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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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SUPPLEMENT NO. 2

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For which of $U(n)$, $SU(n)$, $SO(n)$, $SP(n)$ does $\rho \in \Lambda$?

$SU(n)$, $SP(n)$ are simply-connected so $\rho \in \Lambda$.

$SO(3)$

$\mathfrak{t}$ $\quad \xi = \begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b \in \mathbb{R}$

Type $A_1$ roots $\pm\alpha$, $\alpha(\xi) = \sqrt{-1}\, b$
unit lattice is $\xi_e$ with $b = 2\pi m$, $m \in \mathbb{Z}$
$\rho(\xi_e) = \sqrt{-1}\,\pi m \qquad \rho \notin \Lambda$.

$SO(4)$

$\mathfrak{t}$ $\quad \xi = \begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & -c & 0 \end{pmatrix}$

Type $A_1 \oplus A_1$ roots $\pm\alpha, \pm\beta$ $\qquad \alpha(\xi) = \sqrt{-1}\,(b-c)$
$\beta(\xi) = \sqrt{-1}\,(b+c)$

$\rho(\xi) = \sqrt{-1}\, b$

unit lattice consists of $\xi_e$ with $b = 2\pi m$, $c = 2\pi n$ $\quad m, n \in \mathbb{Z}$
$\rho(\xi_e) = \sqrt{-1}\; 2\pi m \qquad \rho \in \Lambda$

$SO(2l+1)$, $l \geq 2$

$\mathfrak{h}$ $\quad S = \mathrm{diag}\left(\begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}, \ldots\ldots, \begin{pmatrix} 0 & b_l \\ -b_l & 0 \end{pmatrix}\right) \quad b_i \in \mathbb{R}$

Type $B_l$ roots $\pm\alpha_i$, $\pm\beta_{ij}$ $i<j$, $\pm\gamma_{ij}$ $i<j$

$$\alpha_i(S) = -\sqrt{-1}\, b_i$$

$$\beta_{ij}(S) = -\sqrt{-1}(b_i - b_j), \quad \gamma_{ij}(S) = -\sqrt{-1}(b_i + b_j)$$

$$\rho(S) = -\sqrt{-1}\sum_{i,j=i+1}^{l} b_i - \tfrac{1}{2}\sqrt{-1}\sum_j b_j$$

$\Lambda_e$ $\quad S_e$ with $b_1 = 2\pi m_1, \ldots, b_l = 2\pi m_l$, $m_i \in \mathbb{Z}$

With $m_2 = \ldots = m_l = 0$ we have $\rho(S_e) = -\sqrt{-1}\, 3\pi m_1$. $\quad \rho \notin \Lambda$.

$SO(2l)$, $l \geq 3$

$\mathfrak{h}$ $\quad S = \mathrm{diag}\left(\begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}, \ldots\ldots, \begin{pmatrix} 0 & b_l \\ -b_l & 0 \end{pmatrix}\right) \quad b_i \in \mathbb{R}$

Type $D_l$ roots $\pm\alpha_{ij}$ $i<j$, $\pm\beta_{ij}$ $i<j$

$$\alpha_{ij}(S) = -\sqrt{-1}(b_i - b_j), \quad \beta_{ij}(S) = -\sqrt{-1}(b_i + b_j)$$

$$\rho(S) = -\sqrt{-1}\sum_{i,j=i+1}^{l} b_i$$

$\Lambda_e$ $\quad S_e$ with $b_1 = 2\pi m_1, \ldots, b_l = 2\pi m_l$, $m_i \in \mathbb{Z}$

$$\rho(S_e) = -\sqrt{-1}\cdot 2\pi \sum_{i,j} m_i. \qquad \rho \in \Lambda.$$

$U(n) \supset SU(n)$

$$\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \sqrt{-1}\,\mathbb{R} I_n$$

$n=2$. $\quad S_0 = \begin{pmatrix} \sqrt{-1}a & 0 \\ 0 & -\sqrt{-1}a \end{pmatrix}$, $\quad S_1 = \sqrt{-1}\, b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

exponentiate to tori $H_0$, $H_1$

$\exp S_0 = \exp S_1$ $\quad a, b \in [0, 2\pi)$ implies that $a = b$, $e^{\sqrt{-1}\,2a} = 1 \Rightarrow a = 0, \pi$.

