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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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LECTURE NOTES - PART II

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5. Momentum maps and coadjoint orbits.

We have seen in the previous sections how we can modify a group action of G on a symplectic manifold (M,ω) preserving ω to one where there is a Hamiltonian $\lambda : \mathfrak{g} \to C^\infty(M)$ which is a homomorphism of Lie algebras. This we now assume as the basic set-up, and also that G and M are connected.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \xrightarrow{i} & C^\infty(M) & \xrightarrow{j} & \mathrm{Ham} & \longrightarrow & 0 \\ & & & & \nwarrow^{\lambda} & & \uparrow^{\rho} & & \\ & & & & & & \mathfrak{g} & & \end{array}$$

If $\mathfrak{g}^*$ denotes the vector-space dual of $\mathfrak{g}$ then G acts on $\mathfrak{g}$ by the adjoint representation and on $\mathfrak{g}^*$ by its contragredient Ad^*. We denote the pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$ by $\langle\, ,\rangle$ so

$$\langle (Ad^*g)f, \xi\rangle = \langle f, Ad\, g^{-1}(\xi)\rangle \qquad f \in \mathfrak{g}^*,\ \xi \in \mathfrak{g}.$$

Given a Hamiltonian G-space (M,ω,λ) we define a map $P : M \to \mathfrak{g}^*$ by

$$(\circledast) \qquad \langle P(x), \xi\rangle = \lambda(\xi)(x).$$

Theorem 1. If (M,ω,λ) is a Hamiltonian G-space and $P : M \to \mathfrak{g}^*$ is defined as above then P is equivariant for the given action on M and Ad^* on $\mathfrak{g}^*$.

Proof. Since G is connected it is sufficient to verify equivariance for 1-parameter subgroups $\exp t\xi$.

Consider the curve

$$f(t) = Ad^* \exp t\xi \cdot P(\exp -t\xi \cdot x)$$

of elements of $\mathfrak{g}^*$.

$$\langle f(t), \eta \rangle = \langle P(\exp -t\xi \cdot x), \mathrm{Ad}\exp -t\xi(\eta) \rangle$$
$$= \lambda(\mathrm{Ad}\exp -t\xi(\eta))(\exp -t\xi \cdot x).$$

Then

$$\frac{d}{dt}\langle f(t), \eta \rangle = -\lambda(\mathrm{Ad}\exp -t\xi[\xi,\eta])(\exp -t\xi \cdot x) + \tilde{\xi}_{\exp -t\xi \cdot x}(\lambda(\mathrm{Ad}\exp -t\xi(\eta)))$$
$$= -\lambda(\mathrm{Ad}\exp -t\xi[\xi,\eta])(\exp -t\xi \cdot x) + \{\lambda(\xi), \lambda(\mathrm{Ad}\exp -t\xi(\eta))\}(\exp -t\xi \cdot x)$$
$$= 0$$

since λ is a homomorphism. Since this holds for all η,

$$\frac{d f(t)}{dt} = 0 \quad \Rightarrow \quad f(t) = f(0) = P(x).$$

Thus

$$\mathrm{Ad}^*\exp t\xi \, P(\exp -t\xi \cdot x) = P(x) \qquad \forall t, \xi$$

so

$$\mathrm{Ad}^*_g P(x) = P(g \cdot x) \qquad \forall g \in G.$$

■

A partial converse is true. If we have a symplectic G-space (M, ω) then a map $P : M \to \mathfrak{g}^*$ is called a <u>momentum map</u> if

(i) $i_{\tilde{\xi}}\,\omega = d\langle P, \xi \rangle \quad \forall \xi \in \mathfrak{g}$;

(ii) $\mathrm{Ad}^*_g \circ P = P \circ g, \quad \forall g \in G$.

Given a momentum map P we may define λ by ⊛ and then (i) implies $\tilde{\xi} = X_{\lambda(\xi)}$ so

$$\{\lambda(\xi), \lambda(\eta)\}(x) = \tilde{\xi}_x(\lambda(\eta))$$
$$= \frac{d}{dt}\Big|_0 \lambda(\eta)(\exp -t\xi \cdot x)$$

$$= \frac{d}{dt}\Big|_0 \langle P(\exp -t\xi \cdot x), \eta \rangle$$
$$= \frac{d}{dt}\Big|_0 \langle \mathrm{Ad}^*\exp -t\xi \, P(x), \eta \rangle$$
$$= \frac{d}{dt}\Big|_0 \langle P(x), \mathrm{Ad}\exp t\xi(\eta) \rangle$$
$$= \langle P(x), [\xi, \eta] \rangle$$
$$= \lambda([\xi,\eta])(x),$$

showing λ is a homomorphism. Thus Hamiltonian G-spaces are the same as symplectic G-spaces which have a momentum map P.

The reason for the terminology is the following:

<u>Example 1.</u> Consider $\mathbb{R}^n$ acting on $\mathbb{R}^{2n}$ by translations:

$$a \cdot (p, q) = (p, q + a).$$

Then for $\xi \in \mathbb{R}^n$,

$$\tilde{\xi}_{(p,q)} = -\sum_i \xi^i \frac{\partial}{\partial q^i}$$

so

$$i_{\tilde{\xi}}\,\omega = \sum_i \xi^i dp_i = d\left(\sum \xi^i p_i\right).$$

Then

$$\lambda(\xi)(p,q) = \sum_i \xi^i p_i$$

so

$$\langle P(p,q), \xi \rangle = \sum_i \xi^i p_i$$

or

$$P(p,q) = p.$$

Thus the momentum map for translation of position is the momentum itself.

<u>Example 2.</u> $SO(3)$ acting on $\mathbb{R}^6$ by rotations

$$g(p,q) = (gp, gq) \qquad g \in SO(3).$$

If $\xi \in so(3)$

$$\xi_{(p,q)} f = \frac{d}{dt}\Big|_0 f(\exp -t\xi p, \exp -t\xi q)$$

$$= \sum_i -(\xi p)_i \frac{\partial f}{\partial p_i} - (\xi q)^i \frac{\partial f}{\partial q^i}.$$

So

$$\xi_{(p,q)} = \sum_i -(\xi p)_i \frac{\partial}{\partial p_i} - (\xi q)^i \frac{\partial}{\partial q^i}$$

and

$$i_\xi \omega = \sum_i -(\xi p)_i \, dq^i + (\xi q)^i \, dp_i$$

$$= d \sum p_i (\xi q)^i .$$

Thus

$$\langle P(p,q), \xi \rangle = {}^t p \, \xi q .$$

If

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\xi = \xi_1 L_1 + \xi_2 L_2 + \xi_3 L_3$ then

$$\langle P(p,q), \xi \rangle = (p_1, p_2, p_3) \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix}$$

$$= \xi_1 (q^2 p_3 - q^3 p_2) + \xi_2 (q^3 p_1 - q^1 p_3) + \xi_3 (q^1 p_2 - q^2 p_1)$$

so

$$P(p,q) = q \times p .$$

Hence this time the momentum map is angular momentum.

The notion of elementary particle in quantum mechanics is one transforming under an irreducible representation of the symmetry group. We can introduce a similar idea in classical mechanics by requiring the symmetry group G to act transitively on the phase space (M, ω). In general we have

$$\omega_x(\tilde{\xi}_x, \tilde{\eta}_x) = d\lambda(\xi)_x(\tilde{\eta}_x) = \tilde{\eta}_x(\lambda(\xi)) = \{\lambda(\eta), \lambda(\xi)\}(x)$$

$$= \lambda([\eta, \xi])(x)$$

$$= \langle P(x), [\eta, \xi] \rangle .$$

If G is transitive on M, it will be transitive on $P(M) \subset g^*$ which is thus an orbit of the coadjoint representation or <u>coadjoint orbit</u>. The calculation above suggests the following definition: Let $O \subset g^*$ be a coadjoint orbit and $f \in O^*$ then the bilinear form

$$\langle f, [\eta, \xi] \rangle$$

on $g \times g$ is alternating and has as kernel all ξ such that for all η

$$0 = \langle f, [\eta, \xi] \rangle =$$

$$= \frac{d}{dt}\Big|_0 \langle f, \mathrm{Ad} \exp -t\xi \, \eta \rangle$$

$$= \frac{d}{dt}\Big|_0 \langle \mathrm{Ad}^* \exp t\xi f, \eta \rangle .$$

Thus if

$$\hat{\xi}_f = \frac{d}{dt}\Big|_0 \mathrm{Ad}^* \exp -t\xi \cdot f$$

then

$$\hat{\xi}_f = 0 \iff \langle f, [\eta, \xi] \rangle = 0 \quad \forall \eta \in g .$$

It follows

$$\omega^O_f(\hat{\xi}_f, \hat{\eta}_f) = \langle f, [\eta, \xi] \rangle$$

defines a 2-form ω^O on O the Kostant-Kirillov-Souriau 2-form. If we also define

$$\lambda^O(\xi)(f) = \langle f, \xi \rangle$$

so that the corresponding map $\mathcal{O} \to \mathfrak{g}^*$ is just the inclusion map, then

$$(i_{\hat\xi}\,\omega^{\mathcal{O}})_f\,(\hat\eta_f) = \langle f, [\eta,\xi]\rangle$$

$$= \frac{d}{dt}\Big|_0 \langle f, \mathrm{Ad}\exp t\eta\ \xi\rangle$$

$$= \frac{d}{dt}\Big|_0 \langle \mathrm{Ad}^*\exp -t\eta\ f, \xi\rangle$$

$$= \frac{d}{dt}\Big|_0 \lambda^{\mathcal{O}}(\xi)(\mathrm{Ad}^*\exp -t\eta\ f)$$

$$= \hat\eta_f(\lambda^{\mathcal{O}}(\xi)).$$

Since $\{\hat\eta_f : \eta \in \mathfrak{g}\} = T_f\mathcal{O}$ for a homogeneous space, we have

$$\oplus \qquad i_{\hat\xi}\,\omega^{\mathcal{O}} = d\,\lambda^{\mathcal{O}}(\xi).$$

<u>Theorem 2.</u> $d\,\omega^{\mathcal{O}} = 0$.

<u>Proof.</u> Since $\mathcal{O}$ is a homogeneous space it suffices to check on vectors of the form $\hat\xi, \hat\eta, \hat\zeta$. But

$$d\omega^{\mathcal{O}}(\hat\xi,\hat\eta,\hat\zeta) = \hat\xi\,\omega^{\mathcal{O}}(\hat\eta,\hat\zeta) - \hat\eta\,\omega^{\mathcal{O}}(\hat\xi,\hat\zeta) + \hat\zeta\,\omega^{\mathcal{O}}(\hat\xi,\hat\eta) - \omega^{\mathcal{O}}([\hat\xi,\hat\eta],\hat\zeta) + \omega^{\mathcal{O}}([\hat\xi,\hat\zeta],\hat\eta) - \omega^{\mathcal{O}}([\hat\eta,\hat\zeta],\hat\xi).$$

Now

$$\hat\xi_f\,\omega^{\mathcal{O}}(\hat\eta,\hat\zeta) = \frac{d}{dt}\Big|_0 \langle \mathrm{Ad}^*\exp -t\xi\cdot f, [\zeta,\eta]\rangle$$

$$= \langle f, [\xi,[\zeta,\eta]]\rangle$$

so

$$\hat\xi\,\omega^{\mathcal{O}}(\hat\eta,\hat\zeta) = -\lambda([\xi,[\eta,\zeta]])$$

The first three terms in $d\omega$ now cancel by virtue of the Jacobi identity for $\mathfrak{g}$.

Similarly

$$\omega^{\mathcal{O}}([\hat\xi,\hat\eta],\hat\zeta) = \omega^{\mathcal{O}}([\xi,\eta]^\wedge,\hat\zeta)$$

$$= \lambda([[\xi,\eta],\zeta])$$

and so the second group of 3 terms also cancel using the Jacobi identity. ■

Now $\omega^{\mathcal{O}}$ is non-degenerate by construction, and we have shown $d\omega^{\mathcal{O}}=0$. Thus $(\mathcal{O},\omega^{\mathcal{O}})$ is a symplectic manifold. Moreover

$$g_*\,\hat\xi_f = \frac{d}{dt}\Big|_0 \mathrm{Ad}^*_g\,\mathrm{Ad}^*\exp -t\xi\cdot f$$

$$= \frac{d}{dt}\Big|_0 \mathrm{Ad}^*(g\exp -t\xi)\cdot f$$

$$= \frac{d}{dt}\Big|_0 \mathrm{Ad}^*(\exp -t\,\mathrm{Ad}_g\xi\ g)\cdot f$$

$$= (\mathrm{Ad}_g\xi)^\wedge_{\mathrm{Ad}^*_g\cdot f}.$$

Thus

$$(g^*\omega^{\mathcal{O}})_f(\hat\xi_f,\hat\eta_f) = \omega^{\mathcal{O}}_{\mathrm{Ad}^*_g\cdot f}(g_*\hat\xi_f, g_*\hat\eta_f)$$

$$= \omega^{\mathcal{O}}_{\mathrm{Ad}^*_g\cdot f}\left((\mathrm{Ad}_g\xi)^\wedge_{\mathrm{Ad}^*_g\cdot f}, (\mathrm{Ad}_g\eta)^\wedge_{\mathrm{Ad}^*_g\cdot f}\right)$$

$$= \langle \mathrm{Ad}^*_g\cdot f, [\mathrm{Ad}_g\eta, \mathrm{Ad}_g\xi]\rangle$$

$$= \langle f, \mathrm{Ad}_g^{-1}[\mathrm{Ad}_g\eta, \mathrm{Ad}_g\xi]\rangle$$

$$= \langle f, [\eta,\xi]\rangle$$

$$= \omega^{\mathcal{O}}_f(\hat\xi_f,\hat\eta_f).$$

Thus

$$g^*\omega^{\mathcal{O}} = \omega^{\mathcal{O}}.$$

Thus G acts symplectically on $\mathcal{O}$. ⊕ shows that the action is almost Hamiltonian, whilst the inclusion map $\mathcal{O} \subset \mathfrak{g}^*$ is necessarily equivariant, so $\lambda^{\mathcal{O}} : \mathfrak{g} \to C^\infty(\mathcal{O})$ is a homomorphism. Hence $(\mathcal{O}, \omega^{\mathcal{O}}, \lambda^{\mathcal{O}})$ is a Hamiltonian G-space. This is

<u>Theorem 3.</u> (Kirillov, Kostant, Souriau) $(\mathcal{O}, \omega^{\mathcal{O}}, \lambda^{\mathcal{O}})$ is a Hamiltonian G-space.

We can now continue to examine the general Hamiltonian G-space (M, ω, λ). The momentum map $P : M \longrightarrow \mathcal{O} \subset \mathfrak{g}^*$ is equivariant, so $P_* \tilde{\xi}_x = \hat{\xi}_{P(x)}$, thus

$$\begin{aligned} \omega_x(\tilde{\xi}_x, \tilde{\eta}_x) &= \langle P(x), [\eta, \xi] \rangle \\ &= \omega^{\mathcal{O}}_{P(x)}(\hat{\xi}_{P(x)}, \hat{\eta}_{P(x)}) \\ &= \omega^{\mathcal{O}}_{P(x)}(P_* \tilde{\xi}_x, P_* \tilde{\eta}_x) \\ &= (P^* \omega^{\mathcal{O}})_x (\tilde{\xi}_x, \tilde{\eta}_x). \end{aligned}$$

Thus for homogeneous Hamiltonian G-spaces

$$\omega = P^* \omega^{\mathcal{O}}.$$

We have

<u>Theorem 4.</u> If (M, ω, λ) is a homogeneous Hamiltonian G-space and the momentum map P has image $\mathcal{O}$, then $\omega = P^* \omega^{\mathcal{O}}$, $\lambda(\xi) = \lambda^{\mathcal{O}}(\xi) \circ P$ and P is a covering map.

<u>Proof.</u> We have seen $\omega = P^* \omega^{\mathcal{O}}$, whilst

$$\begin{aligned} \lambda^{\mathcal{O}}(\xi) \circ P(x) &= \lambda^{\mathcal{O}}(\xi)(P(x)) = \langle P(x), \xi \rangle \\ &= \lambda(\xi)(x) \end{aligned}$$

by definition. P is a covering map because $\omega = P^* \omega^{\mathcal{O}}$ shows P is an immersion and is equivariant. ■

Thus the only homogeneous Hamiltonian G-spaces are the covering spaces of the coadjoint orbits of G. This is the reason for the importance of coadjoint orbits. We can combine Theorem 5 of §4 with Theorem 4 above to conclude

<u>Theorem 5.</u> If G acts symplectically and transitively on (M, ω) then M is a covering space of a coadjoint orbit of $\hat{G}_0$ on $\hat{\mathfrak{g}}_0^*$. If $H^2(\mathfrak{g}) = 0$ M covers an orbit in $\mathfrak{g}^*$.

Thus transitive symplectic actions can be classified quite explicitly. There are many examples described in the literature, so we shall limit ourselves to one simple example

<u>Example 3</u> $G = SO(3)$.

The Lie algebra is spanned by L_1, L_2, L_3 and

$$\xi_1 L_1 + \xi_2 L_2 + \xi_3 L_3 = \begin{bmatrix} 0 & -\xi_3 & +\xi_2 \\ +\xi_3 & 0 & -\xi_1 \\ -\xi_2 & +\xi_1 & 0 \end{bmatrix}.$$

The invariant form

$$(\xi, \eta) = -\mathrm{Tr}(\xi, \eta)$$

allows us to identify $\mathfrak{g}$ and $\mathfrak{g}^*$ so that Ad and Ad* agree.

