and cyclic permutations.

3. $M = T^*C$, $\omega = d\theta$. Let G be any group of diffeomorphisms of C. There is an induced action on M:

$$g \cdot \beta = \beta \circ g^{-1}_* \quad , \quad \beta \in T^*C.$$

Then $\pi(g \cdot \beta) = g\,\pi(\beta)$, so

$$(g^*\theta)_\beta = \theta_{g\cdot\beta} \circ g_* = g\cdot\beta\ \pi_* g_*$$

$$= \beta \circ g_*^{-1}\pi_* g_*$$

$$= \beta \circ \pi_*$$

$$= \theta_\beta .$$

Thus

$$g^*\theta = \theta$$

and so

$$g^*\omega = \omega .$$

Hence the induced action on T^*C of any group of motions of configuration space C is always symplectic. If $\xi \in \mathfrak{g}$ and $\hat{\xi}$ is the corresponding vector field on C, $\tilde{\xi}$ that on T^*C, then $\pi_* \tilde{\xi} = \hat{\xi}$ and we have

$$i_{\tilde{\xi}}\,\omega = i_{\tilde{\xi}}\,d\theta = \mathcal{L}_{\tilde{\xi}}\theta - d(\theta(\tilde{\xi})).$$

But θ is invariant so $\mathcal{L}_{\tilde{\xi}}\theta = 0$ and hence

$$i_{\tilde{\xi}}\,\omega = d(-\theta(\tilde{\xi})).$$

Thus the action is almost Hamiltonian with Hamiltonian function

$$\lambda(\xi) = -\theta(\tilde{\xi})$$

or

$$\lambda(\xi)(\beta) = -\beta(\pi_* \tilde{\xi}) = -\beta(\hat{\xi}_{\pi(\beta)}).$$

I leave it as an exercise to verify that this action is in fact Hamiltonian.

4. Almost Hamiltonian and Hamiltonian G-spaces.

Let (M,ω) be an almost Hamiltonian G-space so we have our basic diagram

$$\begin{array}{ccccccc} 0 \to \mathbb{R} \to & C^\infty(M) & \xrightarrow{j} & \mathrm{Ham} & \to 0 \\ & & & \uparrow \dot\sigma & \\ & & & \mathfrak{g} & \end{array}$$

If we choose some linear map $a : \mathfrak{g} \to C^\infty(M)$ with $j \circ a = \dot\sigma$ then the failure of a to be a homomorphism of Lie algebras is measured by

$$c(\xi,\eta) = \{a(\xi), a(\eta)\} - a([\xi,\eta]).$$

Since j is a homomorphism

$$j\,c(\xi,\eta) = [\dot\sigma(\xi), \dot\sigma(\eta)] - \dot\sigma([\xi,\eta]) = 0$$

so $c(\xi,\eta)$ is a constant function, and hence defines a skew-symmetric bilinear map $c : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$. c is thus a 2-cochain for the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$. Choosing another map $a' : \mathfrak{g} \to C^\infty(M)$ the difference $b = a' - a$ satisfies $j \circ b = 0$ so $b(\xi)$ is constant for each ξ and hence b is a 1-cochain. If c' is the failure of a' to be a homomorphism, then

$$\begin{aligned} c'(\xi,\eta) &= \{a'(\xi), a'(\eta)\} - a'([\xi,\eta]) \\ &= \{a(\xi)+b(\xi), a(\eta)+b(\eta)\} - a([\xi,\eta]) - b([\xi,\eta]) \\ &= \{a(\xi), a(\eta)\} - a([\xi,\eta]) - b([\xi,\eta]) \\ &= c(\xi,\eta) + (\delta b)(\xi,\eta) \end{aligned}$$

or $\quad c' = c + \delta b$

where δ is the Lie algebra cohomology differential defined in Professor Gutt's lectures. Further

