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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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STRUCTURE OF NON-COMPACT SEMI-SIMPLE GROUPS AND ALGEBRAS
Part I

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These are preliminary lecture notes, intended only for distribution to participants.

(1.1) Let $\mathfrak{g}$ be a real Lie algebra. Let $\mathrm{Der}(\mathfrak{g})$ denote the set of derivations on $\mathfrak{g}$, namely, endomorphisms $D : \mathfrak{g} \to \mathfrak{g}$ such that $D[X,Y] = [DX, Y] + [X, DY]$ (or, equivalently, $D \circ \mathrm{ad}\,X - \mathrm{ad}\,X \circ D = \mathrm{ad}\,DX$). Let $\mathrm{ad}(\mathfrak{g})$ denote the set $\{\mathrm{ad}\,X \in \mathrm{End}(\mathfrak{g}) ;\ X \in \mathfrak{g}\}$. Then $\mathrm{Der}(\mathfrak{g})$ and $\mathrm{ad}(\mathfrak{g})$ are subalgebras of $\mathrm{End}(\mathfrak{g})$ and $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\mathrm{Der}(\mathfrak{g})$.

Define $\mathrm{Aut}(\mathfrak{g}) = \{ g \in GL(\mathfrak{g}) ;\ g[X,Y] = [gX, gY]$ for all $X, Y \in \mathfrak{g}\}$. It is then easy to check that $\mathrm{Aut}(\mathfrak{g})$ is an algebraic group with Lie algebra $\mathrm{Der}(\mathfrak{g})$. Let $\mathrm{Int}(\mathfrak{g})$ denote the connected subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$. From the fact that $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\mathrm{Der}(\mathfrak{g})$, we deduce that $\mathrm{Int}(\mathfrak{g})$ is normal in $\mathrm{Aut}(\mathfrak{g})$.

(1.2) Lemma: Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then we have (1) $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$

(2) the connected component of identity in $\mathrm{Aut}(\mathfrak{g})$, denoted $\mathrm{Aut}(\mathfrak{g})^{\circ}$, is precisely $\mathrm{Int}(\mathfrak{g})$.

Proof: Let $D \in \mathrm{Der}(\mathfrak{g})$. Then $Z \mapsto \mathrm{tr}(D \circ \mathrm{ad}\,Z)$ is a linear form on $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple (by the Cartan criterion for semisimplicity) the Killing form k is nondegenerate. Hence there exists $X \in \mathfrak{g}$ such that $\mathrm{tr}(D \circ \mathrm{ad}\,Z) = k(X, Z)$ for all $Z \in \mathfrak{g}$. Since D is a derivation, we have $\mathrm{ad}\,DY = D \circ \mathrm{ad}\,Y - \mathrm{ad}\,Y \circ D$ for every $Y \in \mathfrak{g}$. Therefore

$k(DY, Z) = \mathrm{tr}(\mathrm{ad}(DY) \circ \mathrm{ad}\,Z) = \mathrm{tr}(D \circ \mathrm{ad}\,Y \circ \mathrm{ad}\,Z - \mathrm{ad}\,Y \circ D \circ \mathrm{ad}\,Z)$

$= \mathrm{tr}(D \circ [\mathrm{ad}\,Y, \mathrm{ad}\,Z]) = k(X, [Y,Z]) = k([X,Y], Z)$,

by the nondegeneracy of k, we get $DY = \mathrm{ad}\,X(Y)$ for all $Y \in \mathfrak{g}$. This proves (1).

Now (2) follows immediately from (1).

(1.3) Definitions: Let $\mathfrak{g}_c$ be a complex semisimple [illegible]. A real form $\mathfrak{g}_0$ of $\mathfrak{g}_c$ is a real (semisimple) [illegible] $\mathfrak{g}$ such that $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ [illegible]. For [illegible] form of $\mathfrak{sl}_n(\mathbb{C})$. A real form $\mathfrak{g}_u$ [illegible] called compact form of $\mathfrak{g}_c$ if the Killing [illegible] restricted to $\mathfrak{g}_u$ is negative definite. For example, [illegible] is a compact form of $\mathfrak{sl}_n(\mathbb{C})$: It is easily checked that the Killing form on $\mathfrak{sl}_n(\mathbb{C})$ is a positive multiple of the form $k' = \mathrm{tr}(XY)$ and restricting to $\mathfrak{su}(n)$, we have $k'(X,Y) = -\mathrm{tr}(X\overline{Y})$, which is a negative definite Hermitian form on all of $\mathfrak{gl}_n(\mathbb{C})$.

We shall later prove that every complex semisimple Lie algebra $\mathfrak{g}_c$ has a compact form. We now use this theorem.

(1.4) Definitions: Let $\mathfrak{g}$ be a real semisimple Lie algebra. An involution of $\mathfrak{g}$ is an element σ of $\mathrm{Aut}(\mathfrak{g})$ of order 2. Let k be the Killing form on $\mathfrak{g}$. We say that σ is a Cartan involution on $\mathfrak{g}$ if σ is an involution on $\mathfrak{g}$ such that if $\mathfrak{k} = \{X \in \mathfrak{g} ;\ \sigma(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} ;\ \sigma(X) = -X\}$ (then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$), then k is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. It is easily checked that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ (hence $\mathfrak{k}$ is a subalgebra), $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. This shows that $k(X, Y) = 0$ if $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$. Hence $\mathfrak{p} = \mathfrak{k}^{\perp}$ with respect to k. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the Cartan decomposition defined by σ.

Example of a Cartan Involution: Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$. The Killing form is a positive multiple of $\mathrm{tr}(XY)$. Let $\sigma(X) = -{}^{t}X$. Then σ is an involution. Moreover, [illegible] $\sigma(H) = -H$ on $\mathfrak{a}$, we see that $\Phi^- = -\Phi^+$. Write

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha. \quad \text{Clearly } \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^-$$

and it is easily checked that this is an orthogonal decomposition with respect to $\langle , \rangle$. Moreover, Φ^+ has the property that if $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$. Thus it follows that $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\mathrm{ad}(\mathfrak{n})$ consists of nilpotent matrices. Since $\sigma(H) = -H$ on $\mathfrak{a}$, we see that $\mathfrak{n}^- = \sigma(\mathfrak{n}^+)$.

(3.5) Proposition (Iwasawa Decomposition for semisimple Lie Algebras): Let $\mathfrak{g}$ be a real semisimple Lie algebra. In the notation of (3.2) and (3.4) we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Proof: The sum $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ in $\mathfrak{g}$ is direct: $X + Y + Z = 0$ with $X \in \mathfrak{k}$, $Y \in \mathfrak{a}$, $Z \in \mathfrak{n}$ implies, by applying σ, that $X - Y + \sigma(Z) = 0$ i.e. $2Y + Z + \sigma(Z) = 0$ which shows that $Y = Z = 0$ (since $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^-$). Therefore $X = 0$ too.

The map $\mathfrak{m} \oplus \mathfrak{n}^- \to \mathfrak{k}$ given by $X \oplus Y \mapsto X + Y + \sigma(Y)$ is a bijection: $X + Y + \sigma(Y) = 0$ implies (by the decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^-$) that $X = 0$ and $Y = 0$. Hence the map is injective. Now choose $B \in \mathfrak{k}$. Then $B = C + \sigma(C)$ for some $C \in \mathfrak{g}$, and $C = N + M + A + \sigma(N')$ where $N, N' \in \mathfrak{n}$, $M \in \mathfrak{m}$, $A \in \mathfrak{a}$. Hence $B = 2M + (N + N') + \sigma(N + N')$, which shows that the map is surjective. This proves the proposition, by a dimension count.

(3.6) Lemma: Let $\mathfrak{n}_n$ denote the Lie algebra of $n \times n$-upper triangular real matrices with zero on the diagonal and let N_n denote the closed subgroup of $\mathfrak{sl}_n(\mathbb{R})$ of $n \times n$-upper triangular matrices with ones on the diagonal. Then $\exp: \mathfrak{n}_n \longrightarrow N_n$ is a diffeomorphism onto.