If $x \in \mathbb{R}^3$ set $$L(x) = x^1L_1 + x^2L_2 + x^3L_3 = \begin{bmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{bmatrix}.$$

Then $[L_1, L_2] = L_3$, etc. so
$$[L_1, L(x)] = x^2L_3 - x^3L_2, \quad L_1 x = \begin{pmatrix} 0 \\ -x^3 \\ x^2 \end{pmatrix} \quad \text{so}$$
$$[L_1, L(x)] = L(L_1 x)$$
and so on for the other generators. Since this equation is linear, we have
$$[\xi, L(x)] = L(\xi x) \qquad \xi \in so(3),$$
and exponentiating
$$g L(x) g^{-1} = L(gx) \qquad g \in SO(3).$$
If we define a map $\mathbb{R}^3 \longrightarrow so(3)^*$ by
$$x \longmapsto (L(x), \cdot\,)$$
then
$$\begin{aligned} \langle Ad^*_g x, \xi \rangle &= \langle x, Ad_{g^{-1}} \xi \rangle \\ &= -Tr(L(x) g^{-1} \xi g) \\ &= -Tr(g L(x) g^{-1} \xi) \\ &= -Tr(L(gx)\xi) \\ &= \langle gx, \xi \rangle. \end{aligned}$$
Thus $so(3)^* = \mathbb{R}^3$ and the coadjoint action coincides with the usual rotations of $\mathbb{R}^3$. The coadjoint orbits are thus 2-spheres given by
$$S^2_R = \{ x : (x^1)^2 + (x^2)^2 + (x^3)^2 = R^2 \} \qquad R > 0$$
together with the origin. This means we have an $SO(3)$-invariant symplectic structure on each S^2_R.

Acting on S^2_R
$$\hat{\xi}_x = \frac{d}{dt}\Big|_0 e^{-t\xi} x = -(\xi x)^1 \frac{\partial}{\partial x^1} - (\xi x)^2 \frac{\partial}{\partial x^2} - (\xi x)^3 \frac{\partial}{\partial x^3}.$$
Then if $O = S^2_R$
$$\begin{aligned} \omega^O_x(\hat{\xi}_x, \hat{\eta}_x) &= -Tr(L(x)[\eta, \xi]) \\ &= 2x_1(\eta_2\xi_3 - \eta_3\xi_2) + 2x_2(\eta_3\xi_1 - \eta_1\xi_3) + 2x_3(\eta_1\xi_2 - \eta_2\xi_1). \end{aligned}$$

On the other hand
$$\begin{aligned} &(x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2)(\hat{\xi}_x, \hat{\eta}_x) \\ &= x^1\{(\xi x)^2(\eta x)^3 - (\xi x)^3(\eta x)^2\} + x^2\{(\xi x)^3(\eta x)^1 - (\xi x)^1(\eta x)^3\} + x^3\{(\xi x)^1(\eta x)^2 - (\xi x)^2(\eta x)^1\} \\ &= x^1\{(\xi_3 x^1 - \xi_1 x^3)(\eta_1 x^2 - \eta_2 x^1) - (\xi_1 x^2 - \xi_2 x^1)(\eta_3 x^1 - \eta_1 x^3)\} \\ &\quad + x^2\{(\xi_1 x^2 - \xi_2 x^1)(\eta_2 x^3 - \eta_3 x^2) - (\xi_2 x^3 - \xi_3 x^2)(\eta_1 x^2 - \eta_2 x^1)\} \\ &\quad + x^3\{(\xi_2 x^3 - \xi_3 x^2)(\eta_3 x^1 - \eta_1 x^3) - (\xi_3 x^1 - \xi_1 x^3)(\eta_2 x^3 - \eta_3 x^2)\} \\ &= R^2\{x^1(\xi_2\eta_3 - \xi_3\eta_2) + x^2(\xi_3\eta_1 - \xi_1\eta_3) + x^3(\xi_1\eta_2 - \xi_2\eta_1)\}. \end{aligned}$$

Thus
$$\omega^O = -\frac{2}{R^2}(x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2).$$
So up to a constant factor, the invariant 2-form on S^2 is the solid angle. In fact:
$$\int_{S^2} \omega^O = 8\pi R.$$
using the above identifications.

6 Quantization.

Quantum mechanics is described by a Hilbert space V and time evolution by a 1-parameter group

$$U(t) = e^{t\hat{H}/i\hbar}$$

of unitary operators on V, where $\hat{H}$ is the Hamiltonian operator. If F is an observable its time evolution is given by

$$F_t = U(t)^{-1} F\, U(t)$$

so it satisfies the differential equation

$$\frac{dF_t}{dt} = \frac{1}{i\hbar}[F_t, \hat{H}]$$

where $[\ ,\ ]$ denotes commutator of operators.

If we compare this with the classical equation

$$\frac{df_t}{dt} = \{f_t, H\}$$

we see that we can obtain quantum mechanics from classical mechanics by a process of replacing classical observables $f \in C^\infty(M)$ by quantum observables $Q(f)$ (acting on V) and the dynamics will correspond provided

$$\circledast \qquad Q(\{f,g\}) = \frac{1}{i\hbar}[Q(f), Q(g)].$$

We require additionally $Q(1) = I$ together with some irreducibility condition. van Hove showed that these conditions are not compatible on all of $C^\infty(M)$, but usually only a small part is actually needed to correspond with the genuine physical observables.

If we ignore the irreducibility question until later, then we can make quite substantial progress. We already have operators, namely the Hamiltonian vector fields X_f which operate on functions, and

$$\frac{1}{i\hbar}[i\hbar X_f, i\hbar X_g] = i\hbar[X_f, X_g] = i\hbar X_{\{f,g\}}.$$

But $f \mapsto i\hbar X_f$ is not a quantization because it sends constants to zero. The most general formula we might guess which has $Q(1) = I$ and is built from f and X_f is

$$Q(f) = i\hbar X_f + \alpha(X_f) + f$$

for some 1-form α. Then

$$\frac{1}{i\hbar}[Q(f), Q(g)] = i\hbar[X_f, X_g] + X_f(\alpha(X_g)+g) - X_g(\alpha(X_f)+f).$$

So to give a quantization we want

$$X_f(\alpha(X_g)+g) - X_g(\alpha(X_f)+f) = \alpha(X_{\{f,g\}}) + \{f,g\},$$

or

$$X_f(\alpha(X_g)) - X_g(\alpha(X_f)) - \alpha([X_f, X_g]) = -\{f,g\}$$

or

$$d\alpha(X_f, X_g) = \omega(X_f, X_g).$$

Since Hamiltonian vector fields span the tangent spaces $\circledast$ will be satisfied precisely when $\omega = d\alpha$.

Thus when ω [redacted] is exact, any choice of a 1 form

α with $\omega = d\alpha$ will give a quantization $Q(f)$. This applies in particular to the cotangent bundle case T^*C where there is a natural choice $\alpha = \Theta$ the canonical 1-form.

Arguments leading to this choice and its uniqueness are discussed in [B].

In the general case ω is closed but not exact. The Poincaré lemma tells us that ω is locally exact, so can be written $\omega = d\alpha_i$ on some covering $\{U_i\}$ of M by contractible open sets. Then on U_i we can take $i\hbar X_f + \alpha_i(X_f) + f$ and ask if we can piece these formulae together. On $U_i \cap U_j$ we will have

$$d(\alpha_i - \alpha_j) = \omega - \omega = 0$$

so

$$\alpha_i - \alpha_j = da_{ij}$$

for functions a_{ij} on $U_i \cap U_j$. If we set

$$c_{ij} = e^{i a_{ij}/\hbar}$$

then

$$Q_i(f) = i\hbar X_f + \alpha_i(X_f) + f$$

satisfies

$$\begin{aligned} Q_i(f)(c_{ij} g) &= c_{ij} Q_i(f)(g) + i\hbar X_f(c_{ij}) g \\ &= c_{ij} Q_j(f)(g) - X_f(a_{ij}) c_{ij} g \\ &\quad + (\alpha_i - \alpha_j)(X_f) c_{ij} g \\ &= c_{ij} Q_j(f)(g). \end{aligned}$$

Thus viewing the c_{ij} as multiplication operators, on $U_i \cap U_j$ we have

$$Q_j(f) = c_{ij}^{-1} \circ Q_i(f) \circ c_{ij}.$$

If we can choose the α_i and a_{ij} so that

$$\text{(A)} \quad \begin{cases} c_{ii} \equiv 1 & \text{on } U_i \\ c_{ij} c_{ji} \equiv 1 & \text{on } U_i \cap U_j \\ c_{ij} c_{jk} c_{ki} \equiv 1 & \text{on } U_i \cap U_j \cap U_k \end{cases}$$

then we can form a space V as follows: we consider functions $g_i : U_i \to \mathbb{C}$ for each i such that

$$\text{(B)} \qquad g_i = c_{ij} g_j \quad \text{on } U_i \cap U_j.$$

If $s = \{g_i\}$ is such a family of functions then we define

$$Q(f)s = \{Q_i(f) g_i\}.$$

This is well-defined because

$$c_{ij} Q_j(f) g_j = Q_i(f) c_{ij} g_j = Q_i(f) g_i.$$

This gives operators $Q(f)$ on V for each $f \in C^\infty(M)$ and

$$\begin{aligned} [Q(f), Q(g)]s &= \{[Q_i(f), Q_i(g)] g_i\} \\ &= \{i\hbar Q_i(\{f,g\}) g_i\} \\ &= i\hbar Q(\{f,g\}) s \end{aligned}$$

so

$$[Q(f), Q(g)] = i\hbar Q(\{f,g\}).$$

Thus when ω is not exact, if we can find solutions to (A) then we can still define operators on a space V. (A) is the set of consistency conditions in order that (B) have non-trivial solutions. Families of functions satisfying (A) are said to be transition functions, or 1-cocycles. A 1-coboundary is a family of the form

$$b_{ij} = b_i / b_j \quad \text{on } U_i \cap U_j$$

where $b_i : U_i \to \mathbb{C}^*$. The space of cocycles modulo coboundaries is a cohomology group $\check{H}^1(\mathcal{U};\mathbb{C}^*)$ where $\mathcal{U} = \{U_i\}$ is the given covering, and $\mathbb{C}^*$ is the sheaf of zero-free complex functions. Taking a fine enough covering this group becomes independent of $\mathcal{U}$ and coincides with the set of line bundles on M. The correspondence is as follows:

If $\pi : L \to M$ is a complex line bundle, then it is locally trivialized by zero-free sections s_i on some covering $\{U_i\}$. Then on $U_i \cap U_j$

$$s_j = c_{ij} s_i$$

for $c_{ij} : U_i \cap U_j \to \mathbb{C}^*$. These transition functions then satisfy (A). A section s of L will be given by

$$s|_{U_i} = g_i s_i$$

for some $g_i : U_i \to \mathbb{C}$ and then on $U_i \cap U_j$

$$s|_{U_i \cap U_j} = g_i s_i = g_j s_j = g_j c_{ij} s_i$$

so

$$g_i = c_{ij} g_j .$$

Thus condition (B) is specifying the space of sections of

L : $V = \Gamma L$.

Not every ω can be quantized as above, for if we take arbitrary α_i with $\omega|_{U_i} = d\alpha_i$ and then a_{ij} with

$$(\alpha_i - \alpha_j)|_{U_i \cap U_j} = d\, a_{ij}$$

we will have

$$d(a_{ij} + a_{jk} + a_{ki}) = \alpha_i - \alpha_j + \alpha_j - \alpha_k + \alpha_k - \alpha_i = 0$$

on $U_i \cap U_j \cap U_k$, so $c_{ijk} = a_{ij} + a_{jk} + a_{ki}$ is a constant. It is a 2-cocycle for the constant sheaf, so defines an element of $\check{H}^2(M;\mathbb{C})$. To satisfy (A) we need

$$e^{i c_{ijk}/\hbar} = 1$$

or $\quad c_{ijk}/\hbar \in 2\pi \mathbb{Z}$

or $\quad c_{ijk}/h \in \mathbb{Z}$.

We say a class in $\check{H}^2(M;\mathbb{C})$ is integral if it has a representative which has only integer values. We see that we can only carry out the quantization process above when ω/h corresponds with an integral class in $\check{H}^2(M;\mathbb{C})$. In this case we say ω/h is an integral form. We thus have

Theorem 1. If ω/h is an integral form then we can define operators $Q(f)$ on $V = \Gamma L$ for every $f \in C^\infty(M)$ which satisfy

$$\frac{1}{i\hbar}[Q(f), Q(g)] = Q(\{f,g\}).$$

This process is called prequantization.

Before we go on to consider some of its properties let us write $Q(f)$ in a more convenient form. We may define for any vector field X on M an operator D_X on ΓL by

$$D_X s = \{X(g_i) + \frac{1}{i\hbar}\alpha_i(X) g_i\} \quad , \text{ if } \quad s = \{g_i\}.$$

The same calculation as before shows this is well-defined and is linear in s and X, moreover

$$D_{fX} s = f D_X s,$$

whilst

$$D_X(fs) = \{X(fg_i) + \frac{1}{i\hbar}\alpha_i(X) f g_i\}$$

$$= \{X(f)g_i + f(X(g_i) + \frac{1}{i\hbar}\alpha_i(X) g_i)\}$$

$$= X(f)s + f D_X s.$$

This shows D is a covariant differentiation in the line bundle L or a connection. Then

$$Q(f) = i\hbar D_{X_f} + f.$$

The curvature of D is given by

$$([D_X, D_Y] - D_{[X,Y]})s$$

$$= \{[X + \frac{1}{i\hbar}\alpha_i(X), Y + \frac{1}{i\hbar}\alpha_i(Y)]g_i - [X,Y]g_i - \frac{1}{i\hbar}\alpha_i([X,Y])g_i\}$$

$$= \frac{1}{i\hbar}\{(X\alpha_i(Y) - Y\alpha_i(X) - \alpha_i([X,Y]))g_i\}$$

$$= \frac{1}{i\hbar}\{d\alpha_i(X,Y) g_i\} = \frac{1}{i\hbar}\omega(X,Y)s$$

Thus D is a connection with curvature $\frac{1}{i\hbar}\omega$.

The converse is easy to prove: given a line bundle L with a connection D having curvature $\frac{1}{i\hbar}\omega$ then $i\hbar D_{X_f} + f$ defines a prequantization of $C^\infty(M)$.

Then ω/h is the de Rham representative of the Chern class $c_1(L)$ of L.

The main problem with this construction is that the space ΓL on which the operators act is too big, it is essentially functions of p and q in Darboux coordinates and we know that irreducible representations of the commutation relations should be on functions of only half the variables. In general we do not have such coordinates defined globally and so we need a more geometrical way of "removing half the variables".

Such a process can be devised by introducing the idea of a polarization. This is a tangent distribution $F \subset TM$ of half the dimension of M: $\operatorname{rank} F = \frac{1}{2}\dim M$ and on which ω vanishes identically: $\omega(X,Y) = 0$ for all $X, Y \in F$. If in addition F is integrable, so $[X,Y] \in F$ whenever X, Y are vector fields with values in F then a modification of Darboux's theorem says there exist coordinates locally such that $\omega = \sum_i dp_i \wedge dq^i$ and F is spanned by $(\partial/\partial p_1, \dots, \partial/\partial p_n)$. Then looking at functions on M annihilated by F is the same as looking at functions of the q's alone.

More generally if we have a line bundle L with connection D having curvature $\omega/i\hbar$ then we can form the polarized sections

$$\Gamma_F L = \{ s\in \Gamma L \;:\; D_X s = 0 \;,\; \forall X\in F\}.$$

$s\in\Gamma_F L$ will be represented locally by functions $\{g_i\}$ where the g_i depend only on the q's. However it is not true any more that $Q(f)$ operates on $\Gamma_F L$ for every $f\in C^\infty(M)$. Rather there is a Lie subalgebra C^1_F of $C^\infty(M)$ given by

$$C^1_F = \{f\in C^\infty(M) : [X_f, X]\in F \quad \forall X\in F\},$$

and

$$Q(f)\,\Gamma_F L \subset \Gamma_F L \quad \forall \; f\in C^1_F .$$

C^1_F is the Lie algebra of quantizable functions for the polarization F.

Many symplectic manifolds do not have real polarizations F, so it is important to generalize the above to allow F to be complex: $F\subset TM^{\mathbb{C}}$, and we keep the remaining conditions $\mathrm{rank}_{\mathbb{C}} F = \frac{1}{2}\dim M$, $\omega(F,F)=0$ and $[F,F]\subset F$. Such an F is called a complex polarization of (M,ω). It is usually necessary to impose regularity conditions on F ($F+\bar{F}$ should have constant dimension and be integrable) to avoid pathologies which can arise in the complex case.

If we have a symplectic G action it may not preserve the quantization. If however the action is Hamiltonian with Hamiltonian λ and F is G-invariant, then differentiating

$$g_* F_x = F_{gx} \;, \quad \forall g\in G$$

gives

$$[\tilde{\xi}, X]\in F \quad \forall \; X\in F \;, \quad \forall \xi\in\mathfrak{g}$$

so

$$[X_{\lambda(\xi)}, X]\in F \quad \forall \; X\in F \;, \quad \forall \xi\in\mathfrak{g}$$

or

$$\lambda(\xi)\in C^1_F \quad \forall \; \xi\in\mathfrak{g} .$$

Thus

$$\xi \longmapsto \frac{1}{i\hbar}\, Q(\lambda(\xi))$$

will be a homomorphism of $\mathfrak{g}$ into the operators on $\Gamma_F L$. Kostant has shown that this homomorphism always exponentiates to a group representation when M is homogeneous. It is essentially a subrepresentation of an induced representation on ΓL.

