INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392 - I

SMR/161 - 05

COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
(4 November - 6 December 1985)

ELEMENTARY STRUCTURE OF LIE ALGEBRAS
LECTURES 4 AND 5

M. PICARDELLO

(4) Representations of sl_2 of finite dimensions; roots and Cartan subalgebra of a semisimple algebra

Basis for sl_2: $x = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ $y = \begin{pmatrix}0&0\\1&0\end{pmatrix}$ $h = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$[hx] = 2x$ $[hy] = -2y$, $[xy] = h$

Weights and f.d. representations of sl_2.

$L = sl_2$. V L-module. By the last theorem, h acts diagonally $\to$ $\forall \lambda \in F$, $V_\lambda = \{v : hv = \lambda v\}$.
If $V_\lambda \neq 0$, say λ is a weight. Then

$$V = \bigoplus_{V_\lambda \neq 0} V_\lambda$$

Lemma $x : V_\lambda \to V_{\lambda+2}$, $y : V_\lambda \to V_{\lambda-2}$

Proof. $h.(x.v) = [hx].v + x.h.v = (2+\lambda)\, x.v$
and similarly for y.

~~Now let V be irreducible~~ Def. If $V_\lambda \neq 0$, $V_{\lambda+2} = 0$, every nonzero $v \in V_\lambda$ is called a maximal vector for V.
Since $\dim V < \infty$, maximal vectors exist. ~~If V is irreducible, it cannot contain~~

Now, let V be irreducible and v_0 maximal, $v_0 \in V_\lambda$.
Put $v_j = \frac{1}{j!} y^j . v_0$ $(i \geq 0)$, and $v_{-1} = 0$.

Proposition.
$$h.v_j = (\lambda - 2j)\, v_j$$
$$y.v_j = (j+1)\, v_{j+1} \quad \text{(ascent)}$$
$$x.v_j = (\lambda - j + 1)\, v_{j-1} \quad \text{(descent)}$$

Proof. i) is obvious ii) is the definition of v_j.

iii):
$$j\, x.v_j = x.\overset{= j v_j}{(y.v_{j-1})} = [xy].v_{j-1} + y.x.v_{j-1}$$
$$= h.v_{j-1} + y.x.v_{j-1} = (\lambda - 2(j-1))\, v_{j-1} + (\lambda - j + 2)\, y.v_{j-2}$$
induction
$$= (\lambda - 2j + 2)\, v_{j-1} + (j-1)(\lambda - j + 2)\, v_{j-1} =$$
$$= j(\lambda - j + 1)\, v_{j-1}$$

Since $\dim V < \infty$, $\exists$ smallest integer m: $v_m \neq 0$, $v_{m+1} = 0$.

$W = \langle v_0, v_1, \dots, v_m \rangle$ is an L-submodule.

V irreducible $\Rightarrow$ $V = W$.

For $j = m+1$, $x.v_j = (\lambda - j + 1)\, v_{j-1}$
$$0 = (\lambda - m)\, v_m$$

$\Rightarrow$ $\lambda = m$

Thus $h v_0 = m v_0$, $h v_m = -m v_m$.

Finally, $\dim V_j = 1$ if V irreducible, because
$V = \langle v_0, v_1, \dots, v_m \rangle$ (one vector $\forall$ h-eigenspace).

Thus:

Theorem. V finite dim. irreducible sl_2-module $\Rightarrow$

① $\exists$ h-eigenspace V_λ, $\lambda = m, m-2, \dots, -m$, $\dim V_\lambda = 1$ s.t. $V = \bigoplus_{-m}^{m} V_\lambda$

② V has, up to scalar multiples, a unique maximal vector, with maximal weight m

③ In particular, $\exists$ at most 1 irreducible sl_2-module $\forall$ dimension.

4) If V is any sl_2-module of finite dimension, not necessarily irreducible, then the eigenvalues of h are integers each occurring with its negative (the same multiplicity), and $\dim V = \dim V_0 + \dim V_1$.

(5) For each dimension $m+1 \geq 1$, the formulas of the Proposition give a representation of sl_2 on the space $\langle v_0, \ldots v_m \rangle \cong F^{m+1}$.

