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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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KIRILLOV THEORY

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Kirillov Theory

Kirillov theory gives a complete classification of irreducible unitary representations of a simply connected nilpotent Lie group G as well as a decomposition of $L^2(G)$. We will begin the study of irreducible unitary representations in this lecture. As is to be expected most results are proved by induction on dimension on G. One needs also some information on the structure of G. We outline some of the facts we need below.

(1) Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra and H the corresponding connected analytic subgroup. Then $\mathfrak{h}$ has a supplement $\mathfrak{h}'$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ such that the map

$$(X, Y) \longmapsto \exp X \cdot \exp Y \quad , X \in \mathfrak{h}, \ X' \in \mathfrak{h}'$$

is a diffeomorphism of $\mathfrak{g}$ on G. In particular $\exp : \mathfrak{g} \to G$ is a diffeomorphism of $\mathfrak{g}$ onto G.

Also for any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, $\exp \mathfrak{h} = H$ is the (connected) Lie subgroup associated to $\mathfrak{h}$ and H is necessarily closed. Further the homogeneous space $H \backslash G$ (hence also G/H) is diffeomorphic to a euclidean space.

One method of obtaining these results is through

② Ado's theorem. Any simply connected nilpotent Lie group G admits a faithful representation $\sigma : G \to GL(V)$ on a finite dimensional vector space V such that $\sigma(x)$ is unipotent for all $x \in G$.

A representation $\tau : G \to GL(W)$ (W finite dimensional) is unipotent iff $\tau(x)$ is unipotent for all $x \in G$. The adjoint representation Ad is a unipotent representation. If τ is a unipotent representation any sub-quotient of a tensor power of τ is also unipotent.

③ Orbit Spaces. If $\tau : G \to GL(W)$ is a unipotent representation, then the orbit $\{\tau(g) v \mid g \in G\}$ of any vector $v \in W$ is closed. This is proved by using ① treating G as a subgroup of the upper triangular unipotent group in $GL(W)$ for a suitable basis in W. The space of orbits of G in W is thus a T_1-space. It is of course also a space with a countable base. The isotropy I_v for any $v \in G$ is connected.

④ The Centre. Let $\mathfrak{z}$ denote the centre of $\mathfrak{g}$. Then $C = \exp \mathfrak{z}$ is precisely the centre of G. It is thus a connected (closed) subgroup of G.

⑤ Groups with 1-dimensional centre. In the sequel we need some information on the structure of groups with a 1-dimensional centre. The Heisenberg group is such a group. Suppose that G has a centre C of dimension 1. Let $Z \in \mathfrak{z}$ be a non-zero element ($\mathfrak{z}$ = Lie algebra of C). ~~Then~~ Consider the

Lie algebra $\mathfrak{g}' = \mathfrak{g}/\mathfrak{z}$. This Lie algebra being again nilpotent has a non-zero centre. Let $\bar{Y} \in \mathfrak{g}'$ be a non-zero central element and $Y \in \mathfrak{g}$ be an element that projects to $\bar{Y}$ under the natural map $\mathfrak{g} \to \mathfrak{g}'$. Then we have $\operatorname{ad} \bar{Y}(\bar{T}) = 0$ for all $\bar{T} \in \mathfrak{g}'$ so that $\operatorname{ad} Y$ maps all of $\mathfrak{g}$ into $\mathfrak{z}$. As $Y \notin \mathfrak{z}$, $\operatorname{ad} Y$ is surjective. We can therefore find $X \in \mathfrak{g}$ such that $[Y, X] = Z$. The centraliser $z(Y)$ of Y in $\mathfrak{g}$ evidently has codimension 1 and we have

$$\mathfrak{g} = z(Y) + \mathbb{R}X$$

The Heisenberg Lie algebra is evidently isomorphic to the three dimensional subalgebra $\mathfrak{h}' = \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathbb{R}X$ in $\mathfrak{g}$; evidently $\mathfrak{h}' \cap \mathfrak{g}$ is a maximal abelian ~~subalg~~ subalgebra of $\mathfrak{h}'$.

We will now prove the ~~first~~ crucial result on unitary irreducible representations. ~~This~~

<u>Theorem</u> Let G be a simply connected nilpotent Lie group and $\sigma : G \to B(\mathcal{H})$ be an irreducible unitary representation of G. Then σ is equivalent to I_χ where χ is a character on a connected closed subgroup H of G.