Proof: If $X \in N_n$, then the series

$$\log(X) = (X-1) - \frac{(X-1)^2}{2} + \frac{(X-1)^3}{3} - + \cdots$$

is convergent (in fact a polynomial) and defines the [illegible]

(3.7) Proposition: Let $O(n)$ be the group of $n \times n$ orthogonal matrices, $D(n)$ be the group of diagonal matrices with positive real entries and N_n as in (3.6). Then the map $\varphi: O(n) \times D(n) \times N_n \longrightarrow GL_n(\mathbb{R})$, given by $(k, a, n) \mapsto k \cdot a \cdot n$, is a diffeomorphism onto $GL_n(\mathbb{R})$.

Proof: The injectivity is easily checked. The Gram-Schmidt Orthonormalisation Process shows that the map is surjective, in fact it provides a smooth inverse to φ.

(3.8) Proposition: Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to a Cartan involution T. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ [see (3.5)] be an Iwasawa decomposition of $\mathfrak{g}$. Let $\bar{K}$, $\bar{A}$, $\bar{N}$ be the connected Lie subgroups of $\mathrm{Int}(\mathfrak{g})$ corresponding to the subalgebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ respectively. Then

(1) the map $H \mapsto \exp(\mathrm{ad}\, H)$ is a diffeomorphism of $\mathfrak{a}$ onto $\bar{A}$

(2) the map $X \mapsto \exp(\mathrm{ad}\, X)$ is a diffeomorphism of $\mathfrak{n}$ onto $\bar{N}$

(3) the map $\bar{K} \times \bar{A} \times \bar{N} \longrightarrow \mathrm{Int}(\mathfrak{g})$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism of $\bar{K} \times \bar{A} \times \bar{N}$ onto $\mathrm{Int}(\mathfrak{g})$.

Proof: Let $\langle , \rangle$ be the inner product on $\mathfrak{g}$ introduced in (3.1). We use the notation of (3.4). Write $\Phi^+ = \{\alpha_1 < \alpha_2 < \cdots < \alpha_r\}$ and $\Phi^- = \{-\alpha_1 < -\alpha_2 < \cdots < -\alpha_r\}$. Let B_i be an ~~ordered~~ orthonormal basis for $\mathfrak{g}_{\alpha_i}$ $(i = 1, \ldots, r)$.

Let U be an orthonormal basis for $\mathfrak{m} \oplus \mathfrak{a}$ and let C_i^{α} denote
an orthonormal basis for $\mathfrak{g}_{-\alpha_i}$ $(i = 1, \ldots, r)$. We have already remarked that $\mathfrak{n}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}^-$ are all orthogonal with respect to $\langle\, ,\rangle$. In fact $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ if $\alpha \neq \beta$. Hence $\mathcal{B} = \{b_1, \ldots, b_r, U, c_r, c_{r-1}, \ldots, c_1\}$ is an orthonormal basis for $\mathfrak{g}$.

Since $\mathfrak{k}$ consists of skew-symmetric matrices for some orthonormal basis of $\mathfrak{g}$, $\mathfrak{k}$ consists of skew-symmetric matrices with respect to $\mathcal{B}$ also. $\mathfrak{a}$ consists entirely of diagonal matrices wrt $\mathcal{B}$ and $\mathfrak{n}$ consists of upper triangular matrices with zeros on the diagonal. If $n = \dim(\mathfrak{g})$, then $ad(\mathfrak{k}) \subset so(n)$, $ad(\mathfrak{a}) \subset \mathcal{D}(n)$ and $\mathfrak{n} \subset \mathfrak{n}_n$ (see (3.6) for the definition of $\mathfrak{n}_n$; $\mathcal{D}(n)$ is the subspace of $n \times n$ diagonal matrices). Since the map $ad : \mathfrak{g} \to \mathfrak{gl}(n)$ is injective, we see that the map $X \mapsto \exp(ad\, X)$ is an injective diffeomorphism of $\mathfrak{a}$ into $D(n)$ (as in (3.7)). Since $Int(\mathfrak{g})$ and $D(n)$ are closed in $GL_n(\mathbb{R})$, it follows that $\exp(ad(\mathfrak{a})) = \bar{A}$. Hence (1) is proved. From lemma (3.6) it follows that $X \mapsto \exp(ad\, X)$ is an injective diffeomorphism of $\mathfrak{n}$ into N_n. Since $Int(\mathfrak{g})$ and N_n are closed in $GL_n(\mathbb{R})$, we get: $\exp(ad(\mathfrak{n})) = \bar{N}$, proving (2).

We have $\bar{K} \subset O(n)$, $\bar{A} \subset D(n)$ and $\bar{N} \subset N_n$. Since $ad(\mathfrak{a})$ and $ad(\mathfrak{n})$ are closed in $\mathcal{D}(n)$ and $\mathfrak{n}_n$, we have: $\bar{A}$ and $\bar{N}$ are closed in $D(n)$ and $N(n)$. We get the commutative diagram

$$\begin{array}{ccc} O(n) \times D(n) \times N_n & \xrightarrow{\varphi} & GL_n(\mathbb{R}) \\ \uparrow & (k,a,n) \mapsto kan & \uparrow \\ \bar{K} \times \bar{A} \times \bar{N} & \xrightarrow{\bar{\varphi}} & Int(\mathfrak{g}) \\ (k,a,n) & \longmapsto & kan \end{array}$$

By lemma (2.5), $\bar{K}$ is compact.

Since $\bar{K} \times \bar{A} \times \bar{N}$ is closed in $O(n) \times D(n) \times N_n$, we see that $Im(\bar{\varphi})$ is closed in $Int(\mathfrak{g})$. On the other hand, since $d\varphi|_x$
is injective for each $x \in O(n) \times D(n) \times N_n$, ~~we see that~~ (prop. (3.7)), $d\bar{\varphi}|_x$ is injective for each $x \in \bar{K} \times \bar{A} \times \bar{N}$. Therefore, by proposition (3.5), we get: $Im(\bar{\varphi})$ is open in $Int(\mathfrak{g})$. Since $Int(\mathfrak{g})$ is connected we get: $\bar{\varphi}$ is a surjection. The diagram above now proves that $\bar{\varphi}$ is a diffeomorphism.

(3.9) Theorem: Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to a Cartan involution τ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ [as in (3.5)] be an Iwasawa decomposition for $\mathfrak{g}$. Let K, A, N be the connected Lie subgroups of G corresponding to the subalgebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$. Then

(1) the map $H \mapsto \exp(H)$ is a diffeomorphism of $\mathfrak{a}$ onto A

(2) the map $X \mapsto \exp(X)$ is a diffeomorphism of $\mathfrak{n}$ onto N

(3) the map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism of $K \times A \times N$ onto G.

The decomposition $G = KAN$ is called an Iwasawa decomposition of G.

Proof: It is clear that $Ad(G) = Int(\mathfrak{g})$, $Ad(A) = \bar{A}$, $Ad(N) = \bar{N}$ and $Ad(K) = \bar{K}$ (in the notation of (3.8)). The map $Ad : A \to \bar{A}$ is a local diffeomorphism and $Ad \circ \exp = \exp \circ ad : \mathfrak{a} \to \bar{A}$ is a diffeomorphism. Hence $\exp : \mathfrak{a} \to A$ is a diffeomorphism. (2) is proved similarly. (3) is now proved by an argument similar to the proof of theorem (2.6).

$g_{ij} \; e^{\lambda_i - \lambda_j} = g_{ij}$ for all i,j. Since $\dfrac{e^{\lambda_i-\lambda_j}-1}{\lambda_i-\lambda_j}$ is of the form e^{ξ} for some real ξ, it is never zero, hence we get $g_{ij}\,(\lambda_i-\lambda_j) = 0$ i.e. $Y = g\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}g^{-1} = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix} = X$.