Exercises: 7.4 (sl_2-modules ~~realized~~ as spaces of homogeneous polynomials of given degree m); 7.6 (tensor products of sl_2-modules and their decomposition); 7.7 (infinite-dimensional sl_2-modules).

The Cartan decomposition of a semisimple algebra.

L semisimple $\Rightarrow \exists\, x \in L : x_s \neq 0$ [otherwise ~~take exact~~ every $x \in L$ is ad-nilpotent (abstract Jord. dec. = Jordan dec. in $\mathrm{ad}\, L$) hence L nilpotent by Engel's Thm].

But then $\overset{\mathrm{ad}\, x_s \in \mathrm{ad}\, L,\ \text{hence}}{x_s \in L}$ $\Rightarrow$ L has subalgebras of semisimple elements.

Def. A subalgebra consisting of semisimple elements is called "toral".

Lemma. Toral subalgebras are abelian

(Warning: ~~at first~~ a priori, the semisimple elements need not be simultaneously diagonalizable!)

Proof. T toral. ~~Claim~~ To show: $\forall x \in T$, $[x, T] = 0$. Since $\mathrm{ad}_T\, x$ semisimple, this is equivalent to say $\mathrm{ad}_T\, x$ has only the eigenvalue 0. ~~Ot~~ Suppose not, $\exists\, y \in T : [x\, y] = a y$ $(a \neq 0)$ Thus $[y, [y, x]] = 0$ (that is, $[y\, x]$ is eigenvector of $\mathrm{ad}_T\, y$ with eigenvalue 0). But $\mathrm{ad}_T\, y$ is diagonalizable too: thus $x = \sum \beta_i x_i$, x_i eigenvectors of $\mathrm{ad}_T\, y$ (with eigenvalues λ_i). Now $[y\, x] = \sum_{\lambda_i \neq 0} \beta_i \lambda_i x_i$, and

$$[y[y\,x]] = \sum \beta_i \lambda_i^2 x_i \neq 0.$$

(The point is: the λ_i's cannot be all 0 since $[y\,x] = \sum \beta_i \lambda_i x_i$ is $\neq 0$, equals $-ay$ (unless $a = 0$, of course!)

Let H be a maximal toral subalgebra in L.

Then H is abelian, hence $\mathrm{ad}_L(H)$ is a family simultaneously diagonalizable. That is: let $\alpha \in H^*$ and define the "root space" $L_\alpha = \{x \in L : [h\, x] = \alpha(h) x \ \forall h \in H\}$.

Then let Φ be the set of $\alpha \neq 0$ s.t. $L_\alpha \neq 0$. We have

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha . \qquad L_0 = C_L(H) \text{ centralizer.}$$

Proposition 1 1) $\forall \alpha, \beta \in H^*$, $[L_\alpha\, L_\beta] \subseteq L_{\alpha+\beta}$

2) $\forall x \in L_\alpha$, $\alpha \neq 0$, $\mathrm{ad}\, x$ is nilpotent

3) If $\alpha + \beta \neq 0$, $\kappa(L_\alpha, L_\beta) = 0$

Proof. 1) $[h\, [x y]] = [[h\, x]\, y] + [x\, [h y]] = (\alpha(h) + \beta(h)) [x y]$

$x \in L_\alpha$, $y \in L_\beta$

2) obvious by 1) and by $\dim L < \infty$.

3) Choose h: $(\alpha + \beta)(h) \neq 0$. Then $\kappa([h x], y) = -\kappa(x, [h y])$ $\Rightarrow \alpha(h)\, \kappa(x, y) = -\beta(h)\, \kappa(x, y) \Rightarrow \kappa(x, y) = 0$.

Corollary $\kappa|_{L_0 \times L_0}$ is nondegenerate

Proof L semisimple $\Rightarrow \kappa$ nondegenerate. But $L_0 \perp_\kappa L_\alpha$ $\forall \alpha \in \Phi$ by Prop 1(3). Hence $L_0 \not\perp_\kappa L_0$: $\nexists\, z \in L_0 : \kappa(z, L_0) = 0 \Rightarrow z = 0$.

Observation $x, y \in \operatorname{End} V$, $[xy]=0$, y nilpotent $\Rightarrow$ xy nilpotent (hence $\operatorname{tr}(xy)=0$).

