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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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KIRILLOV THEORY (cont.d)

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Lecture 6.

Kirillov Theory (continued).

As hitherto G denotes a connected, simply connected nilpotent Lie group. Let σ be a unitary representation of G on a Hilbert space. A vector $v \in \mathbb{H}$ is a C^∞-vector iff the map $g \longmapsto \sigma(g)v$ of G in $\mathbb{H}$ is C^∞. It is obvious that the collection $\mathbb{H}_\infty$ of C^∞-vectors in $\mathbb{H}$ is a vector subspace.

Claim $\mathbb{H}_\infty$ is dense in $\mathbb{H}$

Proof. Let U_n be a fundamental system of relatively compact neighbourhoods of the identity in G. Let φ_n be a (non-negative) C^∞-function on G with support in U_n and such that $\int_G \varphi_n(g)\,dg = 1$. For $\varphi \in L^1(G)$ recall that $\tilde{\sigma}(\varphi)$ was defined as the bounded operator

$$\tilde{\sigma}(\varphi)v = \int_G \varphi(g)\,\sigma(g)\,v\,dg$$

It is easy to check that for any $v \in \mathbb{H}$, $\tilde{\sigma}(\varphi_n)v$ converges to v as n tends to ∞. On the other hand if $\varphi \in C_c(G)$,

$\mathcal{F}(\varphi)$ is easily seen to be C^∞. This proves the claim.

The Lie algebra $\mathfrak{g}$ of G acts on the space $\mathbb{H}_\infty$: For $X \in \mathfrak{g}$, $v \in \mathbb{H}$, we set $\dot\sigma(X)v = \frac{d}{dt}\sigma(\exp tX)(v)\Big|_{t=0}$ (that $\dot\sigma$ is a Lie algebra homomorphism is easy to see) We denote by $\dot\sigma$ also the extension of $\dot\sigma$ as an algebra homomorphism of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ into the algebra $\mathrm{End}_{\mathbb{C}}(\mathbb{H}_\infty)$.

Proposition. Let σ be an irreducible unitary representation of G and let Ω denote the corresponding G orbit in $\mathfrak{g}^*$. Let $m = \dim\Omega/2$ Then there is an identification of the representation space $\mathbb{H}$ of σ with $L^2(\mathbb{R}^m)$ such that $\mathbb{H}_\infty$ corresponds to the space of all C^∞ functions f in $L^2(\mathbb{R}^m)$ ~~and~~ with the property that $x^\alpha D^\alpha f \in L^2(\mathbb{R}^m)$ for all multiindices $\alpha = (\alpha_1, \ldots, \alpha_m)$ ($x_1, \ldots, x_m$ are the coordinates in $\mathbb{R}^m$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$, $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_m}}{\partial x_m^{\alpha_m}}$) and $\dot\sigma$ maps $U(\mathfrak{g})$ onto the algebra of all differential operators with polynomial coefficients.

(Before we prove the Proposition we need to clarify one point: we have tacitly assumed that m ($= \dim\Omega$) is even. This can be seen as follows. Consider the ~~linear~~ (alternating bilinear) form ϕ_λ ($\lambda \in \mathfrak{g}^*$) on $\mathfrak{g}$ defined by

$$\phi(X,Y) = \lambda([X,Y])$$

If $T \in \mathrm{kernel}\,\phi$, $\lambda([T,Y]) = 0$ for all $Y \in \mathfrak{g}$ i.e. $({}^t\mathrm{ad}\,T)(\lambda) = 0$ ~~[illegible]~~ In other words T belongs to the isotropy Lie algebra $\mathfrak{g}_\lambda$ at λ. ~~[illegible]~~ Now ϕ defines a non-degenerate alternating form on $\mathfrak{g}/\mathfrak{g}_\lambda$ leading to the conclusion that $\dim \mathfrak{g}/\mathfrak{g}_\lambda$ is even. Evidently $\dim \mathfrak{g}/\mathfrak{g}_\lambda$ is precisely the dimension of the orbit ~~[illegible]~~ of λ under G. Note moreover that if we identify $\mathfrak{g}/\mathfrak{g}_\lambda$ with the tangent space at $\dot e$ of G/G_λ (G_λ = isotropy group at λ), ϕ_λ defines a translation invariant alternating 2-form (ω_λ) on the orbit Ω_λ ($= G/G_\lambda$) of λ. It is not difficult to check that ω_λ is closed as well so that Ω_λ is a symplectic manifold and this is the basic fact needed in "quantization")