<u>Proof.</u> We will argue by induction on $\dim G$. Let A be a maximal abelian (connected) normal subgroup of G and $\hat{A}$ its dual. If G is abelian, the theorem is trivial so that we may assume that G is non abelian. If $G' = G/C$ and C' is the centre of G' ~~[illegible]~~, C' is non-trivial. If C'' is a connected 1-dimensional Lie subgroup of C' and B is the inverse image of C'' in G, then B is abelian connected and normal in G so that we see that $\dim A \geq 2$.

Consider now the extension $\bar{\sigma} : L^1(A) \to B(\mathcal{H})$. If $\mathcal{F}$ denotes the Fourier Transform and $\mathcal{L} = \mathcal{F}(L^1(A))$

then $\overset{M=}{\wedge}\tilde{\sigma}\circ\mathcal{F}^{-1} : \mathcal{L} \to B(\mathcal{H})$ is ~~an~~ a $*$ algebra homomorphism of the space $\mathcal{L}$ of continuous functions on $\hat{A}$. It is easy to easy that M defines indeed a $*$ algebra homomorphism of the algebra of bounded measurable functions into $B(\mathcal{H})$. It is further not difficult to see that $\|M(\varphi)\| \leq \sup\{|\varphi(x)| \mid x \in \hat{A}\}$ for all $\varphi \in C_c(\hat{A})$. Finally it is easy to check that $\sigma(x) M(\varphi) \sigma(x)^{-1} = M({}^x\varphi)$. Now we will be in a position to apply Mackey's theorem if only we can prove that the projection valued measure M on $\hat{A}$ is supported on a single G orbit. Suppose now that E is any G-stable Borel set in $\hat{A}$; then $P_E\ (= M(1_E)) = 0$ or Identity. Thus if E_n, $1 \leq n < \infty$ is any countable family of Borel sets, with $P_{E_n} = 1$ for all n, then $P_{\bigcap_{n=1}^{\infty} E_n} = 1$ as well. Now any ~~countable~~ intersection of (a family of) G-stable closed sets is ~~a countable~~ the intersection of a countable sub-family. Consequently

there is a minimal (G-stable) closed subset E_0 of $\hat{A}$ such that $P_{E_0} = $ Identity. We claim that E_0 must be a single orbit. To see this let $\{\Omega_n \mid (\Omega_n \neq \phi)\ 1 \leq n < \infty\}$ be a countable base for (G-stable) open sets of E_0. Then $P_{\Omega_n} = $ Identity (by the minimality of E_0) for all n. It follows that $P_{\bigcap_{n=1}^{\infty}\Omega_n} = $ Identity. On the other hand $\bigcap_{n=1}^{\infty} \Omega_n = \phi$ if E_0 is not an orbit as the orbits of all points in E_0 are closed: given any $x \in E_0$, there is a n such that $\Omega_n \cap \text{Orbit}\, x_0 = \phi$. Thus E_0 is a single orbit. It follows then that M is a measure on G/I_v for some isotropy group I_v of a point $v \in \hat{A}$. It follows that σ is induced by a necessarily irreducible unitary representation π of I_v. If $\dim I_v < \dim G$, by induction hypothesis π is induced by a character χ on a subgroup H of I_v. Evidently then σ is induced by χ as well. Assume then $I_v = G$.

This means that the orbit of v reduces to the single point v itself. The element v is a character on A and it is easily seen that for all $a \in A$, $\sigma(a) = v(a)\cdot$ Identity. ~~Thus~~ (i.e. all of A acts as unitary scalars). Let A_0 be the identity component of kernel v. Then since $\dim A \geq 2$, $\dim A_0 \geq 1$. Evidently $A_0 \subset \ker \sigma = G''$, say. Thus $\dim G'' > 0$. If G' = identity component of G, σ yields a representation $\bar{\sigma}_1$ of G/G', a lower dimensional group to which induction hypothesis may be applied. This proves ~~Kirillov~~ the theorem.