Proposition 2. H is self-centralizing: $H = C_L(H) = L_0$.

Proof. Write $C = C_L(H)$.

1) C contains the semisimple and nilpotent parts of its elements.

$x \in C_L(H) \Leftrightarrow \operatorname{ad} x : H \to 0 \Rightarrow$ $(\operatorname{ad} x)_s$, $(\operatorname{ad} x)_n$ do the same

But $(\operatorname{ad} x)_s = \operatorname{ad} x_s$, $(\operatorname{ad} x)_n = \operatorname{ad} x_n$.

Thus $x_s, x_n \in C$.

2) $x \in C$ semisimple element $\Rightarrow x \in H$

$\langle H, x \rangle$ is again toral. But H is maximal toral!

3) $\kappa|_{H \times H}$ is nondegenerate

Suppose $\kappa(h, H) = 0$. We claim that this yields $\kappa(h, C) = 0$ (hence $h = 0$ by the ~~...~~ nondegeneracy of κ on C). Indeed, it is enough, by (1) and (2), to show that $\kappa(h, x) = 0$ $\forall x$ nilpotent in C. But $x \in C \Rightarrow [xh] = 0$, now $\kappa(h,x) = \operatorname{tr}(\operatorname{ad} h \operatorname{ad} x) = 0$ by the Observation

4) C is nilpotent.

By Engel's thm, it is enough to show that $\operatorname{ad}_C x$ is a nilpotent operator $\forall x \in C$. Now, if x is semisimple, then by (2) $x \in H$ and so $\operatorname{ad}_C x = 0$ since $[C,H]=0$. Thus, in this case, $\operatorname{ad}_C x$ is nilpotent. If x is nilpotent, then obviously $\operatorname{ad}_C x$ is nilpotent. In general, $\operatorname{ad}_C x$ is the sum of two commuting nilpotent operators, hence nilpotent.

5) $H \cap [CC] = 0$

$\kappa(H, [CC]) = \kappa([HC], C) = 0$ since $[HC] = 0$. But $\kappa|_{H \times H}$ is nondegenerate!

6) C is abelian, i.e., $[CC] = 0$

Suppose not. By (4), C is nilpotent, hence $Z(C) \cap [CC] \neq 0$. Pick z there, ume $z \neq 0$. If z were semisimple, then by (2) $z \in H$, and by (5) $z = 0$. Thus $z = s + n$, $n \neq 0$. Now, by (1), $n \in C$. ~~Thus~~ Since $z \in Z(C)$, $z : C \to 0$, and so $n : C \to 0$ (Jordan decomp. properties!), that is, $n \in Z(C)$. But then, by the Observation, $\kappa(n, C) = \operatorname{tr}(\operatorname{ad} n \operatorname{ad} C) = 0$. Thus $n \neq 0 \Rightarrow \kappa|_{C \times C}$ is degenerate ✗ (Corollary...)

7) $H = C$.

If not, by (1),(2) C contains $x \neq 0$ nilpotent. ~~Again~~ Since C is commutative by (6), another application of the Observation gives $\kappa(x, C) = 0$, contradicting nondegeneracy.

Corollary: $H \simeq H^*$ via κ (nondegenerate!). $\phi \in H^* \to t_\phi \in H$: $\phi(h) = \kappa(t_\phi, h)$ $\forall h \in H$ (Riesz representation thm).

Proposition 3.

1) Φ spans H^*

2) $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$

3) $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha} \Rightarrow [xy] = \kappa(x,y)\, t_\alpha$

4) $\alpha \in \Phi$: $[L_\alpha L_{-\alpha}]$ has dim. 1, spanned by t_α

5) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$ $\forall \alpha \in \Phi$

6) $\forall x_\alpha \in L_\alpha$ $\exists!$ $y_\alpha \in L_{-\alpha}$, $h_\alpha = [x_\alpha y_\alpha]$:

$\langle x_\alpha, y_\alpha, h_\alpha \rangle \simeq sl_2$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

7) $h_\alpha = \dfrac{2 t_\alpha}{\kappa(t_\alpha, t_\alpha)}$; $h_\alpha = - h_{-\alpha}$; $\alpha(h_\alpha) = 2$

