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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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ELEMENTARY STRUCTURE OF SEMI-SIMPLE LIE ALGEBRAS

Lecture I

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These are preliminary lecture notes, intended only for distribution to participants.

Trieste

① Semisimple, solvable and nilpotent Lie algebras

Definition of Lie algebra: vector space L with bilinear operation ('bracket') $[\ \ ] : L \times L \to L$ such that

1) $[x\,x] = 0 \quad \forall x \in L$

2) $[x\,[y\,z]] + [y\,[z\,x]] + [z\,[x\,y]] = 0$ (Jacobi identity)

L is defined over a field F of characteristic 0, and $\dim_F L < \infty$

Examples

1) $gl(V)$, $\dim V < \infty$, V vector space over F.

$gl(V) = \mathrm{End}\, V$ as a vector space, but $[xy] = xy - yx$.

Basis e_{ij} Exercise: $[e_{ij}\, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$

2) Classical algebras: subalgebras of $gl(V) = gl(n)$.

4 types:

i) $\boxed{A_\ell} = sl(\ell+1)$; $\dim V = \ell + 1$. Subalgebra of $gl(\ell+1)$ given by $tr(x) = 0$.

Exercise: Basis e_{ij}, $i \neq j$, and $e_{ii} - e_{i+1,i+1}$, $i = 1, \dots \ell$.

ii) $\boxed{C_\ell} = sp(2\ell)$; $\dim V = 2\ell$. f symplectic form, i.e., $f(v,w) = -f(w,v)$. Then, in a suitable basis

$$f = \begin{pmatrix} \cancel{1_\ell} & 1_\ell \\ -1_\ell & \cancel{1_\ell} \end{pmatrix}. \qquad C_\ell \text{ is the stabilizer of } f.$$

$$f(x(v), w) = -f(v, x(w))$$

iii) $\boxed{B_\ell} = o(2\ell+1)$, orthogonal; $\dim V = 2\ell + 1$, $f = \begin{pmatrix} & 1_\ell \\ 1_\ell & \end{pmatrix}$,

iv) $\boxed{D_\ell} = o(2\ell)$ $\qquad f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_\ell \\ 0 & 1_\ell & 0 \end{pmatrix}$

3) $t(\ell)$, upper triangular; $n(\ell)$, strictly upper triangular; $\delta(\ell)$, diagonal (2)

$t = n + \delta$, δ abelian $\Rightarrow$ $[t, t] = n$.

4) Lie algebras of derivations. $\mathcal{U}$ any algebra over F (not Lie, nec.).

δ derivation if $\delta(ab) = a\,\delta(b) + \delta(a)\,b$.

$\mathrm{Der}\,\mathcal{U} \subset \mathrm{End}\,\mathcal{U}$. $[\ ] =$ commutator: it is a Lie algebra.

L Lie alg. $\rightarrow$ $\mathrm{Der}\,L \subset gl(L)$ Lie algebra

Def. $\mathrm{ad}\,x = y \rightarrow [x, y]$ is a derivation of L: indeed $\mathrm{ad}\,x([yz]) = [\mathrm{ad}\,x(y)\ z] + [y\ \mathrm{ad}\,x(z)]$ by Jacobi identity.

$\mathrm{ad}\,L \subset \mathrm{Der}\,L$, 'inner derivations'.

$L \supset K$: Definition: K ideal if $[K, L] \subseteq K$. $K \lhd L$.

If $K \lhd L$, all inner derivations of $K = \mathrm{ad}\,K = \mathrm{ad}_K K$ but $\mathrm{ad}_K\,K$ is an algebra of derivations, maybe larger.

Exercises: 1.3, 1.11.

Examples of ideals: center $Z(L) = \ker \mathrm{ad}$

derived algebra $[L\,L]$

K subspace: $N_L(K) = \{x : \mathrm{ad}\,x(K) \subset K\}$ normalizer

$C_L(K) = \{x : \mathrm{ad}\,x|_K = 0\}$ centralizer

L simple if L nonabelian, and without nontrivial ideals.

$I \lhd L \Rightarrow L/I$ Lie algebra, product $[x+I, y+I] = [xy] + I$.

(does not depend on the choice of representatives since $I \lhd L$).

The standard homomorphism theorems hold.

Automorphisms: isomorphism $L \rightarrow L$

Example: $\mathrm{ad}\,x$ nilpotent (i.e. $(\mathrm{ad}\,x)^m = 0$): $\exp(\mathrm{ad}\,x) = \sum^{m-1} \frac{1}{k!} (\mathrm{ad}\,x)^k$. More generally: $\boxed{\text{exercise}}$: δ nilpotent derivation $\Rightarrow$

Def. $x \in L$ "ad-nilpotent" if $\mathrm{ad}\,x$ is a nilpotent gen...

Claim: L nilpotent $\Leftrightarrow$ $\forall x \in L$ x ad-nilpotent

(1) L nilpotent $\Rightarrow$ $\forall x \in L$, x is ad-nilpotent.

