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34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392 - I

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STEIN PROGRAM - FUNCTIONAL CALCULUS
HOMOGENEOUS GROUPS

A. HULANICKI
Department of Mathematics
University of Wroclaw
Plac Uniwersytecki, 1
50137 Wroclaw
Poland

STEIN PROGRAM FUNCTIONAL CALCULUS

Let (X,m) be a measure space and let $E(\lambda)$ be a positive (i.e. $E(\lambda)=0$ for $\lambda\leq 0$) spectral resolution in $L^2(m)$. For a bounded function K on $\underline{R}^+$ we write

$$T_K f = \int_0^\infty K(\lambda)\,dE(\lambda)f \quad, f\in L^2(m).$$

T_K is a bounded operator on $L^2(m)$.

One of the problems studied by harmonic analysts in variuos specific situations is to find conditions on the function K which imply that the operator T_K is bounded on other $L^p(m)$, $p\neq 2$. At this generality however one cannot expect any sensible answer - one has to know more about the spectral measure, as various easy examples show.

In his book (Stein 1970) Stein proposed to study the following stituation.

Let $\{T_t\}_{t>0}$ be a strongly continuous semi-group of operators defined on all $L^p(m)$, $1\leq p<\infty$ such that (a) T_t is a contraction on every $L^p(m)$, $1\leq p<\infty$ for all $t>0$, (b) T_t is self-adjoint on $L^2(m)$ for all $t>0$. In other words, the assumption on the spectral resolution $E(\lambda)$ is that

$$T_t f = \int_0^\infty e^{-t\lambda}\,dE(\lambda)f$$

is a contraction on all $L^p(m)$, $1\leq p<\infty$, for all $t>0$.

The main theorem of his book says that under this assumption if K is the Laplace transform of a bounded function, i.e., if

$$K(\lambda) = \lambda\int_0^\infty e^{-\lambda t}\,\Psi(t)\,dt \quad ,\lambda\in\underline{R}^+ ,$$

where Ψ is a bounded function on $\underline{R}^+$, then the operator T_K is bounded on all $L^p(m)$ for $1<p<\infty$,(but not on $L^1(m)$). We also note that under this condition K is real analytic and

$$(1)\qquad \lambda^n\left|\frac{d^n}{d\lambda^n}K(\lambda)\right| \leq C \quad \text{for all } n=0,1,\dots .$$

As E.M.Stein point out in his book, many important semi-groups satisfy conditions (a) and (b), specifically such are symmetric convolution semi-groups of probability measures on Lie groups and many other e.g. the ones generated by Schrodinger operators with non-negative potential (cf.Barry Simon).

Here we are going to consider some aspects of Stein program related to semi-groups of the form π_{μ_t}, where $\{\mu_t\}_{t>0}$ is a convolution semi-group on a nilpotent Lie group and π an irreducible unitary representation of the group which defined on $L^2(\mathbb{R}^n)$ turns out to be isometric on all $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

In order to be able to pass from the group to the representation we need a theorem due essentially to C.Herz (cf. Herz and Dlugosz).

For a locally compact group G let m be the right invariant Haar measure and let

$$\rho : G \longrightarrow B(L^2(m))$$

be the right-regular representation :

$$\rho_x f(y) = f(xy).$$

Of course, ρ_x is an isometry on every $L^p(m)$, $1 \leq p < \infty$, and $x \longrightarrow \rho_x$ is strongly continuous.

The group G is called <u>amenable</u> if for every unitary representation π of G we have

(1) $$\|\pi_f\| \leq \|\rho_f\| \quad \text{for every } f \text{ in } L^1(G),$$

where, as before,

$$\pi_f = \int \pi_x f(x)\,dm(x) .$$

Compact and solvable groups are amenable. Also if N is a normal subgroup of G and N and G/N are amenable, so is G.

By Cv^p we denote the Banach algebra of bounded operators on $L^p(m)$ which commute with left translations (i.e. if $\lambda_x f(y) = f(x^{-1}y)$, then $\lambda_x T = T\lambda_x$ for T in Cv^p). Such operators we call <u>convolutors</u>. Of course for $f \in L^1(m)$, $\rho_f \in Cv^p$.