Thus we have: each integral coadjoint orbit $\mathcal{O}$ of a Lie group G which has an invariant polarization F gives rise to a representation of G on the polarized sections $\Gamma_F L$ of a line bundle L which has a connection D having curvature $\omega/i\hbar$. $\mathfrak{g}$ acts by quantization of λ.

This is the method of orbits for constructing representations.

7. Examples and further generalizations.

1. The standard example: $\mathbb{R}^{2n}$, $\omega = \sum dp_i \wedge dq^i$ then one chart covers $\mathbb{R}^{2n}$, $\omega = d\alpha$ where $\alpha = \frac{1}{2}\sum_i (p_i dq^i - q^i dp_i)$. The line bundle L is trivial, so $\Gamma L = C^\infty(\mathbb{R}^{2n})$ and

$$D_X g = X(g) + \frac{1}{i\hbar}\alpha(X) g,$$

$$Q(f) = i\hbar D_{X_f} + f$$

then gives

$$Q(p_i) = -i\hbar \frac{\partial}{\partial q^i} + \frac{1}{2}p_i \quad , \quad Q(q^i) = i\hbar \frac{\partial}{\partial p_i} + \frac{1}{2}q^i .$$

This is obviously not the Schrödinger quantization. However if we take F spanned by $\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}$, then g will be polarized if

$$0 = D_{\frac{\partial}{\partial p_i}} g = \frac{\partial g}{\partial p_i} - \frac{1}{2i\hbar} q^i g$$

$$\Rightarrow \qquad g = e^{\frac{1}{2i\hbar}\sum p_i q^i}\varphi(q)$$

and for such g's

$$Q(p_i) g = e^{\frac{1}{2i\hbar}\sum p_i q^i}\left(-i\hbar \frac{\partial \varphi}{\partial q^i}\right),$$

$$Q(q^i) g = e^{\frac{1}{2i\hbar}\sum p_i q^i}\left(q^i \varphi\right),$$

so the usual Schrödinger quantization appears on the polarized sections $\Gamma_F L$. The problem is that C^1_F is now small:

$$C^1_F = \{ f = \sum_i p_i \varphi^i(q) + \varphi(q) \}$$

which is only the linear polynomials in $p_1, \dots, p_n$. It does not include functions quadratic in the p's. A second polarization we might take is to define

$$z_j = p_j + i q^j$$

and take F spanned by $\frac{\partial}{\partial \bar z_1}, \dots, \frac{\partial}{\partial \bar z_n}$. Then

$$D_{\frac{\partial}{\partial \bar z_j}} f = \frac{\partial f}{\partial \bar z_j} + \frac{z_j}{4\hbar} f$$

so f is polarized if $\frac{\partial f}{\partial \bar z_j} + \frac{z_j}{4\hbar} f = 0 \ \forall j$, or

$$f(z, \bar z) = e^{-\sum |z_j|^2/4\hbar}\, \varphi(z_1, \dots, z_n)$$

where φ is holomorphic. Then

$$Q(p_i) f = e^{-\sum |z_j|^2/4\hbar}\left\{ \hbar \frac{\partial \varphi}{\partial z_j} + \frac{z_i}{2}\varphi \right\}$$

$$Q(q_i) f = e^{-\sum_j |z_j|^2/4\hbar}\left\{ i\hbar \frac{\partial \varphi}{\partial z_i} - i\frac{z_i}{2}\varphi \right\},$$

and so on these polarized sections we get the Bargmann-Segal-Fock quantization of the CCR on holomorphic functions.

2. $M = T^*C$, $\omega = d\theta$ is also globally exact, so we can take

$$Q(f) = i\hbar X_f + \theta(X_f) + f$$

acting on functions on T^*C. This is again too big, so we can take a polarization F which is spanned by $\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}$ for coordinates arising from coordinates on C. Then F is all tangents to fibres of T^*C. $D_X = X + \frac{1}{i\hbar}\theta(X)$

and $\theta = \sum p_i dq^i$, so $\theta(X) = 0$ for $X \in F$. Hence $D_X f = 0$ implies $Xf = 0$ so f is a function $\varphi\circ\pi$ only on C. We thus get quantization on functions on C, and C'_F consists of functions linear in $p_1, \ldots, p_n$. Such functions are of the form

$$f = \lambda(X) + \varphi\circ\pi$$

where $X \in \mathfrak{X}(C)$ and

$$\lambda(X)(p) = p(X).$$

Thus this quantization also has the limitation that it quantizes functions only linear in p's.

Coadjoint orbits.

This is the case of interest for representation theory. Now the symplectic form ω need not be exact so we have to consider non-trivial line bundles.

Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit and $f \in \mathcal{O}$. Then G acts transitively on $\mathcal{O}$ and

$$G_f = \{ g \in G : \mathrm{Ad}^*_g f = f \}$$

is the stabiliser of f. It has Lie algebra $\mathfrak{g}_f$ given by

$$\mathfrak{g}_f = \{ \xi \in \mathfrak{g} : f \circ \mathrm{ad}\,\xi = 0 \}$$
$$= \{ \xi \in \mathfrak{g} : \langle f, [\xi, \eta] \rangle = 0 \}.$$

Hence f as a map $\mathfrak{g}_f \to \mathbb{R}$ is a homomorphism of Lie algebras, so

$$f/i\hbar : \mathfrak{g}_f \to i\mathbb{R}$$

is a homomorphism from $\mathfrak{g}_f$ to the Lie algebra $u(1)$ of $U(1)$.

Then we have

Theorem 1. (Kostant) $\mathcal{O}, \omega^{\mathcal{O}}$ is integral if and only if $f/i\hbar$ exponentiates to a homomorphism $\chi : G_f \to U(1)$.

If this is the case, then we can construct a line bundle L^χ over $\mathcal{O}$ as follows: define an action of G_f on $G \times \mathbb{C}$ by

$$(g, c)\cdot h = (gh, \chi(h^{-1})c)$$

and let L^χ be the quotient space $(G\times\mathbb{C})/G_f$. Define

$$\pi : L^\chi \to \mathcal{O}$$

by

$$\pi(g, c) = \mathrm{Ad}^*_g f.$$

This is well-defined since G_f fixes f. It makes L^χ into a line bundle over $\mathcal{O}$. The space of sections s of L coincides with the functions $\varphi : G \to \mathbb{C}$ satisfying

$$\varphi(gh) = \chi(h^{-1})\varphi(g) \qquad h \in G_f,\ g \in G$$

because given such a function $(g, \varphi(g))\cdot G_f$ is a well-defined point in L^χ over $\mathrm{Ad}^*_g f \in \mathcal{O}$. Defining

$$D_\xi \varphi = \xi\varphi + \langle \tfrac{f}{i\hbar}, \xi \rangle \varphi$$

gives a covariant differentiation on such functions on G and so on sections of L, which has curvature $\omega/i\hbar$. It thus determines a prequantization. This can easily be seen to be the induced representation $\mathrm{ind}_{G_f}^{G}\chi$.

A polarization $F \subset T^{\mathbb{C}}$ which is G-invariant is determined by F_f, and this in turn by

$$\mathfrak{p} = \{ \xi \in \mathfrak{g}^{\mathbb{C}} : \xi_f \in F_f \}.$$

We clearly have

$$\mathfrak{g}_f^{\mathbb{C}} \subset \mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$$

and $\operatorname{codim}_{\mathbb{C}} \mathfrak{p} = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{O}$. $\omega(F,F)=0$ is equivalent to

$$\langle f, [\mathfrak{p},\mathfrak{p}] \rangle = 0$$

and F integrable to $\mathfrak{p}$ being a subalgebra. The regularity condition is $\mathfrak{p}+\bar{\mathfrak{p}}$ a subalgebra. Such an algebra is said to be a polarization subordinate to f.

There is a further completeness condition, the Pukanszky condition, but this is technical and refer to the literature for details. Polarizations with all these conditions are called strongly admissible.

Kirillov showed every coadjoint orbit of a nilpotent Lie group is integral, has a strongly admissible polarization and the corresponding representation of G is irreducible. These representations account for all unitary representations of G: $\hat{G} = \mathfrak{g}^*/G$ if G is simply-connected. The representation is independent of the choice of strongly admissible polarization which can always be chosen real.

The Borel-Weil theorem for compact Lie groups can be viewed as a special case, where F = antiholomorphic tangents at $G^{\mathbb{C}}/\mathfrak{b} = G/T$, so $\mathfrak{p} = \mathfrak{b}$ is the Borel subalgebra. Taking polarized sections in this case is the same as holomorphic sections and the representation obtain is irreducible. There is a unique invariant polarization satisfying a positivity condition this time.

Auslander and Kostant extended the nilpotent case to type I solvable groups, but more complications are involved. The orbits may have non-isomorphic line bundles (i.e.

different characters χ_j ~~with~~ which exponentiate $f/i t_j$ when $\mathcal{O}$ is not simply-connected) and complex polarizations have to be used.

The non-compact semisimple case is not yet fully settled. Examples are known of coadjoint orbits without ~~a~~ invariant polarizations (the Kepler manifold as an orbit of SO(4,2)) or where there are several polarizations and these give different representations. For linear ~~ss~~ s.s. Lie groups a lot of representations can be constructed by this method, for example the discrete series was constructed by W. Schmid by a cohomological extension of this theory, and enough representations can be constructed for the Plancherel theorem. However complementary series do not seem to arise by quantization, and there are often interesting isolated representations which have important properties, but seem difficult to construct by quantization inspired methods.

4. I have said little about the Hilbert space structure. In fact this can be obtained by building in certain square roots of volumes called half-forms in a modified procedure. This use a symplectic analogue of spinors. Two half-forms when placed together give a volume form which can then be integrated to give an inner product on which the canonical transformations act unitarily at the ~~pre~~quantization level if they preserve the polarization. I attach a preprint on this by P. Robinson and myself

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Mp^c STRUCTURES AND GEOMETRIC QUANTIZATION

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§1. INTRODUCTION

The Kostant-Souriau [9] [19] scheme for geometric quantization of a symplectic manifold (X,ω) requires both that $[\frac{\omega}{h}] \in H^2(X;\mathbb{R})$ be integral and $c_1(\omega) \in H^2(X;\mathbb{Z})$ be even; this rules out important cases such as the space $\mathbb{CP}(2n)$ of orbits in the energy surface of a harmonic oscillator in odd dimension $2n+1$. Hess [7] proposed the use of Mp^c structures in geometric quantization; this leads to the quantization rule that $[\frac{\omega}{h}] + \frac{1}{2} c_1(\omega)^{\mathbb{R}}$ be integral and so allows for the quantization of all harmonic oscillators (as does the scheme due to Czyz [5]). Hess made use of a pair of polarizations and constructed the quantization directly without passing through a prequantization stage. In order to be able to compare the results of quantization with respect to different polarizations it is desirable to have a single prequantization module on which all polarizations act.

It is the purpose of this note to modify Hess's approach so as to incorporate Kostant's symplectic spinors in such a prequantization. Our scheme is based on the observation that an Mp^c structure is sufficient for the definition of symplectic spinors - thus it is not necessary to assume a metaplectic structure. Moreover, all symplectic manifolds admit Mp^c structures.

This note is organized as follows: §2 is an account of Mp^c and its metaplectic representation. Mp^c is the group of all those unitary self-equivalences of a nontrivial irreducible unitary representation of the Heisenberg group which project to the symplectic group Sp; as such it is a symplectic analogue of $Spin^c$. Mp^c is a non-split central circle extension of Sp and contains the metaplectic group Mp as the kernel of a distinguished character η. We discuss symplectic spinors E' and the (one-dimensional) vacuum states $(E')^{\Gamma} \subset E'$ annihilated by positive polarizations Γ.

In §3 we describe the general theory of Mp^c structures as objects of study in their own right: an Mp^c structure for a symplectic vector bundle (E,ω) over X is a lift P of the symplectic frame bundle $Sp(E,\omega)$ to the structure group Mp^c. We establish their unconditional existence (as observed by several authors: Forger & Hess [6], Rawnsley, and Plymen [10]) and parametrize the space of their equivalence classes. We describe how symplectic spinors give rise naturally to half-forms. Let $E'(P)$ be the bundle of symplectic spinors associated to the Mp^c structure P via the metaplectic representation of Mp^c on E', let $P(\eta)$ be the Hermitian line bundle associated to P via the unitary character η of Mp^c; for a positive polarization F of (E,ω) denote by K^F the canonical bundle and by $E'(P)^F \subset E'(P)$ the bundle of vacuum states; then the half-form bundle $Q_P^F = E'(P)^F \otimes K^F$ is a square-root of $P(\eta) \otimes K^F$. Half-form pairings are presented for transverse and regular pairs of positive polarizations. The use of Mp^c structures compares favourably with metaplectic structures; in particular, we show:

(a) Mp^c structures always exist (indeed, (E,ω) comes equipped with a canonical class of Mp^c structures); (E,ω) admits metaplectic structures iff the Chern class $c_1(E,\omega) \in H^2(X;\mathbb{Z})$ is even.

(b) Mp^c structures always pass to the symplectic normal $(D^\perp/D,\omega_D)$ of an isotropic subbundle D of (E,ω); in contrast, if (E,ω) is metaplectic then $(D^\perp/D,\omega_D)$ is metaplectic iff the Stiefel-Whitney square $w_1(D)^2 \in H^2(X;\mathbb{Z}_2)$ is zero.

Our geometric quantization scheme for the symplectic manifold (X,ω) is presented in §4. We begin by setting up prequantization data in the form of a prequantized Mp^c structure (P,γ) for (X,ω). The $u(1)$-valued 1-form γ on D corresponds naturally to a metric connection ∇^γ (of curvature $\frac{2\omega}{i\hbar}$) in the Hermitian line bundle $P(\eta)$. (X,ω) admits such prequantized Mp^c structures iff $[\frac{\omega}{\hbar}] + \frac{1}{2} c_1(\omega)^{\mathbb{R}} \in H^2(X;\mathbb{R})$ is integral. As prequantization module we take the space

$\Gamma(X;E'(P))$; the prequantization map δ itself comes from the prequantum form γ. Although our approach to prequantization seems at first sight to be unwieldy, it leads to a natural development of quantization. If F is a positive polarization of (X,ω) then δ restricts to give operators on $E'(P)^F$ defined for functions on X whose Hamiltonian flows preserve F; tensoring with Lie derivative in K^F then gives a representation δ^F of these functions $C^1_F(X)$ on the space $\Gamma_P(X;Q^F_P)$ of polarized sections of the half-form bundle Q^F_P. Our quantization δ^F squares up on $P(\eta) \otimes K^F$ to give the Kostant-Souriau prequantization of $(X,2\omega)$ determined by $(P(\eta),\nabla^\gamma)$ tensored with Lie differentiation in K^F; this observation turns out to be rather useful in practice. The half-form pairing allows for both the construction of Hilbert spaces on which to quantize and the comparison of quantizations arising from different polarizations. For background material on geometric quantization consult [2, 9, 18].

We test our proposed scheme in §5, where we discuss specific examples. As our first example we consider a linear symplectic manifold - essentially $\mathbb{R}^{2m}$ with its flat symplectic structure. The first-principles method adopted in this case indicates how our general quantization results are established in a local setting. The general method is illustrated in our second example - that of a complex projective space. We are able to quantize all complex projective spaces in a uniform manner, and recover the familiar quantization condition on energy; quantization takes place on spaces of homogeneous polynomials, which we realize as holomorphic sections of appropriate tensor powers of the hyperplane section bundle.

In conclusion, it appears that Mp^c structures are more natural than metaplectic structures and that the geometric quantization scheme (incorporating symplectic spinors) to which Mp^c structures give rise is both elegant in theory and effective in practice.

This article constitutes a revision and extension of an earlier (November 1982) Warwick preprint, and is in effect a condensed version of the first author's doctoral thesis [16] developed from the second author's informal notes [14]. The first author acknowledges the financial support of an S.E.R.C. studentship; the second author thanks R. Blattner and R. Plymen for stimulating conversations.

§2. THE METAPLECTIC REPRESENTATION

We present here a brief review of the metaplectic representation. Proofs are omitted: these may be found in [14] [16] except where otherwise indicated. Let (V,Ω) be a 2m-dimensional real symplectic vector space. The symplectic group $Sp(V,\Omega)$ is the group of all real-linear automorphisms g of V which preserve Ω in the sense

$$v_1,v_2 \in V \;\Rightarrow \Omega(gv_1,gv_2) = \Omega(v_1,v_2). \tag{2.1}$$

$Sp(V,\Omega)$ is a connected semisimple Lie group whose Lie algebra we denote by $sp(V,\Omega)$.

The Heisenberg group $N(V,\Omega)$ is the simply-connected Lie group with underlying manifold $V \times \mathbb{R}$ and multiplication given by

$$(v_1,t_1)(v_2,t_2) = (v_1 + v_2, t_1 + t_2 - \tfrac{1}{2}\Omega(v_1,v_2)) \tag{2.2}$$

for $v_1,v_2 \in V$ and $t_1,t_2 \in \mathbb{R}$. The Heisenberg algebra $n(V,\Omega)$ is its Lie algebra with underlying vector space $V \oplus \mathbb{R}$ and bracket given by

$$[v_1 \oplus t_1,\; v_2 \oplus t_2] = -0 \oplus \Omega(v_1,v_2) \tag{2.3}$$

for $v_1,v_2 \in V$ and $t_1,t_2 \in \mathbb{R}$. We naturally identify $n(V,\Omega)$

with the Lie algebra of $N(V,\Omega)$ so that the exponential map becomes the identity on $V \times \mathbb{R}$.