Therefore $\exp: S \to S$ is injective. We now show that exp is an isomorphism of tangent spaces. Let $X \in S$. The tangent space at X is identifiable with S. Let

$\varphi(t) = \exp(X+tY)$ $(t\in\mathbb{R})$. We must show that if $\varphi'(0) = 0$ then $Y = 0$. We compute $\varphi(t)$.

$$\varphi(t) = 1 + \frac{X+tY}{1!} + \frac{(X+tY)^2}{2!} + \cdots =$$

$$= \left(1+\frac{X}{1!}+\frac{X^2}{2!}+\cdots\right) + t\left(\frac{Y}{1!} + \frac{XY+YX}{2!} + \frac{X^2Y+XYX+YX^2}{3!}+\cdots\right) + O(t^2).$$

Let L_X and R_X denote the linear transformations $A \mapsto XA$ and $A \mapsto AX$ on $M_n(\mathbb{R})$. Then L_X and R_X commute and we have formally

$$\varphi(t) = e^X + t\left(\frac{L_X-R_X}{L_X-R_X}\frac{Y}{1!}\right) + \frac{L_X^2-R_X^2}{L_X-R_X}\frac{Y^2}{2!} + \frac{L_X^3-R_X^3}{L_X-R_X}\frac{Y^3}{3!}+\cdots)$$

$$+\; O(t^2).$$

$$= e^X + t\,\frac{e^{L_X}-e^{R_X}}{L_X-R_X}(Y) + O(t^2).$$ This gives

$$\varphi(t) = e^X + t\left\{\frac{e^{\operatorname{ad}X}-1}{\operatorname{ad}X}\right\}(Y)\,e^X + O(t^2)$$

where $\dfrac{e^{\operatorname{ad}X}-1}{\operatorname{ad}X} = 1 + \dfrac{\operatorname{ad}X}{2!} + \dfrac{(\operatorname{ad}X)^2}{3!} + \cdots$

A differentiation now gives:

$$\varphi'(0) = \frac{e^{\operatorname{ad}X}-1}{\operatorname{ad}X}(Y)e^X.$$ Suppose $\varphi'(0) = 0$.

Then $\dfrac{e^{\operatorname{ad}X}-1}{\operatorname{ad}X}(Y) = 0$. We may assume that X is diagonal.

Thus $y_{ij}\left(\dfrac{e^{\lambda_i-\lambda_j}-1}{\lambda_i-\lambda_j}\right) = 0$. As before $\dfrac{e^{\lambda_i-\lambda_j}-1}{\lambda_i-\lambda_j} \neq 0$

and hence $y_{ij} = 0$ i.e. $Y = 0$.

This proves that exp is an isomorphism of tangent spaces, whence P is open and $\exp: S \to P$ is a diffeomorphism which is surjective.

We denote the inverse to exp on P by log.

<u>(2.2) Proposition:</u> The map $\varphi: O(n)\times S \to GL_n(\mathbb{R})$ given by $\varphi(k,X) = k\exp(X)$ is a diffeomorphism onto $GL_n(\mathbb{R})$.

<u>Proof:</u> Let $A \in GL_n(\mathbb{R})$. Then ${}^tAA$ is a positive definite symmetric matrix and hence ${}^tAA = \exp\circ\log {}^tAA =$
$= \left[\exp\left(\frac{1}{2}\log {}^tAA\right)\right]^2$. Let $P_A = \exp\left(\frac{1}{2}\log {}^tAA\right)$.

Then we have ${}^tAA = P_A^2$ and for all $v\in\mathbb{R}^n$

$$(AP_A^{-1}v, AP_A^{-1}v) = (v, P_A^{-1}\,{}^tAA\,P_A^{-1}v) = (v, P_A^{-1}P_A^2P_A^{-1}v) = (v,v)$$

where $(\ ,\ )$ is the standard inner product on $\mathbb{R}^n$.

Hence $A = (AP_A^{-1})\cdot P_A$ where $AP_A^{-1}\in O(n)$, $P_A \in P$.

Let $\psi(A) = \left(A\exp\left(-\frac{1}{2}\log A^tA\right), \ \frac{1}{2}\log A^tA\right)$.

Then ψ is an explicit smooth inverse for φ.

We now obtain a similar decomposition for arbitrary semisimple connected Lie groups.

(2.3) <u>Lemma</u>: Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition with respect to a Cartan involution σ. Let $\overline{K} \subset \mathrm{Int}(\mathfrak{g})$ be the connected Lie subgroup of $\mathrm{Int}(\mathfrak{g})$ with the subalgebra $ad(\mathfrak{k})$ of $ad(\mathfrak{g}) = \mathrm{Lie}\ \mathrm{Int}(\mathfrak{g})$. Then $\overline{K}$ is a compact subgroup of $\mathrm{Int}(\mathfrak{g})$.

Proof. Let $\tilde{K} = \{ g \in \mathrm{Aut}(\mathfrak{g});\ g(\mathfrak{k}) \subset \mathfrak{k}\}$. Then $\tilde{K}$ is defined by algebraic equations and is hence a closed subgroup of $GL(\mathfrak{g})$. Moreover $\sigma \in \tilde{K}$. hence the Lie algebra $\tilde{\mathfrak{k}}$ of $\tilde{K}$ is stable under σ and we can write $\tilde{\mathfrak{k}} = ad(\mathfrak{k}) \oplus ad(\mathfrak{p}) \cap \tilde{\mathfrak{k}}$.

Let $ad(X) \in ad(\mathfrak{p}) \cap \tilde{\mathfrak{k}}$. Then $ad\,X(\mathfrak{k}) \subset \mathfrak{k}$ (since $ad\,X \in \tilde{\mathfrak{k}}$). But $ad\,X(\mathfrak{k}) \subset \mathfrak{p}$ since $X \in \mathfrak{p}$. Thus $ad\,X(\mathfrak{k}) = 0$.

Further

$0 = k(\mathfrak{p},\ ad\,X(\mathfrak{k})) = -k(ad\,X(\mathfrak{p}), \mathfrak{k})$, and therefore $ad\,X(\mathfrak{p}) \subset \mathfrak{p}$. But since $X \in \mathfrak{p}$, $ad\,X(\mathfrak{p}) \subset [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ which proves that $X = 0$ i.e. $\tilde{\mathfrak{k}} = ad\,\mathfrak{k}$. Hence $\overline{K}$ is an open subgroup of $\tilde{K}$ which shows that $\overline{K}$ is closed in $\tilde{K}$ and in $GL(\mathfrak{g})$.

On the other hand $\overline{K}$ preserves the inner product

$\langle X, Y\rangle = -k(X, \sigma(Y))$ on $\mathfrak{g}$. Therefore $\overline{K}$ is a closed subgroup of $O(\mathfrak{g}, \langle\ ,\ \rangle)$ and is hence compact.

(2.4) <u>Proposition</u>: Let $\mathfrak{g}$ be a real semisimple Lie algebra, σ a Cartan involution on $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition with respect to σ. Let $\overline{K}$ be the connected Lie subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to the subalgebra $ad(\mathfrak{k})$ of the Lie algebra of $\mathrm{Int}(\mathfrak{g})$ (identified with $ad\,\mathfrak{g}$). Then the map

$f: K \times \mathfrak{p} \to \mathrm{Int}(\mathfrak{g})$ given by $(k, X) \mapsto k\exp(ad\,X)$

is a diffeomorphism of $K \times \mathfrak{p}$ onto $\mathrm{Int}(\mathfrak{g})$.

<u>Proof</u>. Consider the inner product

$\langle X, Y\rangle = -k(X, \sigma(Y))$ on $\mathfrak{g}$. Let $Z \in \mathfrak{g}$. Then

$\langle ad\,Z(X), Y\rangle = -k\langle ad\,Z(X), \sigma(Y)\rangle = k(X,\ ad\,Z \circ \sigma(Y))$

$= k(X,\ \sigma \circ \sigma^{-1} \circ ad\,Z \circ \sigma\,(Y))$.

Now $\sigma^{-1} \circ ad\,Z \circ \sigma\,(Y) = \sigma^{-1}[Z, \sigma Y] = [\sigma^{-1} Z, Y] = ad(\sigma Z)(Y)$ which shows that

$\langle ad\,Z(X), Y\rangle = k(X,\ \sigma\ ad\,\sigma Z(Y)) = -\langle X, ad(\sigma Z)(Y)\rangle$.

