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ELEMENTARY STRUCTURE OF SEMI-SIMPLE LIE ALGEBRAS
Lecture II

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These are preliminary lecture notes, intended only for distribution to participants.

$= \mathrm{ad}\,[x_s, x_n] = 0$!).

<u>Proposition</u>: $\mathcal{U}$ algebra, finite dimens. over F, $\Longrightarrow$ $\mathrm{Der}\,\mathcal{U}$ contains the semisimple and nilpotent parts of all its elements.

<u>Proof</u> $\delta \in \mathrm{Der}\,\mathcal{U}$, $\delta = \sigma + \nu$. $\mathcal{U} = \sum^{\oplus} \mathcal{U}_a$,

$\mathcal{U}_a = \bigcup_k \{\ker(\delta - a\mathbb{1})^k\}$, generalised eigenspace.

$\sigma|_{\mathcal{U}_a} = a\,\mathbb{1}_{\mathcal{U}_a}$. Now $\mathcal{U}_a\mathcal{U}_b \subseteq \mathcal{U}_{a+b}$

(because $(\delta - (a+b)\mathbb{1})^n(xy) = \sum_{i=0}^{n} \binom{n}{i} (\delta - a\mathbb{1})^{n-i}(x)\cdot(\delta - b\mathbb{1})^i(y)$, <u>exercise</u>)

Thus, $x \in \mathcal{U}_a$, $y \in \mathcal{U}_b \Rightarrow \sigma(xy) = (a+b)\,xy$ as $xy \in \mathcal{U}_{a+b}$.

But $(\sigma x)y + x(\sigma y) = a\,xy + b\,xy = (a+b)\,xy$

Hence, by splitting $\mathcal{U} = \sum^{\oplus} \mathcal{U}_a$, the result is proved: $\sigma \in \mathrm{Der}\,\mathcal{U}$ (hence ν too)!

<u>Cartan's criterion for solvability.</u>

<u>Idea</u>: $[LL]$ nilpotent $\Longrightarrow$ L solvable

But $[LL]$ nilpotent $\Longleftrightarrow$ $\mathrm{ad}_{[LL]}\,x$ nilpotent $\forall x \in [LL]$.

<u>Lemma</u>. $A, B \subseteq gl(V)$ subspaces. $M = \{x : {\scriptstyle \mathrm{ad}^{gl(V)}} [x, B] \subseteq A\}$. If $x \in M$ and $\mathrm{tr}(xy) = 0$ $\forall y \in M$, then x is nilpotent.

<u>Proof</u>. $x = s + n$, $s = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$. E vector subsp. of F over $\mathbb{Q}$ given by $E = \mathrm{span}_{\mathbb{Q}}\langle a_1, \dots a_n\rangle$. We claim $s = 0$. This amounts to $E = 0$, hence $E^* = \{f$ linear functionals $f : E \to \mathbb{Q}\} = 0$.

② Solvable algebras and Cartan's criterion; the Killing form

Suppose F algebraically closed, char $F = 0$.

Theorem (Lie's theorem). $L \le gl(V)$ solvable $\Rightarrow$ L has a common eigenvector in V.

Corollary 1. For a suitable basis, $L \le t(V)$.

[illegible]

Corollary 2. $x \in [LL]$, L solvable $\Rightarrow$ ad x nilpotent
(proof: $[t, t] = n$)

$x \in \operatorname{End} V$ "semisimple" if diagonalizable.

Theorem (Jordan decomposition) $\forall\, x \in \operatorname{End} V$:

1) $\exists !\ x_s, x_n \in \operatorname{End} V$, x_s semisimple, x_n nilpotent, $x = x_s + x_n$ and $[x_s, x_n] = 0$
2) $\exists$ polynomials $\underbrace{p, q}_{\text{without constant term}}$: $x_s = p(x)$, $x_n = q(x)$
In particular, $x_s, x_n \in x''$ (double commutant)
3) If $V \supseteq B \supseteq A$, and $x : B \to A$, then $x_s, x_n : B \to A$.
4) On $V_a = \ker (x - a\mathbb{1})^{\operatorname{mult}(a)}$ x_s has the only eigenvalue a (is scalar).

Proof: in the problems session.

Note: x nilpotent $\Rightarrow$ ad x nilpotent
Exercise: x semisimple $\Rightarrow$ ad x semisimple
(indeed, $x = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ $\Rightarrow$ ad $x\,(e_{ij}) = (a_i - a_j)\, e_{ij}$)

Corollary: $x \in \operatorname{End} V$, $x = x_s + x_n$ $\Rightarrow$ ad x = ad x_s + ad x_n is the Jordan decomposition of ad x in End(End V). Proof: ...

Fix $f \in E^*$. Let $y = \begin{pmatrix} f(a_1) & & \\ & \ddots & \\ & & f(a_n) \end{pmatrix}$. Claim: $y \in M$.