Proof We argue by induction on dim G. If the given σ is trivial on a connected central subgroup σ factors through a lower dimensional group to which induction hypothesis would apply. We may assume thus that dim $C = 1$. We then have a decomposition $\mathfrak{g} = z(Y) \oplus \mathbb{R}X$ as we have seen before, $z(Y)$ the centraliser of $Y \in \mathfrak{g}$ is a codimension 1 ideal and $[X,Y] = Z$ is a non-zero element in $\mathfrak{c}$. We also know that σ is induced by a character χ on a connected closed subgroup H of G contained in $Z(Y)$ with $Y \in \mathfrak{h}$ as well. (Let $\lambda \in \mathfrak{g}^*$ be a linear form on $\mathfrak{g}$ such that $\chi(\exp T) = \exp i\lambda(T)$ for all $T \in \mathfrak{h}$. Then σ corresponds to the orbit of λ in $\mathfrak{g}^*$). Let τ denote the representation of $Z(Y)$ induced by χ. Then σ is induced by τ so that τ is irreducible. ~~Then~~ Now if λ' denotes the restriction of λ to $z(Y)$, ~~then~~ and Ω' is the orbit of λ', τ corresponds to Ω' and is thus realised on $L^2(\mathbb{R}^{m'})$, $2m' = \dim \Omega'$ in such a manner that $\dot{\tau}(z(Y))$ is the algebra of all differential operators with polynomial coefficients on $\mathbb{R}^{m'}$. Since σ is induced by τ, the representation space of σ can be identified with space of square integrable functions on $\mathbb{R}$ with values in $L^2(\mathbb{R}^{m'})$ (We use the semidirect product decomposition of G as $\mathbb{R}\ (=\{\exp tX \mid t \in \mathbb{R}\}) \cdot Z(Y)$). By a standard identification we may identify then the representation space of σ with $L^2(\mathbb{R}^{m'+1})$. Let $m = m'+1$; then one checks easily that the orbit Ω_λ of λ has dimension m. We denote the coordinates on $\mathbb{R}^m$ by $(x, y_1, \ldots, y_{m'})$ chosen so that each f in $L^2(\mathbb{R}^m)$ is identified with the (measurable) function $x \mapsto f(x, y_1, \ldots, y_{m'})$ of $\mathbb{R}$ in $L^2(\mathbb{R}^{m'})$. With this identification and the definition of the induced representation, one checks easily that

$$\dot{\sigma}(X) = \frac{\partial}{\partial x}$$

while if one uses the fact $[X,Y] = Z$, one finds that

$$\dot\sigma(X) f = \lambda(Z). x f.$$

Let $\mathcal{P}$ be the algebra of all polynomial differential operators in $\mathbb{R}^m$ and $\mathcal{P}'$ the subalgebra generated ~~consisting of~~ operators of the form y_i (multiplication by y_i) and $\frac{\partial}{\partial y_j}$, $1 \le i, j \le m'$. Evidently $\mathcal{P}$ is generated by $\mathcal{P}'$, x and $\frac{\partial}{\partial x}$. Thus to prove the proposition it suffices to show that $\mathcal{P}' \subset \dot\sigma(U(\mathfrak{g}))$. Suppose now that $T \in U(z(Y))$. Then it is easy to see that we have

$$(\dot I_\tau(T) =)\ \dot\sigma(T) f(x, y_1 \dots y_{m'}) = \dot\tau(\mathrm{Ad}\exp x X(T)) f(x, y_1, \dots y_{m'})$$

Here for $g \in G$, $T \in U(z(Y))$, $\mathrm{Ad}\, g(T)$ is the transform of T under the extension of the adjoint action of T on its Lie algebra $\mathfrak{g}$. Since G is nilpotent one has

$$\mathrm{Ad}\exp -xX\ (T) = \sum_{1 \le i \le m} P_i(x)\, T_i$$

where $T_i \in U(z(Y))$ and $P_i(x)$ are polynomials in x. It follows that we have

$$T = \sum_{1 \le i \le m} P_i(x) \{ \mathrm{Ad}\exp xX\ (T_i) \}.$$

Since $\dot I_\tau(T_i) = \dot\sigma(T_i) = \dot\tau(\mathrm{Ad}\exp x X(T_i))$ and the operator (multiplication by) $P_i(x) \in \dot\sigma(U(\mathfrak{g}))$, we conclude that $\dot\tau(T) \in \dot\sigma(U(\mathfrak{g}))$ for all $T \in z(Y)$. Thus $\mathcal{P}' \subset \dot\sigma(U(\mathfrak{g}))$ and hence $\mathcal{P} = \sigma(U(\mathfrak{g}))$. This proves the proposition.

<u>Corollary</u>. If $\varphi \in L^1(G)$, $\tilde\sigma(\varphi)$ is a compact operator.