$$\begin{aligned} (\delta c)(\xi,\eta,\zeta) &= -c([\xi,\eta],\zeta) + c([\xi,\zeta],\eta) - c([\eta,\zeta],\xi) \\ &= -\{a([\xi,\eta]), a(\zeta)\} + a([[\xi,\eta],\zeta]) \\ &\quad + \{a([\xi,\zeta]), a(\eta)\} - a([[\xi,\zeta],\eta]) \\ &\quad - \{a([\eta,\zeta]), a(\xi)\} + a([[\eta,\zeta],\xi]) \\ &= -\{\{a(\xi),a(\eta)\},a(\zeta)\} + \{\{a(\xi),a(\zeta)\},a(\eta)\} \\ &\quad - \{\{a(\eta),a(\zeta)\},a(\xi)\} \\ &\quad + a([[\xi,\eta],\zeta] - [[\xi,\zeta],\eta] + [[\eta,\zeta],\xi]) \\ &= 0 \end{aligned}$$

using the Jacobi identities for $C^\infty(M)$ and $\mathfrak{g}$. Thus c is a cocycle whose class in $H^2(\mathfrak{g};\mathbb{R}) = Z^2/B^2$ is independent of the choice of a. This class $[c]$ we denote by $c(M,\omega,\mathfrak{g})$ and it is intrinsically associated with the almost Hamiltonian action. If there were a homomorphism $\lambda : \mathfrak{g} \to C^\infty(M)$ making the action Hamiltonian then taking $a = \lambda$ gives $c = 0$ so $c(M,\omega,\mathfrak{g}) = 0$ in $H^2(\mathfrak{g};\mathbb{R})$. The converse is also true, for if $c(M,\omega,\mathfrak{g}) = 0$, we have $c = \delta b$ for some 1-cochain b. This means

$$\{a(\xi), a(\eta)\} - a([\xi,\eta]) = (\delta b)(\xi,\eta) = -b([\xi,\eta])$$

Thus if $\lambda(\xi) = a(\xi) - b(\xi)$, then b constant implies $j\circ\lambda = \dot\sigma$ still, whilst

$$\begin{aligned} \{\lambda(\xi), \lambda(\eta)\} &= \{a(\xi), a(\eta)\} = a([\xi,\eta]) - b([\xi,\eta]) \\ &= \lambda([\xi,\eta]), \end{aligned}$$

so the action is Hamiltonian. We have thus proved:

Theorem 1. If (M,ω) is an almost Hamiltonian G-space then the class $c(M,\omega,\mathfrak{g}) \in H^2(\mathfrak{g};\mathbb{R})$ vanishes if and only if there is a homomorphism λ making (M,ω,λ) a Hamiltonian G-space.

λ, when it exists, may not be unique. If λ, λ' are two homomorphisms the difference $\lambda' - \lambda = b$ will satisfy $j \circ b = 0$, so $b(\xi)$ is a constant function for each ξ and hence is a 1-cochain with values in $\mathbb{R}$. But

$$\begin{aligned} \delta b(\xi,\eta) &= -b([\xi,\eta]) \\ &= \lambda([\xi,\eta]) - \lambda'([\xi,\eta]) \\ &= \lambda([\xi,\eta]) - \{\lambda'(\xi), \lambda'(\eta)\} \\ &= \lambda([\xi,\eta]) - \{\lambda(\xi) + b(\xi), \lambda(\eta) + b(\eta)\} \\ &= \lambda([\xi,\eta]) - \{\lambda(\xi), \lambda(\eta)\} \\ &= 0. \end{aligned}$$

Thus b is a 1-cocycle. Conversely adding a 1-cocycle to a Hamiltonian λ gives another, and since there are no 1-coboundaries for the trivial ~~re~~ representation we see the non-uniqueness of λ is measured precisely by $H^1(\mathfrak{g};\mathbb{R})$. Thus:

Theorem 2. If (M,ω,λ) is a Hamiltonian G-space then so is $(M,\omega,\lambda+b)$, $b \in H^1(\mathfrak{g};\mathbb{R})$ and all Hamiltonian actions on (M,ω) arise this way.