We will now investigate how to obtain more precisely the subgroups and characters which yield irreducible representation on induction. Observe first that a character χ on a closed connected subgroup H of G, (determines and) is determined by a linear form $\lambda : \mathfrak{h} \longrightarrow \mathbb{R}$ of the Lie algebra $\mathfrak{h}$ of H in $\mathbb{R}$, λ being a Lie algebra homomorphism i.e. $\lambda([\mathfrak{h},\mathfrak{h}]) = 0$: λ is given by $\chi(\exp X) = \exp i\lambda(X)$ for all $X \in \mathfrak{h}$. Evidently λ is the restriction of a linear form $\tilde{\lambda}$ on $\mathfrak{g}$.

This leads us to make the following definition

<u>Definition</u>. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is <u>subordinate</u> to a linear form $\lambda \in \mathfrak{g}^*$ iff $\lambda([\mathfrak{h},\mathfrak{h}]) = 0$.

A linear form λ on $\mathfrak{g}^*$ thus defines a character on the Lie subgroup H corresponding to any Lie subalgebra $\mathfrak{h}$ subordinate to it. ~~Starting~~

An irreducible unitary representation, as we have seen above is induced by some character corresponding to a linear form λ on $\mathfrak{g}$ and a subgroup whose Lie algebra is subordinate to λ. ~~Kirillov gives a charac~~

It turns out that every $\lambda \in \mathfrak{g}^*$ determines an irreducible representation which is induced by the character determined by λ on a suitable subgroup H whose Lie algebra $\mathfrak{h}$ is subordinate to λ. More precisely one has

Theorem. Let $\lambda \in \mathfrak{g}^*$ and $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of maximal possible dimension subordinate to λ. ~~Then~~ Let H be the corresponding Lie subgroup and $\underline{\lambda}$ the character $\underline{\lambda}(\exp X) = \exp i\lambda(X)$, $X \in \mathfrak{h}$ on H. Then the representation σ_λ induced by $\underline{\lambda}$ is irreducible and its (unitary) equivalence class is independent of the choice of $\mathfrak{h}$. ((justifying our notation)) Moreover any irreducible representation of G is equivalent to σ_λ for some λ.

Proof Suppose first that ~~$\dim C \geq 2$. Then~~ the kernel of $\lambda|_{\mathfrak{z}}$ is non-zero. Let $\mathfrak{z}' = \{X \in \mathfrak{z} \mid \lambda(X) = 0\}$. If $\mathfrak{h}$ is subordinate to λ so is $(\mathfrak{h}+\mathfrak{z})$ and if $\mathfrak{h}$ is of maximal possible dimension, $\mathfrak{h}+\mathfrak{z} = \mathfrak{h}$ ie $\mathfrak{z} \subset \mathfrak{h}$. Thus we have $\exp \mathfrak{z}'\ (= C', \text{say}) \subset H \subset G$. The representation σ_λ factors through G/C' and is

induced by $\underline{\lambda}'$ the character $\underline{\lambda}$ considered as a character on H/C' through which it factors. Also the Lie algebra $\mathfrak{h}'$ of $H' = H/C'$ is necessarily a maximal dimensional subalgebra of $\mathfrak{g}' = \mathfrak{g}/\mathfrak{z}'$ subordinate to λ' the linear form determined by λ on $\mathfrak{g}'$. By an induction hypothesis we can assume that $\underline{\lambda}'$ induces an irreducible representation $\sigma_{\lambda'}$ of G/C'; but σ_λ is evidently equivalent to $\sigma_{\lambda'} \circ \pi$ where $\pi : G \to G'$ is the natural map. We see thus the first assertion ~~holds~~ of the theorem holds (for G) whenever it holds for groups of dimension less than G and ($\lambda|_{\mathfrak{z}}$ has a non zero kernel (in particular if $\dim C \geq 2$)). Assume then that $\dim C = 1$. and that λ is non-zero on $\mathfrak{z}$; we have seen before $\mathfrak{g} = \mathfrak{z}(Y) \oplus \mathbb{R}X$ with $[X,Y] = Z$ a generator of $\mathfrak{z}$. Let $\mathfrak{h}$ be a subalgebra of maximal dimension subordinate to λ in $\mathfrak{g}$. We claim that there is a subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$ with $\dim \mathfrak{h}' = \dim \mathfrak{h}$