<u>Proof</u>. We assume σ realised on $L^2(\mathbb{R}^m)$ as in the Proposition. Let D denote the differential operator $\left(1 + \sum_{1\le i\le m} x_i^2\right)^N \left(1 - \sum_{1 \le i \le m} \partial^2/\partial x_i^2\right)^N$, $N > 0$ an integer. If N is taken suitably large, D^{-1} is a compact – in fact a Hilbert Schmidt operator in $L^2(\mathbb{R}^m)$. Let $P \in U(\mathfrak{g})$ be chosen such that $\sigma(P) = D$. Now we can write, for $\varphi \in C_c^\infty$ $\tilde\sigma(\varphi) = \tilde\sigma(\varphi).\dot\sigma(D).\dot\sigma(D)^{-1}$ and $\tilde\sigma(\varphi)\dot\sigma(D) = \sigma(D'\varphi)$ where $D' \in U(\mathfrak{g})$ is an element determined by D: this follows from the fact ~~that~~ $\tilde\sigma(\varphi)\dot\sigma(X) = \tilde\sigma(-X\varphi)$ for $X \in \mathfrak{g}$, $\varphi \in C_c^\infty(G)$ as is easily checked. Since $\dot\sigma(D)^{-1}$ is compact, we see that $\tilde\sigma(\varphi)$ is compact if

$\varphi \in C_c^\infty$. (In fact the argument shows that if $\varphi \in C^\infty$ and with rapid decrease at ∞, $\tilde{\sigma}(\varphi)$ is ~~of~~ Hilbert Schmidt since that holds for D^{-1}). ~~[illegible]~~ If $\varphi_n \to \varphi$ in $L^1(G)$, then $\tilde{\sigma}(\varphi_n)$ tends to $\tilde{\sigma}(\varphi)$ in norm. Thus we conclude from the density of $C_c(G)$ in $L^1(G)$, that $\tilde{\sigma}(\varphi)$ is compact for all $\varphi \in L^1(G)$.

Corollary. Let φ be a C^∞ function on G with rapid decrease at ∞. (This means that $u = \varphi \circ \exp$ is a C^∞ function such that $P(x) D^\alpha u(x)$ tends to zero as x tends to ∞ for all polynomials P on $\mathfrak{g}$ and differential operators $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$, $x_1, \dots, x_n$ being coordinates on $\mathfrak{g}$ referred to basis of $\mathfrak{g}$). Let $\hat{\varphi}$ denote the Fourier transform of $\varphi \circ \exp$ — $\hat{\varphi}$ is a C^∞ function with rapid decrease on $\mathfrak{g}^*$. Let $\Omega \subset \mathfrak{g}^*$ be a G-orbit and σ the corresponding ~~rep~~ unitary representation. Then $\tilde{\sigma}(\varphi)$ is of trace class and $\operatorname{Trace} \tilde{\sigma}(\varphi) = \int_\Omega \hat{\varphi}(f)\, d\omega(f)$ where $d\omega(f)$ is a suitable G-invariant measure on the orbit Ω.

Proof. This is again by induction on dim G. If kernel σ ~~has~~ contains a nontrivial connected central subgroup C_0 with $\mathfrak{c}_0$ as its Lie algebra, the all ~~elements~~ linear forms λ in Ω are trivial on $\mathfrak{c}_0$ so that λ defines a linear form λ' on $\mathfrak{g}/\mathfrak{c}_0 = \mathfrak{g}'$ the Lie algebra of G/C_0. The natural inclusion of $\mathfrak{g}'^*$ in $\mathfrak{g}^*$ is compatible with the G action on the two vector spaces and induces a diffeomorphism of the orbit Ω' of λ' on Ω preserving the G-invariant measures on the two manifolds. Moreover if φ is a function on G with rapid decrease we have

$$\int_G \varphi(g)\rho(g) v \, dg = \int_{G'} dg' \int_{C_0} \varphi(gc) \rho(gc) v \, dc$$

where for $g' \in G'$, g is chosen in G such that ~~the image of g in~~ g' is its image. (~~[illegible]~~ $\int_{C_0} \varphi(gc)\rho(gc) v$ defines a function on G')

But $\sigma(gc) = \sigma(g)$ so that

$$\int_G \varphi(g) \rho(g) v \, dg = \int_{G'} \psi(g') \sigma(g') v \, dg'$$

where $\psi(g') = \int_{C_0} \varphi(gc)\,dc$ with $g \in G$ mapping into g'. By induction hypothesis $\tilde{\sigma}(\psi)$ is of trace class and trace $\tilde{\sigma}(\psi) = \int_{\Omega'} \hat{\psi}(f)\,d\omega'(f)$. Thus we have only to show that $\hat{\psi}$ coincides with $\hat{\varphi}$ on Ω' $(= \Omega)$. Now for $\alpha \in \mathfrak{g}^*$, we have (setting $\varphi = \varphi_0 \circ \exp$),

$$\hat{\varphi}(\alpha) = \int_{\mathfrak{g}} \exp i\langle \alpha, x\rangle\, \varphi(x)\,dx$$

$$= \int_{\mathfrak{g}/\mathfrak{c}_0} dx' \int_{\mathfrak{c}_0} \exp i\langle \alpha, x'+c\rangle\, \varphi(x'+c)\,dc$$