In particular we have

Theorem 3. If $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$ then every almost Hamiltonian G-space (M,ω) admits a unique homomorphism $\lambda : \mathfrak{g} \to C^\infty(M)$ making (M,ω,λ) a Hamiltonian G-space.

This applies in particular when ~~g~~ G is semisimple by the Whitehead Lemmas.

What can be done if the obstruction $c(M,\omega,\mathfrak{g})$ does not vanish? To answer this we make a diversion into the theory of extensions. If $\mathfrak{g}$ and $\mathfrak{a}$ are Lie algebras we say $\hat{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ by $\mathfrak{a}$ if there are homomorphisms

$$\mathfrak{a} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{j} \mathfrak{g}$$

with i injective, j surjective and the image of i precisely the kernel of j. Such a sequence of Lie algebras is said to be <u>exact</u>. A basic example for us is

$$\mathbb{R} \to C^\infty(M) \to \mathrm{Ham}$$

which interprets as saying $C^\infty(M)$ is an extension of Ham by $\mathbb{R}$. In the general case we say the ~~ext~~ extension is central if $i(\mathfrak{a})$ is contained in the centre of $\hat{\mathfrak{g}}$. In this case $\mathfrak{a}$ must be abelian, and so is just a vector space. We regard it as a trivial representation of $\mathfrak{g}$.

Thus we suppose we have a central extension

$$\mathfrak{a} \xrightarrow{i} \hat{\mathfrak{h}} \xrightarrow{j} \mathfrak{h}$$

of a Lie algebra $\mathfrak{h}$ and a homomorphism $k : \mathfrak{g} \to \mathfrak{h}$. This gives our standard picture

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "i"] & \hat{\mathfrak{h}} \arrow[r, "j"] & \mathfrak{h} \\ & & \mathfrak{g} \arrow[u, "k"'] \end{tikzcd}$$

We ~~may~~ may form the direct sum Lie algebra $\hat{\mathfrak{h}} \oplus \mathfrak{g}$ and inside pick out the subalgebra

$$\hat{\mathfrak{g}} = \{ (\hat{\xi}, \eta) \in \hat{\mathfrak{h}} \oplus \mathfrak{g} \;:\; j(\hat{\xi}) = k(\eta) \}.$$

There are homomorphisms

$$i' : \mathfrak{a} \longrightarrow \hat{\mathfrak{g}} \qquad i'(\xi) = (i(\xi), 0)$$
$$j' : \hat{\mathfrak{g}} \longrightarrow \mathfrak{g} \qquad j'(\hat{\xi}, \eta) = \eta$$

which may be checked to be well-defined and give a central extension

$$\mathfrak{a} \xrightarrow{i'} \hat{\mathfrak{g}} \xrightarrow{j'} \mathfrak{g}$$

of $\mathfrak{g}$. We call it the <u>induced</u> central extension. It is clear there is a homomorphism $\lambda : \hat{\mathfrak{g}} \to \hat{\mathfrak{h}}$, namely

$$\lambda(\hat{\xi}, \eta) = \hat{\xi}$$

and if we set $k' = k \circ j'$, then we have

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "i"] & \hat{\mathfrak{h}} \arrow[r, "j"] & \mathfrak{h} \\ & & \hat{\mathfrak{g}} \arrow[u, "k'"'] \arrow[ul, "\lambda"] \end{tikzcd}$$

Thus replacing $\mathfrak{g}$ by the induced extension $\hat{\mathfrak{g}}$ gives us tautologically a homomorphism λ.