(proof: $L^k = 0 \Rightarrow \mathrm{ad}\,x_1 \cdots \mathrm{ad}\,x_k(y) = 0$ $\forall x_1, \dots x_k$ $\Rightarrow (\mathrm{ad}\,x)^k \equiv 0$)

not relevant [(2) $x \in gl(V)$ nilpotent operator $\Rightarrow$ x ad-nilpotent

(proof: $\mathrm{ad}\,x = \lambda_x - \rho_x$ as $\mathrm{ad}\,x(y) = xy - yx$. Now $[\lambda_x, \rho_x] = 0$ and $\lambda_x^k = \lambda_{x^k} = 0 = (\rho_x)^k$ $\Rightarrow (\lambda_x - \rho_x)^k = 0$)]

(3) $L \subseteq gl(V)$: if $\forall x \in L$ x is a nilpotent operator (hence ad-nilpotent, by (2)), then L is nilpotent

[Engel's theorem: exercise]

More precisely: under this condition, $\exists$ common eigenvector $v \neq 0$ of L, with eigenvalue 0. By passing to quotients, and by induction on dimension, we get

Corollary: L nilpotent $\subseteq gl(V)$ $\Rightarrow$ for a suitable basis $L \subseteq n(V)$

Indeed, let v be the last basis vector... $L =$

Now form the quotient $L/\langle v \rangle$, and apply the same procedure: $L =$

and so on.

Exercises: ~~Engel's theorem~~ (1) L nilpotent, $0 \neq K \lhd L \Rightarrow K \cap Z(L)$

(proof: $\mathrm{ad}\,L$ acts on K $\exists$ com...

$\Rightarrow \exp \delta = \sum \delta^n/n! \quad \in \mathrm{Aut}\, L$

$\boxed{\text{Exercise}}$: $\mathrm{Int}\, L = \langle \exp \mathrm{ad}\, L \rangle$, inner automorphisms;

$\mathrm{Int}\, L \lhd \mathrm{Aut}\, L$

[indeed, $\forall \phi \in \mathrm{Aut}\, L$, $\phi(\exp \mathrm{ad}\, x)\phi^{-1} = \exp \mathrm{ad}(\phi(x))$]

Solvable Lie algebras.

Derived series: $L^{(0)} = L$, $L^{(i+1)} = [L^{(i)} L^{(i)}]$

L solvable if this series is finite. In this case, $L^{(k)} = 0$ and $\boxed{L^{(k-1)} \neq 0 \text{ is abelian}}$.

Exercise: every term of the series is a characteristic ideal (invariant under all derivations, not only inner).

Homomorphism thms: $\Rightarrow$ if I, J solvable ideals, then $I + J$ solvable ideal.

$\mathrm{Rad}\, L \equiv$ maximal solvable ideal.

$\mathrm{Rad}\, L = 0 \Leftrightarrow$ no solvable ideal $\Leftrightarrow$ no abelian ideal

↳ Def: L is semisimple.

$L/\mathrm{Rad}\, L$ is always semisimple.

Nilpotent Lie algebras

lower central series: $L^0 = L$, $L^{i+1} = [L L^i]$.

L nilpotent if this series is finite. Exercise: all the terms are characteristic ideals.

$L^i \subset L^{(i)} \Rightarrow$ solvable implies nilpotent.

L nilpotent $\Rightarrow L^k = 0$ and $L^{k-1} \neq 0$, $\subseteq Z(L)$.

② Ex 3.1; 3.8, 3.9 (existence of outer derivations of nilpotent alg.)

Proof of Engel's thm.

Induction on $\dim L$. $K \subsetneq L$ subalgebra: K acts by ad as a Lie algebra of nilpotent operators, by ①, on L, hence on L/K since K is an invariant subspace.

By induction, $\exists\, x$: $x + K \neq K$, $\mathrm{ad}_{L/K} K\,(x+K) = 0$.

I.e., $[y\,x] \in K$ $\forall y \in K$, but $x \notin K$. Hence $N_L(K) \supsetneq K$. Now choose K maximal. Hence $N_L(K) = L \Rightarrow K \lhd L \Rightarrow \mathrm{codim}\, K = 1$, otherwise K not maximal (choose a 1-dim subalgebra in L/K and lift!). Hence $L = K + Fz$, $(z \in L - K)$.

Induction again: $W = \{v : Kv = 0\} \neq 0$. $K \lhd L \Rightarrow$ $L W \subseteq W$, indeed $y(xw) = x\underbrace{yw}_{0} - \underbrace{[xy]w}_{0} = 0$ for $x \in L$, $y \in K$.

Now z is a nilpotent ~~auto~~morphism acting on W, it must have an eigenvector $v \in W$. Thus $zv = 0$, $Kv = 0 \Rightarrow Lv = 0$.