Let h be a positive real number and write $\hbar = h/2\pi$. Let

$$W:N(V,\Omega) \to \text{Aut}\ \mathbb{H} \tag{2.4}$$

be an irreducible unitary representation of the Heisenberg group on a Hilbert space $\mathbb{H}$ (with inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$) having central character given by

$$W(0,t) = \exp\left(-\frac{1}{i\hbar}t\right) I,\quad t \in \mathbb{R} \tag{2.5}$$

According to the classical theorem of Stone & von Neumann, W is unique up to unitary equivalence; see [15] [21]. We denote by $Mp^c(V,\Omega)$ the set of all unitary operators U on $\mathbb{H}$ which satisfy

$$(v,t) \in N(V,\Omega) \Rightarrow UW(v,t)U^{-1} = W(gv,t) \tag{2.6}$$

for some g in $Sp(V,\Omega)$ and define

$$\sigma:Mp^c(V,\Omega) \to Sp(V,\Omega):U \to g \tag{2.7}$$

where (2.6) holds.

Proposition 2.1:

$Mp^c(V,\Omega) \subset \text{Aut}\ \mathbb{H}$ is a Lie group and

$$1 \to U(1) \to Mp^c(V,\Omega) \overset{\sigma}{\to} Sp(V,\Omega) \to 1 \tag{2.8}$$

a central short exact sequence of Lie groups which does not split. □

The representation μ of $Mp^c(V,\Omega)$ on $\mathbb{H}$ coming from inclusion $Mp^c(V,\Omega) \subset \text{Aut}\ \mathbb{H}$ is known as the metaplectic representation. Whereas μ is of course both faithful and unitary it is not irreducible, being instead the sum of two irreducibles; see [20].

Proposition 2.2:

There exists a unique unitary character $\eta:Mp^c(V,\Omega) \to U(1)$ whose restriction to $U(1) \subset Mp^c(V,\Omega)$ is the squaring map. The kernel of η is a connected double cover of $Sp(V,\Omega)$ which we denote by $Mp(V,\Omega)$ and call the metaplectic group. □

The representation W differentiates on its (dense W-stable) space $E \subset \mathbb{H}$ of smooth vectors to give a representation

$$\dot{W}:n(V,\Omega) \to \text{End}\ E \tag{2.9}$$

of the Heisenberg algebra which satisfies the canonical commutation relations

$$[\dot{W}(v_1 \otimes t_1), \dot{W}(v_2 \otimes t_2)] = \frac{1}{i\hbar}\,\Omega(v_1, v_2) \tag{2.10}$$

for $v_1 \otimes t_1$, $v_2 \otimes t_2 \in n(V,\Omega)$. $\dot{W}$ extends to a representation of the universal enveloping algebra $N(V,\Omega)$ of $n(V,\Omega)$ and the seminorms

$$E \to \mathbb{R} : f \to \|\dot{W}(u)f\|_{\mathbb{H}},\quad u \in N(V,\Omega) \tag{2.11}$$

endow E with the structure of a Fréchet space.

Let E' be the space of all conjugate-linear functionals on E which are continuous in the Fréchet topology and equip E' with the weak-star topology. $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ gives us an embedding of $\mathbb{H}$ in E' and we have a rigged Hilbert space $E \subset \mathbb{H} \to E'$. The representations W (of $N(V,\Omega)$ on $\mathbb{H}$) and $\dot{W}$ (of $n(V,\Omega)$ on E) and $\dot{W}$ (of $n(V,\Omega)$ on E) admit unique continuous extensions to E' (denoted by the same symbols) which are compatible in the sense that the extension of $\dot{W}$ is the derivative of the extension of W. We further extend $\dot{W}$ by complex linearity to obtain a representation

$$\dot{W}^{\mathbb{C}} : n(V,\Omega)^{\mathbb{C}} \to \mathrm{End}\ E' \tag{2.12}$$

of the complex Heisenberg algebra $n(V,\Omega)^{\mathbb{C}}$ on the space E' of symplectic spinors. The space of smooth vectors for the

metaplectic representation μ is also E. We may differentiate and extend μ to obtain a representation

$$\dot{\mu}^{\mathbb{C}} : mp^{c}(V,\Omega)^{\mathbb{C}} \to \mathrm{End}\ E' \tag{2.13}$$

of the complexification of the Lie algebra $mp^{c}(V,\Omega)$ of $Mp^{c}(V,\Omega)$.

We now proceed to describe how the metaplectic representation interacts with (positive) polarizations of (V,Ω). Let $(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ be the 2m-dimensional complex symplectic vector space obtained by complexification of V and complex-bilinear extension of Ω. A polarization of (V,Ω) is a complex Lagrangian subspace of $(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ - thus, a complex m-dimensional subspace Γ of $V^{\mathbb{C}}$ which satisfies

$$v_1, v_2 \in \Gamma \implies \Omega^{\mathbb{C}}(v_1, v_2) = 0\ . \tag{2.14}$$

The symplectic group $Sp(V,\Omega)$ acts naturally on the space $Lag(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ of all polarizations of (V,Ω) by complexification:

$$g\cdot\Gamma = \{g^{\mathbb{C}}(v)\,|\,v \in \Gamma\} \tag{2.15}$$

for $\Gamma \in Lag(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ and $g \in Sp(V,\Omega)$. The canonical line corresponding to $\Gamma \in \mathrm{Lag}\,(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ is the one-dimensional complex subspace $K^{\Gamma} \subset \Lambda^{m}(V^{\mathbb{C}})^{*}$ defined by

$$K^{\Gamma} = \Lambda^{m}\Gamma^{o} \tag{2.16}$$

where $\Gamma^{o} \subset (V^{\mathbb{C}})^{*}$ is the annihilator of $\Gamma \subset V^{\mathbb{C}}$.

Let $\Gamma \in \mathrm{Lag}\,(V^{\mathbb{C}},\Omega^{\mathbb{C}})$. The stabilizers

$$\mathrm{Sp}(V,\Omega;\Gamma) = \{g \in \mathrm{Sp}(V,\Omega) \mid g^{\mathbb{C}}\Gamma = \Gamma\} \tag{2.17}$$

and

$$\mathrm{sp}(V,\Omega)^{\mathbb{C}}_{\Gamma} = \{\xi \in \mathrm{sp}(V,\Omega)^{\mathbb{C}} \mid \xi\Gamma \subset \Gamma\} \tag{2.18}$$

of Γ for the natural representations on $V^{\mathbb{C}}$ stabilize the complex line $K^{\Gamma} \subset \Lambda^{m}(V^{\mathbb{C}})^{*}$ in the natural representations on $\Lambda^{m}(V^{\mathbb{C}})^{*}$. The characters thus determined are

$$\mathrm{Det}_{\Gamma}: \mathrm{Sp}(V,\Omega;\Gamma) \to \mathbb{C}^{*}: g \to \mathrm{Det}_{\mathbb{C}}(g^{\mathbb{C}}|\Gamma) \tag{2.19}$$

and

$$\mathrm{Tr}_{\Gamma}: \mathrm{sp}(V,\Omega)^{\mathbb{C}} \to \mathbb{C}: \xi \to \mathrm{Tr}_{\mathbb{C}}(\xi|\Gamma). \tag{2.20}$$

The polarization Γ of (V,Ω) is said to be positive if

$$v \in \Gamma \Rightarrow i\Omega^{\mathbb{C}}(v,\bar{v}) \geq 0 \tag{2.21}$$

where $v \to \bar{v}$ denotes conjugation in $V^{\mathbb{C}}$ over V; Γ is strictly positive iff the inequality in (2.21) is strict for nonzero $v \in \Gamma$. We denote by $\mathrm{Lag}_{+}(V,\Omega)$ (respectively, $\mathrm{Lag}_{++}(V,\Omega)$) the space of all positive (respectively, strictly positive) polarizations of (V,Ω). Note that if $\Gamma \in \mathrm{Lag}_{+}(V,\Omega)$ then

$$\Gamma \in \mathrm{Lag}_{++}(V,\Omega) \Longleftrightarrow \Gamma \cap \bar{\Gamma} = 0. \tag{2.22}$$

The action (2.15) stabilizes both $\mathrm{Lag}_{+}(V,\Omega)$ and $\mathrm{Lag}_{++}(V,\Omega)$; indeed, the orbits of $\mathrm{Sp}(V,\Omega)$ on $\mathrm{Lag}_{+}(V,\Omega)$ are parametrized by the dimensions $r(\Gamma)$ of $\Gamma \cap \bar{\Gamma}$ for $\Gamma \in \mathrm{Lag}_{+}(V,\Omega)$.

We say that the polarizations Γ_1 and Γ_2 of (V,Ω) are transverse iff

$$\Gamma_1 \cap \bar{\Gamma}_2 = 0. \tag{2.23}$$

If Γ_1 and Γ_2 lie in $\mathrm{Lag}_{+}(V,\Omega)$ then according to [3] we always have

$$\Gamma_1 \cap \bar{\Gamma}_2 = (\Gamma_1 \cap \bar{\Gamma}_1) \cap (\Gamma_2 \cap \bar{\Gamma}_2), \tag{2.24}$$

whence $\Gamma_1 \cap \bar{\Gamma}_2$ is the complexification of an isotropic subspace of (V,Ω) and strictly positive polarizations are transverse to all positive polarizations.

By virtue of (2.14) we can regard the polarization Γ of (V,Ω) as an abelian (complex) Lie algebra embedded in $n(V,\Omega)^{\mathbb{C}}$ and so provide E' with the structure of Γ-module by restricting the representation $\dot{W}^{\mathbb{C}}$ (2.12).

Proposition 2.3:

If Γ is a positive polarization of (V,Ω) then the vacuum state

$$(E')^{\Gamma} \subset E' \text{ of } \Gamma \text{ defined by}$$

$$(E')^{\Gamma} = \{f \in E' \mid v \in \Gamma \Rightarrow \dot{W}^{\mathbb{C}}(v \oplus 0)f = 0\} \tag{2.25}$$

is a complex line. □

If $\Gamma \in Lag(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ is not positive then $(E')^{\Gamma}$ is zero and in order to recover a complex line we must pass to higher Lie algebra cohomology $H^*(\Gamma;E')$ of Γ with coefficients in E'; see [4].

Let $\Gamma \in Lag_+(V,\Omega)$ and denote by $Mp^c(V,\Omega;\Gamma)$ the full preimage of $Sp(V,\Omega;\Gamma)$ under $\sigma: Mp^c(V,\Omega) \to Sp(V,\Omega)$. Differentiation of (2.6) reveals that the metaplectic action of $U \in Mp^c(V,\Omega)$ maps $(E')^{\Gamma}$ to $(E')^{\sigma(U).\Gamma}$. In particular, the metaplectic action of $Mp^c(V,\Omega;\Gamma)$ stabilizes $(E')^{\Gamma}$ and so defines a character

$$\tau_{\Gamma}: Mp^c(V,\Omega;\Gamma) \to \mathbb{C}^{\times} \tag{2.26}$$

Proposition 2.4:

If $\Gamma \in Lag_+(V,\Omega)$ then the characters τ_{Γ}, $Det_{\Gamma}\circ\sigma$, η of $Mp^c(V,\Omega;\Gamma)$ satisfy the relation

$$(\tau_{\Gamma})^2 \cdot Det_{\Gamma}\circ\sigma = \eta. \quad \square \tag{2.27}$$

Denote by $mp^c(V,\Omega)^{\mathbb{C}}_{\Gamma}$ the full preimage of $sp(V,\Omega)^{\mathbb{C}}_{\Gamma}$ under the complexified derivative $\sigma_*^{\mathbb{C}}: mp^c(V,\Omega)^{\mathbb{C}} \to sp(V,\Omega)^{\mathbb{C}}$ of σ. Differentiation of (2.6) reveals the equality

$$[\dot{\mu}^{\mathbb{C}}(x), \dot{W}^{\mathbb{C}}(v \oplus t)] = \dot{W}^{\mathbb{C}}(\sigma_*^{\mathbb{C}}x(v)) \tag{2.28}$$

of operators on E' whenever $x \in mp^c(V,\Omega)^{\mathbb{C}}$ and $v \oplus t \in n(V,\Omega)^{\mathbb{C}}$. In particular, the $\dot{\mu}^{\mathbb{C}}$-action of $mp^c(V,\Omega)^{\mathbb{C}}_{\Gamma}$ stabilizes $(E')^{\Gamma}$ and so defines a character

$$\dot{\tau}_{\Gamma}: mp^c(V,\Omega)^{\mathbb{C}}_{\Gamma} \to \mathbb{C}. \tag{2.29}$$

Proposition 2.5:

If $\Gamma \in Lag_+(V,\Omega)$ then the character $\dot{\tau}_{\Gamma}$ of $mp^c(V,\Omega)^{\mathbb{C}}_{\Gamma}$ is given by

$$2\dot{\tau}_{\Gamma} = \eta_*^{\mathbb{C}} - Tr_{\Gamma}\circ\sigma_*^{\mathbb{C}}. \quad \square \tag{2.30}$$

An $Sp(V,\Omega)$-invariant pseudo-Hermitian form $\langle\cdot,\cdot\rangle_K$ is defined on $\Lambda^m(V^{\mathbb{C}})^*$ by

$$k_1,k_2 \in \Lambda^m(V^{\mathbb{C}})^* \Rightarrow \langle k_1,k_2\rangle_K \lambda_\Omega = i^m(k_1 \wedge \bar{k}_2) \tag{2.31}$$

where the Liouville form $\lambda_\Omega \in \Lambda^{2m}V^* \subset \Lambda^{2m}(V^{\mathbb{C}})^*$ is defined by

$$m!\lambda_\Omega = (-1)^{\frac{1}{2}m(m-1)}\Omega^m \tag{2.32}$$

If (Γ_1,Γ_2) is a transverse pair of polarizations of (V,Ω) then $\langle\cdot,\cdot\rangle_K$ restricts to give a canonical nonsingular sesquilinear pairing of the canonical lines K^{Γ_1} and K^{Γ_2} into $\mathbb{C}$. A similar property holds true for vacuum states:

<u>Proposition 2.6:</u>

The inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ on $\mathbb{H}$ extends to give a canonical nonsingular sesquilinear pairing

$$\langle\cdot,\cdot\rangle_{E'} : (E')^{\Gamma_1} \times (E')^{\Gamma_2} \to \mathbb{C}$$

of vacuum states for each transverse pair (Γ_1,Γ_2) of positive polarizations of (V,Ω). □

(2.24) suggests that in order to deal with non-transverse pairings we should consider how the metaplectic representation interacts with isotropic subspaces.

The subspace L of V is $(\Omega-)$ isotropic iff wholly contained in its $(\Omega-)$ orthogonal

$$L^\perp = \{v \in V \mid \Omega(v,\ell) = 0, \ \ \forall \ell \in L\} \tag{2.34}$$

The restriction of Ω to $L^\perp$ then has kernel precisely L and so descends to a symplectic form Ω_L on the quotient $L^\perp/L$. The resulting symplectic vector space $(L^\perp/L,\Omega_L)$ is the symplectic normal of $L \subset (V,\Omega)$

Let L be an isotropic subspace of (V,Ω) and define

$$Sp(V,\Omega;L) = \{g \in Sp(V,\Omega) \mid gL = L\} \tag{2.35}$$

There is a natural Lie group epimorphism

$$\nu_L : Sp(V,\Omega;L) \to Sp(L^\perp/L,\Omega_L) : g \to g_L \tag{2.36}$$

given by

$$g \in Sp(V,\Omega;L) \Rightarrow \pi_L \circ g|L^\perp = g_L \circ \pi_L \tag{2.37}$$

where $\pi_L : L^\perp \to L^\perp/L$ is the projection map.

Fix an irreducible unitary representation W_L of $N(L^{\perp}/L,\Omega_L)$ on a Hilbert space $\mathbb{H}_L$ such that

$$W_L(0,t) = \exp\{-\frac{1}{i\hbar}t\} I, \ \forall t \in \mathbb{R}. \tag{2.38}$$

Denote by $Mp^c(L^{\perp}/L,\Omega_L)$ the corresponding automorphism group and by $\mathcal{E}_L \subset \mathbb{H}_L \subset \mathcal{E}_L'$ the resulting rigged Hilbert space. Define

$$(\mathcal{E}')^L = \{f \in \mathcal{E}' | \dot{W}(\ell\oplus 0)f = 0, \ \forall \ell \in L\} \tag{2.39}$$

and denote by $Mp^c(V,\Omega;L)$ the full preimage of $Sp(V,\Omega;L)$ under $\sigma: Mp^c(V,\Omega) \to Sp(V,\Omega)$. Assume $L \neq 0, L^{\perp}$.

<u>Proposition 2.7:</u>

There exists a canonical topological linear isomorphism

$$R_L: (\mathcal{E}')^L \to \mathcal{E}_L' \tag{2.40}$$

which intertwines W and W_L. A Lie group epimorphism

$$\hat{\nu}_L: Mp^c(V,\Omega;L) \to Mp^c(L^{\perp}/L,\Omega_L) \tag{2.41}$$

lifting ν_L is then defined by

$$\hat{\nu}_L(U) = |Det(\sigma U|L)|^{\frac{1}{2}} R_L U R_L^{-1} \tag{2.42}$$

and satisfies

$$\eta\hat{\nu}_L(U) = \eta(U) \ sign(Det(\sigma U|L)) \tag{2.43}$$

for $U \in Mp^c(V,\Omega;L)$. $\square$

We remark that there exists no lift $Mp(V,\Omega;L) \to Mp(L^{\perp}/L,\Omega_L)$ of ν_L to metaplectic double covers.