We may do this for an almost Hamiltonian action of G on (M, ω):

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "i"] & C^{\infty}(M) \arrow[r, "j"] & \mathrm{Ham} \arrow[r] & 0 \\ & & & \mathfrak{g} \arrow[u, "\sigma"'] \end{tikzcd}$$

The induced extension of $\mathfrak{g}$ by $\mathbb{R}$ we denote by $\hat{\mathfrak{g}}$ and obtain a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "i"] & C^{\infty}(M) \arrow[r, "j"] & \mathrm{Ham} \arrow[r] & 0 \\ & & & \hat{\mathfrak{g}} \arrow[u, "\sigma'"'] \arrow[ul, "\lambda"] \end{tikzcd}$$

Here

$$\hat{\mathfrak{g}} = \{ (f, \xi) \in C^{\infty}(M) \oplus \mathfrak{g} \;:\; X_f = \tilde{\xi} \}$$

and

$$\sigma'(f, \xi) = \tilde{\xi} = X_f \quad , \quad \lambda(f, \xi) = f.$$

If $\hat{G}$ is the simply-connected group with Lie algebra $\hat{\mathfrak{g}}$ there is a homomorphism $\rho : \hat{G} \to G$ whose differential ρ is $j' : \hat{\mathfrak{g}} \to \mathfrak{g}$. By means of ρ $\hat{G}$ acts on (M, ω) symplectically with Hamiltonian λ. Thus (M, ω, λ) is a Hamiltonian $\hat{G}$-space.

The problem with this construction is that $\hat{G}$ depends on (M, ω). What we need is an extension which is universal and so that such a homomorphism λ will exist. It can be found by a trick from cohomology theory. Let

$$\mathfrak{a} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{j} \mathfrak{g}$$

be any central extension, then we ~~can~~ may chose a linear map $a : \mathfrak{g} \to \hat{\mathfrak{g}}$ with $j \circ a = \mathrm{id}$.

$$j([a(\xi), a(\eta)] - a([\xi, \eta])) = [\xi, \eta] - [\xi, \eta] = 0$$

is

$$[a(\xi), a(\eta)] - a([\xi, \eta]) = i(c(\xi, \eta))$$

for some alternating bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$. Just as before

c is a 2-cochain for $\mathfrak{g}$ with values in the trivial representation on $\mathfrak{a}$, and a calculation analogous to the one for almost Hamiltonian actions shows $\delta c = 0$ and c changes by a coboundary if a is changed to a different linear map. Hence the extension $\hat{\mathfrak{g}}$ determines a cohomology class $c(\hat{\mathfrak{g}}) = [c] \in H^2(\mathfrak{g};\mathfrak{a})$. If two extensions are said to be equivalent if there is a diagram of the form

$$\begin{tikzcd} & \hat{\mathfrak{g}}_1 \arrow[d] \arrow[dr] & \\ \mathfrak{a} \arrow[ur] \arrow[dr] & & \mathfrak{g} \\ & \hat{\mathfrak{g}}_2 \arrow[ur] & \end{tikzcd}$$

which commutes, then it is easy to see $c(\hat{\mathfrak{g}})$ depends only on the equivalence class. This provides a map from equivalence classes of extensions by $\mathfrak{a}$ to $H^2(\mathfrak{g};\mathfrak{a})$. It is in fact a bijection:

<u>Theorem 4</u> $\hat{\mathfrak{g}} \longrightarrow c(\hat{\mathfrak{g}}) \in H^2(\mathfrak{g};\mathfrak{a})$ is a bijection from equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{a}$ to the Lie algebra cohomology group $H^2(\mathfrak{g};\mathfrak{a})$.

The reverse implication is proved by exhibiting an extension of $\mathfrak{g}$ by $\mathfrak{a}$ given $c \in Z^2$: namely on the vector space direct sum ~~define~~ $\mathfrak{a}\oplus\mathfrak{g}$ define a bracket by

$$[(\xi,\eta),(\xi',\eta')]_c = (c(\eta,\eta'), [\eta,\eta']).$$

The cocycle condition for c is the Jacobi identity for $[\ ,\ ]_c$.
If we take $a(\eta) = (0,\eta)$ then

$$[a(\eta), a(\eta')]_c - a([\eta,\eta']) = (c(\eta,\eta'), 0)$$

showing that the class of the extension is $[c]$.