and $\mathfrak{h}' \subset z(Y)$ and such that $\sigma_\lambda \sim \sigma'_\lambda$ where σ'_λ is the representation induced by the character $\underline{\lambda}'$ determined by λ on $H' = \exp \mathfrak{h}'$. To see this let $\mathfrak{a} = z(Y) \cap \mathfrak{h}$. Then either $\mathfrak{a} = \mathfrak{h}$ and then the assertion is obvious or $\mathfrak{a}$ has codimension one in $z(Y)$. If $z(Y)/\mathfrak{a}$ is of dimension 1, we see that $X = X' + E$ where $X' \in \mathfrak{h}$ and $E \in z(Y)$. Since $[X', Y] = [X, Y]$ we may replace X by X'. We assume thus (that) $X \in \mathfrak{h}$ (changing our X if necessary). It follows then that $Y \notin \mathfrak{h}$ (if $Y \in \mathfrak{h}$, $\lambda([X,Y]) = 0$, a contradiction since $\lambda(Z) \neq 0$ for $Z \in \mathfrak{c}$). Since $Y \notin \mathfrak{h}$, $Y \notin \mathfrak{a}$; moreover $[Y, \mathfrak{a}] = 0$ so that $\mathfrak{h}' = \mathfrak{a} + \mathbb{R}Y$ is subordinate to λ and evidently $\dim \mathfrak{h} = \dim \mathfrak{h}'$. Now $\mathbb{R}X \oplus \mathfrak{a} \oplus \mathbb{R}Y$ is a Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$. Let G' be the corresponding subgroup of G.

G contains both the groups $H (= \exp \mathfrak{h})$ and $H' (= \exp \mathfrak{h}')$. Let $\underline{\lambda}$ (resp $\underline{\lambda}'$) be the character on H (resp H') determined by λ. Let τ_λ (resp. $\tau_{\lambda'}$) denote the representation of G' induced by the character $\underline{\lambda}$ (resp $\underline{\lambda}'$) on H (resp H'). Evidently σ_λ (resp σ'_λ) is induced by the representation τ_λ (resp $\tau^*_{\lambda'}$) of G'. Thus it suffices to show that τ_λ and $\tau_{\lambda'}$ are equivalent. Let $\mathfrak{a}_0^* = \ker(\lambda|_{\mathfrak{a}})$. Then $\mathfrak{a}_0$ is evidently an ideal in $\mathfrak{a}$. Also $[Y, \mathfrak{a}_0] = 0$ since $\mathfrak{a}_0 \subset z(Y)$. Finally $z(Y)$ is an ideal in $\mathfrak{g}$ so that $[X, \mathfrak{a}_0] \subset \ker \lambda \cap \mathfrak{a} = \mathfrak{a}_0$. Thus $\mathfrak{a}_0$ is an ideal in $\mathfrak{g}'$ and λ factors through to a linear $\bar{\lambda}$ on $\mathfrak{g}'/\mathfrak{a}_0$; and $\mathfrak{g}'/\mathfrak{a}_0$ is the Heisenberg Lie algebra with $\bar{X}, \bar{Y}, \bar{Z}$, the images of X, Y, Z as a basis, $[\bar{Y}, \bar{X}] = \bar{Z}$, $\bar{Z}$ central in $\mathfrak{g}'/\mathfrak{a}_0$. The representations $\tau_\lambda, \tau_{\lambda'}$ of G' are of the form $\bar{\tau}_{\bar{\lambda}} \circ p$, $\bar{\tau}_{\bar{\lambda}'} \circ p$ respectively

