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COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
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PETER-WEYL THEOREM AND REPRESENTATIONS OF COMPACT GROUPS

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Representing a Lie group G as a group of linear operators usually concerns us with continuous homomorphisms $\Pi : G \longrightarrow GL(V)$ where V is a separable (possibly infinite dimensional Hilbert space, $GL(V)$ is the group of bounded invertible operators on V (open in the Banach space of bounded ~~invertible~~ operators on V, with the operator norm). We like a unitary structure on V so that often Π is into $U(V)$.

The theory of unitary group representat[illegible] ..... the representation theory of a compact Lie group.

If G is compact any finite dimensional representation is unitarisable, completely reducible and the irreducible representations of G are finite-dimensional. As is the case for a finite group we will see that each irreducible representation of G compact occurs in the left regular representation with multiplicity equal to the degree (dimension of ~~the representation~~ V) (c.f. the Peter-Weyl theorem)

The theory of unitary group representations in Hilbert space includes Fourier analysis, spectral theory and the representation theory of finite groups as special cases, and may be regarded as a unification of these. In these lectures I will describe the representation theory of a compact Lie group.

Historical notes:

Finite group representations were introduced by Frobenius in 1896 following Weber who defined the character of a finite abelian group in 1881. They were influenced by the work of Lagrange and Gauss in number theory, and many arguments in ~~the~~ nineteenth century number theory may be interpreted as Fourier analysis on finite abelian groups.

G finite group $\qquad A$ finite abelian group

a representation is a homomorphism $\pi : G \longrightarrow GL(V)$, a character $\lambda : A \longrightarrow S^1$. Any complex valued function f on A may be written uniquely in the form

$$f = \sum_{\lambda \in \hat{A}} c_\lambda \lambda \quad \text{where} \quad c_\lambda = \frac{1}{o(A)} \sum_{x \in A} f(x)\overline{\lambda(x)}$$

$o(A)$ is the order of A, $\hat{A}$ is the group of characters.

The expansion theorem on S^1

$$f = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\,dx \qquad (\dagger\dagger)$$

convergence in the mean, was introduced by Fourier in his famous paper on heat conduction submitted to the French academy in 1807 (this was at first rejected by Lagrange among others).

The early development of group representations by Frobenius, Burnside and Schur was a branch of pure algebra with applications to the structure theory of finite groups (eg the theorem of Burnside that a group whose order is divisible by two primes is soluble). The fact that there is a relationship between $(\dagger)$ and $(\dagger\dagger)$ does not seem to have been noticed formally until the middle 1920's when the theory was extended so as to apply to compact Lie groups by the work of Hurwitz, Schur, Cartan and Weyl with applications to physics in quantum mechanics.

In 1924 Schur applied earlier ideas of Hurwitz on integration over manifolds to define integration of continuous functions on compact Lie groups; and determined the characters of the orthogonal groups. In the next three years 1924-27 H. Weyl determined the ~~characters of the orthogonal~~ irreducible representations, and their characters, of all the classical compact semi-simple Lie groups; and with F. Peter proved the Peter-Weyl theorem which asserts that one can form an orthonormal basis for the square integrable functions $L^2(G)$ on a compact Lie group G consisting of matrix elements of the irreducible representations of G. When the group is commutative the matrix elements are necessarily linear combinations of characters.

In fact a large part of the theory of unitary group representations is concerned with generalizations of the Fourier expansion theorem to a separable locally compact group G. Weyl first gave a group theoretical interpretation of classical harmonic analysis e.g. the classical theory of expansions in spherical harmonics.

$SO(3)$ acts on the 2-sphere $S^2 = SO(3)/SO(2)$ gives rise to an action of $SO(3)$ on the space of complex valued smooth functions on S^2, $(g.f)(x) = f(g^{-1}x)$. For J an integer take the $2J+1$ dimensional subspace H_J of homogeneous polynomials of degree J; these are the eigenspaces of the Laplacian, the spherical harmonics, then

$$L^2(S^2) = \sum_J{}^{\oplus} H_J$$ complete orthogonal direct sum (or Hilbert space direct sum) of irreducible representations of $SO(3)$

This was developed further by E. Cartan in his work on Riemannian symmetric spaces in the late 1920's. Also H. Weyl made heavy use of Cartan's 'infinitesimal' theory of complex semi-simple Lie algebras and their representations in his own work on representations.