$H_0 \cap H_1 = \{I, e^{\sqrt{-1}\pi} I\}$ cyclic group of order 2.

Roots $\pm\alpha$, $\alpha(S_0) = \sqrt{-1}\, 2a$, $\alpha(S_1) = 0$ $\qquad \rho = \frac{1}{2}\alpha$

As a character of $H_0$, $e^{\rho}$ is $e^{\rho}(\exp S_0) = e^{\sqrt{-1}a}$; this is non-trivial on $H_0 \cap H_1$ thus $\rho \notin \Lambda$.

$n$ $\quad S_0 = \sqrt{-1}\,\mathrm{diag}(a_1, \ldots, a_n)$, $\quad S_1 = \sqrt{-1}\, a\, I_n$ $\quad \sum_i a_i = 0$, $a_i \in \mathbb{R}$, $a \in \mathbb{R}$

exponentiate to tori $H_0$, $H_1$.

$\exp S_0 = \exp S_1$ $\qquad a, a_i \in [0, 2\pi)$

implies that $a_i = a$, $e^{\sqrt{-1}\, n a} = 1 \Rightarrow a = \frac{2k\pi}{n}$ $n^{th}$ root of unity.

$H_0 \cap H_1 = \{e^{\sqrt{-1}\, 2k\pi/n} I\}$ cyclic group of order $n$.

Root system type $A_{n-1}$

$$\alpha_{ij}(S_0) = \sqrt{-1}(a_i - a_j), \quad \alpha_{ij}(S_1) = 0 \quad i \neq j$$

$$\rho = \frac{1}{2}\sum_{i<j}\alpha_{ij}$$

$n=3$ $\quad (12) \quad (13) \quad (23)$ $\qquad (12)+(23)=(13)$

$\rho = (13) \qquad e^{\rho} \in \hat{H}_0, \quad e^{\rho}(\exp \xi_0) = e^{\sqrt{-1}(a_1 - a_3)}$ this is trivial on $H_0 \cap H_1$ so $\rho \in \Lambda$.

$n=4$

$$\begin{array}{ccc} (12) & (13) & (14) \\ & (23) & (24) \\ & & (34) \end{array}$$

$(12)+(23)+(34) = (14)$

$(13)+(24)(\xi_0) = \sqrt{-1}\, 2(a_1+a_2)$

$n=5$

$$\begin{array}{cccc} (12) & (13) & (14) & (15) \\ & (23) & (24) & (25) \\ & & (34) & (35) \\ & & & (45) \end{array}$$

summing along diagonals

$2\rho(\xi_0) = \sqrt{-1}\, 2\,(15)(\xi_0) + \sqrt{-1}\, 2(a_1+a_2-a_4-a_5)$

$n=6$

$$\begin{array}{ccccc} (12) & (13) & (14) & (15) & (16) \\ & (23) & (24) & (25) & (26) \\ & & (34) & (35) & (36) \\ & & & (45) & (46) \\ & & & & (56) \end{array}$$

$$\begin{aligned} 2\rho(\xi_0) &= \sqrt{-1}\; 2\,(16)(\xi_0) \\ &\quad + \sqrt{-1}\, 2(a_1+a_2-a_5-a_6) \\ &\quad + \sqrt{-1}\, 2(a_1+a_2+a_3) \end{aligned}$$

$n=7$

$$\begin{array}{cccccc} (12) & (13) & (14) & (15) & (16) & (17) \\ & (23) & (24) & (25) & (26) & (27) \\ & & (34) & (35) & (36) & (37) \\ & & & (45) & (46) & (47) \\ & & & & (56) & (57) \\ & & & & & (67) \end{array}$$

sum of diagonals $(\xi_0)$ $\qquad (17)(\xi_0)$

$\sqrt{-1}(a_1+a_2\} - a_6 - a_7)$

$\sqrt{-1}(a_1+a_2+a_3-a_5-a_6-a_7)$

$\sqrt{-1}(a_1+a_2+a_3-a_5-a_6-a_7)$

$\sqrt{-1}(a_1+a_2-a_6-a_7)$

$(17)(\xi_0)$

$n$

$$\begin{array}{ccc} (12) & \cdots\cdots & (1n) \\ & (23) \;\cdots\cdots & (2n) \\ & \ddots & \vdots \\ & & (n\text{-}1\; n) \end{array}$$