Main ingredients: nondegeneracy of κ; $Z(L) = 0$

Proposition 4. 1) $\forall \alpha \in \Phi$ $\dim L_\alpha = 1$. Thus $S_\alpha = L_\alpha \oplus L_{-\alpha} + \underset{H_\alpha}{\overset{\shortparallel}{[L_\alpha L_{-\alpha}]}}$

and $\forall x_\alpha \neq 0 \in L_\alpha$, $\exists!\ y_\alpha \in L_{-\alpha} : [x_\alpha, y_\alpha] = h_\alpha$

2) The only scalar multiples of $\alpha \in \Phi$ which are roots are $\pm\alpha$

3) $\alpha, \beta \in \Phi \Rightarrow \underbrace{\beta(h_\alpha)}_{\text{"Cartan integer"}} \in \mathbb{Z}$, and $\beta - \beta(h_\alpha)\alpha \in \Phi$

4) $\alpha, \beta, \alpha+\beta \in \Phi \Rightarrow [L_\alpha L_\beta] = L_{\alpha+\beta}$

5) $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. ~~$\Rightarrow$~~ Let r, q largest integers s.t. $\beta - r\alpha$, $\beta + q\alpha \in \Phi$. Then the string $\beta + j\alpha$, $-r \le j \le q$ $\in \Phi$, and $\beta(h_\alpha) = r - q$

6) L = ~~$\oplus L$~~ Lie algebra generated by L_α, $\alpha \in \Phi$.

Proof. Use the action of $sl_2 \subset L$ on L, adjoint.

$\forall \alpha \in \Phi \to S_\alpha \simeq sl_2$. Describe ~~the~~ L as an S_α-module (adjoint action):

Let $M = H \oplus \bigoplus_{c \in F} L_{c\alpha}$. M is an S_α-submodule (S_α shifts the root spaces in M). ($\operatorname{ad} x_\alpha : L_{c\alpha} \to L_{(c+1)\alpha}$) Weights of h_α in M: 0 (on H) and $c\,\alpha(h_\alpha) = 2c$ (for $c \in F$). Thus $2c \in \mathbb{Z}$.

Now $H = \operatorname{Ker}\alpha \oplus F h_\alpha$. $S_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle$.

On $h \in \operatorname{Ker}\alpha$ h_α acts by 0 (as on all of H), and $[x_\alpha h] = -\underset{0}{\alpha(h)} x_\alpha = [y_\alpha h]$. Thus S_α acts trivially on $\operatorname{Ker}\alpha$.

But $S_\alpha \subset M$ is an irreducible S_α-submodule. How many times we have the weight 0? In $\operatorname{Ker}\alpha = \dim H - 1$, In $S_\alpha \subseteq M = 1$ $\Big\} = \dim H$.

On the other hand ~~if h ... has a weight ... as ... L~~ ~~... the weight 0 ... the~~ if $\exists$ other submodule, $\exists$ other weight 0 on H. ~~on H ... L ... it has the weight $\beta(h_\alpha)$~~

~~But~~ we have exhausted the weight 0 of h_α on H. Thus

$H \supseteq$ ... Ker ... $\langle x, \ldots \rangle$. But then ...

Proof. 1) If not, $\exists\ 0 \neq h \in H$: $\alpha(h) = 0$ $\forall \alpha \in \Phi$ $\rightarrow [h, L_\alpha] = 0$ $\forall \alpha$; ~~since~~ $[h, H] = 0$, $\rightsquigarrow h \in Z(L) = 0$ $*$

2) $\alpha \in \Phi$. If $-\alpha \notin \Phi$, $\kappa(L_\alpha, L_\beta) = 0$ $\forall \beta \in H^*$ (Prop. 1.(3))
$\Rightarrow \kappa(L_\alpha, L) = 0$, contradicting nondegeneracy

3) $x \in L_\alpha$, $y \in L_{-\alpha}$, $h \in H \Rightarrow \kappa(h, [xy]) = \kappa([hx], y) =$
$= \alpha(h)\,\kappa(x,y) = \underset{\alpha(h)}{\underbrace{\kappa(t_\alpha, h)}}\,\kappa(x,y) = \kappa(\kappa(x,y)t_\alpha, h) =$
$= \kappa(h, \kappa(x,y)t_\alpha) \Longrightarrow H \perp_\kappa [xy] - \kappa(x,y)t_\alpha \Rightarrow$
$\Rightarrow [xy] = \kappa(x,y)t_\alpha$ by nondegeneracy