Therefore, with respect to the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, we have $^t(ad\,Z) = -\sigma \circ ad\,Z \circ \sigma^{-1} = -ad(\sigma Z)$

which means that $ad(\mathfrak{g})$ is stable under transposes. Moreover, $^t g = \sigma g^{-1})\sigma^{-1} \in \mathrm{Int}(\mathfrak{g})$ for $g \in \mathrm{Int}(\mathfrak{g})$. Hence $^t\sigma = \sigma$ in particular.

Fix $g \in \mathrm{Int}(\mathfrak{g})$. Then $g\,{}^t g = e^W$, where W is a $\langle\ ,\ \rangle$ symmetric transformation in $\mathrm{End}(\mathfrak{g})$. On the other hand, $e^{W}, e^{2W}, \ldots$ belong to $\mathrm{Int}(\mathfrak{g})$ and therefore the Zariski closure of this set in $\mathrm{Int}(\mathfrak{g})$ contains e^{tW} $(t \in \mathbb{R})$, whence $W = ad(Z)$ for some $Z \in \mathfrak{g}$. Moreover,

$ad(Z) = W = {}^t W = -ad(\sigma(Z))$, which shows that $Z \in \mathfrak{p}$, hence $p = \exp(\frac{1}{2} ad\,Z) \in \mathrm{Int}(\mathfrak{g})$. Let $k = gp^{-1}$. Then $k \in O(\mathfrak{g}, \langle\ ,\ \rangle) \cap \mathrm{Int}(\mathfrak{g})$, as in proposition (2.2).

Further $^t k = \sigma k^{-1} \sigma^{-1} = k^{-1}$. We compute $\langle k(X), Y\rangle$ for $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$:

$\langle k(X), Y\rangle = \langle X, {}^t k(Y)\rangle = \langle X, \sigma k^{-1}\sigma^{-1}(Y)\rangle = -\langle \sigma X, k^{-1} Y\rangle =$

$= -\langle X, k^{-1}(Y)\rangle = -\langle kX, Y\rangle$.

This proves that $\langle k(X), Y\rangle = 0$, hence $k(\mathfrak{k}) \subset \mathfrak{k}$ i.e. $k \in \tilde{K}$ as in lemma (2.3). This proves that $O(\mathfrak{g}, \langle\ ,\ \rangle) \cap \mathrm{Int}(\mathfrak{g}) \subset \tilde{K}$.

Let $\gamma(t)$ be an arc in $\mathrm{Int}(\mathfrak{g})$ connecting 1 to g. Then, by proposition (2.2), we have $\gamma(t) = k(t) \exp\left(\mathrm{ad}\,\frac{Z(t)}{2}\right)$, where $k(t)$ is an arc in $O(\mathfrak{g}, \langle\,,\rangle) \cap \mathrm{Int}(\mathfrak{g}) \subset \tilde{K}$ joining 1 to k and $Z(t)$ is an arc in $\mathfrak{p}$ joining 0 to Z. Hence k belongs to the connected component $\bar{K}$ of $\tilde{K}$ (from Lemma (2.3)). Therefore $g = kp$, with $k \in K$ and

$$\psi(g) = \left(g \exp\left(-\tfrac{1}{2}\log g^t g\right),\ \mathrm{ad}^{-1}\,\tfrac{1}{2}\log g^t g\right)$$

gives an explicit smooth inverse for φ, on $\mathrm{Int}(\mathfrak{g})$.

(2.5) Lemma: Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan involution σ and write the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to σ. Let G be a connected real Lie group with Lie algebra $\mathfrak{g}$ and let $K \subset G$ be the connected Lie subgroup corresponding to the subalgebra $\mathfrak{k}$. Then K contains the centre Z of G.

Proof: We first note that the centre Z is discrete. Moreover, the map $\mathrm{Ad}: G \to \mathrm{Ad}(\mathfrak{g})$ has kernel precisely Z and $\mathrm{Ad}(G) = \mathrm{Int}(\mathfrak{g})$. Let $H = \mathrm{Ad}^{-1}(\mathrm{Ad}(\bar{K}))$, $h \in H$ and $\gamma(t)$ an arc in G connecting 1 to h. We have, from proposition (2.4),

$$\mathrm{Ad}\,\gamma(t) = \bar{k}(t) \exp(\mathrm{ad}\, Z(t)), \quad \mathrm{Ad}(h) = \bar{k}$$

where $Z(t)$ is ~~an arc connecting 0 to 0~~ a loop in $\mathfrak{p}$ at 0, $\bar{k}(t)$ is an arc in $\bar{K}$ (in the notation of (2.3)), connecting 1 to $\bar{k}$. But $\exp \mathrm{ad}\, Z(t) = \mathrm{Ad} \circ \exp Z(t)$ and therefore $\mathrm{Ad}\,\gamma(t) \exp(-Z(t))$ ~~is an arc in H~~ $= \bar{k}(t)$. On the other hand, $\bar{k}(t) \in \bar{K} = \mathrm{Ad}(K)$, which shows that $\gamma(t) \exp(-Z(t))$ is an arc in H connecting 1 to h. Hence $H = KZ$ is a connected subgroup of G, and by the discreteness of Z, we see that $\mathrm{Lie}(H) = \mathrm{Lie}(K)$ which implies $K = H \supset Z$.

(2.6) Theorem: Let $\mathfrak{g}$ be a real semisimple Lie algebra, σ a Cartan involution on $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the decomposition given by σ. Let G be a connected real Lie group with Lie algebra $\mathfrak{g}$ and let $K \subset G$ be the connected subgroup with Lie algebra $\mathfrak{k}$. Then the map

$$\varphi : K \times \mathfrak{p} \longrightarrow G \quad \text{given by} \quad (k, X) \longmapsto k \exp(X)$$

is a diffeomorphism of $K \times \mathfrak{p}$ onto G.

Proof: Fix $g \in G$. Now $\mathrm{Ad}(g) = \bar{k} p$, where $p = \exp(\mathrm{ad}\, X)$ for some $X \in \mathfrak{p}$, by proposition (2.4) and $\bar{k} \in \bar{K} = \mathrm{Ad}(K)$. Thus, $\mathrm{Ad}(g) = \mathrm{Ad}(k_1 \exp(X))$, hence $g = (z k_1) \exp(X)$ with $X \in \mathfrak{p}$ and $z k_1 \in ZK \subset K$ (by lemma (2.5)). This shows that φ is surjective.

Suppose $k_1 \exp(X_1) = k_2 \exp(X_2)$ with $k_1, k_2 \in K$ and $X_1, X_2 \in \mathfrak{p}$. Therefore $\mathrm{Ad}(k_1) \exp(\mathrm{ad}\, X_1) = \mathrm{Ad}(k_2) \exp(\mathrm{ad}\, X_2)$ and by Proposition (2.4), we get $\mathrm{ad}\, X_1 = \mathrm{ad}\, X_2$ i.e. $X_1 = X_2$ and $k_1 = k_2$. Hence φ is injective.

Consider the commutative diagram

$$\begin{tikzcd}
K \times \mathfrak{p} \arrow[r, "\varphi"] \arrow[d, "\mathrm{Ad} \times \mathrm{Id}"'] & G \arrow[d, "\mathrm{Ad}"] \\
\bar{K} \times \mathfrak{p} \arrow[r, "\bar{\varphi}"] & \mathrm{Int}(\mathfrak{g}) \\
(\bar{k}, x) \arrow[r, mapsto] & \bar{k} \exp(\mathrm{ad}\, x)
\end{tikzcd}$$

The vertical maps are covering maps and $\bar{\varphi}$ is a diffeomorphism by proposition (2.4). Therefore φ is an isomorphism on tangent spaces, which shows that φ is a surjective diffeomorphism.

(2.7) Corollary: The normalizer of K in G is precisely K.

Proof: Suppose $g \in G$ normalizes K. ~~By the~~ theorem (2.6) we can assume that $g = p = \exp(X), X \in \mathfrak{p}$. This implies in particular that $[X, \mathfrak{k}] \subset \mathfrak{k}$. On the other hand, X being in $\mathfrak{p}$, we have $[X, \mathfrak{k}] \subset \mathfrak{p}$, hence $[X, \mathfrak{k}] = 0$. Moreover $K(\mathrm{ad}(X)(\mathfrak{p}), \mathfrak{k}) = K(\mathfrak{p}, \mathrm{ad}\, X(\mathfrak{k}))$
$= K(\mathfrak{p}, 0) = 0$, hence $\mathrm{ad}\, X(\mathfrak{p}) \subset \mathfrak{p}$; on the other hand, X being in $\mathfrak{p}$, we have $[X, \mathfrak{p}] \subset \mathfrak{k}$ whence $[X, \mathfrak{p}] = 0$. By semisimplicity, $X = 0$.