<u>THEOREM (C.Herz)</u>. Let π be an isometric representation of an amenable group G on $L^p(X)$, where X is a measure space and p is a number $1 < p < \infty$. Then the contraction

$$L^1(m) \ni f \longrightarrow \pi_f \in B(L^p(X))$$

has a unique extension to a contraction

(2) $$Cv^p \ni T \longrightarrow \pi_T \in B(L^p(X)) .$$

<u>COROLLARY</u>. Let π be a representation of an amenable group on $L^p(X)$, where X is a measure space, such that for $1 \leq p < \infty$ π_x is an isometry . Let E be a spectral measure on $L^2(m)$ such that the projections $E(M)$ commute with left translations. Then there exists a unique spectral measure E^{π} on $L^2(X)$ such that $E^{\pi}(M) = \pi_{E(M)}$, as defined by (2) (for $p = 2$) . Moreover, if for a function K on $\mathbb{R}$

$$T_K f = \int_{-\infty}^{\infty} K(\lambda)\,dE(\lambda) f$$

is a bounded operator on $L^p(m)$ (i.e. $T_K \in Cv^p$), then

$$\pi_{T_K} \phi = \int_{-\infty}^{\infty} K(\lambda)\,dE^{\pi}(\lambda)\phi$$

is a bounded operator on $L^p(X)$.

In particular, if $T_K = f * k$, where $k \in L^1(m)$, then $\pi_{T_K} = \int \pi_x k(x)\,dx$.

<u>Example</u>. Let G be a locally compact group and let H be a subgroup of G such that $G/H = X$ has a G-invariant measure. Let χ be a multiplicative character of H, ie. $\chi(xy) = \chi(x)\chi(y)$, $x, y \in H$ and $|\chi(x)| = 1$. The induced representation $\pi = \mathrm{Ind}_H^G \chi$ is of the form

(3) $$\pi_x \phi(u) = a(x,u)\phi(x^{-1}u) \quad , x \in G \;, u \in X ,$$

where $a(x,u)$ is a Borel function on G X such that $|a(x,u)| = 1$ and satisfies the condition $a(xy,u) = a(x,u)a(y,x^{-1}u)$ which implies that $\pi_x\pi_y = \pi_{xy}$. Then, of course, π_x is an isometry on every $L^p(X)$, $1 \leq p < \infty$. From the Lebesgue bounded convergence theorem it follows that π is a strongly continuous representation on $L^p(X)$,if $\lim_{x \to e} a(x,u) = 1$ for almost all u.

In particular, if G is a nilpotent simply connected Lie group and π is an irreducible unitary representation of G, then, by Kirillov theory, π is of the form (3), where $a(x,u)$ is continuous and $a(e,u) = 1$. Consequently, π is a strongly continuous isometric representation on all $L^p(G/H)$, for all $1 \leq p < \infty$.

Suppose now we are given a semi-group of probability measures $\{\mu_t\}_{t>0}$ on a nilpotent Lie group G such that $\mu_t = \tilde{\mu}_t$. Let A be the infinitesimal generator of $\{\mu_t\}_{t>0}$. As we know

$$f * \mu_t = \int_0^{\infty} e^{-\lambda t}\,dE(\lambda) f , \quad f \in L^2(m),$$

where $E(\lambda)$ is the spectral resolution of the self-adjoint operator A.

QUESTION. For which functions K on $\mathbb{R}^+$ we have $T_K f = f * k$, where $k \in L^1(m)$, where $T_K f = \int_0 K(\lambda) dE(\lambda) f$?

We note that although the assumptions on the spectral measure E made by Stein are here satisfied, we require that the operator T_K be bounded on all $L^p(m)$, including $p = 1$, and thus our aim is slightly more demanding. As we shall see later, for some semi-groups we shall show that condition (1) with only $n = 0,1,\ldots, N$ for some fixed N is sufficient for the operator T_K to be bounded on $L^p(m)$, $1 < p < \infty$.

Now we are going to specify a number of conditions on a semi-group $\{\mu_t\}_{t>0}$ of probability measures which on one hand side are satisified by the most important semi-groups from the point of view of applications to Schrödinger operators, and on the other hand provide a large class of fucntions K on $\mathbb{R}^+$ for which T_K are given by convolution on the right by L^1 functions.