Thus to build extensions we need to find cocycles. Since the coefficients $\mathfrak{a}$ are acted on trivially by $\mathfrak{g}$ we have

$$H^2(\mathfrak{g};\mathfrak{a}) \cong H^2(\mathfrak{g};\mathbb{R})\otimes\mathfrak{a},$$

so if $\mathfrak{a}_0 = H^2(\mathfrak{g};\mathbb{R})^*$ then

$$H^2(\mathfrak{g};\mathfrak{a}_0) \cong H^2(\mathfrak{g};\mathbb{R})\otimes H^2(\mathfrak{g};\mathbb{R})^* \cong \operatorname{End} H^2(\mathfrak{g};\mathbb{R}).$$

In $\operatorname{End} H^2(\mathfrak{g};\mathbb{R})$ we have the identity map 1 and this then corresponds with $\sum_i [c_i]\otimes\alpha_i$ in $H^2(\mathfrak{g};\mathbb{R})\otimes H^2(\mathfrak{g};\mathbb{R})^*$ where $[c_i]$ is a basis for $H^2(\mathfrak{g};\mathbb{R})$ and α_i the dual basis. If we choose ~~a cocy~~ cocycles $c_i \in [c_i]$ then

$$c_0(\xi,\eta) = \sum c_i(\xi,\eta)\alpha_i$$

defines a 2-cocycle on $\mathfrak{g}$ with values in $\mathfrak{a}_0$ whose class $[c_0]$ corresponds with $1 \in \operatorname{End} H^2(\mathfrak{g};\mathbb{R})$. We call it the <u>tautological class</u>. The corresponding extension $\hat{\mathfrak{g}}_0$ of $\mathfrak{g}$ by $\mathfrak{a}_0$ is called the <u>universal extension</u>:

$$\mathfrak{a}_0 \xrightarrow{i_0} \hat{\mathfrak{g}}_0 \xrightarrow{j_0} \mathfrak{g}.$$

If $\hat{G}_0$ denotes the simply-connected group with Lie algebra $\hat{\mathfrak{g}}_0$ then j_0 induces a homomorphism $\rho_0 : \hat{G}_0 \to G$ with $\rho_{0*} = j_0$. If now we take an almost Hamiltonian G space (M,ω) then we make $\hat{G}_0$ act by

$$\hat{g}\cdot x = \rho_0(\hat{g})\cdot x \qquad \hat{g}\in\hat{G}_0,\ x \in M.$$

Then (M,ω) is an almost Hamiltonian $\hat{G}_0$ space and the ~~map~~ homomorphism $\dot{j}_0 : \hat{\mathfrak{g}}_0 \longrightarrow \mathrm{Ham}$ is given by $\dot{\sigma} \circ j_0$.

<u>Theorem 5</u> Every almost Hamiltonian G-space (M, ω) is Hamiltonian as a $\hat{G}_0$ space.

<u>Proof.</u> We must write down a homomorphism $\lambda : \hat{\mathfrak{g}}_0 \longrightarrow C^\infty(M)$. To do this let $[c_i]$, α_i be the dual bases for $H^2(\mathfrak{g}; \mathbb{R})$, $H^2(\mathfrak{g}, \mathbb{R})^*$ as above and expand $c(M, \omega, \mathfrak{g}) \in H^2(\mathfrak{g}; \mathbb{R})$ in the basis

$$c(M, \omega, \mathfrak{g}) = \sum_i k_i [c_i].$$

Choose a linear map $a : \mathfrak{g} \longrightarrow C^\infty(M)$ and put

$$c(\xi, \eta) = \{ a(\xi), a(\eta) \} - a([\xi, \eta])$$

so that $[c] = c(M, \omega, \mathfrak{g})$. Then c and $\sum_i k_i c_i$ are in the same cohomology class, so there is a 1-cochain b with