§3. The Iwasawa Decomposition of a Semisimple Lie Group

(3.1) Lemma: Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan involution σ and the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\langle X, Y \rangle = -K(X, \sigma(Y))$ where K is the Killing form on $\mathfrak{g}$. Then $\mathrm{ad}\, Z$ is skew symmetric with respect to $\langle\, ,\, \rangle$ for $Z \in \mathfrak{k}$ and $\mathrm{ad}\, Z$ is symmetric with respect to $\langle\, ,\, \rangle$ for $Z \in \mathfrak{p}$.

Proof: Easy to check.

(3.2) Definition: Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. If $X \in \mathfrak{a}$, then lemma (3.1) implies that $\mathrm{ad}\, X$ is diagonalisable as an operator on $\mathfrak{g}$. Since $\mathfrak{a}$ is abelian, the elements of $\mathrm{ad}(\mathfrak{a})$ is simultaneously diagonalisable with real eigenvalues. If $\lambda \in \mathfrak{a}^*$, we set $\mathfrak{g}_\lambda = \{X \in \mathfrak{g};\ \mathrm{ad}(H)(X) = \lambda(H) X \text{ for all } H \in \mathfrak{a}\}$,
$\mathfrak{m} = \{X \in \mathfrak{k};\ \mathrm{ad}(H)(X) = 0 \text{ for all } H \in \mathfrak{a}\}$ and
$\Phi = \{\alpha \in \mathfrak{a}^* - \{0\};\ \mathfrak{g}_\alpha \neq 0\}$. Then Φ is called the restricted root system of $\mathfrak{g}$ relative to $\mathfrak{a}$.

(3.3) Lemma: We have $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

Proof: Clearly $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. If $H \in \mathfrak{a}$, then $\sigma(H) = -H$. Therefore $\mathfrak{g}_0$ is stable under σ, which implies $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p})$. Now $\mathfrak{g}_0 \cap \mathfrak{k} = \mathfrak{m}$ by definition. If $X \in \mathfrak{g}_0 \cap \mathfrak{p}$, then $\mathbb{R}X + \mathfrak{a}$ is an abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$, whence by maximality of $\mathfrak{a}$, $X \in \mathfrak{a}$. Hence $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$.

(3.4) Definitions: Let $H_1, \dots, H_\ell$ be a basis of $\mathfrak{a}$. On $\mathfrak{a}^*$ define an ordering $<$ as follows:

$\lambda < \mu$ if $\lambda(H_1) = \mu(H_1), \dots, \lambda(H_{i-1}) = \mu(H_{i-1})$ and $\lambda(H_i) < \mu(H_i)$ for some $i \geq 1$. Let $\Phi^+ = \{\alpha \in \Phi;\ \alpha > 0\}$, $\Phi^- = \{\alpha \in \Phi;\ \alpha < 0\}$.

$\sigma(H) = -H$ on $\mathfrak{a}$, we see that $\Phi^- = -\Phi^+$. Write

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha. \quad \text{Clearly } \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^-$$

and it is easily checked that this is an orthogonal decomposition with respect to $\langle , \rangle$. Moreover, Φ^+ has the property that if $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$. Thus it follows that $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\mathrm{ad}(\mathfrak{n})$ consists of nilpotent matrices. Since $\sigma(H) = -H$ on $\mathfrak{a}$, we see that $\mathfrak{n}^- = \sigma(\mathfrak{n}^+)$.

(3.5) Proposition (Iwasawa Decomposition for semisimple Lie Algebras): Let $\mathfrak{g}$ be a real semisimple Lie algebra. In the notation of (3.2) and (3.4) we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Proof: The sum $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ in $\mathfrak{g}$ is direct: $X + Y + Z = 0$ with $X \in \mathfrak{k}$, $Y \in \mathfrak{a}$, $Z \in \mathfrak{n}$ implies, by applying σ, that $X - Y + \sigma(Z) = 0$ so $2Y + Z + \sigma(Z) = 0$ which shows that $Y = Z = 0$ (since $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^-$). Therefore $X = 0$ too.

The map $\mathfrak{m} \oplus \mathfrak{n}^- \to \mathfrak{k}$ given by $X \oplus Y \mapsto X + Y + \sigma(Y)$ is a bijection: $X + Y + \sigma(Y) = 0$ implies (by the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^-$) that $X = 0$ and $Y = 0$. Hence the map is injective. Now choose $B \in \mathfrak{k}$. Then $B = C + \sigma(C)$ for some $C \in \mathfrak{g}$, and $C = N + M + A + \sigma(N')$ where $N, N' \in \mathfrak{n}$, $M \in \mathfrak{m}$, $A \in \mathfrak{a}$. Hence $B = 2M + (N + N') + \sigma(N + N')$, which shows that the map is surjective. This proves the proposition, by a dimension count.

(3.6) Lemma: Let $\mathfrak{n}_n$ denote the Lie algebra of $n \times n$-upper triangular real matrices with zeros on the diagonal and let N_n denote the closed subgroup of $SL_n(\mathbb{R})$ of $n \times n$-upper triangular matrices with ones on the diagonal. Then $\exp : \mathfrak{n}_n \to N_n$ is a diffeomorphism onto.

Proof: If $X \in N_n$, then the series

$$\log(X) = (X-1) - \frac{(X-1)^2}{2} + \frac{(X-1)^3}{3} - \cdots$$

is convergent (in fact a polynomial) and defines an inverse.

(3.7) Proposition: Let $O(n)$ be the group of $(n \times n)$ orthogonal matrices, $D(n)$ be the group of diagonal matrices with positive real entries and N_n as in (3.6). Then the map $\varphi : O(n) \times D(n) \times N_n \to GL_n(\mathbb{R})$, given by $(k, a, n) \mapsto k \cdot a \cdot n$, is a diffeomorphism onto $GL_n(\mathbb{R})$.

Proof: The injectivity is easily checked. The Gram-Schmidt orthonormalisation process shows that the map is surjective; in fact it provides a smooth inverse to φ.

(3.8) Proposition: Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to a Cartan involution σ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ [see (3.5)] be an Iwasawa decomposition of $\mathfrak{g}$. Let $\bar{K}, \bar{A}, \bar{N}$ be the connected Lie subgroups of $\mathrm{Int}(\mathfrak{g})$ corresponding to the subalgebras $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ respectively. Then

(1) the map $H \mapsto \exp(\mathrm{ad}\, H)$ is a diffeomorphism of $\mathfrak{a}$ onto $\bar{A}$

(2) the map $X \mapsto \exp(\mathrm{ad}\, X)$ is a diffeomorphism of $\mathfrak{n}$ onto $\bar{N}$

(3) the map $\bar{K} \times \bar{A} \times \bar{N} \to \mathrm{Int}(\mathfrak{g})$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism of $\bar{K} \times \bar{A} \times \bar{N}$ onto $\mathrm{Int}(\mathfrak{g})$.

Proof: Let $\langle , \rangle$ be the inner product on $\mathfrak{g}$ introduced in (3.1). We use the notation of (3.4). Write $\Phi^+ = \{\alpha_r < \alpha_{r-1} < \cdots < \alpha_1\}$ and $\Phi^- = \{-\alpha_1 < -\alpha_2 < \cdots < -\alpha_r\}$.

