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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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LECTURE III
ELEMENTARY STRUCTURE OF LIE ALGEBRAS

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These are preliminary lecture notes, intended only for distribution to participants.

Trace of c_ϕ: $\quad \operatorname{tr} c_\phi = \sum \operatorname{tr}(\phi(x_i)\,\phi(y_i)) = \sum \beta(x_i, y_i) =$

$= \dim L$

If ϕ is irreducible, then c_ϕ is a scalar: $c_\phi = \frac{\dim L}{\dim V} \mathbb{1}_V$

In general, c_ϕ ~~does~~ does not depend on the basis ~~[illegible]~~.

$\forall T \in \operatorname{Aut} V$, let $x_i' = T x_i$, $\quad y_j' = (T^*)^{-1} y_j$; then

$$\sum_i \phi(x_i')\,\phi(y_i') = \sum_{i,j,\ell} t_{ij}\, t^{-1}_{\ell i}\, \phi(x_j)\,\phi(y_\ell) =$$

$$= \sum_{j,\ell} (T^{-1}T)_{\ell j}\, \phi(x_j)\,\phi(y_\ell) = \sum_{j,\ell} \delta_{j\ell}\, \phi(x_j)\,\phi(y_\ell) = c_\phi$$

Exercise: compute for sl_2, sl_3 (Ex. 6.1).

If ϕ is **not** **faithful**, then $L = \bigoplus_{i=1}^{m} L_i$, and

$\operatorname{Ker} \phi = \bigoplus_{i=k+1}^{m} L_i$, say. Then ~~$\phi$ let~~

$L/\ker\phi \simeq \bigoplus_{i=1}^{k} L_i = L'$. $\quad \phi|_{L'}$ is faithful.

Define c_ϕ as $c_{\phi|_{L'}}$. Then c_ϕ commutes with $\phi(L') = \phi(L)$.

Complete reducibility of finite dimensional representations

Lemma. $\phi: L \to gl(V)$ representation of L semisimple

$\Rightarrow \phi(L) \subseteq sl(V)$ (in particular, if $\dim V = 1$ then $\phi = 0$).

Proof. L semisimple $\Rightarrow L = [LL] \Rightarrow \phi(L) = [\phi(L)\,\phi(L)]$

$\subseteq [gl(V)\ gl(V)]$ ... (Exercise)

is solvable. Thus, since ϕ is an isomorphism, Rad β is a solvable ideal in L. Since L is semisimple, Rad $\beta = 0$.

In other words: β is non-degenerate

Thus we can define dual bases. $\{x_1, \dots, x_n\}$ basis in L

$\longrightarrow \{y_1, \dots, y_n\}$ dual basis (w.r.t. β) means $\beta(x_i, y_j) = \delta_{ij}$

$x \in L \qquad [x\, x_i] = \sum_{j=1}^{n} a_{ij}\, x_j \qquad [x\, y_i] = \sum b_{ij}\, y_j$

What is the relation between the $\{a_{ij}\}$ and $\{b_{ij}\}$?

$$a_{ik} = \sum_j a_{ij}\,\delta_{jk} = \sum a_{ij}\,\beta(x_j, y_k) = \beta([x x_i], y_k) =$$

$$= -\beta([x_i\, x], y_k) = -\beta(x_i, [x y_k]) = -\sum_j b_{kj}\,\beta(x_i, y_j)$$

β is associative!

$= -b_{ki}$

Now define the Casimir operator (w.r.t. the given basis).

$$c_\phi(\beta) = \sum_i \phi(x_i)\,\phi(y_i) \qquad \text{(dual bases)}$$

Since (exercise) $\quad [x, yz] = [x,y]z + y[x,z]$

[that is, $\operatorname{ad} x$ is a derivation of the associative algebra $\operatorname{End} V$ (ad x defined by the commutator action)]

one has, $\forall x \in L$:

$$[\phi(x), c_\phi] = \sum [\phi(x), \phi(x_i)]\,\phi(y_i) + \sum \phi(x_i)\,[\phi(x), \phi(y_i)] =$$

$$= \sum_{i,j} a_{ij}\,\phi(x_j)\,\phi(y_i) + \sum_{i,j} b_{ij}\,\phi(x_i)\,\phi(y_j) = 0$$

That is, c_ϕ commutes with $\phi(L)$.