Finally we consider in more detail the case of strictly positive polarizations.

A Hilbert structure for (V,Ω) is a real-linear automorphism J of V such that $J^2 = -I$ and such that the real-bilinear form

$$V \times V \to \mathbb{R}: (v_1,v_2) \to \Omega(Jv_1,v_2) \tag{2.44}$$

is symmetric and positive-definite.

$$\langle v_1,v_2\rangle_J = \Omega(Jv_1,v_2) + i\Omega(v_1,v_2) \tag{2.45}$$

then defines a Hermitian inner product on the complex vector space V_J (having V as underlying real vector space and $iv = J(v)$ for $v \in V$). The unitary group

$$U(V,\Omega;J) = \{g \in Sp(V,\Omega) \mid gJ = Jg\} \quad (2.46)$$

of the Hilbert space $(V_J, \langle\cdot,\cdot\rangle_J)$ is a maximal compact subgroup of $Sp(V,\Omega)$; moreover, all maximal compact subgroups of $Sp(V,\Omega)$ arise in this way. The $(+i)$-eigenspace

$$\Gamma_J = (I - iJ^{\mathbb{C}})V \quad (2.47)$$

of $J^{\mathbb{C}}$ is a strictly positive polarization of (V,Ω); moreover, all strictly positive polarizations of (V,Ω) arise in this way. Note that

$$U(V,\Omega;J) = Sp(V,\Omega;\Gamma_J) \quad (2.48)$$

when (2.47) holds. Fix a Hilbert structure J for (V,Ω) and for convenience write $U(V)$ in place of $U(V,\Omega;J)$. Denote by $MU^c(V)$ the full preimage of $U(V)$ under $\sigma: Mp^c(V,\Omega) \to Sp(V,\Omega)$. In contrast to Proposition 2.1 we have:

Proposition 2.8:

The central short exact sequence of Lie groups

$$1 \to U(1) \to MU^c(V) \xrightarrow{\sigma} U(V) \to 1 \quad (2.49)$$

splits; indeed, $MU^c(V)$ is the direct product of $U(1)$ and $U(V)$. □

Remark:

The group $Mp^c(V,\Omega)$ and its metaplectic representation are of course dependent on W. In theory this dependence is natural with respect to intertwining operators for W; in practice the particular form of W is important. In the Schrödinger model $V = \mathbb{R}^{2m}$, $\mathbb{H} = L^2(\mathbb{R}^m)$, $\mathcal{E}$ is the Schwartz space $S(\mathbb{R}^m)$, and $\mathcal{E}'$ the space $S'(\mathbb{R}^m)$ of tempered distributions; unfortunately, the metaplectic representation has only been written down explicitly on certain generating subgroups in this model. In the Bargmann-Segal model on Fock space (as presented by Rawnsley [14]) $V = \mathbb{C}^m$ and $\mathcal{E}$, $\mathbb{H}$, $\mathcal{E}'$ are all spaces of entire functions on $\mathbb{C}^m$ subject to certain growth conditions; the metaplectic representation can be written down explicitly and in particular vacuum states and their pairings become transparent. See [14, 16].

§3. Mp^c STRUCTURES AND SYMPLECTIC SPINORS

Let (E,ω) be a real symplectic vector bundle of rank $2m$ over the manifold X.

The symplectic frame bundle of (E,ω) modelled on (V,Ω) is the principal $Sp(V,\Omega)$ bundle $Sp(E,\omega)$ on X having as fibre over $x \in X$ the set of all real-linear isomorphisms $b:V \to E_x$ satisfying

$$\omega_x(bv_1,bv_2) = \Omega(v_1,v_2),\ v_1,v_2 \in V \qquad (3.1)$$

and on which $Sp(V,\Omega)$ acts on the right by composition.

An Mp^c structure for (E,ω) is a principal $Mp^c(V,\Omega)$ bundle P over X together with a σ-equivariant morphism $P \to Sp(E,\omega)$ of principal bundles. The Mp^c structures P_1 and P_2 are equivalent iff there exists an isomorphism $P_1 \to P_2$ of principal $Mp^c(V,\Omega)$ bundles which commutes with the respective projections on $Sp(E,\omega)$. We denote by $Mp^c[E,\omega]$ the set of equivalence classes [P] of Mp^c structures P for (E,ω). Our first result in this section is the unconditional existence of Mp^c structures.

Proposition 3.1:

(E,ω) always admits Mp^c structures; indeed $Mp^c[E,\omega]$ has a natural base point.

Proof:

Since $U(V)$ is maximal compact in $Sp(V,\Omega)$, there exists a $U(V)$-reduction B of $Sp(E,\omega)$. According to Proposition 2.8 there exists a splitting of $\sigma:MU^c(V) \to U(V)$, by means of which B extends to a principal $MU^c(V)$ bundle B^c. B^c extends via inclusion $MU^c(V) \subset Mp^c(V,\Omega)$ to an Mp^c structure P_B for (E,ω). The class $[P_B] \in Mp^c[E,\omega]$ is independent of B, since all maximal compact reductions are equivalent. □

Remark 3.2:

See also [10]. We may refer to the distinguished element of $Mp^c[E,\omega]$ as the neutral class.

We now introduce certain fibre product constructions whose immediate function is to yield a deeper understanding of $Mp^c[E,\omega]$.

Let P be an Mp^c structure for (E,ω) and let Y be a principal $U(1)$ bundle on X. The fibre product

$$Y \dot{\times} P = \{(y,p) \in Y \times P \mid \pi(y) = \pi(p)\} \qquad (3.2)$$

is a principal $U(1) \times Mp^c(V,\Omega)$ bundle over X to which is associated via the morphism

$$U(1) \times Mp^c(V,\Omega) \to Mp^c(V,\Omega):(\lambda,U) \to \lambda U \qquad (3.3)$$

an Mp^c structure P^Y for (E,ω) having the obvious projection on $Sp(E,\omega)$.

Let P_1 and P_2 be Mp^c structures for (E,ω). The fibre product

$$P_1 \dot{\times} P_2 = \{(p_1,p_2) \in P_1 \times P_2 \mid \sigma(p_1) = \sigma(p_2)\} \tag{3.4}$$

of P_1 and P_2 over $Sp(E,\omega)$ is a principal bundle on X having structure group

$$Mp^{cc}(V,\Omega) = \{(U_1,U_2) \in Mp^c \times Mp^c \mid \sigma U_1 = \sigma U_2\}. \tag{3.5}$$

Associated to $P_1 \dot{\times} P_2$ via the unitary character

$$Mp^{cc}(V,\Omega) \to U(1) : (U_1,U_2) \to U_1^{-1}U_2 \tag{3.6}$$

is a principal U(1) bundle on X which we may denote by $P_1^{-1}P_2$.

The following relationships between these constructions are readily established.

__Proposition 3.3:__

Let P, P_1 and P_2 be Mp^c structures for (E,ω) and let Y be a principal U(1) bundle on X.

(i) The Mp^c structures $P_1^{(P_1^{-1}P_2)}$ and P_2 are canonically equivalent.

(ii) The principal U(1) bundles $P^{-1}P^Y$ and Y are canonically isomorphic.

(iii) There is a canonical bijection between equivalences $P^Y \to P$ and trivializations of Y. □

Recall that isomorphism classes of principal U(1) bundles on X naturally constitute the cohomology group $H^1(X;\underline{U(1)})$ and that the Chern class gives an isomorphism

$$c: H^1(X;\underline{U(1)}) \to H^2(X;\mathbb{Z}). \tag{3.7}$$

As a routine consequence of Proposition 3.3 we have the following description of $Mp^c[E,\omega]$:

__Proposition 3.4:__

$Mp^c[E,\omega]$ is naturally a principal $H^2(X;\mathbb{Z})$ space for the action

$$(c[Y],\ [P]) \to [P^Y] \tag{3.8}$$

□

__Remark 3.5:__

More is true. It is clear from Propositions 3.1 and 3.4 that $Mp^c[E,\omega]$ is naturally an abelian group isomorphic to $H^2(X;\mathbb{Z})$. We shall see this again in Proposition 3.13.

Our primary reason for introducing Mp^c structures is that they enable us to define bundles of symplectic spinors.

Let P be an Mp^c structure for (E,ω). Associated to P via the metaplectic representation of $Mp^c(V,\Omega)$ on the rigged Hilbert space $\mathcal{E} \subset \mathbb{H} \subset \mathcal{E}'$ are vector bundles $\mathcal{E}(P) \subset \mathbb{H}(P) \subset \mathcal{E}'(P)$ of infinite rank over X. We may refer to $\mathcal{E}'(P)$ - or to any of its subbundles - as a bundle of symplectic spinors for (E,ω).

Let $N(E,\omega)$ be the bundle of Lie groups on X with fibre $N(E_x,\omega_x)$ over $x \in X$ and let $n(E,\omega)$ be the bundle of Lie algebras on X with fibre $n(E_x,\omega_x)$ over $x \in X$. These Heisenberg bundles are canonically associated to $Sp(E,\omega)$ via the natural actions of $Sp(V,\Omega)$ on $N(V,\Omega)$ and $n(V,\Omega)$. By association, the representation (2.4) gives rise to a bundle of representations W of $N(E,\omega)$ on $\mathbb{H}(P)$; we likewise obtain a bundle of representations $\dot{W}^{\mathbb{C}}$ of the Lie algebra bundle $n(E,\omega)^{\mathbb{C}}$ on $\mathcal{E}'(P)$.

A polarization of (E,ω) is a complex subbundle F of $E^{\mathbb{C}}$ having rank m and satisfying

$$x \in X;\ v_1,v_2 \in F_x \Rightarrow \omega_x^{\mathbb{C}}(v_1,v_2) = 0. \tag{3.9}$$

We shall always assume F to be positive:

$$x \in X;\ v \in F_x \qquad i\omega_x^{\mathbb{C}}(v,\bar{v}) \geq 0 \tag{3.10}$$

- for the general case we refer to [4]. The canonical bundle K^F of F is the complex line bundle defined by

$$K^F = \Lambda^n F^o \tag{3.11}$$

where $F^o \subset (E^{\mathbb{C}})^*$ is the annihilator of $F \subset E^{\mathbb{C}}$. We remark that positive polarizations have isomorphic canonical bundles (as follows from the existence of the pairings (3.18)); this enables us to define the Chern class $c_1(E,\omega) \in H^2(X;\mathbb{Z})$ of (E,ω) by

$$c_1(E,\omega) = c[K^F] \tag{3.12}$$

for any positive polarization F of (E,ω). We say that F is regular iff $F \cap \bar{F}$ is a subbundle of $E^{\mathbb{C}}$ (or, has constant rank); in this case a choice Γ of positive polarization of (V,Ω) such that dim $(\Gamma \cap \bar{\Gamma})$ equals rank $(F \cap \bar{F})$ determines a reduction

$$Sp(E,\omega;F) = \{b \in Sp(E,\omega) \mid b^{\mathbb{C}}\Gamma = F\} \tag{3.13}$$

of $Sp(E,\omega)$ to structure group $Sp(V,\Omega;\Gamma)$ to which K^F is associated via Det_Γ.

Let P be an Mp^c structure for (E,ω), and F be a positive polarization of (E,ω). For $x \in X$ we define $E'(P)^F_x$ to be the vacuum states for $F_x \in Lag_+(E_x,\omega_x)$ in the representation

$$\dot{W}^{\mathbb{C}}_x : n(E,\omega)^{\mathbb{C}}_x \to \mathrm{End}\ E'(P)_x. \tag{3.14}$$

It is apparent from Proposition 2.3 that $E'(P)^F$ is a complex line bundle on X. $E'(P)^F$ is related to the hermitian line bundle $P(\eta)$ (associated to P via the unitary character) and the canonical bundle K^F, as follows:

Proposition 3.6:

There exists a canonical isomorphism of complex line bundles

$$E'(P)^F \otimes E'(P)^F \otimes K^F \xrightarrow{\sim} P(\eta) \tag{3.15}$$

Proof:

Suppose F to be regular and choose a model $\Gamma \in lag_+(V,\Omega)$. The part P^F of P lying over $Sp(E,\omega;F)$ is a principal $Mp^c(V,\Omega;\Gamma)$ bundle to which $E'(P)^F$, K^F, $P(\eta)$ are associated via τ_Γ, $\mathrm{Det}_\Gamma \circ \sigma, \eta$. In this case (3.15) comes directly from (2.27). The general case (in which the rank of $F \cap \bar{F}$ may vary) follows from a closer study of the metaplectic representation. See [16].

Remark 3.7:

Define the half-form bundle Q^F_P by

$$Q^F_P = E'(P)^F \otimes K^F$$

As a corollary of Proposition 3.6 there exists a canonical isomorphism of complex line bundles

$$Q^F_P \otimes Q^F_P \xrightarrow{\sim} P(\eta) \otimes K^F \tag{3.17}$$

- otherwise said, Q^F_P is a canonical square-root of $P(\eta) \otimes K^F$.

Let (F,G) be a pair of positive polarizations of (E,ω). We say that (F,G) is transverse if $F \cap \bar{G} = 0$. There is then a canonical nonsingular sesquilinear pairing

$$\langle\cdot,\cdot\rangle_K : K^F \times K^G \to \underline{\mathbb{C}} \tag{3.18}$$

into the product line bundle $\underline{\mathbb{C}} = X \times \mathbb{C}$ given by

$$\alpha \in K^F,\ \beta \in K^G \Rightarrow \langle\alpha,\beta\rangle_K\ \lambda_\omega = i^m(\alpha \wedge \bar{\beta}) \tag{3.19}$$

where $\lambda_\omega \in \Gamma(X;\Lambda^{2m}(E^{\mathbb{C}})^*)$ is the Liouville volume

$$\lambda_\omega = (-1)^{\frac{1}{2}m(m-1)} \frac{\omega^m}{m!} \tag{3.20}$$

For vacuum states and half-forms we have the following analogue.

Proposition 3.8:

Let (F,G) be a transverse pair of positive polarizations of (E,ω). There exist canonical nonsingular sesquilinear pairings

$$E'(P)^F \times E'(P)^G \to \underline{\mathbb{C}} \tag{3.21}$$

$$Q_P^F \times Q_P^G \to \underline{\mathbb{C}}. \tag{3.22}$$

Proof:

If $x \in X$ and $p \in P_x$ then $(p^{-1}F_x, p^{-1}G_x)$ is a transverse pair of positive polarizations of (V,Ω); (3.21) is defined over $x \in X$ by transport of the pairing $\langle\cdot,\cdot\rangle_{E'} : (E')^{p^{-1}F_x} \times (E')^{p^{-1}G_x} \to \mathbb{C}$ guaranteed by Proposition 2.6. (3.22) comes from (3.18) and (3.21). See [14] [16] for details. □

In order to deal with non-transverse pairings we consider isotropic subbundles of (E,ω) and invoke Proposition 2.7.

Let D be an isotropic subbundle of (E,ω); thus, D is contained in

$$D^{\perp} = \{v \in E \mid \omega(v,d) = 0,\ \forall d \in D\} \tag{3.23}$$

and $D^{\perp}/D$ inherits a symplectic form ω_D. If L is an isotropic subspace of (V,Ω) with $\dim L = \operatorname{rank} D$ then

$$Sp(E,\omega;D) = \{b \in Sp(E,\omega) \mid bL = D\} \tag{3.24}$$

is an $Sp(V,\Omega;L)$-reduction of $Sp(E,\omega)$ to which $Sp(D^{\perp}/D,\omega_D)$ is associated via ν_L (2.36). If F is a positive polarization of (E,ω) such that $D^{\mathbb{C}} \subset F$ then $F_D = F/_D\mathbb{C}$ is a positive polarization of the symplectic normal $(D^{\perp}/D,\omega_D)$. If $\alpha \in \mathbb{R}$ then the bundle $\mathcal{D}^{\alpha}(D)$ of α-densities on D is associated to the frame bundle of D via the character $|\mathrm{Det}|^{-\alpha}$ of the general linear group.

Mp^c structures always pass to symplectic normals. Let P be an Mp^c structure for (E,ω) and let D be an isotropic subbundle of (E,ω) with $D \neq 0, D^{\perp}$. Writing $E'(P)^D$ for the subbundle of $E'(P)$ annihilated by D under $\dot{W}$, we have:

Proposition 3.9:

$(D^{\perp}/D,\omega_D)$ inherits an Mp^c structure P_D and there exists a canonical isomorphism

$$E'(P)^D \xrightarrow{\sim} E'(P_D) \otimes \mathcal{D}^{\frac{1}{2}}(D) \tag{3.25}$$

which restricts to a canonical isomorphism of complex line bundles

$$E'(P)^{F} \xrightarrow{\sim} E'(P_D)^{F_D} \otimes \mathcal{D}^{\frac{1}{2}}(D) \qquad (3.26)$$

whenever F is a positive polarization of

(E,ω) such that $D^{\mathbb{C}} \subset F$.

<u>Proof</u>:

A consequence of Proposition 2.7. P_D is associated to that part of P which lies over Sp(E,ω;D) via the lift $\hat{\nu}_L$ (2.41) of ν_L (2.36). The isomorphism (3.25) is clear from the definition (2.42) of $\hat{\nu}_L$. That (3.25) restricts to (3.26) follows from the fact that R_L intertwines W and W_L. □

We say that the pair (F,G) of positive polarizations of (E,ω) is regular iff $F \cap \bar{G}$ is a subbundle of $E^{\mathbb{C}}$ (or, has constant rank); in this case, $F \cap \bar{G} = D^{\mathbb{C}}$ for some isotropic subbundle D of (E,ω), (F_D, G_D) is a transverse pair of positive polarizations of $(D^{\perp}/D, \omega_D)$, and we have a canonical nonsingular sesquilinear pairing

$$K^F \times K^G \to \mathcal{D}^{-2}(D); \qquad (3.27)$$

see [12, 16].