Let B_i be an ~~ordered~~ orthonormal basis for $\mathfrak{g}_{\alpha_i}$ ($i = 1, \ldots, r$), Let $\mathcal{V}$ be an orthonormal basis for $\mathfrak{m} \oplus \mathfrak{a}$ and let $C_i^{\,k_i}$ denote an orthonormal basis for $\mathfrak{g}_{-\alpha_i}$ $(i = 1, \dots, r)$. We have already remarked that $\mathfrak{n}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}^-$ are all orthogonal with respect to $\langle\ ,\ \rangle$. In fact $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ if $\alpha \neq \beta$. Hence $\mathcal{B} = \{B_1, \dots, B_r, \mathcal{V}, C_r, C_{r-1}, \dots, C_1\}$ is an orthonormal basis for $\mathfrak{g}$. Since $\mathfrak{k}$ consists of skew-symmetric matrices for some orthonormal basis of $\mathfrak{g}$, $\mathfrak{k}$ consists of skew-symmetric matrices with respect to $\mathcal{B}$ also. $\mathfrak{a}$ consists entirely of diagonal matrices wrt $\mathcal{B}$ and $\mathfrak{n}$ consists of upper triangular matrices with zeros on the diagonal. If $n = \dim(\mathfrak{g})$, then $ad(\mathfrak{k}) \subset so(n)$, $ad(\mathfrak{a}) \subset \mathcal{D}(n)$ and $\mathfrak{n} \subset \mathfrak{N}_n$ (see (3.0) for the definition of $\mathfrak{N}_n$; $\mathcal{D}(n)$ is the subspace of $n \times n$ diagonal matrices). Since the map $ad : \mathfrak{g} \to \mathfrak{gl}(n)$ is injective, we see that the map $X \mapsto \exp(ad\,X)$ is an injective diffeomorphism of $\mathfrak{a}$ into ~~M(n)~~ $D(n)$ (as in (3.7)). Since $Int(\mathfrak{g})$ and $D(n)$ are closed in $GL_n(\mathbb{R})$, it follows that $\exp(ad\,\mathfrak{a}) = \bar{A}$. Hence (1) is proved. From lemma (3.6) it follows that $X \mapsto \exp(ad\,X)$ is an injective diffeomorphism of $\mathfrak{n}$ into N_n. Since $Int(\mathfrak{g})$ and ~~D(n)~~ N_n are closed in $GL_n(\mathbb{R})$, we get: $\exp(ad(\mathfrak{n})) = \bar{N}$, proving (2).

We have $\bar{K} \subset O(n)$, $\bar{A} \subset D(n)$ and $\bar{N} \subset N_n$. Since $ad(\mathfrak{a})$ and $ad(\mathfrak{n})$ are closed in $\mathcal{D}(n)$ and $\mathfrak{N}_n$, we have: $\bar{A}$ and $\bar{N}$ are closed in $D(n)$ and $N(n)$. We get the commutative diagram

$$\begin{tikzcd}
O(n) \times D(n) \times N_n \arrow[r, "\varphi", "{(k,a,n) \mapsto kan}"'] & GL_n(\mathbb{R}) \\
\bar{K} \times \bar{A} \times \bar{N} \arrow[u, hook] \arrow[r, "\bar{\varphi}", "{(k,a,n) \mapsto kan}"'] & Int(\mathfrak{g}) \arrow[u, hook]
\end{tikzcd}$$

By lemma (2.3), $\bar{K}$ is compact.

Since $\bar{K} \times \bar{A} \times \bar{N}$ is closed in $O(n) \times D(n) \times N_n$, we see that $Im(\bar{\varphi})$ is closed in $Int(\mathfrak{g})$. On the other hand, since $d\varphi|_x$ is injective for each $x \in O(n) \times D(n) \times N_n$, ~~we see that~~ (prop. (3.7)) $d\bar{\varphi}|_x$ is injective for each $x \in \bar{K} \times \bar{A} \times \bar{N}$. Therefore, by proposition (3.5), we get: $Im(\bar{\varphi})$ is open in $Int(\mathfrak{g})$. Since $Int(\mathfrak{g})$ is connected we get: $\bar{\varphi}$ is a surjection. The diagram above now proves that $\bar{\varphi}$ is a diffeomorphism.

<u>(3.9) Theorem:</u> Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to a Cartan involution τ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ [as in (3.5)] be an Iwasawa decomposition for $\mathfrak{g}$. Let K, A, N be the connected Lie subgroups of G corresponding to the subalgebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$. Then

(1) the map $H \mapsto \exp(H)$ is a diffeomorphism of $\mathfrak{a}$ onto A

(2) the map $X \mapsto \exp(X)$ is a diffeomorphism of $\mathfrak{n}$ onto N

(3) the map $K \times A \times N \longrightarrow G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism of $K \times A \times N$ onto G.

The decomposition $G = KAN$ is called an <u>Iwasawa decomposition</u> of G.

<u>Proof:</u> It is clear that $Ad(G) = Int(\mathfrak{g})$, $Ad(A) = \bar{A}$, $Ad(N) = \bar{N}$ and $Ad(K) = \bar{K}$ (in the notation of (3.5)). The map $Ad : A \to \bar{A}$ is a local diffeomorphism and $Ad \circ \exp = \exp \circ ad : \mathfrak{a} \to \bar{A}$ is a diffeomorphism. Hence $\exp : \mathfrak{a} \to A$ is a diffeomorphism. (2) is proved similarly. (3) is now proved by an argument similar to the proof of theorem (2.6).

§4. The Proof of The Existence of a compact form $\mathfrak{g}_u$ of a Complex Semisimple Lie Algebra $\mathfrak{g}_{\mathbb{C}}$

(4.1) Lemma: Let $\alpha, \beta \in \Phi$ and $\beta - n\alpha, \beta-(n-1)\alpha, \dots, \beta-\alpha, \beta, \dots, \beta+m\alpha$ be the α-root string through β. If $X_\alpha \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ and $Z \in \mathfrak{g}_\beta$, then we have:

$$[Y, [X, Z]] = \frac{m(1+n)}{2} (\alpha, \alpha)\, k(X, Y)\, Z.$$

Proof: Let $U \in \mathfrak{g}$. Then the linear operators $R_{ad U}$ and $L_{ad U}$ on $End(\mathfrak{g})$, defined by right multiplication and left multiplication by $ad U$ respectively, commute with each other. Therefore, if $V \in \mathfrak{g}$ and $p > 0$ is an integer, then

$$R^p_{ad U}(ad V) = (R_{ad U} - L_{ad U} + L_{ad U})^p (ad V) =$$

$$= \sum_{k=1}^{p} \binom{p}{k} (R_{ad U} - L_{ad U})^k L^{p-k}_{ad U} (ad V) + (ad U)^p (ad V).$$

Therefore, taking $p = (n+1)$, $U = X$ and $V = Y$, we get:

$$ad Y \circ (ad X)^{n+1} = (ad X)^{n+1} \circ ad Y + \binom{n+1}{1} (ad X)^n \circ ad [Y, X] +$$

$$+ \binom{n+1}{2} (ad X)^{n-1} \circ ad [[Y, X], X].$$

Now, it is a fact from the ~~theory of semisimple Lie algebras~~ representation theory of sl_2, that $Z = (ad X)^n (Z')$ for some $Z' \in \mathfrak{g}_{\beta - n\alpha}$. Then:

$$[Y, [X, Z]] = ad Y \circ ad X (Z) = ad Y \circ (ad X)^{n+1} (Z') =$$

$$= (ad X)^{n+1} \circ ad Y (Z') + \binom{n+1}{1} (ad X)^n \, ad [Y, X] (Z') +$$

$$+ \binom{n+1}{2} (ad X)^{n-1} \circ ad [[Y, X], X] \, (Z').$$

Now, since $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, we have $[X, Y] \in \mathfrak{h}$. Further for $H \in \mathfrak{h}$, we have

$$k([X, Y], H) = k(X, [Y, H]) = \alpha(H)\, k(X, Y)$$

which shows that $\gamma([X, Y]) = (\gamma, \alpha)\, k(X, Y)$ for γ

Hence we have $[Y, [X, Z]] =$

$$(ad X)^{n+1} \, ad Y (Z') + \binom{n+1}{1} (ad X)^n (\beta - n\alpha, \alpha)(-1)\, k(X, Y) Z' +$$

$$+ \binom{n+1}{2} (ad X)^{n-1} (-\alpha, \alpha)\, k(X, Y)\, ad X (Z') =$$

$$= 0 + (n+1)(n\alpha - \beta, \alpha)\, k(X, Y)\, Z + \frac{(n+1)n}{2} (-\alpha, \alpha)\, k(X, Y)\, Z$$

$$= \frac{(n+1)(\alpha, \alpha)}{2} \left\{ \frac{n}{1} - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \right\} k(X, Y)\, Z.$$

Now, by standard facts in root systems, we have

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = n - m,$$

which proves the lemma.