Now ad $s\,(e_{ij}) = (a_i - a_j)\, e_{ij}$; $\Rightarrow$ ad $y\,(e_{ij}) = (f(a_i) - f(a_j))\, e_{ij}$

$\exists$ r polynomial without constant term s.t. $r(a_i - a_j) = f(a_i) -$
(no compatibility conflict, since $a_i - a_j = a_k - a_l$ $\Rightarrow$ $f(a_i) - f(a_j) = f(a_k) - f(a_l)$ by linearity).

Now ad $y = r(\text{ad } s)$. But ad $s = p(\text{ad } x)$
$\Rightarrow$ ad y = polynom. in ad x without constant term.

Now ad $x : B \to A$ $\Rightarrow$ ad $y : B \to A$, i.e., $y \in$

Thus $\operatorname{tr}(xy) = 0$, that is, $\sum a_i f(a_i) = 0$.
Apply f: we get $\sum f(a_i)^2 = 0$. But $f(a_i) \in$
$\Rightarrow f(a_i) = 0\ \forall i$ $\Rightarrow$ $f \equiv 0$

Observation: $\operatorname{tr}([xy]\, z) = \operatorname{tr}(x\, [yz])$ (Exercise)

Theorem (Cartan's criterion). $L \le gl(V)$, $\operatorname{tr}(xy) = 0$ $\forall\, x \in [LL]$, $y \in L$ $\Rightarrow$ L solvable.

Proof. We intend to show that $[LL]$ is nilpotent, that is, $\forall\, x \in [LL]$ is nilpotent endomorphism.
In the lemma (with $B = L$, $A = [LL]$), we get $M = \{x \in gl(V) : \text{ad } x(L) \subseteq [LL]\} \supseteq L$. Now we let $x \in [LL]$. But to apply the lemma we need to know $\operatorname{tr}(xy) = 0$ $\forall\, y \in M$.
We only know $\operatorname{tr}(xy) = 0$ $\forall y \in L$. On the other

Hence, $tr([xy]\,z) = tr(x\,[yz]) = 0$ (4)

$[xy] \in [LL]$, $z \in M$, $[yz] \in [LL]$ by def. of M!

Corollary. $tr(\mathrm{ad}\,x\ \mathrm{ad}\,y) = 0 \quad \forall\ x \in [LL],\ y \in L \Rightarrow$
$\Rightarrow L$ solvable.

Proof. By the thm, $\mathrm{ad}\,L$ solvable. $L \simeq \mathrm{ad}\,L/\ker \mathrm{ad}\,L$
$\simeq \mathrm{ad}\,L/Z(L)$ is also solvable (homom. image of solvable)

Killing form and semisimple algebras

Def. $K(x,y) = tr(\mathrm{ad}\,x\ \mathrm{ad}\,y)$. It is associative:
$K([xy], z) = K(x,[yz])$ (by the observation of the last page).

Fact. $I \trianglelefteq L \Rightarrow K_I = K_L|_{I\times I}$

Proof. $W \subseteq V$, $\phi \in End\,V$, $\phi: V \to W \Rightarrow$
$\Rightarrow tr\,\phi = tr\,(\phi|_W)$

$$\left[\ \phi = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \}\,W \quad \text{in suitable basis}\ \right]$$

But $x, y \in I \Rightarrow \mathrm{ad}\,x\ \mathrm{ad}\,y : L \to I$
$\Rightarrow tr(\mathrm{ad}\,x\ \mathrm{ad}\,y) = tr\,(\mathrm{ad}\,x\ \mathrm{ad}\,y)|_I = tr(\mathrm{ad}_I\,x\ \mathrm{ad}_I\,y)$.

Def. β symmetric bilinear form. β nondegenerate if $S = \mathrm{Rad}\,\beta = 0$
$\mathrm{Rad}\,\beta = \{x : \beta(x,y) = 0\ \forall y\}$. Associativity $\Rightarrow \mathrm{Rad}\,\beta$
is an ideal: $\beta([zx], y) = \beta(x, [zy]) = 0$!

Now, L semisimple iff $\mathrm{Rad}\,L = 0$ (same as $\nexists$ abelian ideals).

Now: 1) $\mathrm{Rad}\,K$ is a solvable ideal, hence $\subseteq \mathrm{Rad}\,L$

Proof: Indeed, $\forall x \in \mathrm{Rad}\,K$, $y \in L$, $tr(\mathrm{ad}\,x\ \mathrm{ad}\,y) = K(x,y) =$
By the corollary to Cartan's criterion, $\mathrm{Rad}\,K$ is solvable.

2) $\mathrm{Rad}\,K \subsetneq \mathrm{Rad}\,L$ in general; however, $\mathrm{Rad}\,K = 0$
$\Rightarrow \mathrm{Rad}\,L = 0$

Proof. Enough to show that $I \trianglelefteq L$ abelian $\Rightarrow I \subseteq \mathrm{Rad}\,K$
(hence $I = 0 \rightsquigarrow \mathrm{Rad}\,L = 0$). $x \in I$, $y \in L \Rightarrow$
$\mathrm{ad}\,x\ \mathrm{ad}\,y : L \to I \Rightarrow (\mathrm{ad}\,x\ \mathrm{ad}\,y)^2 : L \to [II] =$
$\Rightarrow \mathrm{ad}\,x\ \mathrm{ad}\,y$ is nilpotent, hence $tr(\mathrm{ad}\,x\ \mathrm{ad}\,y) = 0$
$\rightsquigarrow x \in \mathrm{Rad}\,K$.