ASSUMPTIONS. Let $\{\mu_t\}_{t>0}$ be a semi-group of probability measures on a nilpotent Lie group and let A be the infinitesimal generator of $\{\mu_t\}_{t>0}$.

We assume

(i) $\mu_t = \mu_t^\sim$ for all $t>0$

(4) (ii) $Af = f * F^\sim$, $f \in C_c^\infty(G)$, where F has compact support,

(iii) For some $\lambda_0 \geq 1$ and $S>0$ $(\lambda_0 - A)^{-S} f = f * k_{\lambda_0}^{*S}$ with $k_{\lambda_0}^{*S} \in L^2(m)$.

THEOREM (Functional calculus). Let ω be a polynomial weight on a nilpotent Lie group G and let $\{\mu_t\}_{t\geq 0}$ satisfy assumptions (4). Then there exists a number N such that if K $C^N(\mathbb{R}^+)$ and

$$(*) \qquad \lim_{\lambda\to\infty} (1+\lambda)^{N+S} \frac{d^j}{d\lambda^j} K(\lambda) = 0 \quad \text{for } j = 0,\ldots N ,$$

then the operator

$$T_K f = \int_0^\infty K(\lambda) dE(\lambda) f$$

is of the form

$$T_K f = f * k , \text{ where } k \in L^1_\omega .$$

Proof. By proposition 6, section 2, for $\lambda > 0$ we have

$$(5) \qquad k_\lambda = \int_0^\infty e^{-\lambda t} \mu_t \, dt ,$$

Consequently $k_\lambda \in M(G)$.

Let μ_γ be as in proposition 10, section 2!

Lemma 1 If for a submultiplicative function ϕ on G

$$(6) \qquad \langle \mu_\gamma , \phi \rangle < \infty ,$$

then there exists a C such that for all $\lambda > C$ we have $\langle k_\lambda, \phi \rangle < \infty$.

Proof of the lemma. By proposition 1, section 3, (6) implies that there is a constatnt C such that

$$\langle \mu_t , \phi \rangle \leq C e^{tC} \quad \text{for all } t>0.$$

Hence, for $\lambda > C$,

$$\langle k_\lambda , \phi \rangle = \int_0^\infty e^{-\lambda t} \langle \mu_t , \phi \rangle dt \leq C \int_0^\infty e^{-(\lambda - C)t} dt = C(\lambda - C)^{-1}.$$

Corollary. (4) imply that for every submultiplicative function ϕ there is a $C>0$ such that for $\lambda > C$ $k_\lambda^{*S} = f$ belongs to $L^2(m) \cap L^1_\phi$ and $f = f^\sim$.

Lemma 2. Let G be a connected nilpotent Lie group. There exists a number Q such that for every symmetric compact neighbourhood of e in G the Haar measure $m(U^n)$ of U^n satisfies

$$m(U^n) \leq C_U n^Q \quad \text{for all } n = 1,2,\ldots, \quad C = C_U .$$

The proof of lemma 2 is not difficult but uses a little more of the structure of nilpotent groups. (cf.e.g. Dixmier)

As before, we fix a symmetric compact neighbourhood U of e and we write

$$\tau(x) = \min\{n : x \in U^n\}.$$

For a function $f \in L^1_\phi$ we define

$$e(f) = \sum_{k=1}^\infty \frac{i^k f^{*k}}{k!} .$$

Since ϕ is submultiplicative, we have

$$(7) \qquad \| e(f) \|_{L^1_\phi} \leq \exp \| f \|_{L^1_\phi} .$$

Lemma 3. If $f = f^\sim \in L^2 \cap L^1_\phi$, where $\phi = e^{\alpha\tau}$ for some $\alpha > 0$ and $\omega = (1+\tau)^\beta$, then

$$\| e(nf) \|_{L^1_\omega} \leq C_f |n|^q \quad \text{for all } n \in \mathbb{Z} ,$$

where

$$q = 1 + \beta + Q/2 .$$

<u>Proof of lemma 3.</u> For a positive integer m we have

$$\|e(nf)\|_{L^1_\omega} = \int_{U^m} |e(nf)|\,\omega\, dm + \int_{G\setminus U^m} |e(nf)|\, e^{-\alpha\tau}\omega e^{\alpha\tau}\, dm$$

$$= I_1 + I_2 .$$

Since $\omega(x) \leq (1+m)^{\beta}$ for $x \in U^m$, by Schwarts inequality, the integral I_1 is estimated by

$$I_1 \leq m(U)^{m/2}(1+m)^{\beta}\|e(nf)\|_{L^2} \leq C\, m^{Q/2+\beta}\|e(nf)\|_{L^2} .$$