(sum of diagonals) $(\xi_0)$

$(1n)(\xi_0)$ , $(1n)(\xi_0)$

$\sqrt{-1}\; 2\sum_{i=1}^{j} a_i - a_{n+1-i}, \; j=2,\dots,\left[\frac{n}{2}\right]$

$n$ odd

For $n$ even, add $\sqrt{-1}\; 2\sum_{i=1}^{n/2} a_i$

$U(n)$ $\qquad n$ even $\qquad \rho \notin \Lambda$

$\qquad\quad\;\; n$ odd $\qquad \rho \in \Lambda$

K compact, connected

$$\begin{tikzcd} & \mathrm{Spin}(k) \arrow[d] \\ K \arrow[ur, dotted] \arrow[r, "\mathrm{Ad}"'] & SO(k) \end{tikzcd}$$

Isotropy representation of K lifts to Spin iff $\rho \in \Lambda$.

(cf J.F. Adams Lectures on Lie groups)

$\mathfrak{g}$ semi-simple

The degree of the irreducible representation of $\mathfrak{g}$ of highest weight $\lambda$, $\phi_\lambda$, is

$$d(\lambda) = \prod_{\alpha\in R^+} \frac{\langle \lambda+\rho, \alpha\rangle}{\langle \rho, \alpha\rangle}$$

$\rho$ is half the sum of the +ve roots, which is equal to the sum of the fundamental weights.

Also
$$d(\lambda) = \prod_{\alpha\in R^+} \left(\frac{\langle\lambda,\alpha\rangle}{\langle\rho,\alpha\rangle} + 1\right)$$

$$= \prod_{\alpha\in R^+} \left(\frac{\lambda_\alpha}{\rho_\alpha} + 1\right) \qquad \lambda_\alpha = \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$$

## $A_2$


$$R^+ = \{\alpha, \beta, \alpha+\beta\} \qquad \begin{array}{l}\rho_\alpha = 1\\ \rho_\beta = 1\end{array}$$

(in fact in general if $\alpha$ is a simple root $W_\alpha\rho = \rho - \alpha$,
$\|W_\alpha\rho\|^2 = \|\rho\|^2 - 2\langle\rho,\alpha\rangle + \|\alpha\|^2$ ie $\rho_\alpha = 1$ )

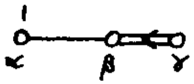
For $A_2$, $\langle\alpha,\alpha\rangle = \langle\beta,\beta\rangle$ so

$$d(\lambda) = (\lambda_\alpha+1)(\lambda_\beta+1)\left(\frac{\lambda_\alpha+\lambda_\beta}{2}+1\right)$$

ie

$$d(n,m) = \tfrac{1}{2}(n+1)(m+1)(n+m+2)$$

## $B_3$




Cartan matrix $\begin{pmatrix} 2 & -1 & 0\\ -1 & 2 & -2\\ 0 & -1 & 2\end{pmatrix}$

$$R^+ = \{\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma, \beta+2\gamma, \alpha+\beta+\gamma, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma\}$$

$$\langle\alpha,\alpha\rangle = \langle\beta,\beta\rangle = 2\langle\gamma,\gamma\rangle$$

$2\langle\rho,\alpha\rangle = \langle\alpha,\alpha\rangle$, $\quad 2\langle\rho,\beta\rangle = \langle\beta,\beta\rangle$, $\quad 2\langle\rho,\gamma\rangle = \langle\gamma,\gamma\rangle$

$2\langle\rho,\alpha+\beta\rangle = 2\langle\alpha,\alpha\rangle$, $\quad 2\langle\rho,\alpha+\beta+\gamma\rangle = 5/2\langle\alpha,\alpha\rangle$

$2\langle\rho,\alpha+\beta+2\gamma\rangle = 3\langle\alpha,\alpha\rangle$, $\quad 2\langle\rho,\alpha+2\beta+2\gamma\rangle = 4\langle\alpha,\alpha\rangle$

$$d = 2\left(\tfrac{1}{2}+1\right)\left(\tfrac{2}{5}+1\right)\left(\tfrac{1}{3}+1\right)\left(\tfrac{1}{4}+1\right)$$
$$= 2\ \tfrac{3}{2}\ \tfrac{7}{5}\ \tfrac{4}{3}\ \tfrac{5}{4}$$
$$= 7$$