Frobenius	1896	Finite group representations
Weber	1881	Characters of finite abelian groups
Fourier	1807	Expansion theorem
Schur	1924	Integration on compact Lie groups (also Hurwitz) Irreducible characters of the orthogonal group.
Weyl	1924-27	Irreducible representations (and characters) of all the classical compact semi simple Lie groups.
- Peter		Peter-Weyl theorem, group theoretical interpretation of classical harmonic analysis.
Cartan	1894, 1914	Classification of the simple Lie algebras over $\mathbb{C}$, $\mathbb{R}$; irreducible finite-dimensional representations

Compact Lie Groups

(0) Elementary Representation theory

(1) Harmonic analysis on compact Lie groups (Peter-Weyl theorem, Schur orthogonality relations for irreducible characters)

(2) Basic structure theory of compact Lie groups and semi-simple Lie algebras over $\mathbb{R}$, $\mathbb{C}$ (E. Cartan)

(3) Classification of the irreducible representations (Cartan-Weyl theorem of highest weight)

(4) Geometric construction of the irreducible representations

(0) A global Lie group G is a smooth manifold such that $G \times G \longrightarrow G$, $(x,y) \longmapsto xy^{-1}$ is smooth.

eg V a finite-dim vector space over $\mathbb{R}$ ($\mathbb{C}$) ; $GL(V)$, the automorphisms of V, is the open set of $\mathfrak{gl}(V)$, the endomorphisms of V, given by $\det \neq 0$, so is a smooth manifold. The product map is smooth since it is given by ~~the~~ polynomials $\sum a_{ij} b_{jk}$ and the inverse map is smooth since it is given by polynomials divided by the determinant. For $V = \mathbb{R}^n$ or $\mathbb{C}^n$ write $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$.

A very important fact about a Lie group is that the structure of G (and also representations) is determined to a large extent by that of its Lie algebra $\mathfrak{g}$. (General: a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$ is a vector space (here finite-dim) with a bracket $[\,,\,]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which is bilinear, skew-symmetric and satisfies the Jacobi identity.) the left invariant vector fields (ie $\xi : G \longrightarrow TG$, $\xi(g) \in T_gG$ and $\xi(g) = l_{g*}\xi(e)$ where left translations $l_g : G \longrightarrow G$, tangent map l_{g*}. $\mathfrak{g}$ is identified with T_eG by $\xi \longmapsto \xi(e)$)

A 1-parameter subgroup of G is a continuous homomorphism $\alpha : \mathbb{R} \longrightarrow G$; then the exponential mapping $\exp : \mathfrak{g} \longrightarrow G$ is defined by $\alpha(t) = \exp t\xi$ is the unique 1-parameter subgroup with $\alpha'(0) = \xi$ (Picard's theorem, and ξ is complete); this is a local diffeomorphism.

To find the Lie algebra of $G = GL(n,\mathbb{C})$:

$T_1G = \mathbb{R}^{2n^2} = \mathfrak{gl}(n,\mathbb{C})$. If X is an $n \times n$ matrix, for small t, $\gamma(t) = 1 + tX$ is a curve in G (so taking the tangent vector at $t=0$) gives an element of T_1G. Define

$$\tilde{X}(g)f = X(f \circ l_g) \qquad f \text{ a smooth function on } G$$
$$= \frac{d}{dt} f(l_g(1+tX))\Big|_{t=0}$$
$$= \frac{d}{dt} f(g + tgX)\Big|_{t=0} \quad \text{directional derivative of } f \text{ at } g \text{ along } gX.$$

$\tilde{X} \in \mathfrak{g}$ with $\tilde{X}(1) = X$. Now

$$[\tilde{X}\,\tilde{Y}](1)f = X(\tilde{Y}f) - Y(\tilde{X}f)$$
$$= \frac{d}{dt}\tilde{Y}(1+tX)f\Big|_{t=0} - \frac{d}{ds}\tilde{X}(1+sY)f\Big|_{s=0}$$

$$= \frac{\partial}{\partial t}\frac{\partial}{\partial s} f(1+tX+s(1+tX)Y)\Big|_{s=t=0} - \frac{\partial}{\partial s}\frac{\partial}{\partial t} f(1+sY+t(1+sY)X)\Big|_{s=t=0}$$

$$= (XY-YX)f$$

ie Thus $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ with the commutator $[XY] = XY - YX$

Considerable simplification arises if one is dealing just with linear Lie groups; these are the ones having a faithful finite-dimensional representation ie the closed subgroups $G \subseteq GL(V)$ (always true for $\mathfrak{g}$, Ado's theorem, but not for G eg $\widetilde{SL}(2,\mathbb{R})$ is not a matrix group). Any Compact Lie group is linear. Then one defines the Lie algebra

$$\mathfrak{g} = \{\xi \in \mathfrak{gl}(V);\ e^{t\xi} \in G \text{ for all } t \in \mathbb{R}\}$$

where exponentiation of matrices

$$e^{\xi} = \sum_{n=0}^{\infty} \frac{\xi^n}{n!}$$ converges in the operator norm.