But

$$\|e(nf)\|_{L^2} \leq \|T_n f\|_{L^2} \leq \|T_n\|\,\|f\|_{L^2} \quad , \text{where}$$

$$T_n\varphi = \varphi * \sum_{k=1}^{\infty} \frac{(inf)^{*k-1}}{k!} .$$

Since $f = f^{\sim}$, the operator T_n is normal and, by spectral theorem,

$$\|T_n\| \leq \sup\left\{\left|\frac{e^{in\lambda}-1}{\lambda}\right| : \lambda \in \mathbb{R}\right\} = n.$$

Hence

$$I_1 \leq C\, m^{\beta+Q/2}\, n\, \|f\|_{L^2} .$$

To estimate I_2 we note that for $x \in G\setminus U^m$ we have $\tau(x) \geq m$, whence $\omega(x)e^{-\tau(x)} \leq C\exp(-\alpha m/2)$. Thus, by (7)

$$I_2 \leq C \exp(n\|f\|_{L^1_\phi} - \alpha m/2) .$$

Hence, putting m = integral part$(\, 2n\,\alpha^{-1}\|f\|^{-1}_{L^1_\phi}\,) + 1$ we complete the proof of lemma 3.

Now we consider the operator

$$S_n\phi = \phi * e(nk^{*S}_{\lambda_0}) = \int_0^{\infty} \exp(in(\lambda_0+\lambda)^{-S}) - 1\; dE(\lambda)\phi .$$

By lemma 3 the operator S_n is a convolution operator by a function in L^1_ω with L^1_ω-norm $\leq C|n|^{1+\beta+Q/2}$. Consequently the norm of the operator S_n on every $L^p(m)$, $1 \leq p < \infty$, is $\leq C|n|^{1+Q/2}$. Let F be a function on $(-\pi, \pi]$ with $F(t) = 0$ for $t \leq 0$ which extends to a periodic function in $C^k(\mathbb{R})$ for some $k \geq 2$. Then

$$F(t) = \sum_n \hat{F}(n)e^{int} = \sum_n \hat{F}(n)(e^{int} - 1) ,$$

since $F(0) = \sum_n \hat{F}(n)$ and $F(0) = 0$. Moreover, $|\hat{F}(n)| \leq C|n|^{-k}$.

<u>Lemma 4</u>. If $N = 3 + \beta + Q/2$ the theorem holds.

<u>Proof of lemma 4</u>. Suppose a function K satisfies the assumptions of the theorem with $N = 3 + \beta + Q/2$. We put

$$F(\lambda) = \begin{cases} K(\lambda^{-1/S} - \lambda_0) & \text{if} \quad 0 \leq \lambda \leq 1 \\ 0 & \text{if} \quad -\pi \leq \lambda \leq 0 \end{cases}$$

and we extend F to a periodic C^N function on $\mathbb{R}$. We note that condition (*) is sufficient for

$$\lim_{\lambda\to\infty} \frac{d^j}{d\lambda^j} F(\lambda) = 0 \quad \text{for all } j = 0, 1, \ldots, N.$$

We have

$$F((\lambda_0+\lambda)^{-S}) = K(\lambda).$$

Moreover, since the operator norm of S_n on every $L^p(m)$, $1 \leq p < \infty$, is at most $C|n|^{1+\beta+Q/2}$ and, since $F \in C^N$, $|\hat{F}(n)| \leq C|n|^{-N}$, $n \neq 0$, we see that the operator

$$\sum_n \hat{F}(n) S_n$$

is bounded on every $L^p(m)$ and, as a matter of fact, it is given by a convolution by a function in L^1_ω .

On the other hand,

$$\sum_n \hat{F}(n)S_n = \int_0^{\infty} \sum_n \hat{F}(n)\left[\exp(in(\lambda_0+\lambda)^{-S}) - 1\right] dE(\lambda) = \int_0^{\infty} K(\lambda)\, dE(\lambda) ,$$

which completes the proof of the theorem.