(4.2) Lemma. If $\alpha \in \Phi$, choose E_α in $\mathfrak{g}_\alpha$ such that $k(E_\alpha, E_{-\alpha}) = 1$. If $\alpha, \beta, \alpha+\beta \in \Phi$, then define $N_{\alpha,\beta}$ by $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$. If $\alpha, \beta \in \Phi$ but $\alpha+\beta \notin \Phi$, then define $N_{\alpha,\beta} = 0$. Then:

(1) if $\alpha, \beta, \gamma \in \Phi$ and $\alpha+\beta+\gamma = 0$, then

$$N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}.$$

(2) if $\alpha, \beta, \gamma, \delta \in \Phi$ and $\alpha+\beta+\gamma+\delta = 0$, then

$$N_{\alpha,\beta} N_{\gamma,\delta} + N_{\beta,\gamma} N_{\alpha,\delta} + N_{\gamma,\alpha} N_{\beta,\delta} = 0,$$

provided no two of the $\alpha, \beta, \gamma, \delta$ sum to zero.

Proof. (1), we have $N_{\alpha,\beta} = k([E_\alpha, E_\beta], E_\gamma) =$

$$= k(E_\alpha, [E_\beta, E_\gamma]) = N_{\beta,\gamma} = k([E_\gamma, E_\alpha], E_\beta) = N_{\gamma,\alpha}.$$

(2) We have $N_{\alpha,\beta}N_{\gamma,\delta} = k([E_\alpha,E_\beta],[E_\gamma,E_\delta])$

$= k(E_\alpha,[E_\beta,[E_\gamma,E_\delta]])$, which, by the Jacobi identity, is

$= -k(E_\alpha,[E_\gamma,[E_\delta,E_\beta]]) - k(E_\alpha,[E_\delta,[E_\beta,E_\gamma]]) =$

$= -k([E_\alpha,E_\gamma],[E_\delta,E_\beta]) - k([E_\alpha,E_\delta],[E_\beta,E_\gamma]) =$

$= -N_{\gamma,\alpha}N_{\beta,\delta} - N_{\alpha,\delta}N_{\beta,\gamma}$.

(4.3) Definition: If $\mathfrak g_{\mathbb C}$ is a complex semisimple Lie algebra, $\mathfrak h \subset \mathfrak g_{\mathbb C}$ is a Cartan subalgebra and $\mathfrak g_{\mathbb C} = \mathfrak h \oplus \sum_{\alpha\in\Phi}\mathfrak g_\alpha$ the root space decomposition, then write $\mathfrak h_{\mathbb R} = \mathrm{Hom}(\sum_{\alpha\in\Phi}\mathbb R\alpha, \mathbb R)$. We have Φ^+ and Δ as before.

Let $H_1,\dots,H_l$ be a basis of $\mathfrak h_{\mathbb R}$ such that $\alpha(H_i)>0$ for $\alpha\in\Delta$ and $i=1,\dots,l$. We introduce the lexicographic ordering on $\mathfrak h^*_{\mathbb R}$: write, for $\lambda,\mu\in\mathfrak h_{\mathbb R}$, $\lambda>\mu$ if $\lambda(H_1)=\mu(H_1),\dots,\lambda(H_{i-1})=\mu(H_{i-1})$ and $\lambda(H_i)>\mu(H_i)$ for some $i\geq 1$. Then Φ^+ consists of elements α with $\alpha>0$. If $\alpha\in\Phi$, define $|\alpha| = \alpha$ if $\alpha\in\Phi^+$ and $-\alpha$ if $\alpha\in\Phi^-$. For $\beta\in\Phi^+$, let $\Phi_\beta = \{\alpha\in\Phi;\ |\alpha|<\beta\}$.

(4.4) Remark: Suppose A is an automorphism of $\mathfrak g_{\mathbb C}$ such that $A|\mathfrak h = -\mathrm{Id}$, then it is easily checked that $A(\mathfrak g_\alpha) = \mathfrak g_{-\alpha}$. We choose $E_\alpha\in\mathfrak g_\alpha$ such that $k(E_\alpha,E_{-\alpha})=1$. Then $A(E_\alpha) = c_\alpha E_{-\alpha}$. The sequence $\{c_\alpha;\ \alpha\in\Phi\}$ has the properties:

(1) for all $\alpha\in\Phi$, $c_\alpha c_{-\alpha}=1$: $1 = k(E_\alpha,E_{-\alpha}) = k(AE_\alpha, AE_{-\alpha}) = k(c_\alpha E_{-\alpha}, c_{-\alpha}E_\alpha) = c_\alpha c_{-\alpha}$.

(2) if $\alpha,\beta,\alpha+\beta\in\Phi$, then $c_\alpha c_\beta N_{-\alpha,-\beta} = c_{\alpha+\beta}N_{\alpha,\beta}$;

for, $E_{-\alpha-\beta}N_{\alpha,\beta}c_{\alpha+\beta} = N_{\alpha,\beta}A(E_{\alpha+\beta}) = A(N_{\alpha,\beta}E_{\alpha+\beta}) = A[E_\alpha,E_\beta] = c_\alpha c_\beta[E_{-\alpha},E_{-\beta}] = c_\alpha c_\beta N_{-\alpha,-\beta}E_{-\alpha-\beta}$

Conversely, suppose there exist $\{c_\alpha\}_{\alpha\in\Phi}$ such that for $\alpha,\beta\in\Phi$, the conditions (1) and (2) are satisfied. Define $A(E_\alpha) = c_\alpha E_{-\alpha}$ and $A(H) = -H$ for $H\in\mathfrak h$, $\alpha\in\Phi$. Then A is an automorphism of $\mathfrak g_{\mathbb C}$ such that $A|\mathfrak h = -\mathrm{Id}$.

(4.5) Proposition: There exists an automorphism A of $\mathfrak g_{\mathbb C}$ such that $A|\mathfrak h = -\mathrm{Id}$.

Proof: We have to prove the existence of a sequence $\{c_\alpha\}_{\alpha\in\Phi}$ of complex numbers satisfying (1) and (2) of (4.4).

Fix $\beta\in\Phi^+$. Suppose there exists a sequence $\{c_\alpha\}_{\alpha\in\Phi_\beta}$ such that (1) holds for all $\varphi\in\Phi_\beta$ and (2) holds for all $\varphi,\psi\in\Phi_\beta$ with $\varphi+\psi\in\Phi_\beta$. Let β' denote the smallest positive not greater than β. We now proceed to show that there exists a sequence $\{c_\alpha\}_{\alpha\in\Phi_{\beta'}}$ such that (1) holds for all $\varphi\in\Phi_{\beta'}$ and (2) holds for all $\varphi,\psi\in\Phi_{\beta'}$ with $\varphi+\psi\in\Phi_{\beta'}$. This will complete the proof.

Case 1: β is not of the form $\varphi+\psi$ for $\varphi,\psi\in\Phi_\beta$. Then set $c_\beta = c_{-\beta} = 1$. Hence $c_\varphi c_{-\varphi} = 1$ for all $\varphi\in\Phi_{\beta'}$. Moreover, if $\varphi,\psi,\varphi+\psi\in\Phi_{\beta'}$ it is easy to see that $\varphi,\psi,\varphi+\psi\in\Phi_\beta$ and hence (2) $c_\varphi c_\psi N_{-\varphi,-\psi} = c_{\varphi+\psi}N_{\varphi,\psi}$.