The regular pairing of vacuum states and half-forms is as follows.

<u>Proposition 3.10:</u>

Let P be an Mp^c structure for (E,ω). If (F,G) is a regular pair of positive polarizations of (E,ω) with $F \cap \bar{G} = D^{\mathbb{C}}$ then there exist canonical nonsingular sesquilinear pairings

$$E'(P)^F \times E'(P)^G \to \mathcal{D}^1(D) \qquad (3.28)$$

$$\Omega_P^F \times \Omega_P^G \to \mathcal{D}^{-1}(D) \qquad (3.29)$$

<u>Proof</u>:

(3.29) comes from (3.27) and (3.28). To define (3.28) we pass to the sympletic normal (Proposition 3.9), apply the transverse pairing to (F_D, G_D) (Proposition 3.8) and self-pair $\mathcal{D}^{\frac{1}{2}}(D)$ naturally into $\mathcal{D}^1(D)$. See [14, 16] for amplification. □

<u>Remark 3.11:</u>

(3.27) and the Hermitian structure on P(η) give a pairing

$$(P(\eta) \otimes K^F) \times (P(\eta) \otimes K^G) \to \mathcal{D}^{-2}(D) \qquad (3.30)$$

which is the square of (3.29) in the sense determined by Remark 3.7. The Liouville density $|\lambda_\omega| \in \mathcal{D}^1(E)$ gives rise to a canonical isomorphism $\mathcal{D}^{-1}(D) \tilde{\to} \mathcal{D}^1(E/D)$ so that (3.29) can be considered as a pairing

$$Q_P^F \times Q_P^G \to \mathcal{D}^1(E/D) \tag{3.31}$$

see [3, 12].

Our next result tells us how the symplectic spinors associated to an Mp^c structure transform under the twisting $(Y,P) \to P^Y$ of an Mp^c structure P for (E,ω) by a principal U(1) bundle Y to which is associated a Hermitian line bundle L via the standard action of U(1) on $\mathbb{C}$.

<u>Proposition 3.12</u>:

There is a canonical isomorphism

$$\mathcal{E}'(P^Y) \xrightarrow{\sim} \mathcal{E}'(P) \otimes L \tag{3.32}$$

which restricts to an isomorphism

$$\mathcal{E}'(P^Y)^F \xrightarrow{\sim} \mathcal{E}'(P)^F \otimes L \tag{3.33}$$

for each positive polarization F of (E,ω).

<u>Proof</u>:

$U(1) \subset Mp^c(V,\Omega)$ is central, acts trivially on $n(V,\Omega)^{\mathbb{C}}$, and acts by scalars in the metaplectic representation on $\mathcal{E}'$. □

In Remark 3.5 we saw that $Mp^c[E,\omega]$ is naturally an abelian group isomorphic to $H^2(X;\mathbb{Z})$; let us now see this explicitly in terms of symplectic spinors.

<u>Proposition 3.13</u>:

A canonical isomorphism of principal $H^2(X;\mathbb{Z})$ spaces

$$\kappa: Mp^c[E,\omega] \to H^2(X;\mathbb{Z}): [P] \to c[\mathcal{E}'(P)^F] \tag{3.34}$$

is defined independently of the positive polarization F of (E,ω). $Mp^c[E,\omega]$ is thus naturally an abelian group isomorphic to $H^2(X;\mathbb{Z})$.

<u>Proof</u>:

That $c[\mathcal{E}'(P)^F]$ is independent of F and depends only on [P] is clear from Proposition 3.8 (after fixing a strictly positive polarization G of (E,ω)). The equivariance of κ is a consequence of Proposition 3.12. Since $Mp^c[E,\omega]$ and $H^2(X;\mathbb{Z})$ are principal, (3.34) must perforce be an isomorphism. □

<u>Remark 3.14</u>:

We note that the Mp^c structure P for (E,ω) belongs to the neutral class iff the bundle $\mathcal{E}'(P)^F$ of vacuum states is trivial for any (equivalently, some) positive polarization F of (E,ω).

We close this section by comparing Mp^c structures with metaplectic structures.

Metaplectic structures for (E,ω) are defined after the fashion of Mp^c structures but with $Mp(V,\Omega)$ in place of $Mp^c(V,\Omega)$. We denote by $Mp[E,\omega]$ the space of equivalence classes of metaplectic structures for (E,ω) (which may be empty).

Remark 3.15:

(E,ω) admits metaplectic structures iff $c_1(E,\omega)$ is even iff the second Stiefel-Whitney class $w_2(E) = \mathrm{mod}_2 c_1(E,\omega)$ is zero; see [12] . Compare Proposition 3.1. When nonempty, $Mp[E,\omega]$ is naturally a principal space for $H^1(X;\mathbb{Z}_2)$, [12]. In contrast with Proposition 3.1, however, $Mp[E,\omega]$ has no preferred base-point in general.

Remark 3.17:

If D is an isotropic subbundle of (E,ω) then

$$\mathrm{mod}_2 c_1(E,\omega) = \mathrm{mod}_2 c_1(D^{\perp}/D,\omega_D) + w_1(D)^2 \qquad (3.35)$$

- see [16]. In view of Remark 3.15 it is now clear that if (E,ω) admits metaplectic structures then $(D^{\perp}/D,\omega_D)$ will

admit metaplectic structures iff $w_1(D)^2 = 0$ (when we say that D is metalinear). Thus metaplectic structures do not generally pass down to symplectic normals - in marked contrast with the case (Proposition 3.9) for Mp^c structures.

§4. GEOMETRIC QUANTIZATION : THEORY

Let (X,ω) be a connected symplectic manifold of dimension 2m. We denote by $C(X)$ the associative algebra of smooth complex functions on X and by $\mathcal{X}(X)$ the Lie algebra of complex vector fields on X. The Hamiltonian vector field $\xi_\phi \in \mathcal{X}$ of $\phi \in C(X)$ is given by

$$\xi_\phi \lrcorner\, \omega^{\mathbb{C}} = d\phi \tag{4.1}$$

The Poisson bracket on $C(X)$ is then defined by

$$\{\phi,\psi\} = \xi_\phi \psi \tag{4.2}$$

for $\phi,\psi \in C(X)$ and gives $C(X)$ the structure of a complex Lie algebra: the Poisson algebra $C(X,\omega)$. The map

$$\xi : C(X,\omega) \to \mathcal{X}(X) : \phi \to \xi_\phi \tag{4.3}$$

is a homomorphism of Lie algebras.

A prequantized Mp^c structure for (X,ω) is a pair (P,γ) with P an Mp^c structure for (TX,ω) and γ a $u(1)$-valued 1-form on P satisfying

$$a \in Mp^c(V,\Omega) \Rightarrow R_a^*\gamma = \gamma \tag{4.4}$$

$$z \in mp^c(V,\Omega) \Rightarrow \gamma(\tilde{z}) = \frac{1}{2}\eta_* z \tag{4.5}$$

$$d\gamma = \pi^* \frac{\omega}{i\hbar} \tag{4.6}$$

where R_a is right multiplication by a, $\tilde{z}$ is the fundamental vector field generated by z, and $\pi : P \to X$ is the bundle projection. We say that P is prequantizable when (P,γ) exists and that γ is a prequantum form. The prequantized Mp^c structures (P_1,γ_1) and (P_2,γ_2) for (X,ω) are equivalent iff there exists an equivalence $f : P_1 \to P_2$ of Mp^c structures such that $f^*\gamma_2 = \gamma_1$. The concept of a prequantized Mp^c structure is due to Hess [7].

Let us immediately relate prequantized Mp^c structures to (the more familiar) Hermitian line bundles with connection.

Proposition 4.1:

If P is an Mp^c structure for (TX,ω) then there is a canonical bijection between prequantum forms γ on P and Hermitian connections ∇^γ of curvature $\frac{2\omega}{i\hbar}$ in $P(\eta)$.

Proof:

Let Y be the principal $U(1)$ bundle associated to P via η with associating morphism $f : P \to Y$; Y is naturally the unitary frame bundle of $P(\eta)$. The bijection asserted in the Proposition is effected by

$$f^*\alpha^\gamma = 2\gamma \tag{4.7}$$

where α^γ denotes the principal connection in Y corresponding to ∇^γ in $P(\eta)$. □

Write $c_1(\omega) = c_1(TX,\omega)$ and denote by $c^{\mathbb{R}}$ the real cohomology class arising from the integer cohomology class c under change of coefficients $H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{R})$.

<u>Proposition 4.2</u>:

The Mp^c structure P for (TX,ω) is prequantizable iff

$$\kappa([P])^{\mathbb{R}} = [\tfrac{\omega}{h}] - \tfrac{1}{2}\, c_1(\omega)^{\mathbb{R}} \tag{4.8}$$

<u>Proof</u>:

A consequence of Weil's theorem [18], Propositions 3.6 and 4.1, and definitions (3.12) and (3.34). □

As a corollary of Propositions 3.13 and 4.2 we have the following existence criterion (first derived by Hess [7] using different methods):

<u>Proposition 4.3</u>:

(X,ω) admits prequantized Mp^c structures iff the real cohomology class

$$[\tfrac{\omega}{h}] - \tfrac{1}{2}\, c_1(\omega)^{\mathbb{R}} \tag{4.9}$$

is integral (when we say that (X,ω) is quantizable). □

A flat $U(1)$ bundle (Y,α) on X is a principal $U(1)$ bundle Y equipped with a flat connection α. The flat $U(1)$ bundles (Y_1,α_1) and (Y_2,α_2) are equivalent iff there exists an isomorphism $f:Y_1 \to Y_2$ such that $f^*\alpha_2 = \alpha_1$. We naturally identify the space of equivalence classes of flat $U(1)$ bundles on X with Čech cohomology $H^1(X;U(1))$ of X with (locally constant) coefficients in $U(1)$.

In order to describe the space of equivalence classes of prequantized Mp^c structures for (X,ω) we adapt the various constructions that were introduced to establish Proposition 3.4.

Thus: let (P,γ) be a prequantized Mp^c structure and (Y,α) a flat $U(1)$ bundle. The fibre sum $\alpha \dotplus \gamma$ is the $u(1)$-valued 1-form defined on $Y \dot{\times} P$ (3.2) by

$$(\alpha \dotplus \gamma)(\zeta \dot{\times} \xi) = \alpha(\zeta) + \gamma(\xi) \tag{4.10}$$

for $\zeta \dot{\times} \xi \in T(Y \dot{\times} P)$. Associated to $\alpha \dotplus \gamma$ on $Y \dot{\times} P$ is a prequantum form γ^α on P^Y. The twisting

$$((Y,\alpha),(P,\gamma)) \to (P^Y,\gamma^\alpha) \tag{4.11}$$

passes to the level of equivalence classes to yield:

Proposition 4.4:

The set of equivalence classes of prequantized Mp^c structures for the quantizable (X,ω) is naturally a principal $H^1(X;U(1))$ space for the action

$$[Y,\alpha]\cdot[P,\gamma] = [P^Y,\gamma^\alpha] \tag{4.12}$$

Proof:

Along similar lines to that of Proposition 3.4 - we omit the details. □

Remark 4.5:

Recall from [9] that the polarization-independent part of the Kostant-Souriau quantization scheme involves both a prequantum $U(1)$ bundle (Y,β) (thus, a principal $U(1)$ bundle Y equipped with a connection β of curvature $\frac{\omega}{i\hbar}$) and a metaplectic structure P_o and thus requires both that $[\frac{\omega}{h}]$ be integral and that $c_1(\omega)$ be even. It is clear from Proposition 4.3 that (X,ω) is quantizable (in our sense) whenever the Kostant-Souriau scheme applies: indeed, a twisting akin to (4.11) produces a prequantized Mp^c structure from the prequantum $U(1)$ bundle (Y,β) and the Mp^c structure P associated to P_o via inclusion $MP(V,\Omega) \subset Mp^c(V,\Omega)$.

Let us now outline our geometric quantization scheme. The first step is to construct a representation of the Poisson algebra $C(X,\omega)$.

Assume (X,ω) to be quantizable and let (P,γ) be a prequantized Mp^c structure for (X,ω); observe that γ is a connection (Mp^c-invariant and of curvature $\pi^*\frac{\omega}{i\hbar}$) in the principal $U(1)$ bundle $P \to Sp(TX,\omega)$.

As prequantization module we take the space $\Gamma(X;E'(P))$ of smooth symplectic spinors. We identify sections $s \in \Gamma(X;E'(P))$ with functions $\tilde{s}:P \to E'$ which transform as

$$p \in P,\ a \in Mp^c(V,\Omega) \Rightarrow \tilde{s}(R_a p) = \mu(a)^{-1}\tilde{s}(p) \tag{4.13}$$

If $\phi \in C(X)$ then Lie differentiation along ξ_ϕ annihilates $\omega^{\mathbb{C}}$ so that ξ_ϕ lifts to a complex vector field $\tilde{\xi}_\phi$ on the symplectic frame bundle $Sp(TX,\omega)$. The γ-horizontal lift $\hat{\xi}_\phi$ of $\tilde{\xi}_\phi$ is then a complex vector field on P along which we can differentiate $\tilde{s}$ for $s \in \Gamma(X;E'(P))$. Since γ is Mp^c invariant we have

$$\hat{\xi}_\phi\tilde{s} = (D_\phi s)^\sim \tag{4.14}$$

for some $D_\phi s \in \Gamma(X;E'(P))$. This gives a map

$$D: C(X,\omega) \to \mathrm{End}\ \Gamma(X;E'(P)) \tag{4.15}$$

which satisfies

$$D_\phi(\psi S) = \psi D_\phi S + \{\phi,\psi\} S \tag{4.16}$$

$$D_{\{\phi,\psi\}} S = [D_\phi, D_\psi] s + \frac{1}{i\hbar}\{\phi,\psi\} s \tag{4.17}$$

for ϕ, $\psi \in C(X)$ and $s \in \Gamma(X;E'(P))$. (4.16) holds since $\hat{\xi}_\phi$ and ξ_ϕ are π-related for the bundle projection $\pi: P \to X$; (4.17) holds since γ has curvature $\pi^* \frac{\omega}{i\hbar}$ in $P \to \mathrm{Sp}(TX,\omega)$ and $\phi \to \bar{\xi}_\phi$ is bracket-preserving.

As a straightforward consequence of (4.16) and (4.17) we now deduce:

<u>Proposition 4.6</u>:

A Lie algebra morphism $\delta: C(X,\omega) \to \mathrm{End}\ \Gamma(X;E'(P))$ (4.18)
is defined by the prescription

$$\delta_\phi s = D_\phi s + \frac{1}{i\hbar}\phi s \tag{4.19}$$

for $\phi \in C(X,\omega)$ and $s \in \Gamma(X;E'(P))$. □

We refer to the Lie algebra morphism δ as prequantization of (X,ω) relative to (P,γ).

It is desirable to have available a local description of prequantization. According to the Darboux theorem on the existence of local symplectic coordinates, every symplectic manifold is locally linear. The case of a linear symplectic manifold, treated in outline in §5 and at depth in [16], thus provides a convenient framework for the local picture. Of course, every symplectic manifold admits locally both prequantum U(1) bundles and metaplectic structures; this is reflected in the local picture in the light of Remark 4.5 and [9] (see Remark 5.3).

The bundle representation $\dot{W}^{\mathbb{C}}$ of $n(TX,\omega)^{\mathbb{C}}$ on $E'(P)$ - see (3.14) - induces an action of $X(X) \subset \Gamma(X;n(TX,\omega)^{\mathbb{C}})$ on $\Gamma(X;E'(P))$. This map interacts with D (4.15) in the following manner (which should be compared with (2.28)):

<u>Proposition 4.7</u>:

If $\phi \in C(X)$ and $\xi \in X(X)$ then

$$[D_\phi, \dot{W}^{\mathbb{C}}(\xi)] = \dot{W}^{\mathbb{C}}([\xi_\phi,\xi]) \tag{4.20}$$

<u>Proof</u>:

A local verification is sufficient; for this we refer to [16]. □

A polarization of (X,ω) is a polarization F of the symplectic vector bundle (TX,ω) which is involutive as a subbundle of $TX^{\mathbb{C}}$, thus

$$[\zeta_1,\zeta_2] \in \Gamma(X;F), \quad \forall\ \zeta_1,\zeta_2 \in \Gamma(X;F) \tag{4.21}$$

As before we suppose F to be positive. Let U be an open subset of X. We denote by $C_F(U)$ the set of all $\psi \in C(U)$ for which $\xi_\psi \in \Gamma(U;F)$ and by $C_F^1(U)$ the set of all $\phi \in C(U)$ for which

$$\zeta \in \Gamma(U;F) \Rightarrow [\xi_\phi,\zeta] \in \Gamma(U;F). \tag{4.22}$$

$C_F^1(U)$ is a subalgebra of the Poisson algebra $C(U,\omega|U)$, and $C_F(U)$ is an abelian ideal in $C_F^1(U)$. In view of (4.21) and (4.22) it is clear that if $\phi \in C_F^1(X)$ then Lie differentiation in $\wedge^m T^*X^{\mathbb{C}}$ along ξ_ϕ stabilizes K^F.