$$c = \sum_i k_i c_i + \delta b .$$

Now $\hat{\mathfrak{g}}_0$ is defined as the sum $\sigma_0 \oplus \mathfrak{g}$ with bracket

$$[(\alpha, \xi), (\beta, \eta)]_{c_0} = (c_0(\xi, \eta), [\xi, \eta]).$$

Set

$$\lambda(\alpha, \xi) = a(\xi) + \sum_i k_i \, \alpha([c_i]) - b(\xi).$$

Then

$$\begin{aligned} \{\lambda(\alpha, \xi), \lambda(\beta, \eta)\} &= \{ a(\xi), a(\eta) \} \\ &= c(\xi, \eta) + a([\xi, \eta]). \end{aligned}$$

On the other hand

$$\begin{aligned} \lambda([(\alpha, \xi), (\beta, \eta)]) &= \lambda((c_0(\xi, \eta), [\xi, \eta])) \\ &= a([\xi, \eta]) + \sum_i k_i \, c_0(\xi, \eta)([c_i]) - b([\xi, \eta]) \\ &= a([\xi, \eta]) + \sum_{ij} k_i \, c_j(\xi, \eta) \, \alpha_j([c_i]) - b([\xi, \eta]) \\ &= a([\xi, \eta]) + \sum_i k_i \, c_i(\xi, \eta) - b([\xi, \eta]) \\ &= a([\xi, \eta]) + c(\xi, \eta). \end{aligned}$$

Hence λ is a homomorphism as required.

This theorem shows that a group G has a central extension $\hat{G}_0$ (determined by $H^2(\mathfrak{g}; \mathbb{R})$) such that the almost Hamiltonian spaces of G form a subset of the Hamiltonian spaces of $\hat{G}_0$. For this reason we can restrict attention to this case.

<u>Example.</u> Consider the first example of section 3. If $\mathbb{R}^{2n}$ acts on itself by translations, then we saw

$$i_\xi \omega = d(\xi_1 q^1 + \cdots + \xi_n q^n - \xi_{n+1} p_1 - \cdots - \xi_{2n} p_n), \quad \xi \in \mathbb{R}^{2n}$$

$$a(\xi) = \xi_1 q^1 + \cdots + \xi_n q^n - \xi_{n+1} p_1 - \cdots - \xi_{2n} p_n$$

a choice of section of j, and

$$\begin{aligned} c(\xi, \eta) &= \{ a(\xi), a(\eta) \} - a([\xi, \eta]) \\ &= \sum_i -\xi_i \eta_{i+n} + \xi_{i+n} \eta_i \\ &= {}^t\xi J \eta . \end{aligned}$$

Thus the extension by $\mathbb{R}$ of $\mathbb{R}^{2n}$ defined by this action is given by

$$[(\xi, s), (\eta, t)] = (0, {}^t\xi J \eta).$$

This is the Heisenberg Lie algebra and is a non-trivial extension by $\mathbb{R}$.

The universal extension is found by calculating $H^2(\mathbb{R}^{2n};\mathbb{R})$. Since $\mathbb{R}^{2n}$ is abelian, the differential $\delta = 0$, so ~~the~~ $H^2(\mathbb{R}^{2n};\mathbb{R})$ is the same as the 2-cochains, which is the space of bilinear skew symmetric maps $\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}$. The dual of this is $\Lambda^2\mathbb{R}^{2n}$, so $\mathfrak{a}_0 = \Lambda^2\mathbb{R}^{2n}$. The extension $\hat{\mathfrak{g}}_0$ is then $\Lambda^2\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$ with the bracket

$$[(A,\xi),(B,\eta)] = (\xi\wedge\eta,\ 0).$$

The Hamiltonian for the action of $\hat{\mathfrak{g}}_0$ on $\mathbb{R}^{2n}$ is then determined by

$$\lambda(\eta\wedge\zeta,\ \xi) = a(\xi) + c(\eta,\zeta)$$

as can be checked by direct calculation.