where $p : G' \longrightarrow$ Heisenberg gp is the projection determined by $\mathfrak{g}' \longrightarrow \mathfrak{g}'/\mathfrak{a}_0$ and $\bar{\tau}_{\bar{\lambda}}$ (resp $\bar{\tau}_{\bar{\lambda}'}$) is the representation induced by the character on $\exp(\mathbb{R}\bar{X} \oplus \mathbb{R}\bar{Z})$ (resp. $\exp \mathbb{R}\bar{Y} \oplus \mathbb{R}\bar{Z}$) determined by $\underline{\lambda}$. These representations are both realised on $L^2(\mathbb{R})$ and the Fourier Transform sets up an equivalence of $\bar{\tau}_{\bar{\lambda}}$ and $\bar{\tau}_{\bar{\lambda}'}$. Thus $\tau_{\underline{\lambda}}$ and $\tau_{\underline{\lambda}'}$ are equivalent and hence τ_λ is equivalent to τ'_λ.
Also since any maximal dimensional subalgebra of $z(Y)$ subordinate to $\lambda|_{z(Y)}$, the representations of $Z(Y)$ ($=\exp z(Y)$) induced by the character determined by λ on the two ~~any such subalgebras~~ subgroups corresponding to any ~~any~~ two such subalgebra are — by induction hypothesis equivalent. Thus τ_λ is independent of the choice of the (maximal dimensional) subalgebra of $\mathfrak{g}$ subordinate to λ. It remains still for us to prove that σ_λ is irreducible. We may evidently assume that $\mathfrak{f}$ is chosen such that $Y \in \mathfrak{f} \subset z(Y)$.
(Let the) ~~The~~ representation space for the representation π of $Z(Y)$ induced by the character $\underline{\lambda}$ on $H (= \exp \mathfrak{f})$ be the Hilbert space $\mathcal{H}$. Then the representation σ_λ is the same as the one induced by π so that its representation space can be identified ~~is~~ the space of square integrable functions on $\mathbb{R} = \{\exp tX \mid t \in \mathbb{R}\}$ ($G = \{\exp tX \mid t \in \mathbb{R}\} \cdot Z(Y)$) with values in $\mathcal{H}$.
~~Further one checks easily that~~ Let $Y' = Y + \alpha Z$ be so chosen that $\lambda(Y') = 0$. Then it is easy to see that $\sigma_{\underline{\lambda}}(\exp tY')\, f(x) = e^{it\lambda([Y,Z])x} \cdot f(x)$.
It follows that if K is a bounded operator that commutes with all the $\sigma_{\underline{\lambda}}(g)$, $g \in G$, it commutes with the operators $\{$Multiplication by $e^{itx} \mid t \in \mathbb{R}\}$
It is not difficult to see that such an operator

on $L^2(\mathbb{R}, \mathcal{H})$ is necessarily of the form

$$(K \cdot f)(x) = K(x) f(x)$$

where $x \mapsto K(x)$ is a measurable function of $\mathbb{R}$ in $B(\mathcal{H})$ (= bdd operators on $\mathcal{H}$) such that $\|K(x)\|$ is in $L^\infty(\mathbb{R})$. Since ~~xxx~~ $\exp tX$, $t \in \mathbb{R}$ act as left translations on $L^2(\mathbb{R}, \mathcal{H})$, we conclude that K is translation invariant i.e. $K(x) = K$ for all $x \in \mathbb{R}$. That K commutes with $\sigma_{\underline{\lambda}}(g)$ for $g \in Z(Y)$ leads us to the condition that for ~~all~~ $g \in Z(Y)$ and $f \in L^2(\mathbb{R}, \mathcal{H})$

$$K\, \pi(x^{-1} g x) f(x) = \pi(x^{-1} g x) K f(x)$$

for almost all x. Varying f over a countable dense set in $L^2(\mathbb{R}, \mathcal{H})$ suitably and choosing x suitably one concludes that K commutes with $\pi(g)$ for all $g \in Z(Y)$; and by induction hypothesis π is irreducible. Thus K is ~~the identity~~ a scalar operator. This proves that the first assertion in the theorem is true.

To prove the second assertion we begin with the result proved earlier viz. that any irreducible representation σ of G is induced by a character on a closed connected subgroup H. It is clear that we may assume that the character is induced by a linear form $\lambda : \mathfrak{g} \to \mathbb{R}$ which on the Lie algebra $\mathfrak{h}$ of H is a Lie algebra homomorphism. We need only prove that $\dim \mathfrak{h}$ is maximal possible subject to the condition that ~~for all~~ all subalgebras considered be subordinate to λ. We prove this again by induction. If $\ker \lambda$ ~~xxx~~ has a non-zero intersection with $\mathfrak{c}$ an induction hypothesis applied to the group $G' = G/C'$, $C' = \exp \mathfrak{c}'$ yields the necessary result. Thus we have only to consider the case when $\dim \mathfrak{c} = 1$ and $\lambda(\mathfrak{c}) \neq 0$. Arguing as before one constructs a subalgebra $\mathfrak{h}'$