As a corollary of Proposition 4.7 we deduce

Proposition 4.8:

If $\phi \in C_F^1(X)$ and $s \in \Gamma(X;E'(P)^F)$ then $D_\phi s$ and $\delta_\phi s$ lie in $\Gamma(X;E'(P)^F)$. □

Prequantization thus restricts to define a representation of $C_F^1(X)$ on the space of sections of the vacuum state bundle

$E'(P)^F$, which upon tensoring with Lie differentiation in K^F yields a Lie algebra morphism

$$\delta^F : C_F^1(X) \to \mathrm{End}\ \Gamma(X;Q_P^F) \tag{4.23}$$

$$\phi \in C_F^1(X) \Rightarrow \delta_\phi^F = \delta_\phi \otimes I + I \otimes L_{\xi_\phi} \tag{4.24}$$

where Q_P^F is the half-form bundle $E'(P)^F \otimes K^F$. In similar fashion D (4.15) gives rise to

$$D^F : C_F^1(X) \to \mathrm{End}\ \Gamma(X;Q_P^F) \tag{4.25}$$

satisfying the analogues of (4.16) and (4.17). δ^F and D^F are clearly related by

$$\phi \in C_F^1(X) \Rightarrow \delta_\phi^F = D^F + \frac{1}{i\hbar}\phi \tag{4.26}$$

Having reduced the algebra of observables from $C(X,\omega)$ to $C_F^1(X)$ and the representation module from $\Gamma(X;E'(P))$ to $\Gamma(X;Q_P^F)$, we further cut down the representation module to the space of polarized sections of the half-form bundle, as follows.

The section s of Q_P^F is said to be polarized iff

$$\psi \in C_F(U) \Rightarrow D^F_\psi s = 0 \qquad (4.27)$$

whenever U is an open set in X. We write $\Gamma_F(X;Q^F_P)$ for the space of all polarized sections of Q^F_P.

<u>Proposition 4.9:</u>

Let $U \subset X$ be open. If $\phi \in C^1_F(U)$ and $s \in \Gamma_F(U;Q^F_P)$ then $\delta^F_\phi s \in \Gamma_F(U;Q^F_P)$.

<u>Proof:</u>

Let $\psi \in C_F(U)$; then using (4.16) and (4.17)

$$D^F_\psi(\delta^F_\phi s) = D^F_\psi D^F_\phi s + D^F_\psi(\frac{1}{i\hbar}\phi s)$$

$$= (D^F_\phi D^F_\psi s + D^F_{\{\psi,\phi\}}s - \frac{1}{i\hbar}\{\psi,\phi\}s)$$

$$+ (\frac{1}{i\hbar}\phi D^F_\phi s + \frac{1}{i\hbar}\{\psi,\phi\}s)$$

which vanishes since $C_F(U)$ is an ideal in $C^1_F(U)$. □

<u>Remark 4.10:</u>

In particular $\Gamma_F(X;Q^F_P)$ is naturally a $C_F(X)$-module since δ^F_ψ acts on $\Gamma_F(X;Q^F_P)$ as multiplication by $\frac{1}{i\hbar}\psi$ when $\psi \in C_F(X)$.

By restriction we therefore have a Lie algebra morphism

$$\delta^F : C^1_F(X) \to \text{End } \Gamma_F(X;Q^F_P) \qquad (4.28)$$

which we call quantization of (X,ω) relative to the quantization data $(P,\gamma;F)$. As has come to be expected of a quantum bundle, Q^F_P has real Chern class

$$c[Q^F_P]^{\mathbb{R}} = [\frac{\omega}{h}] + \frac{1}{2}c_1(\omega)^{\mathbb{R}} \qquad (4.29)$$

in view of Proposition 4.2.

As we have presented it thus far, our geometric quantization scheme is perhaps rather abstract; let us therefore cast it into a more familiar form.

Recall from Remark 3.7 that there is a canonical isomorphism of complex line bundles

$$Q^F_P \otimes Q^F_P \xrightarrow{\sim} P(\eta) \otimes K^F \qquad (4.30)$$

In Q^F_P we have the operator D^F (4.25), in $P(\eta)$ the metric connection ∇^γ of Proposition 4.1, and in K^F the Lie derivative along Hamiltonian vector fields. Relative to (4.30) these satisfy the following Leibnitz rule (which should be compared with (2.30)):

Proposition 4.11:

If $\phi \in C_F^1(X)$ then

$$D_\phi^F \otimes I + I \otimes D_\phi^F = \nabla_{\xi_\phi}^\gamma \otimes I + I \otimes L_{\xi_\phi} \qquad (4.31)$$

Proof:

It suffices to establish the formula locally; this is done in [16] to which we refer for details. □

This result has a number of important consequences.

Remark 4.12:

Our scheme can be phrased in terms of flat partial connections (for which see [11]). Indeed $\nabla^\gamma \otimes I + I \otimes L$ gives a flat F-connection in $P(n) \otimes K^F$ and so uniquely determines a flat F-connection ∇^F in Q_P^F via the Leibnitz rule. The operators $\nabla_{\xi_\psi}^F$ and D_ψ^F agree whenever $\psi \in C_F(X)$. Note that polarized sections of Q_P^F are determined by ∇^F after the usual fashion for flat partial connections: $s \in \Gamma(X;Q_P^F)$ is polarized iff

$$\xi \in \Gamma(X;F) \Rightarrow \nabla_\xi^F s = 0. \qquad (4.32)$$

Remark 4.13:

If we are given both $(P(n),\nabla^\gamma)$ and the square-root Q_P^F of $P(n) \otimes K^F$, then in order to compute the quantization we

simply pass $\nabla^\gamma \otimes I + I \otimes L$ to Q_P^F (uniquely) via the Leibnitz rule and add the appropriate multiplication operator. This technique is particularly effective, especially when the topology is simple; see §5.

Remark 4.14:

As a special case suppose (X,ω) to be the symplectic manifold which underlies a Kähler manifold and suppose F to be the bundle of antiholomorphic tangents; C_F then consists of the holomorphic functions and K^F the holomorphic m-forms. If P is a prequantizable Mp^c structure, then each prequantum form γ on P endows Q_P^F with a flat F-connection ∇^F (as in Remark 4.12), and according to [13] there is a unique holomorphic structure in Q_P^F which is compatible with ∇^F in the sense that the (local) holomorphic sections are precisely the (local ∇^F-) polarized sections; P(n) is likewise given a holomorphic structure, and (4.30) is a holomorphic isomorphism when K^F has the canonical holomorphic structure.

Once having set up the quantization map (4.28) the subsequent development of our scheme differs little from that of the usual scheme of Blattner, Kostant, Sternberg [2]. Let us outline the procedure. Choose a positive polarization F of (X,ω) with $F \cap \bar{F} = D^{\mathbb{C}}$ and assume

(a) D is fibrating (thus, the leaf space X/D is a manifold and the projection $X \to X/D$ a submersion);

(b) Blattner's obstruction for D vanishes (see [3,12].).

Referring to Remark 3.11 we have a self-pairing $\langle\langle\cdot,\cdot\rangle\rangle_F$ of the quantum bundle Q_P^F into $D^1(TX/D)$. If s and t lie in $\Gamma_P(X;Q_P^F)$ then $\langle\langle s,t\rangle\rangle_F$ descends to a density $\langle s,t\rangle_F$ on the leaf-space X/D. Denote by $H_F \subset \Gamma_F(X;Q_P^F)$ the space of those s for which $\langle s,s\rangle_F$ has compact support (and so may be integrated over X/D). H_F is a pre-Hilbert space with inner product $\int\langle\cdot,\cdot\rangle_F$ and is stable under δ_ϕ^F for $\phi \in C_F^1(X)$ since δ_ϕ^F is support-decreasing. The completion $\mathbf{H}_F = \mathbf{H}_F(X,\omega;P,\gamma)$ of H_F is the Hilbert space on which we quantize $C_F^1(X)$.

Our scheme allows for the comparison of quantizations arising from a pair of positive polarizations; indeed the pairing $\langle\langle\cdot,\cdot\rangle\rangle_{F,G}$ of half-form bundles given in (3.31) will give rise to a pairing $\int\langle\cdot,\cdot\rangle_{F,G}$ of $\mathbf{H}_F$ and $\mathbf{H}_G$ in sufficiently regular situations.

§5. GEOMETRIC QUANTIZATION : EXAMPLES

In our final section we demonstrate how our geometric quantization scheme applies in two specific cases - those of a linear symplectic manifold and a complex projective space. Since every symplectic manifold is locally linear, a study of linear symplectic manifolds yields a local picture of our scheme. Complex projective spaces arise as orbit spaces for the energy surfaces of harmonic oscillators; our scheme provides a uniform treatment of harmonic oscillators irrespective of dimensional parity whereas the Kostant-Souriau scheme is unable to deal with the odd-dimensional harmonic oscillators. See [16] for more detail.

Linear symplectic manifolds

Let (V,Ω) be a 2m-dimensional real symplectic vector space. X will denote V endowed with its natural manifold structure. For each $x \in X$ the real-linear isomorphism

$$b_x : V \to T_xX : v \to v_x \tag{5.1}$$

$$f \in C(X) \Rightarrow v_x f = \frac{d}{dt} f(x+tv)\Big|_{t=0} \tag{5.2}$$

induces $\omega_x \in \Lambda^2 T_x^* X$ according to

$$\omega_x(bv_1,bv_2) = \Omega(v_1,v_2), \quad v_1,v_2 \in V. \tag{5.3}$$

We refer to (X,ω) as the linear symplectic manifold modelled on (V,Ω). Note that $Sp(TX,\omega)$ is canonically trivialized by

$$B: X \times Sp(V,\Omega) \to Sp(TX,\omega): (x,g) \to b_x \circ g \tag{5.4}$$

Choose a symplectic basis $(e_1,\ldots,e_m,f_1,\ldots,f_m)$ for (V,Ω) - thus

$$\begin{aligned} \Omega(e_j,e_k) &= \Omega(f_j,f_k) = 0 \\ \Omega(e_j,f_k) &= \delta_{jk} \end{aligned} \tag{5.5}$$

This has the effect of identifying (V,Ω) with $\mathbb{R}^{2m}$ and $Sp(V,\Omega)$ with the real symplectic group $Sp(m;\mathbb{R})$. The dual basis $(p_1,\ldots,p_m,q_1,\ldots,q_m)$ for V^* then forms a global symplectic coordinate chart for (X,ω):

$$\omega = \sum_{j=1}^{m} dp_j \wedge dq_j$$

and the Hamiltonian vector field of $\phi \in C(X)$ is

$$\xi_\phi = \sum_{j=1}^{m} \left(\frac{\partial\phi}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial\phi}{\partial p_j}\frac{\partial}{\partial q_j}\right) \tag{5.7}$$

Since $H^2(X;\mathbb{Z}) = 0$ it follows from Proposition 3.4 that all Mp^c structures for (TX,ω) are equivalent. We shall work with the product Mp^c structure: $P = X \times Mp^c(V,\Omega)$ with projection $B \circ (I \times \sigma): P \to Sp(TX,\omega)$.

From Propositions 4.3 and 4.4 it is clear that (X,ω) is quantizable and has precisely one class of prequantized Mp^c structures. Denote by α the natural flat connection in the principal $Mp^c(V,\Omega)$ bundle $\pi: P \to X$; a routine exercise in basic forms establishes the following description of prequantum forms on P.

<u>Proposition 5.1</u>:

The $u(1)$-valued 1-form γ on P is a prequantum form for (X,ω) iff

$$\gamma = \frac{1}{2}\eta_*\alpha + \frac{1}{i\hbar}\pi^*\theta \tag{5.8}$$

for some primitive θ of ω (so: $\omega = d\theta$). □

Let α^b denote the (complexification of the) connection in $Sp(TX,\omega) \to X$ induced from the natural flat connection in $X \times Sp(V,\Omega) \to X$ via B (5.4). A map

$$z: C(X) \to C(X) \otimes sp(V,\Omega)^{\mathbb{C}} \tag{5.9}$$

is defined by

$$\phi \in C(X) \Rightarrow z_\phi = \alpha^b(\tilde{\xi}_\phi) \circ b \qquad (5.10)$$

In terms of our chosen symplectic coordinates, if $\phi \in C(X)$ then $z_\phi : X \to sp(V,\Omega)^{\mathbb{C}}$ corresponds to the function matrix

$$\left[\begin{array}{c|c} P_\phi & Q_\phi \\ \hline R_\phi & S_\phi \end{array}\right]$$

where $(P_\phi)_{jk} = \frac{\partial^2\phi}{\partial p_k \partial q_j}$, $(Q_\phi)_{jk} = \frac{\partial^2\phi}{\partial q_k \partial q_j}$,

$$(R_\phi)_{jk} = -\frac{\partial^2\phi}{\partial p_k \partial p_j}, \quad (S_\phi)_{jk} = -\frac{\partial^2\phi}{\partial q_k \partial p_j}.$$

Since P is the product $X \times Mp^c(V,\Omega)$ we have a canonical trivialization of $E'(P)$ and hence a canonical identification

$$s : C(X) \otimes E' \to \Gamma(X; E'(P)) : f \to s_f \qquad (5.12)$$

of E'-valued functions with smooth symplectic spinors.

Regarding $sp(V,\Omega)$ as the Lie algebra of $Mp(V,\Omega)$ we have the following formula for prequantization relative to $(P, \gamma = \frac{1}{2}\eta_* \alpha + \frac{1}{i\hbar}\pi^*\theta)$.

Proposition 5.2:

If $\phi \in C(X)$ and $f \in C(X) \otimes E'$ then

$$\delta_\phi(s(f)) = s\left(\frac{1}{i\hbar}(\phi + \theta\xi_\phi)f + \xi_\phi f - \dot{\mu}^{\mathbb{C}}(z_\phi)f\right) \qquad (5.13)$$

Proof:

A matter of identifying the γ-horizontal lift $\hat{\xi}_\phi$ of $\tilde{\xi}_\phi$ in terms of z_ϕ. See [16]. □

Remark 5.3:

This formula for prequantization compares with the local formulae in [9], modulo notational conventions and the term in $\frac{1}{i\hbar}(\theta\xi_\phi)$ (which arises from the structure of a prequantum $U(1)$ bundle).

Since $\Lambda^m T^*X^{\mathbb{C}}$ is canonically isomorphic to $X \times \Lambda^m(V^{\mathbb{C}})^*$ we have a canonical identification

$$s : C(X) \otimes \Lambda^m(V^{\mathbb{C}})^* \to \Gamma(X; \Lambda^m T^*X^{\mathbb{C}}) : k \to s_k \qquad (5.14)$$

of $\Lambda^m(V^{\mathbb{C}})^*$-valued functions with complex m-forms, and in terms of the natural representation of $sp(V,\Omega)^{\mathbb{C}}$ on $\Lambda^m(V^{\mathbb{C}})^*$ we have the formula

$$L_{\xi_\phi}\, s(k) = s(\xi_\phi k - z_\phi . k) \qquad (5.15)$$

when $\phi \in C(X)$ and $k \in C(X) \otimes \Lambda^m(V^{\mathbb{C}})^*$.

For an account of quantization with respect to an arbitrary positive polarization of (X,ω) see [16]; we are here content to deal with a translation-invariant polarization.

Let $\Gamma \in Lag_+(V,\Omega)$; the linear polarization of (X,ω) modelled on Γ is the translation-invariant positive polarization F of (X,ω) defined by

$$x \in X \Rightarrow F_x = b_x^{\mathbb{C}}\Gamma \tag{5.16}$$

Let $f \in C(X) \otimes E'$ have constant value $f_\Gamma \in (E')^\Gamma$ and let $k \in C(X) \otimes \Lambda^m(V^{\mathbb{C}})^*$ have constant value $k_\Gamma \in K^\Gamma$; thus $s_f \otimes s_k$ is a zero-free global section of the quantum bundle Q_P^F.

Let $\phi \in C_F^1(X)$. It is readily verified that

$$x \in X \Rightarrow z_\phi(x) \in sp(V,\Omega)_\Gamma^{\mathbb{C}} . \tag{5.17}$$

From Propositions 2.5 and 5.2 it follows that

$$f \equiv f_\Gamma \Rightarrow \delta_\phi s_f = (\frac{1}{i\hbar}(\phi + \theta\xi_\phi) + \frac{1}{2}\operatorname{Tr}_\Gamma z_\phi)s_f \tag{5.18}$$

From (5.15) and since $sp(V,\Omega)_\Gamma^{\mathbb{C}}$ acts on K^Γ via Tr_Γ it follows that

$$k \equiv k_\Gamma \Rightarrow L_{\xi_\phi} s_k = (-\operatorname{Tr} z_\phi)s_k \tag{5.19}$$

We therefore have the following explicit formula for quantization of (X,ω) relative to $(P,\gamma;F)$ in terms of the translation-invariant section $s_o = s_f \otimes s_k$ of Q_P^F:

<u>Proposition 5.4:</u>

If $\phi \in C_F^1(X)$ and $\psi \in C(X)$ then

$$\delta_\phi^F(\psi \cdot s_o) = ((\frac{1}{i\hbar}(\phi + \theta\xi_\phi) - \frac{1}{2}\operatorname{Tr}_\Gamma z_\phi)\psi + \{\phi,\psi\})s_o \tag{5.20}$$

□

<u>Remark 5.5:</u>

It can be checked that

$$X \in C_F(X) \Rightarrow \operatorname{Tr}_\Gamma z_X = 0 \tag{5.21}$$

Consequently the section $\psi \cdot s_o$ of Q_P^F is polarized iff ψ satisfies the differential equations

$$X \in C_F(U) \Rightarrow \xi_X\psi + \frac{1}{i\hbar}(\theta\xi_X)\psi = 0 \tag{5.22}$$

on each open set $U \subset X$. Since $F + \bar{F} \subset TX^{\mathbb{C}}$ is involutive, there exists $\theta^F \in \Gamma(X;F^o)$ such that $d\theta^F = \omega^{\mathbb{C}}$; since X is cohomologically trivial, there exists $\lambda^F \in C(X)$ such that $\theta^F = \theta + d\lambda^F$. The condition for $\psi \cdot s_o$ to be polarized can now be written in the form

$$\psi \cdot s_o \in \Gamma_F(X; Q_P^F) \iff \psi \exp\left(-\frac{1}{i\hbar}\lambda^F\right) \in C_F(X). \tag{5.23}$$

We close this section with two concrete examples. These are essentially the simplest cases of quantizing: first, a cotangent bundle (T^*R^m) with respect to the vertical polarization, and second, a Kähler manifold with respect to the antiholomorphic polarization.