Standard Constructions of L-modules.

1) ~~Contragredient~~ Dual L-module. V L-module, V^* dual vect. sp.

$\forall f \in V^*$, $v \in V$, $x \in L$ define $(x.f)(v) = -f(x.v)$

Then V^* is a L-module

2) Tensor product. V, W L-modules. On $V \otimes W$ define $x.(v \otimes w) = x.v \otimes w + v \otimes x.w$

(~~derivation~~ ("differentiation of the group action on tensors" ...).

3) $V \otimes V^* \longrightarrow \operatorname{End} V : \; f \otimes v \longrightarrow \boxed{w \to f(w)\, v}$

This extends from generators to the whole of $V \otimes V^*$. Taking into account a basis of V and the 'dual basis' of V^* we see that $V \otimes V^* \xrightarrow{\text{onto}} \operatorname{End} V$. By checking dimensions, $V \otimes V^* \simeq \operatorname{End} V$ (Exercise)

Now suppose V is a L-module. Then $V \otimes V^*$ is a L-module, so $\operatorname{End} V$ is a L-module!

Exercise: check that the action is $(x.T)(v) =$
$= x.(Tv) - T(x.v)$

4) More generally, ~~$V \otimes W^*$~~ $\simeq W \otimes V^* \simeq \operatorname{Hom}(V, W)$ and if V, W are L-modules, so is $\operatorname{Hom}(V,W)$ by the rule $(x.T)(v) = x.Tv - T(x.v)$ (Exercise)

Casimir element of a representation.

L semisimple. Assume first that $\phi : L \to gl(V)$ is faithful (i.e., 1-1).

Let $\beta(x,y) = \operatorname{tr}(\phi(x)\,\phi(y))$ $= \kappa$ if $\phi = ad$. As usual β is associative

hence Rad β is an ideal. By Cartan's criterion ...

3) f.d. Representations of semisimple algebras and sl_2

Representation of L: Lie homomorphism $\phi : L \to$ ~~...~~ $gl(V)$

Equivalently: ϕ repr. of L in V $\Longleftrightarrow$ V L-module

that is, $\exists$ bilinear map $L \times V \to V$
$(x, v) \to x.v$

which respects brackets: $\underbrace{[x,y]}_{\text{bracket in } L}.v = \underbrace{x.y.v - y.x.v}_{\text{commutator in } gl(V)}$

$x.v = \phi(x)\, v$

Assume $\dim V < \infty$.

Homomorphism of L-modules V, W: $\phi : V \to W$ commuting with the action of L, i.e., $\phi(x.v) = x.\phi(v)$

$$\begin{tikzcd} L \times V \arrow[r] \arrow[d, "\phi"] & V \arrow[d, "\phi"] \\ L \times W \arrow[r] & W \end{tikzcd}$$

Isomorphism of L-modules $\rightsquigarrow$ equivalent representations

V, L-module irreducible if it has no nontrivial submodules

V --- completely reducible if direct sum of irreducibles

(Exercise 6.2) $\rightsquigarrow$ $\forall$ submodule $W \subseteq V$, $\exists$ complementary submodule $\tilde{W} \leq V$ $(W \oplus \tilde{W} = V)$.

Direct sum of ~~sub~~ L-modules: $x(v, v') = (xv, xv')$

Schur's lemma: $\phi : L \to gl(V)$ irreducible $\Longrightarrow$ $\forall T \in \operatorname{End} V$: $[T, \phi(x)] = 0$ $\forall x$ is a scalar

Theorem (Weyl). All $\phi : L \to gl(V)$ finite dimensional repns. of L semisimple are completely reducible.