If $\mu$ is a weight of $\phi_\lambda$ and $\alpha$ is a root, recall that the $\alpha$ chain of weights through $\mu$ consists of the $\mu + t\alpha$ with $t$ an integer between $-t'$ and $t''$, where $2\dfrac{\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle} = t' - t''$

then

$\mu - \alpha$ is a weight iff $\quad 2\dfrac{\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle} + t''\langle\mu,\alpha\rangle > 0 \qquad (+)$

Also

if $\mu$ is a weight then $\mu = \lambda - \sum_i n_i\alpha_i$ uniquely.

To find the weights of $\phi_\lambda$, start with the highest weight $\lambda$ and subtract simple roots from it using (+).

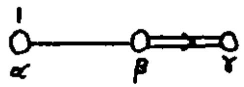
$A_2$  : dimension is 3

highest weight is $\lambda_1 = \frac{1}{3}\alpha + \frac{2}{3}\beta$

Consider (†) with $\lambda_1$, here $t''=0$ for $\alpha,\beta$ and $\lambda_2 = \lambda_1 - \beta = \frac{1}{3}\alpha - \frac{1}{3}\beta$

$\lambda_{2\alpha} > 0$ so $\lambda_3 = \lambda_2 - \alpha = -\left(\frac{2}{3}\alpha + \frac{1}{3}\beta\right)$

Weights $\left\{ \frac{1}{3}\alpha + \frac{2}{3}\beta,\ \frac{1}{3}\alpha - \frac{1}{3}\beta,\ -\left(\frac{2}{3}\alpha+\frac{1}{3}\beta\right)\right\}$ ~~and~~

$C_3$  Cartan matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$

dimension is 6

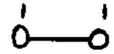
highest weight is $\lambda_1 = \alpha + \beta + \frac{1}{2}\gamma$

$\lambda_2 = \beta + \frac{1}{2}\gamma$

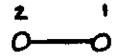
| | $\mu_{\alpha_i}$ | $t''(\mu,\alpha_i)$ | |
|---|---|---|---|
| $-\alpha+\beta+\frac{1}{2}\gamma$ | $-1$ | $1$ | – |
| $\frac{1}{2}\gamma$ | $1$ | $0$ | $\lambda_3$ |
| $\beta-\frac{1}{2}\gamma$ | $0$ | $0$ | – |
| $-\frac{1}{2}\gamma$ | $1$ | $0$ | $\lambda_4$ |
| $-\beta+\frac{1}{2}\gamma$ | $-1$ | $1$ | – |
| $-\alpha+\frac{1}{2}\gamma$ | $0$ | $0$ | – |
| $-\beta-\frac{1}{2}\gamma$ | $1$ | $0$ | $\lambda_5$ |
| $-\frac{3}{2}\gamma$ | $-1$ | $1$ | – |
| $-\alpha-\frac{1}{2}\gamma$ | $0$ | $0$ | – |
| $-\alpha-\beta-\frac{1}{2}\gamma$ | $1$ | $0$ | $\lambda_6$ |
| $-\beta-\frac{3}{2}\gamma$ | $0$ | $0$ | – |
| $-2\beta-\frac{1}{2}\gamma$ | $-1$ | $1$ | – |

each weight is non-degenerate ie $m_\mu = 1$, $\mu$ a weight.

Sometimes a weight $\mu$ can be non-degenerate ie $m_\mu \geq 2$

eg the zero weight in  dim = 8 which has weights

$\{\alpha+\beta, \alpha, \beta, 0, -\beta, -\alpha, -(\alpha+\beta)\}$ $m_0 = 2$;

or can have $m_\mu \geq 2$ for $\mu \neq 0$ eg in 

The partition function $P(\ )$ is defined on $g$ by $P(\mu)$ is the number of sequences $(a_\alpha)_{\alpha>0}$ of non-negative integers $a_\alpha$ satisfying $\mu = \sum_{\alpha\in R^+} a_\alpha \alpha$ then Kostant's formula gives the multiplicity of a weight $\mu$ of $\phi_\lambda$

$$m_\mu = \sum_{w\in W(R)} (\det w)\, P\left((w\lambda-\mu)+(w\delta-\delta)\right)$$