Thus: Proposition: L semisimple $\Leftrightarrow K$ nondegenerate

Simple ideals

$L = \oplus I_j$, $I_j \trianglelefteq L$, if $L = \oplus I_j$ as vector space
(but $[I_i I_j] \subseteq I_i \cap I_j = 0$ if $i \neq j$, hence the bracket works componentwise ...).

Theorem. L semisimple $\Rightarrow L = \oplus L_j$ simple ideals, and every simple ideal appears here.

Proof. Inductive step: $I \trianglelefteq L \Rightarrow I^\perp = \{x : K(x,y) = 0\ \forall y \in I\} \trianglelefteq L$
(since K is associative!). By Cartan's criterion, $I \cap I^\perp$
is solvable, hence $I \cap I^\perp = 0$. Thus $L = I \oplus I^\perp$.

Now, by induction on $\dim L$, $L = L_1 \oplus L_1^{\perp} = L_1 \oplus L_2 \oplus \cdots \oplus L_k$.

But if we choose L_1 minimal ideal, L_1 is semisimple (because $I \trianglelefteq L_1 \Rightarrow I \trianglelefteq L$, since

But if $K \trianglelefteq I$ and $I \trianglelefteq L$, then $K \trianglelefteq L$, since

$$[i + i^{\perp}, K] = [i, K] + \underbrace{[i^{\perp}, K]}_{I^{\perp} \cap K = 0 \text{ since } K \subset I} \subseteq K \text{ (since } K \trianglelefteq I).$$

all ideals in L are semisimple!

Thus, choose L_1 minimal. Then L_1 semisimple (if it had an abelian ideal, so would L!), hence simple (by minimality and the above decomposition $L_1 = I \oplus I^{\perp}$).

Similarly, $L_1^{\perp}$ is semisimple. Proceed by induction.

Finally, $\forall I \overset{\text{simple}}{\trianglelefteq} L$, $[IL]$ is an ideal in L. Thus

$L = \oplus L_i \Rightarrow I = [IL] = \oplus [IL_i]$ — hence $[IL] = I$ since $[IL] \neq 0$ as $Z(L) = 0$.

Thus all summands but one are zero and I appears in the decomposition.

Corollary: $L = [LL]$

Observation. $\forall$ derivation δ, $[\delta, \mathrm{ad}\, x] = \mathrm{ad}(\delta x)$ (Exercise)

Theorem. L semisimple $\Rightarrow \mathrm{Der}\, L = \mathrm{ad}\, L$.

Proof. L semisimple $\Rightarrow Z(L) = 0 \Rightarrow$ ~~hence~~ ad is a Lie algebra isomorphism $\Rightarrow M = \mathrm{ad}\, L$ has nondegenerate Killing form. Now, let $D = \mathrm{Der}\, L$. By the observation, $[D, M] \subseteq M$.

We want to prove $M^{\perp} = 0$

Thus $M \trianglelefteq D$, and $K_M = K_D|_{M \times M}$. Let $M^{\perp} \subseteq D$ orthogonal to M under K_D: then $M \cap M^{\perp} = 0$ since K_M is nondegenerate. Since $M, M^{\perp} \trianglelefteq D$, $[M\, M^{\perp}] \subseteq M \cap M^{\perp} = 0$.

Now let $\delta \in M^{\perp}$: it follows $\mathrm{ad}(\delta x) = [\delta, \mathrm{ad}\, x] = 0$ $\forall x \in L$. Since ad is an isomorphism, $\delta x = 0$ $\forall x$ hence $\delta = 0$, and $M^{\perp} = 0$, $M = D$.

The last Theorem allows us to carry the Jordan decomposition to any semisimple Lie algebra.

L semisimple $\Rightarrow L \simeq \mathrm{ad}\, L = \mathrm{Der}\, L$, which is closed under semisimple and nilpotent parts.

Thus $\forall x \in L$, $\mathrm{ad}\, x = \underset{(\mathrm{ad}\, x)_s}{\mathrm{ad}\, s} + \underset{(\mathrm{ad}\, x)_n}{\mathrm{ad}\, n}$. ~~Strokes~~

belong to $\mathrm{ad}\, L = \mathrm{Der}\, L$ separately!

Since ad is 1-1, $\exists\, s, n \in L$: $\mathrm{ad}\, x = \mathrm{ad}\, s + \mathrm{ad}\, n$

and we have $x = s + n$, $[sn] = 0$, s ad-semisimple, n ad-nilpotent

What if L was already a matrix algebra? Another Jordan decomp.? In that case, we shall see that the two Jordan decompositions coincide.