(1) Proof. 1) Assume first that $\exists$ L-submodule $W \subset V$, codim $W = 1$.

$W \subset V$ codim. 1

By the lemma, $L : V/W \to 0$. Thus we have an Exact sequence of L-modules $0 \to W \to V \to F \to 0$

(image contained in Ker $\to$: arrows are module-homomorphisms).

reduction to W irreducible: Now proceed by induction on $\dim W$. Let $W' \subsetneq W$ nonzero submodule: $L.W' \subseteq W'$. Then $L . V/W' / W/W' \to 0$,

and we have another exact sequence $0 \to \frac{W}{W'} \to \frac{V}{W'} \to F \to 0$

By induction, this splits: $\exists$ 1-dim. submodule $\tilde{W}/W'$ of V/W' s.t. $W/W' \oplus \tilde{W}/W' = V/W'$. By lifting back, we get a third exact sequence $0 \to W' \to \tilde{W} \to F \to 0$

(indeed, $\tilde{W}/W' \simeq F$, and $L . \tilde{W}/W' = 0$ by the lemma)

However, $\dim \tilde{W}$ (new') $< \dim W$. Thus, again by induction, also this sequence splits: $\exists$ 1-dim. submodule $X \leq \tilde{W}$ s.t. $\tilde{W} = W' \oplus X$. Now $X \leq \tilde{W}$ and $\tilde{W} \cap W \leq W'$ (since $W/W' \oplus \tilde{W}/W' = V/W'$). Thus $X \cap W \leq W'$.

But X is disjoint from W', since $\tilde{W} = W' \oplus X$.

Thus $V = W \oplus X$, since dimensions add up to $\dim V$.

Therefore we can assume W irreducible.

required complement is given by Ker c

Also assume w.l.o.g. ϕ faithful, otherwise $\phi = \phi|_{\ker\phi^{\perp}} \oplus 0|_{\ker\phi}$

c Casimir element of ϕ. $[c, \phi(L)] = 0 \Rightarrow$

$c\,\phi(x.v) = x.(cv) \Rightarrow c$ L-module endomorphism of V. Thus c preserves the L-invariant subspaces: $c(W) \subseteq W$; and Ker c is a L-submodule of V;

$c \neq 0$ polynomial in elements of $\phi(L)$ [illegible]

But $L . V/W \to 0 \Rightarrow c(V/W) = 0$

meaningful since $c(W) \subseteq W$

Hence $\operatorname{tr} c|_{V/W} = 0$. But W irreducible $\Rightarrow c|_W = \lambda 1_W$

Since $\operatorname{tr}_V c \neq 0$, one must have $\lambda \neq 0$.

$$c = \left(\begin{array}{c|c} \lambda 1_W & /// \\ \hline 0 & \operatorname{tr}_{V/W} = 0 \end{array}\right) \Big\} W$$

1-dimensional, as V/W

Thus $U = \ker c$ has codim. 1 — a L-module!

But $U \cap W = 0$ by this construction, since $c|_W = \lambda 1_W$.

Hence $V = W \oplus U$, and we are done.

general case: 2) general case $W \leq V$ any submodule. $0 \to W \to V \to V/W \to 0$

reduction to previous case by considering Hom(V,W)

Hom(V, W) : linear maps $f : V \to W$, is a L-module (action $x.f(w) = x.f(w) - f(x.w)$)

subspace of Hom(V,W): $Q = \{ f : f|_W \text{ is scalar} \}$.

Q is an L-submodule: $\forall$ $f \in Q$, $f|_W = a 1_W$

$\Rightarrow (x.f)(w) = x.f(w) - f(x.w) = a(x.w) - a(x.w) = 0$.

Let $R \leq Q$, subspace given by $R = \{ f \in Q : f|_W = 0 \}$.

Then: R is an L-submodule, $L.Q \leq R$

But $\dim Q/R = \dim_F F = 1$. Thus we can apply part (1) of the proof.