$A_2$ $R^+ = \{\alpha, \beta, \alpha+\beta\}$ $W(R) = \mathrm{Symm}(3)$
$= \{1, (12), (13), (23), (123), (132)\}$

| $w$ | $\alpha$ $(1,-1,0)$ | $\beta$ $(0,1,-1)$ | $\alpha+\beta$ $(1,0,-1)$ | $\det w$ |
|---|---|---|---|---|
| 1 | | | | 1 |
| (12) | $-\alpha$ | $\alpha+\beta$ | $\beta$ | $-1$ |
| (13) | $-\beta$ | $-\alpha$ | $-(\alpha+\beta)$ | $-1$ |
| (23) | $\alpha+\beta$ | $-\beta$ | $\alpha$ | $-1$ |
| (123) | $-(\alpha+\beta)$ | $\alpha$ | $-\beta$ | $1$ |
| (132) | $\beta$ | $-(\alpha+\beta)$ | $-\alpha$ | $1$ |

$\det w$ is $(-1)^r$ where $r$ is the number of positive roots sent into negative roots by $w$. The Weyl group acts on the diagonal matrices by permuting the diagonal entries.

$$1\rho = \rho \qquad\qquad (23)\rho = \alpha = \rho - \beta$$
$$(12)\rho = \beta = \rho - \alpha \qquad\qquad (123)\rho = -\beta = \rho - (\alpha + 2\beta)$$
$$(13)\rho = -(\alpha+\beta) = \rho - 2(\alpha+\beta) \qquad\qquad (132)\rho = -\alpha = \rho - (2\alpha+\beta)$$

dimension is $d(2,1) = \frac{1}{2}\cdot 3\cdot 2\cdot 5 = 15$

highest weight $\lambda_1 = 5/3\,\alpha + 4/3\,\beta$

$$\lambda_2 = \lambda_1 - \alpha = 2/3\,\alpha + 4/3\,\beta\,,\quad \lambda_3 = \lambda_1 - \beta = 5/3\,\alpha + 1/3\,\beta$$

$$1\lambda_1 = \lambda_1$$

$$(12)\lambda_1 = -5/3\,\alpha + 4/3(\alpha+\beta) = -1/3\,\alpha + 4/3\,\beta = \lambda_4 = \lambda_1 - 2\alpha$$

$$(13)\lambda_1 = -5/3\,\beta - 4/3\,\alpha = \lambda_{12} \qquad = \lambda_1 - 3(\alpha+\beta)$$

$$(23)\lambda_1 = 5/3(\alpha+\beta) - 4/3\,\beta = 5/3\,\alpha + 1/3\,\beta = \lambda_3 = \lambda_1 - \beta$$

$$(123)\lambda_1 = -5/3(\alpha+\beta) + 4/3\,\alpha = -1/3\,\alpha - 5/3\,\beta = \lambda_{11} = \lambda_1 - (2\alpha+3\beta)$$

$$(132)\lambda_1 = 5/3\,\beta - 4/3(\alpha+\beta) = -4/3\,\alpha + \tfrac{1}{3}\beta = \lambda_8 = \lambda_1 - (3\alpha+\beta)$$