~~Suppose~~ Taking $\alpha = \lambda$, we have $\lambda(\mathfrak{c}_0) = 0$ so that

$$\hat{\varphi}(\alpha) = \int_{\mathfrak{g}/\mathfrak{c}_0} dx' \exp i\langle\alpha, x'\rangle \int_{\mathfrak{c}_0} \varphi(x'+c)\,dc$$

and $\int_{\mathfrak{c}_0} \varphi(x'+c)\,dc$ is precisely ψ in the definition above except that we are now treating φ as a function on the Lie algebra.

Our problem is thus reduced to the case when $\dim C = 1$ and $\sigma(C)$ is nontrivial. The corresponding linear form $\lambda\ (\in \Omega)$ is non zero on $\mathfrak{c}$.

~~We will take without ... A~~

As we have seen before, in this case we can find $X, Y \in \mathfrak{g}$ such that $Z = [X, Y]$ is a non zero element of $\mathfrak{c}$, the centralizer of Y in $\mathfrak{g}$ is a ideal of codimension 1, so that $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{z}(Y)$. Let $A = \{\exp tX \mid t \in \mathbb{R}\}$ so that G is expressible as a semidirect product $G = A \times Z(Y)$.

The Haar measure on G is easily seen to be $da \cdot dz$ where da (resp. dz) is the Haar measure on A (resp. G). Now let φ be a C^∞ function of G with rapid decrease. Then if $g = a \cdot z$, $a \in A$, $z \in Z(Y)$, we have for $v \in \mathcal{H}_\sigma$ = representation space of σ,

$$\hat{\sigma}(\varphi)v = \int_G \varphi(g)\sigma(g)v\,dg = \int_A da \int_{Z(Y)} \varphi(az)\,\sigma(az)v\,dz$$

Now σ is induced by the character $\underline{\lambda}$ on a subgroup $H \subset Z(Y)$ whose Lie algebra $\mathfrak{h}$ is subordinate to λ and contains Y. Let τ be the representation of $Z(Y)$ induced by $\underline{\lambda}$. Then σ is induced by τ. We can then identify $\mathcal{H}_\sigma$ with the space of square-integrable functions on $\mathbb{R}$ with values in $\mathcal{H}_\tau$. If $f : A \longrightarrow \mathcal{H}_\tau$ is such a function we have from the definition of induced representations, for $x \in A$,

$$\Big[\int_G \varphi(g)\sigma(g) f \, dg\Big](x) = \int_A da \int_{Z(Y)} dz \; \varphi(az)\tau(x^{-1}aza^{-1}x)f(a^{-1}x)$$

$$\int_A \int_{Z(Y)} \varphi(xza^{-1})\,\tau(z)\,f(a)\,dz\,da$$

(we use here the invariance of dz under $\mathrm{Int}\, g$, $g \in G$ as also the fact that the measure da is invariant under $a \mapsto a^{-1}$). Now for fixed $x, a \in A$, $\psi_{x,a}(z) = \varphi(xza^{-1})$ is a C^∞ function on $Z(Y)$ with rapid decrease so that

$$\int_{Z(Y)} \varphi(xza^{-1})\,\tau(z)\,dz = \tilde{\tau}(\psi_{x,a}) \;(= K(x,a), \text{say})$$

is an operator of trace class on $\mathcal{H}_\tau$. Thus $\tilde{\sigma}(\varphi)$ is given by the (operator valued) kernel $K(x,a)$

One deduces from this easily that

$$\mathrm{Trace}\,\tilde{\sigma}(\varphi) = \int_A k(a,a)\,da$$

where $k(a,a) = \mathrm{Trace}\,K(a,a)$; and by induction hypothesis applied to $Z(Y)$,

$$\mathrm{Trace}\,K(a,a) = \mathrm{Trace}\,\tilde{\tau}(\psi_{aa}) = \int_{\Omega'} \hat{\psi}_{aa}(f)\,d\omega'(f)$$

where Ω' is the orbit of the restriction λ' of λ to $z(Y)$. We have thus

$$\mathrm{Trace}\,\tilde{\sigma}(\varphi) = \int_A da \int_{\Omega'} \hat{\psi}_{aa}(f)\,d\omega'(f).$$

It is not difficult to analyse now the orbit of λ carefully and show that the integral on the right is indeed $\int_\Omega \hat{\varphi}(f)\,d\omega(f)$. ∎