Using the fact that the exponential mapping is a local diffeomorphism one shows that the 1-parameter subgroups in $GL(V)$ are given by $\alpha(t) = e^{t\xi}$ for $\xi \in \mathfrak{gl}(V)$, ξ is called the infinitesimal generator, $\xi(v) = \frac{d}{dt}\alpha(t)v\big|_{t=0}$ (since $\alpha(t)$ is differentiable in fact analytic.) An application of the Campbell-Baker-Hausdorff formula shows that $\mathfrak{g}$ is a vector space and is closed under the commutator. CBH relates the structures of $\mathfrak{g}$ and G, for ξ, η in a nbd of 0 in $\mathfrak{g}$

$$\exp\xi \exp\eta = \exp\left(\xi+\eta+\tfrac{1}{2}[\xi\eta] + \text{h.o.t}\right)$$

(holds for general G)

Trotter product formula

$$\exp(\xi+\eta) = \lim_{n\to\infty}\left(\exp \xi/n \exp \eta/n\right)^n$$

commutator formula

$$\exp[\xi\eta] = \lim_{n\to\infty}\left[\exp\frac{\xi}{n}\exp\frac{\eta}{n}\right]^{n^2}$$

where $[gh] = ghg^{-1}h^{-1}$ the group commutator.

Geometric picture of the relationship between $\mathfrak{g}$ and G: making the smooth change of coordinates $\xi \longmapsto e^{\xi}$ in a nbd of 0, we see that $\mathfrak{g}$ is the tangent space at I of this surface.

A representation of a Lie group G is a pair (U, Π) where U is a real (or complex) separable Hilbert space and

$$\Pi : G \longrightarrow GL(U)$$

is a ~~homomorphism~~ 'continuous' homomorphism into the group of bounded invertible operators on U (open in the Banach space $B(U)$) with the weak, strong or uniform operator topology on $GL(U)$ ie requiring the continuity of the map

$g \longmapsto \langle \Pi(g)v, w\rangle$ from G into $\mathbb{R}$ (or $\mathbb{C}$) weakly continuous

$g \longmapsto \Pi(g)v$ from G into U for each $v \in U$ strongly continuous

$g \longmapsto \Pi(g)$ from G into $GL(U)$ uniformly continuous.

Π is orthogonal (unitary) if $\Pi(g)$ is orthogonal (unitary) $\forall g \in G$.

As G is locally compact, if Π is unitary, then weak $\Leftrightarrow$ strong ($\Leftrightarrow$ uniform for V finite-dimensional)

If $\phi : G \longrightarrow H$ is a continuous homomorphism between Lie groups (implies smooth in fact analytic) then the differential $d\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a homomorphism of Lie algebras

eg inner automorphism $a_g : G \longrightarrow G$, $a_g(h) = ghg^{-1}$; get the adjoint representation of G, $Ad : G \longrightarrow GL(\mathfrak{g})$, $Ad\, g = da_g$, $g \in G$.

For a linear Lie group G the adjoint representation is given by

$$Ad\, g\, \xi = g \xi g^{-1} \quad \text{for } \xi \in \mathfrak{g}.$$

Now $Ad\,(\exp t\xi)$ = a 1-parameter subgroup

$$= e^{tA} \quad \text{some } A \in gl(\mathfrak{g})$$

infinitesimal generator $A(\eta) = \frac{d}{dt} Ad(\exp t\xi)\eta \Big|_{t=0}$, $\eta \in \mathfrak{g}$

$$= \frac{d}{dt}\left(e^{t\xi}\, \eta\, e^{-t\xi}\right)\Big|_{t=0}$$

$$= \xi\eta - \eta\xi = ad\,\xi(\eta)$$

where the adjoint representation of $\mathfrak{g}$ is $ad\,\xi(\eta) = [\xi\eta]$.

Representation $\Pi : G \longrightarrow GL(V)$, V finite-dim, gives a representation $d\Pi : \mathfrak{g} \longrightarrow gl(V)$ satisfying

$$\Pi(\exp \xi) = e^{d\Pi(\xi)} \quad \text{for } \xi \in \mathfrak{g}.$$

Proof

$\Pi(\exp t\xi)$ is a 1-parameter subgroup in $GL(V)$

$= e^{tA}$ and define $d\Pi(\xi) = A$.