| | $M_{\alpha_i}$ | $t''(M,\alpha_i)$ | |
|---|---|---|---|
| $\lambda_2 - \alpha = -1/3\,\alpha + 4/3\,\beta$ | 0 | 1 | $\lambda_4$ |
| $\lambda_2 - \beta = 2/3\,\alpha + 1/3\,\beta$ | 2 | | $\lambda_5$ |
| $\lambda_3 - \alpha = 2/3\,\alpha + 1/3\,\beta$ | | | – |
| $\lambda_3 - \beta = 5/3\,\alpha - 2/3\,\beta$ | -1 | 1 | – |
| $\lambda_4 - \alpha = -4/3\,\alpha + 4/3\,\beta$ | -2 | 2 | – |
| $\lambda_4 - \beta = -1/3\,\alpha + 1/3\,\beta$ | 3 | | $\lambda_6$ |
| $\lambda_5 - \alpha = -1/3\,\alpha + 1/3\,\beta$ | | | – |
| $\lambda_5 - \beta = 2/3\,\alpha - 2/3\,\beta$ | 0 | 1 | $\lambda_7$ |
| $\lambda_6 - \alpha = -4/3\,\alpha + 1/3\,\beta$ | -1 | 2 | $\lambda_8$ |
| $\lambda_6 - \beta = -1/3\,\alpha - 2/3\,\beta$ | 1 | | $\lambda_9$ |
| $\lambda_7 - \alpha = -1/3\,\alpha - 2/3\,\beta$ | | | – |
| $\lambda_7 - \beta = 2/3\,\alpha - 5/3\,\beta$ | -2 | 2 | – |
| $\lambda_8 - \alpha = -7/3\,\alpha + 1/3\,\beta$ | -3 | 3 | – |
| $\lambda_8 - \beta = -4/3\,\alpha - 2/3\,\beta$ | 2 | | $\lambda_{10}$ |
| $\lambda_9 - \alpha = -4/3\,\alpha - 2/3\,\beta$ | | | – |
| $\lambda_9 - \beta = -1/3\,\alpha - 5/3\,\beta$ | -1 | 2 | $\lambda_{11}$ |
| $\lambda_{10} - \alpha = -7/3\,\alpha - 2/3\,\beta$ | -2 | 3 | $\cancel{\lambda_{12}}$ |
| $\lambda_{10} - \beta = -4/3\,\alpha - 5/3\,\beta$ | 0 | 1 | $\lambda_{12}$ |
| $\lambda_{11} - \alpha = -4/3\,\alpha - 5/3\,\beta$ | | | – |
| $\lambda_{11} - \beta = -1/3\,\alpha - 8/3\,\beta$ | -3 | 3 | – |

Let $m_i$ be the multiplicity of the weight $\lambda_i$, then from Kostant's formula

$m_1 = P(0) = 1$

$\lambda_2 = \lambda_1 - \alpha$

$m_2 = P(\alpha) = 1$

$\lambda_3 = \lambda_1 - \beta = (23)\lambda_1$

$m_3 = P(\beta) = 1$

$\lambda_4 = \lambda_1 - 2\alpha = (12)\lambda_1$

$m_4 = P(2\alpha) = 1$

$\lambda_5 = \lambda_1 - (\alpha+\beta)$

$m_5 = P(\alpha+\beta) = 2$

$\lambda_6 = \lambda_1 - (2\alpha+\beta)$

$m_6 = P(2\alpha+\beta) = 2$

$\lambda_7 = \lambda_1 - (\alpha+2\beta)$

$m_7 = P(\alpha+2\beta) - P(\beta) = 2-1 = 1$

$\lambda_8 = \lambda_1 - (3\alpha+\beta) = (132)\lambda_1$

$m_8 = P(3\alpha+\beta) - P(\beta) = 2-1 = 1$

$\lambda_9 = \lambda_1 - 2(\alpha+\beta)$

$m_9 = P(2\alpha+2\beta) - P(2\alpha) = 2-1 = 1$

$\lambda_{10} = \lambda_1 - (3\alpha+2\beta)$

$m_{10} = P(3\alpha+2\beta) - P(2\beta) - P(3\alpha) = 3-1-1 = 1$

$\lambda_{11} = \lambda_1 - (2\alpha+3\beta) = (123)\lambda_1$

$m_{11} = P(2\alpha+3\beta) - P(2\alpha+\beta) = 3-2 = 1$

$\lambda_{12} = \lambda_1 \cancel{-(4\alpha+2\beta)}$

$m_{12} = P\cancel{(4\alpha+2\beta)} - P(\alpha+2\beta) - P(4\alpha) = 0$

$\lambda_{12} = \lambda_1 - 3(\alpha+\beta) = (13)\lambda_1$

$m_{12} = P(3\alpha+3\beta) - P(3\beta) - P(3\alpha+\beta) = 4-1-2 = 1$

$\lambda_5, \lambda_6$ have multiplicity 2, all other weights are nondegenerate.

Exercise :-

Let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $\phi_\lambda$ an irreducible representation of $\mathfrak{g}_{\mathbb{C}}$. From the C-D diagram of $\phi_\lambda$, how can one ~~test if~~ determine if $\phi_\lambda$ is the differential of $\pi_\lambda$ in $\hat{G}$ ? eg for $B_\ell$, $D_\ell$.

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