4) By (3), $[L_\alpha L_{-\alpha}] = \begin{cases} 0 \\ F t_\alpha \end{cases}$. Enough now to show $[L_\alpha L_{-\alpha}] \neq 0$. Let $x \neq 0 \in L_\alpha$. If $\kappa(x, L_{-\alpha}) = 0$, then $\kappa(x, L) = 0$ (Prop. 1.(3)), contradicting nondegeneracy. Thus $\exists y$: $\kappa(x,y) \neq 0$. By (3), $[xy] = \kappa(x,y)\,t_\alpha$.

5) Suppose $\alpha(t_\alpha) = 0$. Then $[t_\alpha x] = 0 = [t_\alpha y]$ $\forall x \in L_\alpha, y \in L_{-\alpha}$ ($\overset{\shortparallel}{\alpha(t_\alpha)x}$)
Choose, as in (4), x and y s.t. $\kappa(x,y) \neq 0$.
By ~~and~~ dilating x and y, we can assume $\kappa(x,y) = 1$. Then $[xy] = t_\alpha$, $[x t_\alpha] = [y t_\alpha] = 0$. Let $S = \langle x, y, t_\alpha \rangle$: then S is ~~solvable~~ nilpotent (Heisenberg!). Now $\operatorname{ad}_L S$ is nilpotent since ad is an isomorphism. Thus $\operatorname{ad}_L t_\alpha$ is both nilpotent and semisimple. Hence $\operatorname{ad}_L t_\alpha = 0 \Rightarrow t_\alpha \in Z(L)$ $*$

6) Take $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$. Then $[x_\alpha, y_\alpha] = \kappa(x_\alpha, y_\alpha)\,t_\alpha$
$[t_\alpha, x_\alpha] = \alpha(t_\alpha)\,x_\alpha$
$[t_\alpha, y_\alpha] = -\alpha(t_\alpha)\,y_\alpha$

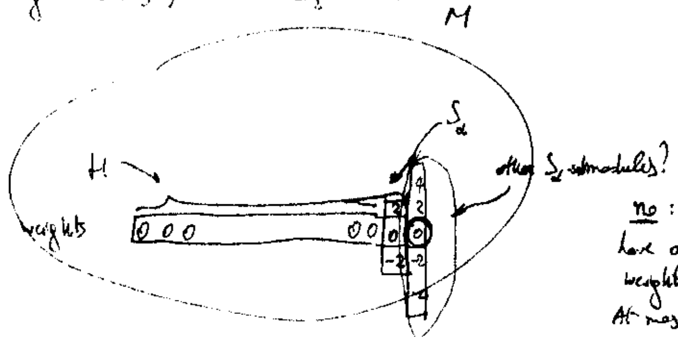
To have $S_\alpha = \langle x_\alpha, y_\alpha, t_\alpha \rangle \simeq sl_2$, just choose $h_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ and $(\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha))$ $\alpha(h_\alpha) = 2$

y_α s.t. $\kappa(x_\alpha, y_\alpha) = \dfrac{2}{\kappa(t_\alpha, t_\alpha)}$

(7) Finally, t_α is s.t. $\kappa(t_\alpha, h) = \alpha(h)$. Thus $t_{-\alpha} = -t_\alpha$.

dim L_{-2} = dim L_2 = 1

If the drawing were this, then dim L_2 = 2 ...



no: we can't have any other weights 0 in H! At most dim H (with multiplicities)

Therefore h_α does <u>not</u> have the weight 1 in M. Thus $M = \sum_\alpha \oplus$ ...

In particular, dim $L_\alpha = 1$, and the multiples of α which $\in \Phi$ are $\pm\alpha$.

Now set $K = \sum_{j \in \mathbb{Z}} L_{\beta + j\alpha}$. K is an S_α-module; weight spaces $L_{\beta+j\alpha} \rightarrow$ $\boxed{\text{of dimension 1}}$ with weights $\beta(h_\alpha) + j\,\alpha(h_\alpha) = \beta(h_\alpha) + 2j$

$\Rightarrow$ Either 0 or 1 may be weights, but not both. Thus K is $\boxed{\text{irreducible}}$ (since the weight 0, or the weight 1, have multiplicity 1)

~~K irreducible~~.