Need to show that $d\Pi$ is a homomorphism. For this use CBH;

$$\Pi(\exp(\xi+\eta)) = \lim_{n\to\infty} \left(\Pi(\exp \tfrac{\xi}{n})\, \Pi(\exp \tfrac{\eta}{n})\right)^n$$

$$e^{d\Pi(\xi+\eta)} = \lim_{n\to\infty} \left(e^{d\Pi(\xi)/n}\, e^{d\Pi(\eta)/n}\right)^n = e^{d\Pi(\xi)+d\Pi(\eta)}$$

also

$$\Pi(\exp[\xi\eta]) = \lim_{n\to\infty} \left[\Pi(\exp \tfrac{\xi}{n})\, \Pi(\exp \tfrac{\eta}{n})\right]^{n^2}$$

$$e^{d\Pi[\xi\eta]} = \lim_{n\to\infty} \left[e^{d\Pi(\xi)/n}\, e^{d\Pi(\eta)/n}\right]^{n^2} = e^{[d\Pi(\xi)\, d\Pi(\eta)]}$$

$g \exp \xi\, g^{-1} = \exp Ad\,g\,\xi$ for $g \in G$, $\xi \in \mathfrak{g}$. $\det e^A = e^{\operatorname{tr} A}$, A $n \times n$.

Closed subgroups of $GL(n, \mathbb{C})$	Lie algebra
Unitary group $U(n)$ $A^{-1} = \bar{A}^t$	skew hermitian matrices $\mathfrak{u}(n)$ $A + \bar{A}^t = 0$
Special linear group $SL(n, \mathbb{C})$ $\det A = 1$	matrices of trace zero $\mathfrak{sl}(n, \mathbb{C})$ $\operatorname{tr} A = 0$
Orthogonal group $O(n, \mathbb{C})$ $A^{-1} = A^t$	skew symmetric matrices $\mathfrak{o}(n, \mathbb{C})$ $A + A^t = 0$
Symplectic group $SP(n, \mathbb{C})$ $A^t J A = J$, $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$	symplectic Lie algebra $A^t J + J A = 0$

eg for symplectic groups from $A \exp Z\, A^{-1} = \exp(AZA^{-1})$, $(\exp Z)^t = \exp Z^t$

$$\exp t(J^{-1} Z^t J) = \exp -tZ \text{ iff } Z^t J + J Z = 0$$

$\Pi : G \longrightarrow GL(V)$ (unitary) representation.

Such Π_1, Π_2 are equivalent if there is a (unitary) intertwining operator A from V_1 into V_2 ie $A\Pi_1(g) = \Pi_2(g)A$, $g \in G$.

A ~~closed~~ subspace U of V is said to be invariant if for each $g \in G$ $\Pi(g)U \subseteq U$. Π is irreducible if the only closed invariant subspaces of V are V and $\{0\}$. Exercise: V finite-dim, G connected then Π is irreducible iff $d\Pi$ is irreducible. Adjoint rep is irreducible iff $\mathfrak{g}$ is simple.

<u>Schur's Lemma</u>

Suppose ~~Let~~ (U, Π_1), (V, Π_2) are ~~be~~ finite-dimensional irreducible representations of G. Then (U, V complex)

$$\mathrm{Hom}_G(U,V) = \begin{cases} \{0\} & \text{if } U \text{ and } V \text{ are not equivalent.} \\ \mathbb{C}I & \text{if } U \text{ is equivalent to } V. \end{cases}$$

<u>Proof</u>

If $A \in \mathrm{Hom}_G(U,V)$ then $\mathrm{Ker}\,A$ ~~and Im A~~ is an invariant subspace of U and $\mathrm{Im}\,A$ is an invariant subspace of V. So for A non-zero, must have $\mathrm{Ker}\,A = \{0\}$ and $\mathrm{Im}\,A = V$ ie U and V are equivalent; with ~~let~~ V_λ be ~~the~~ a non-zero λ-eigenspace, $V_\lambda = V$, $A = \lambda I$. □

<u>Lemma</u>

A finite-dimensional unitary representation (V, Π) of G is a unitary direct sum of irreducible representations.

<u>Proof</u>

By induction on the dimension of V. If 1-dim, clear, otherwise $V = U \oplus U^{\perp}$ for U an invariant subspace. □

For G compact, any finite-dimensional representation is equivalent to one that is unitary and any irreducible representation is finite-dimensional

<u>Lemma</u>

(V, Π) finite-dim, irreducible unitary representation of G. Then any invariant hermitian form on V is a real multiple of the Hilbert space structure of V.