with $Y \in f' \subset z(Y)$, ~~such that~~ $(\dim f' = \dim f)$, $\lambda([f',f']) = 0$, and the representation induced by the character $\underline{\lambda'}$ of $H' = \exp f'$ is equivalent to σ. ~~Thus~~ If σ is irreducible, $\underline{\lambda'}$ induces an irreducible representation of $Z(Y)$. By induction hypothesis f' is a maximal dimensional subalgebra of $Z(Y)$ subordinate to λ. Suppose now f_0 is a maximal dimensional subalgebra of $\mathfrak{g}$ subordinate to λ. Then one has also a subalgebra of $z(Y)$ subordinate to λ of ~~maximal possible~~ the same dimension. This shows that f' and hence f^* is of maximal dimension subordinate to λ.

The next result determines when two ~~of~~ representations σ_λ and $\sigma_{\lambda'}$ for $\lambda, \lambda' \in \mathfrak{g}^*$ are equivalent.

<u>Theorem</u> $\sigma_\lambda \sim \sigma_{\lambda'}$ iff λ and λ' are in the same G orbit

<u>Proof</u>. If λ, λ' are in the same G-orbit, evidently σ_λ and $\sigma_{\lambda'}$ are equivalent. Assume then that ~~λ and λ' are in the same G orbit~~. σ_λ and $\sigma_{\lambda'}$ are equivalent. If the kernel of $\sigma_\lambda|_C$ $(= \ker \sigma_{\lambda'}|_C)$ has dimension > 0, passing to the quotient by the connected component of the kernel, the problem is reduced to that of groups of lower dimension. We may thus assume that $\dim C = 1$ and that $\sigma_\lambda(x) = \chi(x)\,\mathrm{Id} = \sigma_{\lambda'}(x)$ for all $x \in C$. This means that λ and λ' coincide on the centre. Further ~~also~~ Since $\dim C = 1$, we have the decomposition $\mathfrak{g} = z(Y) \oplus \mathbb{R}X$ with $[X,Y] =$ generating $\mathfrak{c}$ as described above. Further we can find a subalgebra f (f') of $\mathfrak{g}$ of maximal dimensions subordinate to λ (resp λ') such that $Y \in f \subset z(Y)$, (resp. $Y \in f' \subset z(Y)$). Conjugating Y by a suitable

element x (resp x') $= \exp tX$ (resp. $\exp t'X$) one can assume that $\lambda(Y)$ (resp $\lambda'(Y) = 0$). ~~Suppose Then the~~ Let $\mathcal{H}_\lambda$ (resp. $\mathcal{H}^*_{\lambda'}$) denote the representation* space of the representation τ_λ (resp. $\tau_{\lambda'}$) of $Z(Y)$ induced by the character ~~[illegible]~~ $\underline{\lambda}$ (resp $\underline{\lambda}'$) of ~~[illegible]~~ $H = \exp f$ (resp $H' = \exp f'$) Evidently σ_λ (resp $\sigma_{\lambda'}$) is induced by τ_λ (resp. $\tau_{\lambda'}$). The representation space of σ_λ (resp. $\sigma_{\lambda'}$) can then be identified with $L^2(\mathbb{R}, \mathcal{H}_\lambda)$ (resp. $L^2(\mathbb{R}, \mathcal{H}'_\lambda)$) using the semidirect product decomposition $G = \{\exp tX \mid t \in \mathbb{R}\} \cdot Z(Y)$, and identifying $\mathbb{R}$ with $\{\exp tX \mid t \in \mathbb{R}\}$. Since σ_λ is assumed equivalent $\sigma_{\lambda'}$, one has an operator $K : L^2(\mathbb{R}, \mathcal{H}_\lambda) \longrightarrow L^2(\mathbb{R}, \mathcal{H}_{\lambda'})$ such that for $g \in G$.