<u>Example 5.6</u>:

Let Γ be the (real) polarization of (V,Ω) having basis $(e_1,\ldots,e_m)$. If $\phi \in C(X)$, then $\phi \in C_F(X)$ iff $\frac{\partial\phi}{\partial p_j} = 0$ for $1 \le j \le m$ (thus: " ϕ is a function of the q's alone") and $\phi \in C_F^1(X)$ iff $\frac{\partial^2\phi}{\partial p_j \partial p_k} = 0$ for $1 \le j, k \le m$ (thus " ϕ is a function of the q's, linear in the p's"). If $\phi \in C_F^1(X)$ then

$$Tr_\Gamma z_\phi = \sum_{j=1}^{m} \frac{\partial^2\phi}{\partial p_j \partial q_j} \tag{5.24}$$

Let the prequantum form $\gamma = \frac{1}{2}\eta_* \alpha + \frac{1}{i\hbar}\pi^*\theta$ be determined by the primitive

$$\theta = \sum_{j=1}^{m} p_j dq_j \tag{5.25}$$

The section $\psi \cdot s_o$ of Q_P^F is polarized iff $\psi \in C_F(X)$. Quantization (4.28) appears as

$$\delta_\phi^F(\psi \cdot s_o) =$$

$$\left(\frac{1}{i\hbar}\left(\phi - \Sigma p_j \frac{\partial\phi}{\partial p_j}\right) - \frac{1}{2}\Sigma\frac{\partial^2\phi}{\partial p_j \partial q_j}\right)\psi - \Sigma \frac{\partial\phi}{\partial p_j}\frac{\partial\psi}{\partial q_j}\cdot s_o \tag{5.26}$$

for $\phi \in C_F^1(X)$ and $\psi \in C_F(X)$.

<u>Example 5.7</u>:

Let Γ be the (strictly positive) polarization of (V,Ω) having basis $(e_1 + if_1,\ldots,e_m + if_m)$. If $\phi \in C(X)$, then $\phi \in C_F(X)$ iff the Cauchy-Riemann equations

$$1 \le j \le m \Rightarrow \frac{\partial\phi}{\partial p_j} + i\,\frac{\partial\phi}{\partial p_j} = 0 \tag{5.27}$$

hold and $\phi \in C_F^1(X)$ iff

$$1 \le j, k \le m \Rightarrow \begin{cases} \frac{\partial^2\phi}{\partial p_j \partial q_k} + \frac{\partial^2\phi}{\partial q_j \partial p_k} = 0 \\ \frac{\partial^2\phi}{\partial p_j \partial q_k} + \frac{\partial^2\phi}{\partial q_j \partial p_k} = 0 \end{cases} \tag{5.28}$$

If $\phi \in C^1_F(X)$ then

$$\mathrm{Tr}_\Gamma z_\phi = i \sum_{j=1}^{m} \frac{\partial^2 \phi}{\partial q_j \partial q_j} \tag{5.29}$$

Let the prequantum form $\gamma = \frac{1}{2}\eta_*\alpha + \frac{1}{i\hbar}\tau^*\theta$ be determined by the primitive

$$\theta = \frac{1}{2} \sum_{j=1}^{m} (p_j dq_j - q_j dp_j) \tag{5.30}$$

The section $\psi \cdot s_o$ of Q^F_p is polarized iff $\psi \cdot \exp(\frac{1}{4\hbar} \Sigma(p_j^2 + q_j^2)) \in C_F(X)$. Quantization (4.28) appears as

$$\delta^F_\phi(\psi \cdot s_o) =$$

$$\left(\frac{1}{i\hbar}\left(\phi - \frac{1}{2}\Sigma\left(p_j \frac{\partial\phi}{\partial p_j} + q_j \frac{\partial\phi}{\partial q_j}\right)\right) - \frac{1}{2} i \Sigma \frac{\partial^2\phi}{\partial q_j^2}\right)\psi \tag{5.31}$$

$$+ \Sigma\left(\frac{\partial\phi}{\partial q_j}\frac{\partial\psi}{\partial p_j} - \frac{\partial\phi}{\partial p_j}\frac{\partial\psi}{\partial q_j}\right)) \cdot s_o$$

for $\phi \in C^1_F(X)$ and $\psi \cdot \exp(\frac{1}{4\hbar} \Sigma(p_j^2 + q_j^2)) \in C_F(X)$.

Complex Projective Spaces

Let $(V, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space of dimension

$m = n+1$ with $n > 0$. The projective space $\mathbb{P}(V)$ is the space of all complex lines in V and is naturally a complex manifold of dimension n. The hyperplane section bundle $\pi_H: H_V \to \mathbb{P}(V)$ over $\mathbb{P}(V)$ is the holomorphic line bundle whose total space H_V is the set of all complex-linear functionals on complex lines in V and whose projection π_H assigns to each linear functional the line on which it acts.

Recall that on every complex manifold there is a natural bigrading

$$\Omega^r = \bigoplus_{p+q=r} \Omega^{p,q} \tag{5.32}$$

of complex forms according to type, and a natural decomposition of the exterior derivative into $d = \partial + \bar\partial$ with $\partial: \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar\partial: \Omega^{p,q} \to \Omega^{p,q+1}$.

If $|\cdot|$ denotes the norm on V defined by the Hermitian inner product $\langle\cdot,\cdot\rangle$ then

$$\hat\Phi = \partial\bar\partial \log |\cdot|^2 \in \Omega^{1,1}(V) \tag{5.33}$$

descends to a (1,1)-form Φ on $\mathbb{P}(V)$ under the natural map $\rho_V: V\backslash\{0\} \to \mathbb{P}(V)$ which assigns to each $v \in V\backslash\{0\}$ the complex line through v, thus

$$\hat\Phi = \rho_V^*\Phi \tag{5.34}$$

Choose a unitary basis $(v_1,\ldots,v_m)$ for $(V,\langle\cdot,\cdot\rangle)$, with dual basis $(z^1,\ldots,z^m)$ for V^*. This identifies V with $\mathbb{C}^m$ and $\langle\cdot,\cdot\rangle$ with the standard inner product on $\mathbb{C}^m$, and $\mathbb{P}(V)$ becomes $\mathbb{P}(\mathbb{C}^m) = \mathbb{CP}(n)$. If we define

$$U_j = \{\ell \in \mathbb{P}(V) \;\; |\ell \cap \ker z^j = 0\} \tag{5.35}$$

and

$$w_j^k : U_j \to \mathbb{C} : \rho_V(v) \to \frac{z^k(v)}{z^j(v)} \tag{5.36}$$

then $(w_j^1,\ldots,\hat{w}_j^j,\ldots,w_j^m)$ are holomorphic coordinates on the open set $U_j \subset \mathbb{P}(V)$ and

$$\Phi|U_j = L_j^{-4}(L_j^2 \sum_{k\neq j} dw_j^k \wedge d\bar{w}_j^k - \sum_{r,s\neq j} \bar{w}_j^r w_j^s dw_j^r \wedge d\bar{w}_j^s) \tag{5.37}$$

where the length function $L_j : U_j \to \mathbb{R}_+$ is defined by

$$L_j^2 = 1 + \sum_{k\neq j} |w_j^k|^2 \tag{5.38}$$

In subsequent local formulae we shall generally simplify notation by writing $U = U_j$, $L = L_j$, and $(w^1,\ldots,w^n)$ in place of $(w_j^1,\ldots,\hat{w}_j^j,\ldots,w_j^m)$, for a fixed j in the range $1 \le j \le m$.

For each positive real number E (energy) we define

$$\omega_E = iE\,\Phi. \tag{5.39}$$

$(\mathbb{P}(V),\omega_E)$ is then a symplectic manifold of which the antiholomorphic tangent bundle F is a positive polarization. Up to scalar multiples ω_E is the fundamental Kähler form of the Fubini-Study metric on $\mathbb{P}(V)$.

Before quantizing $(\mathbb{P}(V),\omega_E)$ we digress to recall some basic facts concerning (Hermitian holomorphic) complex line bundles on complex projective space.

We shall denote by $\mathcal{O}(\mathbb{P}(V);L)$ the space of holomorphic sections of the holomorphic line bundle $L \to \mathbb{P}(V)$ over the open set $U \subset \mathbb{P}(V)$

<u>Remark 5.8</u>:

Each $f \in V^*$ gives rise to a global holomorphic section s_f of the hyperplane section bundle H_V according to

$$s_f : \mathbb{P}(V) \to H_V : \quad \ell \to f|\ell \tag{5.40}$$

and we have a natural complex-linear isomorphism of the global sections of H_V with V^*:

$$s : V^* \to \mathcal{O}(\mathbb{P}(V);H_V) : f \to s_f \tag{5.41}$$

Remark 5.9:

More generally, if $P^r(V)$ denotes the space of homogeneous polynomials on V of degree $r \in \mathbb{N} \cup \{0\}$ then there is a natrual complex-linear isomorphism

$$P^r(V) \to \mathcal{O}(\mathbb{P}(V); H_V^r) \qquad (5.42)$$

where $H_V^r = H_V \otimes \ldots \otimes H_V$ (r factors) and if $s < 0$ then $\mathcal{O}(\mathbb{P}(V); H_V^s) = 0$ where $H_V^s = (H_V^{-s})^*$. □

We shall denote by $s_j = s_{z^j} \in \mathcal{O}(\mathbb{P}(V); H_V)$ the holomorphic section of H_V corresponding to $z^j \in V^*$ under (5.41). More generally, for $r \in \mathbb{N}$ we shall write $s^r \in \mathcal{O}(\mathbb{P}(V); H_V^r)$ for the rth. tensor power of $s = s_j$ when j is understood.

Remark 5.10:

By restriction and dualization, $\langle\cdot,\cdot\rangle$ on V induces a Hermitian inner product $\langle\cdot,\cdot\rangle_\ell$ on $\ell^* = \pi_H^{-1}(\ell)$ for each $\ell \in \mathbb{P}(V)$. In this way $\langle\cdot,\cdot\rangle$ gives rise to a Hermitian structure $\langle\cdot,\cdot\rangle_H$ in the hyperplane section bundle H_V. Over $U_j \subset \mathbb{P}(V)$ we have the local formula

$$\langle s_j, s_j\rangle_H = L_j^{-2} \qquad (5.43)$$

Remark 5.11:

Let ∇ denote the unique connection in H_V which is compatible both with the natural holomorphic structure and with the Hermitian structure $\langle\cdot,\cdot\rangle_H$. It is readily verified that over $U_j \subset \mathbb{P}(V)$

$$\xi \in X(U_j) \Rightarrow \nabla_\xi s_j = -L_j^{-2} \sum_{k\neq j} \bar{w}_j^k dw_j^k(\xi) s_j \qquad (5.44)$$

and that ∇ has curvature precisely Φ (5.34):

$$[\nabla_\xi, \nabla_\eta]s - \nabla_{[\xi,\eta]}s = \Phi(\xi,\eta)s \qquad (5.45)$$

whenever $\xi,\eta \in X(\mathbb{P}(V))$ and $s \in \Gamma(\mathbb{P}(V); H_V)$.

As a consequence of (5.45) the hyperplane section bundle has real Chern class

$$c[H_V]^{\mathbb{R}} = [-\frac{\Phi}{2\pi i}] \qquad (5.46)$$

and more generally if $r \in \mathbb{Z}$ then

$$c[H_V^r]^{\mathbb{R}} = [-r\frac{\Phi}{2\pi i}]. \qquad (5.47)$$

Remark 5.12:

The group of isomorphism classes of (holomorphic or complex) line bundles on $\mathbb{P}(V)$ is infinite cyclic, generated by the hyperplane section bundle. Up to isomorphism these line bundles on $\mathbb{P}(V)$ are determined by their real Chern classes.

In particular a careful consideration of transition functions shows that the canonical bundle K^F of holomorphic m-forms is isomorphic to $H_V^{-m} = (H_V^*)^{n+1}$, so that

$$c[K^F]^{\mathbb{R}} = [m \frac{\Phi}{2\pi i}]. \tag{5.48}$$

This concludes our digression; we can now describe our quantization of $(\mathbb{P}(V), \omega_E)$.

Since $H^2(\mathbb{P}(V);\mathbb{Z})$ is infinite cyclic (generated by $c[H_V]$) the group $Mp^c[T\mathbb{P}(V), \omega_E]$ of equivalence classes of Mp^c structures for $(T\mathbb{P}(V), \omega_E)$ is likewise infinite cyclic.

From (5.39) and (5.48) we deduce

$$[\frac{\omega_E}{h}] \pm \frac{1}{2} c_1(\omega_E)^{\mathbb{R}} = [-(\frac{E}{h} \mp \frac{1}{2}m)\frac{\Phi}{2\pi i}] \tag{5.49}$$

and so in view of Proposition 4.3 and Remark 5.12 we have:

Proposition 5.13:

$(\mathbb{P}(V), \omega_E)$ is quantizable iff

$$E = (N + \tfrac{1}{2}m)\hbar, \quad N \in \mathbb{Z}. \tag{5.50}$$

□

Assume henceforth that the quantization condition (5.50) is satisfied.

Since $\mathbb{P}(V)$ is simply-connected it follows that $H^1(X;U(1))$ is trivial and therefore that all prequantized Mp^c structures for $(\mathbb{P}(V), \omega_E)$ are equivalent. Choose and fix the prequantized Mp^c structure (P, γ).

Referring to Remark 4.14 we have canonical structures of holomorphic line bundle on K^F, $P(\eta)$, $E'(P)^F$, Q_P^F, and a canonical isomorphism of holomorphic line bundles

$$Q_P^F \otimes Q_P^F \xrightarrow{\sim} P(\eta) \otimes K^F \tag{5.51}$$

It is a simple matter to deduce from (5.39) (5.48) (5.51) that we have holomorphic isomorphisms

$$K^F \xrightarrow{\sim} H_V^{-m}, \qquad P(\eta) \xrightarrow{\sim} H_V^{2N+m},$$
$$E'(P)^F \xrightarrow{\sim} H_V^{N+m}, \qquad Q_P^F \xrightarrow{\sim} H_V^{N}. \tag{5.52}$$

Since $\mathbb{P}(V)$ is compact these isomorphisms are unique modulo nonzero constants and may be chosen so as to make (5.51) correspond to the standard isomorphism

$$H_V^N \otimes H_V^N \xrightarrow{\sim} H_V^{2N+m} \otimes H_V^{-m} \tag{5.53}$$

<u>Remark 5.14</u>:

The space $\Gamma_F(\mathbb{P}(V);Q_P^F)$ of polarized sections of the quantum bundle $Q_P^F = H_V^N$ is precisely $\mathcal{O}(\mathbb{P}(V);H_V^N)$. In view of Remark 5.9 we therefore recover the familiar quantization condition

$$E = (N + \tfrac{1}{2}m)\hbar;\ N = 0,1,2,... \tag{5.54}$$

for the quantization module to be nonzero.

Assume henceforth that the quantization condition (5.54) holds.

To develop an explicit formula for quantization δ^F of $(\mathbb{P}(V),\omega_E)$ relative to $(P,\gamma;F)$ is now quite straightforward. By means of the isomorphisms (5.52) we pass across the various operators involved in quantization to the appropriate tensor powers of the hyperplane section bundle. In particular $(P(\eta),\nabla^\gamma)$ becomes the $(2N + m)$th. tensor power $(H_V^{2N+m}, \nabla^{2N+m})$ of (H_V,∇).

Observe that we are now in the position covered by Remark 4.13: we have identified both $(P(\eta),\nabla^\gamma) = (H_V^{2N+m}, \nabla^{2N+m})$ and the square-root $Q_P^F = H_V^N$ of $P(\eta) \otimes K^F = H_V^{2N+m} \otimes H_V^{-m} = H_V^{2N}$. After some routine computation the technique suggested by Remark 4.13 results in (see [16]):

<u>Proposition 5.15</u>:

If $\phi \in C_F^1(U)$ then quantization

$$\delta_\phi^F = D_\phi^F + \frac{1}{i\hbar}\phi \tag{5.55}$$

is determined on the canonical holomorphic section $s^N = s_j^N$ of $H_V^N = Q_P^F$ over $U = U_j$ by the formula

$$D_\phi^F\, s^N = \frac{L^2}{(2N+m)i\hbar}\left((\delta^{rs} + w^r\bar{w}^s)\frac{\partial^2\phi}{\partial w^r\partial\bar{w}^s} - 2N\bar{w}^b\frac{\partial\phi}{\partial\bar{w}^b}\right)s^N \tag{5.56}$$

where the indices b,r,s are summed over {1,...,n}. □

<u>Remark 5.16</u>:

By virtue of (5.42) any holomorphic (or, polarized) section of $H_V^N = Q_P^F$ is a polynomial multiple of the canonical section s^N over U: it is clear how such a section can be quantized using (5.56).

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