Case 2: β is of the form $\varphi_0+\psi_0$ for some $\varphi_0,\psi_0\in\Phi_\beta$. We set $c_\beta N_{\varphi_0,\psi_0} = c_{\varphi_0}c_{\psi_0}N_{-\varphi_0,-\psi_0}$ and

$$c_{-\beta} N_{-\varphi_0, -\psi_0} = c_{-\varphi_0} c_{-\psi_0} N_{\varphi_0, \psi_0}.$$ This shows that

$c_\beta c_{-\beta} = 1$, and that for the pair (φ_0, ψ_0), the equation (2) holds. We now show that this definition is consistent: suppose $\varphi + \psi = \beta$ with $\varphi, \psi \in \Phi_\beta$. We must show that

$$c_\beta N_{\varphi,\psi} = c_\varphi c_\psi N_{-\varphi,-\psi}, \quad c_{-\beta} N_{-\varphi,-\psi} = c_{-\varphi} c_{-\psi} N_{\varphi,\psi}, \text{ i.e.}$$

$$(*)\ c_{\varphi_0} c_{\psi_0} N_{-\varphi_0,-\psi_0} N_{\varphi,\psi} = c_\varphi c_\psi N_{-\varphi,-\psi} N_{\varphi_0,\psi_0}$$ (and similarly for $(-\varphi_0, -\psi_0)$ and $(-\varphi, -\psi)$). Now, from (2) of (4.2), we get,

$\cancel{c_{\varphi_0} c_{\psi_0} N_{-\varphi_0,-\psi_0} N_{\varphi,\psi}} =$ by writing $(-\varphi_0) + (-\psi_0) + \varphi + \psi = 0$,

$$c_{\varphi_0} c_{\psi_0} N_{-\varphi_0,-\psi_0} N_{\varphi,\psi} = -c_{\varphi_0} c_{\psi_0} \{ N_{-\psi_0,\varphi} N_{-\varphi_0,\psi} + N_{\varphi,-\varphi_0} N_{-\psi_0,\psi} \}$$

$$= -c_{\varphi_0} c_{\psi_0} N_{-\psi_0,\varphi} N_{-\varphi_0,\psi} - c_{\varphi_0} c_{\psi_0} N_{\varphi,-\varphi_0} N_{-\psi_0,\psi}.$$

The first part of this sum is, by the assumption on Φ_β, equal to:

$$-c_\varphi c_\psi \left(c_{\psi_0} c_{-\varphi} N_{-\psi_0,\varphi} \quad c_{\varphi_0} c_{-\psi} N_{-\varphi_0,\psi} \right) =$$

$$= -c_\varphi c_\psi \left(c_{\psi_0 - \varphi} N_{\psi_0,-\varphi} \quad c_{\varphi_0 - \psi} N_{-\varphi_0,\psi} \right) =$$

$$= -c_\varphi c_\psi N_{\psi_0,-\varphi} N_{-\varphi_0,\psi}$$

Now using the fact that N is skew symmetric, we get this to be $-c_\varphi c_\psi N_{-\varphi,\psi_0} N_{-\psi,\varphi_0}$. Similarly for the second part of the above sum. This proves $(*)$ and the equation obtained from $(*)$ by replacing all the vectors by their negatives. With this definition of c_β, c_β we claim that (1) and (2) hold for $\{c_\varphi\}_{\varphi \in \Phi_{\beta'}}$:

(1) if $\varphi \in \Phi_{\beta'}$, $|\varphi| < \beta$, then $c_\varphi c_{-\varphi} = 1$ by assumption on Φ_β. Moreover $c_\beta c_{-\beta} = 1$, hence $c_\varphi c_{-\varphi} = 1$ for all $\varphi \in \Phi_{\beta'}$.

(2) Suppose $\varphi, \psi, \varphi + \psi \in \Phi_{\beta'}$; ~~and~~ (i) if $\varphi, \psi, \varphi+\psi \in \Phi_\beta$ then (2) holds by assumption. (ii) If $\varphi + \psi = \beta$, then again we have checked that (2) holds. (iii) if $\varphi, \beta, \varphi + \beta = \psi$ are in $\Phi_{\beta'}$ we must check that

$$c_\varphi c_\beta N_{\varphi,-\beta} = c_\psi N_{\varphi,\beta}.$$

Now $\varphi \oplus \beta \oplus (-\psi) = 0$. Hence, from (1) of lemma (4.2),

$$c_\psi N_{\varphi,\beta} = c_\psi N_{-\psi,\varphi}. \quad \text{Also,} \quad (-\varphi) \oplus (-\beta) \oplus \psi = 0.$$

$$c_\varphi c_\beta N_{-\varphi,-\beta} = c_\varphi c_\beta N_{\psi,-\varphi}, \quad \text{with } \beta = \psi - \varphi.$$

Hence, by the definition of c_β, we get

$$c_\varphi c_\beta N_{-\varphi,-\beta} = c_\varphi \frac{c_\psi c_{-\varphi} N_{-\psi,\varphi} N_{\psi,-\varphi}}{N_{\psi,-\varphi}} =$$

$$= c_\psi N_{-\psi,\varphi},$$ which checks (2) for the case under consideration. (iv) we have $\varphi, -\beta, \varphi - \beta = \psi$ are in $\Phi_{\beta'}$. The proof is similar to (iii).

This completes the proof of the proposition.

<u>(4.6) Theorem:</u> Let $\mathfrak{g}_c$ be a complex semisimple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra. We have $\mathfrak{g}_c = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$, and $\mathfrak{h}_{\mathbb{R}}$ as in (4.3). Then there exists a compact form $\mathfrak{g}_u$ of $\mathfrak{g}_c$ such that $i\mathfrak{h}_{\mathbb{R}}$ is a maximal abelian subspace of $\mathfrak{g}_c$.

Proof: Let $A \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ such that $A|_{\mathfrak{h}} = -\mathrm{Id}$. Let $E_\alpha, E_{-\alpha}$ be chosen as in (4.5). Then $A(E_\alpha) = c_\alpha E_{-\alpha}$. Choose $\{a_\alpha\}_{\alpha\in\Phi}$ such that $a_\alpha^2 = c_{-\alpha}$ and $a_\alpha a_{-\alpha} = -1$, write $X_\alpha = a_\alpha E_\alpha$. Define $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ if $\alpha+\beta \in \Phi$ and otherwise define $N_{\alpha,\beta} = 0$. Now $k(X_\alpha, X_{-\alpha}) = a_\alpha a_{-\alpha} k(E_\alpha, E_{-\alpha}) = -1$ and $A(X_\alpha) = a_\alpha c_\alpha E_{-\alpha} = -a_{-\alpha} E_{-\alpha} = -X_{-\alpha}$. Hence

$$-N_{\alpha,\beta} X_{-\alpha-\beta} = A(X_{\alpha+\beta} N_{\alpha,\beta}) = A[X_\alpha, X_\beta] = [X_{-\alpha}, X_{-\beta}] =$$

$$= N_{-\alpha,-\beta} X_{-\alpha-\beta}.$$

We now compute

$$k([X_\alpha, X_\beta], [X_{-\alpha}, X_{-\beta}]) = N_{\alpha,\beta} N_{-\alpha,-\beta} k(X_{\alpha+\beta}, X_{-\alpha-\beta}) =$$

$$= + N^2_{\alpha,\beta}.$$

On the other hand,

$$k([X_\alpha, X_\beta], [X_{-\alpha}, X_{-\beta}]) = \cancel{-k(X_\beta, [X_\alpha, [X_{-\alpha}, X_\beta]])} =$$

$$= -k([X_{-\alpha}, [X_\alpha, X_\beta]], X_{-\beta}) = -\frac{m(n+1)}{2} k(X_\alpha, X_{-\alpha}) \underset{(\alpha,\alpha)}{k(X_\beta, X_{-\beta})}$$

$$= -\frac{m(n+1)}{2} (\alpha, \alpha)$$

(by Lemma (4.1)) is negative. This shows that $N_{\alpha,\beta} = i N'_{\alpha,\beta}$, where $N'_{\alpha,\beta}$ is real. Now let

$$\mathfrak{g}_u = i\mathfrak{h}_{\mathbb{R}} \bigoplus_{\alpha\in\Phi^+} \mathbb{R}(X_\alpha + X_{-\alpha}) \bigoplus_{\alpha\in\Phi^+} i\mathbb{R}(X_\alpha - X_{-\alpha}).$$

It is easy to check that $\mathfrak{g}_u$ is a compact form of $\mathfrak{g}_{\mathbb{C}}$.