$$\sigma_{\lambda'}(g)\, K = K\, \sigma_\lambda(g)$$

Taking $g = \exp tY$, $t \in \mathbb{R}$ one finds that we have ~~[illegible]~~ $K(\exp itx \cdot f(x)) = \exp itx (Kf)(x)$ for all $t \in \mathbb{R}$ ie K commutes with the operations of multiplication by characters on the L^2 function spaces $\mathcal{H}$. follows that K is of the form

$$f(x) \longmapsto K(x) f(x)$$

where $K(x) : \mathcal{H}_\lambda \longrightarrow \mathcal{H}_{\lambda'}$ is a bounded operator such that $x \longmapsto \|K(x)\|$ is a L^∞ function on $\mathbb{R}$. Using now ~~that~~ the fact that $\sigma_{\lambda'}(g) K = K \sigma_\lambda(g)$ for all $g \in Z(Y)$ it is not difficult to see that for almost all x, $K(x)\tau_\lambda(g) = \tau_{\lambda'}(g) K(x)$. In particular τ_λ and $\tau_{\lambda'}$ are equivalent. Induction hypothesis applied to $Z(Y)$ now shows that $r(\lambda)$ and $r(\lambda')$, where $r : \mathfrak{g}^* \longrightarrow \mathfrak{z}(Y)^*$ is the restriction map, are conjugates under $Z(Y)$. In other words we may assume that λ and λ' coincide on $\mathfrak{z}(Y)$. Now $\exp tY(X) = X + t[X, Y]$ so that we can choose $t_0 \in \mathbb{R}$ such that ~~[illegible]~~

$\lambda'(\exp -t_0 Y(X)) = \lambda(X)$. Since $\exp t_0 Y\,(E) = E$ for all $E \in z(Y)$, $\mathrm{Ad}[\exp t_0 Y](\lambda') = \lambda$. This proves the theorem. The following lemma was used above and for completeness we give its proof.

<u>Lemma</u>. Let $\mathbb{H}, \mathbb{H}'$ be Hilbert spaces ~~and~~, G a locally compact group and $T : L^2(G,\mathbb{H}) \to L^2(G,\mathbb{H}')$ be a bounded operator such that $T\chi f = \chi T f$ for all characters $\chi \in \hat{G}$. Then there is a bounded measurable function $\tilde{T} : G \to \text{Ⓑ} \{\text{Bounded operators of } \mathbb{H} \text{ in } \mathbb{H}'\}$ i.e. $x \mapsto \hat{T}(x)$ is measurable and $\|\tilde{T}(x)\| < M$ for all $x \in G$ such that $(Tf)(x) = \tilde{T}(x).\, f(x)$.

<u>Proof</u>. Let $K \subset G$ be any compact set and let φ a continuous function with support in K. Let T_φ be the operator $M_\varphi T M_\varphi$ where M_φ is multiplication by φ. Then $M_\varphi T M_\varphi$ commutes with M_χ, $\chi \in \hat{G}$. Now by the Weierstrass approximation theorem any continuous function f on G can be approximated uniformly on Support K by linear combinations of characters. Using this one sees easily that $T_\varphi M_\psi = M_\psi T_\varphi$ for all ψ continuous with compact support. ~~Let~~ For a compact set K, let 1_K be the characteristic function of K and let $f_K : G \to \text{Ⓑ}$ be defined by $f_K(g)v = T(1_K \cdot v)(g)$; then $f_K \in L^\infty(G,\text{Ⓑ})$. Suppose that $L \supset K$ is any compact set and ψ ~~has sup~~ is a continuous function with support in K, we have for $v \in \mathbb{H}$,

$$\psi \cdot T_\varphi(1_K v) = T_\varphi(\psi \cdot 1_K v) = T_\varphi(\psi\, 1_L v) = \psi T_\varphi(1_L v)$$

Thus $T_\varphi(1_K v) = T_\varphi(1_L v)$ almost everywhere in K and all v. It follows that T_φ is of the form $M_{\xi(\varphi)}$ for some $\xi(\varphi) : G \to \text{Ⓑ}$ in $L^\infty(G,\text{Ⓑ})$, where for $\theta \in L^\infty(G,\text{Ⓑ})$, $M_\theta f(x) = \theta(x) f(x)$, $f \in L^2(G,\mathbb{H})$.

It is not difficult to see that $\varphi T \varphi(1_K v) = \psi T \psi(1_K v)$ for $v \in \mathbb{H}$ and $\varphi, \psi \in C_c(G)$ with $\varphi(K) = \psi(K) = 1$.

It follows that the $\xi(\varphi)$ enable one to define a bounded measurable function $\tilde{T} : G \to B$ having the requisite properties