The following theorem which is a consequence of a theorem by L.Hörmander (cf. Hörmander) exhibits a class of semi-groups $\{\mu_t\}_{t>0}$ which satisfy conditions (i) – (iii) of (4).

THEOREM Let $X_1,\dots,X_k$ be elements of the Lie algebra $\mathfrak{g}$ of G such that the smallest Lie subalgebra of $\mathfrak{g}$ which contains $X_1,\dots,X_k$ is $\mathfrak{g}$. Then

(8) $$-L = X_1^2 + \dots + X_k^2$$

is the infinitesimal generator of a semi-group of probability measures such that for every $\lambda \geq 1$ there exists a S such that (4) (iii) holds. Also, if G is unimodular, it is easy to see that both (i) and (ii) are satisfied.

Proof. A slightly weaker version of Hörmander's theorem (cf. Hörmander) says that if $X_1,\dots,X_k$ are vector fields on a manifold M such that for every point $x \in M$ the tangent space $T_x(M)$ is spanned by $X_1,\dots,X_k$ and the iterated commutators of $X_1,\dots,X_k$, then for every point x there is a constant C and $S > 0$ such that

(9) $$|f(x)| \leq C\|(1+L)^S f\|_{L^2}, \quad f \in C_0^\infty(M),$$

where L is given by (8).

We see that our assumptions concerning the left-invariant fields $X_1,\dots,X_k$ on G are those of Hörmander theorem. Since by proposition 11, section 2, $-L$ is the infinitesimal generator of a convolution semi-group of probability measures which are symmetric, as we easily verify, the resolvent k_λ is a positive measure such that $k_\lambda = \tilde{k_\lambda}$. To prove (iii) we have to show that k_λ^{*S} beloge to $L^2(m)$ for $\lambda \geq 1$. But this follows from (9): for f in $C_c^\infty(G)$ we have

$$\langle f, k_\lambda^{*S}\rangle = f * k_\lambda^{*S}(e) \leq C\|(\lambda+L)^S(f*k_\lambda^{*S})\|_{L^2} = C\|f\|_{L^2}$$

which by Riess theorem implies that $k_\lambda^{*S} \in L^2(m)$.

To summarise this section: if $X_1,\dots,X_k$ generate the Lie algebra of a nilpotent Lie group G, then the semi-group $\{\mu_t\}_{t>0}$ of symmetric probability mesures on G whose infinitesimal generator is given by (9) satsifies (4) and so by the functional calculus we see that functions K which satisfy (*) define operators

$$T_K f = \int_0^\infty K(\lambda)\,dE(\lambda) f$$

which are bounded on every $L^p(m)$.

HOMOGENOUS GROUPS

This notion was introduced by E.M.Stein in his address at the Congress at Nice in 1970 and since then has made a considerable careera. For details of recent book (Folland and Stein).

By a homogenous group we mean a (necessarily) nilpotent Lie group G such that the Lie algebra $\mathfrak{g}$ of G admits a one-parameter group of dilations $\{\delta_t\}_{t>0}$ i.e. for a basis $X_1,\dots,X_n$ of $\mathfrak{g}$

(1) $$\delta_t X_j = t^{d_j} X_j, \quad j = 1,\dots,n, \quad t > 0,$$

with $1 = d_1 \leq \dots \leq d_n$ and extended by linearity they are Lie algebra automorphisms of $\mathfrak{g}$. This means that the linear transformations δ_t of $\mathfrak{g}$ have the property that $\delta_t[X,Y] = [\delta_t X, \delta_t Y]$. If $V_k = \{X \in \mathfrak{g} : \delta_t X = t^k X\}$, then $[V_i,V_j] \subset V_{i+j}$. A basis for which (1) holds is called a homogenous basis.

Example I. If $\mathfrak{g}$ is the Heisenberg algebra with the basis X,Y,Z such that $[X,Y] = Z$, then $\delta_t X = tX$, $\delta_t Y = tY$ and $\delta_t Z = t^2 Z$ are dilations on $\mathfrak{g}$. $\{\delta_t\}_{t>0}$ define also automorphisms of G, where $G = \exp\mathfrak{g}$. We write

$$\delta_t \exp X = \exp\delta_t X$$

and we easily check using Campbell - Hausdorff formula that δ_t is an automorphism of G.