We get $Q = R \oplus T$, $T = \{\lambda f\}$; we can assume $f|_W = 1_W$. Now L kills T by the lemma: hence

$\forall x \in L$, $0 = (x.f)(w) = x.(f(w)) - f(x.w) \Rightarrow$

That is, f is an L-homomorphism (commutes with the action [illegible]

$\operatorname{Ker} f \cap W = 0$, because $f = 1$ on W. Thus
$V = W \oplus \operatorname{Ker} f$.

Application: preservation of Jordan decomposition

Theorem. $L \subset \mathfrak{gl}(V)$ linear semisimple Lie algebra, at $\dim < \infty$. Then L contains the semisimple and nilpotent parts (in $\mathfrak{gl}(V)$) of all its elements: in particular, by uniqueness of the respective Jordan decompositions, the two decompositions coincide (abstract and usual). $\left(\begin{array}{l} \operatorname{ad} x_n \text{ nilpotent in } \operatorname{ad} L \iff x_n \text{ nilpotent in } L \subseteq \mathfrak{gl}(V) \\ \text{because ad is an isomorphism} \end{array} \right)$
Moreover, if ϕ is a f.d. repres. and $x = s + n$, then $\phi(x) = \phi(s) + \phi(n)$ is the usual Jordan decomposition of $\phi(x)$.

Proof : in the problems session.

Exercises: 6.5 (reductive algebras), 6.6 (any two nondegenerate associative symmetric bilinear forms on a simple algebra are proportional), 6.7

Proof of the theorem. $\operatorname{ad} x : L \to L \Rightarrow \operatorname{ad} x_s, \operatorname{ad} x_n : L \to L$.
$\Rightarrow x_s, x_n \in \underbrace{N_{\mathfrak{gl}(V)}(L)}_{= N} \rhd L$. In general $N \neq L$.
We want to show that $x_s, x_n \in L$.
Let W be any L-submodule of V, and $L_W = \{ y \in \mathfrak{gl}(V) : y : W \to W$ and $\operatorname{tr} y|_W = 0 \}$. $L \subset L_W$ $\forall W$, since W L-module and $L = [L L] \Rightarrow L \subset \mathfrak{sl}(V) \Rightarrow$ trace 0. $L_0 = \left(\bigcap_W L_W\right) \cap N$
$N \supseteq L_0 \rhd L$, $\forall x \in L$, $x_s, x_n \in L_W$ $\forall W$, ~~because~~
$x : W \to W \Rightarrow \begin{array}{c} x_s \\ x_n \end{array} : W \to W$. Hence $x_s, x_n \in L_0$.
Claim : $L = L_0$. If not, $L_0 = L \oplus M$. But $[L, L_0] \subseteq L$ since $L_0 \subseteq N$. Hence $[L, M] \subseteq L$. But $[L, M] \subseteq M$ since M is an L-module. Thus

$[L, M] \subseteq L \cap M = 0$. Then, $\forall y \in M$, $[L, y] = 0$. If W is an irreducible L-module, then, by Schur's Lemma, $y|_W = a \mathbb{1}_W$. But $\operatorname{tr} y|_W = 0$ since $y \in M \subseteq L_0$. Thus $y|_W = 0$. Now, by Weyl's thm, $V = \bigoplus W_i$, W_i irreducible L-modules, and $y \in M \Rightarrow y|_{W_i} = 0$
$\Rightarrow y = 0 \Rightarrow M = 0 \Rightarrow L_0 = L$.

Final part: L, as vector space, is spanned by the eigenvectors of $\operatorname{ad} s$, since $\operatorname{ad} s$ is diagonalizable (acting on L). Thus $\phi(L)$ is ~~...~~ spanned by the eigenvectors of $\operatorname{ad}_{\phi(L)} \phi(s) \Rightarrow \operatorname{ad}_{\phi(L)} \phi(s)$ is semisimple. And so on. Now $\phi(x) = \phi(s) + \phi(n)$ is the abstract Jordan decomposition of $\phi(x)$. Apply the first part of the Theorem.