$\beta - r\alpha \quad (H) \quad \cdots \quad \beta + q\alpha$

Highest weight $\beta(h_\alpha) + 2q$

Lowest weight $\beta(h_\alpha) - 2r$

Weights in arithmetic progression, scaled by

<u>"α-string of roots through β"</u>

Weights symmetric around 0 $\Longrightarrow$ $\beta(h_\alpha) - 2r = -(\beta(h_\alpha) + 2q)$

$\Rightarrow \beta(h_\alpha) = r - q$ integer!

Now $\beta - r\alpha, \ldots\ldots, \beta + q\alpha$ are roots, in arithm. progression

So $\underbrace{\beta - r\alpha + q\alpha}_{\beta - \beta(h_\alpha)\alpha} \in \Phi$

$\boxed{\text{Properties of roots.}}$ Consider on H^* the inner product induced by κ:

$\langle \alpha, \beta \rangle = \kappa(t_\alpha, t_\beta)$. Then $\beta(h_\alpha) = \dfrac{2\,\beta(t_\alpha)}{\kappa(t_\alpha, t_\alpha)} =$

$$= \frac{2\,\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = \frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}.$$

Then: <u>"root system axioms"</u>

1) Φ spans H

2) $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$, but no other multiples

3) $\alpha, \beta \in \Phi \Rightarrow \beta - \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \Phi$

(5) Root systems and the Weyl group

> **Aim**: reduce the investigation of the structure and properties of s.s. Lie algebras to the study of geometry of euclidean spaces

Properties of root systems: $\Phi \subset E$ vector space is a root system if:

1) Φ is finite, $0 \notin \Phi$, Φ spans E
2) $\forall \alpha \in \Phi$, its only multiples in Φ are $\pm\alpha$
3) $\forall \alpha \in \Phi$, $\sigma_\alpha(\Phi) = \Phi$, where $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is the reflection around the hyperplane $P_\alpha = \alpha^\perp$.
4) $\langle \beta,\alpha\rangle \equiv \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Def. The group generated by the reflections σ_α, $\alpha \in \Phi$, is called the Weyl group, W. It acts on Φ.

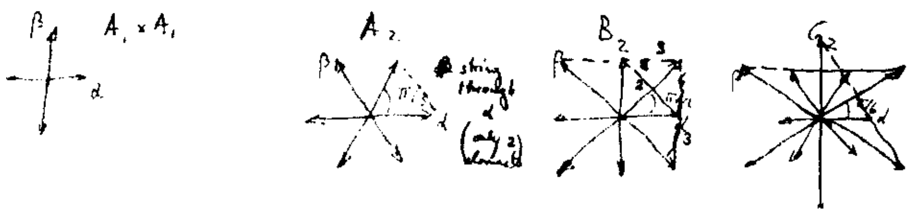
Fact: $\forall \sigma \in GL(E)$: $\sigma(\Phi) \subseteq \Phi$, $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ and $\langle\beta,\alpha\rangle = \langle\sigma(\beta), \sigma(\alpha)\rangle$ $\forall \alpha,\beta \in \Phi$.

$\rightsquigarrow$ automorphisms of E conjugate W.

Property (4) establishes a link between the angle θ formed by two roots (as vectors in E), and the ratio of their norms:

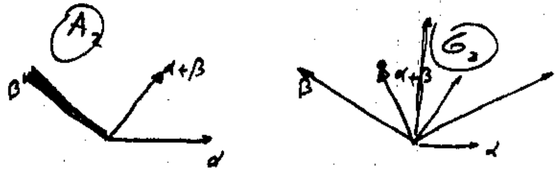
	$\theta = \pi/3,\ 2\pi/3$	ratio $= 1$
these are all possibilities:	$\pi/4,\ \frac{3}{4}\pi$	ratio $= \sqrt{2}$
	$\pi/6,\ \frac{5}{6}\pi$	ratio $= \sqrt{3}$
	$\theta = \pi/2$	undetermined