In other words, if $\delta_t X = t^k X$ for $X \in \mathfrak{g}$, then

(2) $$X(f \circ \delta_t) = t^k Xf \circ \delta_t \quad .$$

Example II. Let $\mathfrak{g}$ be a chain algebra (cf. section 1) and let $X, Y_0,\dots,Y_d$ be the basis of $\mathfrak{g}$. We may define dilations on $\mathfrak{g}$ putting e.g.

$$\delta_t X = tX, \ \delta_t Y_j = t^{1+j}, \ j = 0,\dots,d.$$

Example III. Suppose $X_1,\dots,X_k$ are free generators of a nilpotent free algebra $\mathfrak{g}$ of class c. (By this we mean that $X_1,\dots,X_k$ generate $\mathfrak{g}$ as a Lie algebra and the only relations among $X_1,\dots,X_k$ are the ones which follow from the Jacobi identity, antysymmetry of $[X,Y]$ and $[X_{i_1},[X_{i_2},\dots,[X_{i_c},X_{i_{c+1}}]\dots] = 0$.)

If $1 = d_1 \le \dots \le d_k$ are arbitrary numbers we put

$$\delta_t X_j = t^{d_j} X_j \quad , j = 1,\dots,k$$

and we extend δ_t to dilations of $\mathfrak{g}$.

If $\mathfrak{g}$ is a homogenous Lie algebra and ∂ is an element in the envelloping algebra we define $\delta_t \partial$ as follows: if $\partial = X_{i_1} \dots X_{i_k}$, where $X_{i_1},\dots,X_{i_k}$ are element of a homogenous basis we put

$$\delta_t \partial = t^{d_{i_1} + \dots + d_{i_k}} \partial .$$

Then we extend it by linearity. We see that $\{\delta_t\}_{t>0}$ is a one-parameter group of automorphisms of the envelloping algebra.

Equality (2) implies that if ∂ is a homogenous element of the enevelloping algebra of degree d, then

(3) $$\partial (f \circ \delta_t) = t^d (\partial f) \circ \delta_t \quad , f \in C_o^\infty(G).$$

Proposition 1. If $X_1,\dots,X_k$ are elements of a homogenous Lie algebra all of homogenous degree 1, then if $-L = X_1^2 + \dots + X_k^2$, L is, of course, homogenous of degree 2. Let $\{\mu_t\}_{t>0}$ be the semi-group generated by $-L$. Then

(4) $$\langle f , \mu_{st} \rangle = \langle f \circ \delta_{s^{1/2}} , \mu_t \rangle \qquad \text{for all } f \in C_o(J)$$

in particular,

(5) $$\langle f , \mu_t \rangle = \langle f \circ \delta_{t^{1/2}} , \mu_1 \rangle \quad .$$

Proof. For f in $C_o^\infty(G)$ we have

$$\frac{d}{dt} f * \mu_{st} = -sL(f * \mu_{st})$$

and

$$\frac{d}{dt}[(f \circ \delta_{s^{1/2}}) * \mu_t \circ \delta_{s^{-1/2}}] = -L[(f \circ \delta_{s^{1/2}}) * \mu_t] \circ \delta_{s^{-1/2}}$$
$$= -sL[(f \circ \delta_{s^{1/2}}) * \mu_t \circ \delta_{s^{-1/2}}] .$$

Hence both $u(t,x) = f * \mu_{st}$ and $v(t,x) = (f \circ \delta_{s^{1/2}}) * \mu_t \circ \delta_{s^{-1/2}}$ satisfy the differential equation

$$\frac{\partial}{\partial t} u(t,x) = -sLu(t,x) \quad \text{with} \quad u(0,x) = f(x)$$

and so they are equal. Putting $x = e$, and noticing that μ_t are symmetric, we obtain (4).

For a measure μ in $M(G)$ let μ_t be defined by

$$\langle f , \mu_t \rangle = \langle f \circ \delta_{t^{1/2}} , \mu \rangle , \qquad f \in C_o(G).$$

Then, since $\delta_{t^{1/2}}$ is an automorphism of G, we easily verify that

(6) $$(\mu * \partial)_t = \mu_t * \partial_t$$

for every μ, ∂ in $M(G)$.

The homogenous dimension of G is the number Q defined by

$$dm(\delta_t x) = t^Q dm(x) \qquad t > 0 ,$$

(where m is a Haar measure on G which is equal to the Lebesgue measure on $\mathfrak{g}$ and is both left and right invariant). It is easy to see that

$$Q = d_1 + \dots + d_n .$$

For a function f on G we write

(7) $$f_t(x) = t^{-Q/2} f(\delta_{t^{-1/2}} x) .$$

Proposition 2. If $f \in L^1(m)$ and $\int f \, dm = 1$, then $\{f_t\}$ is an approximate identity as $t \to 0$, ie. for every p, $1 \le p < \infty$ and $g \in L^p(m)$

$$\lim_{t \to 0} \| g * f_t - g \|_{L^p} = 0 .$$

The proof is identical as in the case of the real line and ordinary dilations, (cf. Folland, Stein).

Suppose now that $X_1,\dots,X_k$ generate $\mathfrak{g}$ as a Lie algebra and that they are homogenous of degree 1. Let $-L = X_1^2 + \dots + X_k^2$. Consequently, there is a semi-group of probability measures $\{\mu_t\}_{t>0}$ which satisfies (4), section 4, and let $E(\lambda)$ be the spectral resolution of the self-adjoint operator $\bar{L}$ on $L^2(m)$. Then

$$f * \mu_t = \int_0^\infty e^{-t\lambda} dE(\lambda) f \quad , \qquad f \in L^2(m).$$

We see that the function $\lambda \longrightarrow e^{-t\lambda}$ (t fixed) satisfies condition (*) of the theorem on functional calculus of the previous section. Hence $\mu_t \in L^1(m)$. Let

$$p_t(x)dm(x) = d\mu_t(x)$$

By (5) and (7) we obtain

(8) $$p_t(x) = t^{-Q/2}\, p_1(\delta_{t^{-1/2}} x).$$

Let us write

$$f * k_1^{*S} = (1 + L)^{-S} f = \int_0^\infty (1 + \lambda)^{-S} dE(\lambda) f.$$

We have

$$f * k_1 = f * \int_0^\infty e^{-t} p_t \, dt ,$$

whence, by (8),

$$f * (k_1)_t = \int_0^\infty (1 + t\lambda)^{-1} dE(\lambda) f$$

and so, by (6)

$$f * (k_1^{*S})_t = \int_0^\infty (1 + t\lambda)^{-S} dE(\lambda) f.$$

Again by (6), we see that

$$e(nf_t) = e(nf)_t$$

and so, if K satisfies the condition (*) we see that for the $L^1(m)$ function k defined by

$$f * k = \int_0^\infty K(\lambda) dE(\lambda) f$$

we have

$$f * k_t = \int_0^\infty K(t\lambda) dE(\lambda) f ,$$

moreover, if $K(0) = 1$, we have $\int k(x)dm(x) = 1$ and k_t is an approximate identiy. The last conclution follows immediately from the fact that if $\phi \in L^1(m)$ and $\lim_{t\to 0} \| f * \phi_t - f \|_{L^2} = 0$ for all f in $L^2(m)$, then $\int \phi \, dm = 1$.

Thus we arrive at our main theorem.

MAIN THEOREM. Let G be a homogenous group such that $X_1,\dots,X_k$ generate the Lie algebra $\mathfrak{g}$ of G and $\delta_t X_j = tX_j$, $j = 1,\dots,k$. The operator

$$- L = X_1^2 + \dots + X_k^2$$

is essentially self adjoint on C_c^∞ in $L^2(m)$. Let

$$Lf = \int_0^\infty \lambda \, dE(\lambda) f \qquad , \quad f \quad C_c^\infty(G)$$

be the spectral representation of L. There exist numbers S and N such that if K $C^N(\mathbb{R}^+)$ and

(*) $$\lim_{\lambda\to\infty} (1+\lambda)^{N+S} \frac{d^j}{d\lambda^j} K(\lambda) = 0 \quad \text{for } j = 0,\dots, N,$$

then

$$\int_0^\infty K(\lambda) dE(\lambda) f = f * k,$$

where $k \in L^1(m)$. Moreover,

$$\int_0^\infty K(t\lambda) dE(\lambda) f = f * k_t ,$$

where k_t is defined by (7). If $K(0) = 1$, then

$$\lim_{t\to 0} \| f * k_t - f \|_{L^p} = 0$$

for all f in $L^p(m)$, and all $1 \le p < \infty$.