Two dim. examples:



Lemma. α, β non proportional; $(\alpha,\beta) > 0 \Rightarrow \alpha - \beta \in \Phi$
$(\alpha,\beta) < 0 \Rightarrow \alpha + \beta \in \Phi$

Examples:



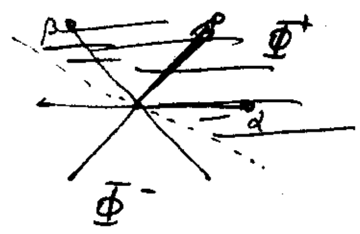
Definition $\Delta \subset \Phi$ is a <u>base</u> if: 1) is a basis of E
2) $\forall \beta \in \Phi$, $\beta = \sum c_\alpha \alpha$ with $\alpha \in \Delta$ and all $c_\alpha \geq 0$ or $c_\alpha \leq 0$

Δ = 'set of 'simple roots'

$\Phi = \Phi^+ \cup \Phi^-$: $\beta \in \Phi^+$ iff $\beta = \sum_\Delta c_\alpha \alpha$ with $c_\alpha \geq 0$.

This gives an ordering: $\beta > \gamma$ if $\beta - \gamma \in \Phi^+$

Example: ordering in A_2



Note: the ~~β~~ base Δ should be as 'open' as possible, ~~but any two basis order~~

The ordering is established by the direction of $\alpha + \beta$. Indeed, it is given by the projection onto $\alpha+\beta$.

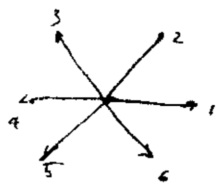
Problem: what happens in higher 'rank' (i.e., $\dim E > 2$?)

Answer: the restriction to the space generated by two roots is a 2-dim. root system!

[Moreover, if Φ' is a subset of Φ s.t. $\Phi' = -\Phi'$ and $\alpha, \beta \in \Phi'$, $\alpha+\beta \in \Phi \Rightarrow \alpha+\beta \in \Phi'$, then ...

Theorem Φ has a base. More precisely, let $\gamma \in E$ be 'regular' (i.e., not in any 'wall' P_α). Then γ induces an ordering of Φ, $\Phi = \Phi^+(\gamma) \cup (-\Phi^+(\gamma))$.
Say α is 'decomposable' if $\alpha = \beta_1 + \beta_2$, $\beta_i \in \Phi^+(\gamma)$.
Then the set of all indecomposable roots in $\Phi^+(\gamma)$ is a base, call it $\Delta(\gamma)$.
(as more 'open' as it can be!)

Example : A_2



bases: (1, 3) (1, 5) (2, 4) (2, 6) (3, 5) (4, 6)

The connected components of $E - \bigcup P_\alpha$ are called 'Weyl chambers'. Weyl chambers are in natural 1-1 correspondance with bases: the walls of a Weyl chamber are orthogonal to the basic vectors (the choice of sign of the vector is unique in $\Phi^+(\gamma)$).

Theorem The Weyl group acts transitively (permutes) on the set of Weyl chambers, and on the set of bases. It is generated by the σ_α with $\alpha \in \Delta$ (base). It acts simply transitively (on the bases and the Weyl chambers).
The matrix of Cartan integers determines Φ up to isomorphism (of E), hence it determines the Lie algebra uniquely.
Every regular γ gives rise to a W-orbit which intersect each chamber in exactly one point. But W is not ~~transit~~ simply transitive on points in general.

$G = \tilde{L}$ acts on L by Ad. (differential of conjugation)
Remark. If $H = L_0$, $\tilde{H}$ preserves all the root spaces, because $[L_0 L_\alpha] = L_\alpha$.
On the other hand, the L_α's are the (~~simultaneous~~) joint ~~generalized~~ eigenspaces of H. Thus the normalizer of $\tilde{H}$ in $G = \tilde{L}$ must permute these eigenspaces. The subgroup which fixes each L_α is the centralizer of H in G. But $C_G(H) = \tilde{H}$, since H is self-centralizing. Thus, $W \simeq N_G(H)/\tilde{H}$

here we denote a Lie algebra by $L, H \dots$ and the corresponding Lie group by $\tilde{L}, \tilde{H} \dots$