Let π be a representation of G on $L^2(X)$ such that for all p, $1 \le p < \infty$ π is a strongly continuous isometric representation on $L^p(X)$. Let

$$\pi(L)\phi = \int_0^\infty \lambda \, dE^\pi(\lambda)\phi ,$$

where ϕ belongs to the Garding space of π in $L^2(X)$. Then if $K \in C^N(\mathbb{R}^+)$ and satisfies (*), then

$$\lim_{t\to 0} \Big\| \int_0^\infty K(t\lambda) dE^\pi(\lambda) f - f \Big\|_{L^p} = 0 \quad \text{for} \quad f \in L^p(m).$$

APPLICATION. Let

$$H = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^m P_j^2$$

be a Schrödinger operator on $\mathbb{R}^n$ where the potential is a sum of squares of polynomials. Assume that $P_1,\dots,P_k$ depend essentially on all variables, ie.

(1) , p.4, holds.

Consider the generalised chain algebra $\mathfrak{g}'$ and let $X_1',\dots,X_n'$, $Y_1',\dots,Y_m'$ be the generators of $\mathfrak{g}'$ such that $\pi'(X_j') = \frac{\partial}{\partial x_j}$, $\pi'(Y_j') = M_{p_j}$.
The generalised chain algebra $\mathfrak{g}'$ may not admit dilations such that $X_1',\dots,Y_m'$ are homogenous of degree one. Therefore we consider the free nilpotent Lie algebra $\mathfrak{g}$ generated freely by $X_1,\dots,X_n$, $Y_1,\dots,Y_m$ and we lift the representation π' to (a representation of) $\mathfrak{g}$ mapping X_j onto X_j' and Y_j onto Y_j'. Now on $\exp \mathfrak{g} = G$ we define dilations $\delta_t X_j = tX_j$, $\delta_t Y_j = tY_j$ and we consider the operator

$$-L = X_1^2+\dots+X_n^2 + Y_1^2 + \dots + Y_m^2 .$$

We note that

$$\pi(L) = H .$$

Since π is irreducible H has discrete spectrum. In fact, $k_1^{*S} \in L^1(m) \cap L^2(m)$, so $\pi(k_1^{*S})$ is Hilbert-Schmidt (cf.p.2).

Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the eigen-values of H and $\phi_1, \phi_2, \dots$ the corresponding orthonormal eigen-functions of H.

Let

$$K(\lambda) = (1-\lambda)_+^N$$

Then , of course K satisfies (*). Moreover

$$\pi_{k_t}\phi = \sum_j (1 - t\lambda_j)_+^N (\phi,\phi_j)\phi_j ,$$

consequently, by the main theorem, for every ϕ in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, we have

$$\lim_{t\to 0} \| \phi - \sum_j (1-t\lambda_j)_+^N (\phi,\phi_j)\phi_j \|_{L^p} = 0 .$$

__REMARKS__. By further refinement of the arguments J.W.Jenkins and the author proved that for a somewhat larger N for every ϕ in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$,

$$\lim_{t\to 0} \sum_j (1 - t\lambda_j)_+^N (\phi,\phi_j)\phi_j(x) = \phi(x) \quad \text{almost everywhere.}$$

E.M.Stein and the author (cf.Folland Stein) proved the following multiplier theorem. If L is as in the main theorem , there exists a (finite !) N such that if a function K belongs to $C^N(\mathbb{R}^+)$ and

$$(**) \qquad \lambda^j \left| \frac{d^j}{d\lambda^j} K(\lambda) \right| \leq C \quad \text{for all } j = 0,\dots,N ,$$

then the operator

$$T_K f = \int_0^\infty K(\lambda) dE(\lambda) f$$

is bounded on $L^p(m)$, $1 < p < \infty$, but not necessarily on $L^1(m)$!

This, by Herz theorem shows that

$$\phi \longrightarrow \sum_{j=1}^{\infty} K(\lambda_j)(\phi,\phi_j)\phi_j$$

is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, provided K satisfies (**).

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