<u>Proof</u>

Hilbert space inner product $\langle , \rangle$. If h is an invariant hermitian form then $h(\Pi(g)v, \Pi(g)w) = h(v,w)$ and there is an hermitian operator A so that $h(v,w) = \langle Av, w\rangle$. Then $A \in \mathrm{Hom}_G(V,V)$ $(\langle A\Pi(g)v, w\rangle = h(\Pi(g)v, w) = h(v, \Pi(g)^{-1}w) = \langle \Pi(g)Av, w\rangle)$ so $A = cI$, $c \in \mathbb{C}$. But since A is hermitian c is real. □

<u>Corollary</u>

If two irreducible unitary finite-dimensional representations are equivalent then they are unitarily equivalent.

<u>Proof</u>

(V_i, Π_i) $i = 1, 2$, A an equivalence. Then $\langle Av, Aw\rangle_2 = c\langle v, w\rangle_1$ with c real positive; $B = c^{-1/2}A$ defines a unitary equivalence □

Contragredient representation: With (V, Π) take V^* and define $(\Pi^*(g)\lambda)(v) = \lambda(\Pi(g)^{-1}v) \longleftrightarrow g^{-1}$ (matrix elements)

Tensor product of (V_1, Π_1), (V_2, Π_2) is $(V_1 \otimes V_2, \Pi_1 \otimes \Pi_2)$, $(\Pi_1 \otimes \Pi_2)(g) = \Pi_1(g) \otimes \Pi_2(g)$.

(V, Π) finite-dimensional, unitary.

Define the character of Π χ, $\chi(g) = \mathrm{tr}\,\Pi(g)$, $g \in G$. Only depends on the equivalence class of Π. Later for G compact we will see that χ determines Π.

$\chi(g) = \sum_i \langle \Pi(g)v_i, v_i\rangle = \sum_i \overline{\langle \Pi(g)^{-1}v_i, v_i\rangle} = \overline{\chi(g^{-1})}$, $g \in G$. $\chi(g^{-1}) = \overline{\chi(g)}$.

A [illegible] $|\chi(g)|^2 = 1$ [illegible] $\chi : G \longrightarrow$ [illegible]

Differential forms: M smooth manifold. p-forms $\Omega^p(M)$ is the space of sections of the p^{th} exterior power of the dual of the tangent bundle of M; $\Omega^p(M) = \Gamma(\Lambda^p T^*M)$, $\Omega^0(M) = C^\infty(M)$ the smooth functions on M. Exterior derivative d

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \longrightarrow \cdots\cdots \longrightarrow \Omega^n(M) \xrightarrow{d} 0$$

$n = \dim M$. $d^2 = 0$ and $df(\xi) = \xi f$, $f \in C^\infty(M)$, $\xi \in \mathcal{X}(M)$.

eg $M = \mathbb{R}^3$

$$C^\infty(\mathbb{R}^3) \xrightarrow{\text{grad}} C^\infty(\mathbb{R}^3)\otimes\mathbb{R}^3 \xrightarrow{\text{curl}} C^\infty(\mathbb{R}^3)\otimes\mathbb{R}^3 \xrightarrow{\text{div}} C^\infty(\mathbb{R}^3)$$

$$\text{curl}(\text{grad}) = 0 \quad , \quad \text{div}\circ\text{curl} = 0 .$$

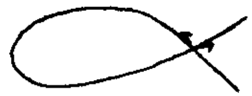
M is orientable if there exists an n-form without zeros. (rank $\Lambda^n T^*M = 1$) If M is connected, oriented and has a Riemannian metric then there exists a preferred n-form, namely the oriented n-form with unit length, called the Riemannian volume form vol. A 2-dim surface being oriented corresponds to there being a smooth map $M \longrightarrow \mathbb{R}^3$ which gives a unit outward normal vector at each point of M eg the 2-sphere is orientable, the Möbius band is not.

M connected, orientable. Can define integration of n-forms over M

$$\int_M \beta \quad , \quad \beta \in \Omega^n(M)$$

Differential forms:

Vector fields do not behave well with respect to smooth maps ie if $f: M \longrightarrow N$ we don't have a natural way of defining a corresponding smooth map from the vector fields of one manifold to the other eg in $\mathbb{R}^3$ the simple curve has



no well defined tangent vector at the intersection. This motivates passing to the dual of the tangent bundle.

A p-form on M is a smooth section of the pth exterior power of the dual of the tangent bundle of M, the 0-forms are the smooth functions on M.

Let (U, ϕ) be a chart on M with $x \in U$ and coordinate functions $(x^1, \ldots, x^n)$ $n = \dim M$. $x^i \in C^\infty(U)$ gives $dx^i \in \Omega^1(U)$ where $(dx^1, \ldots, dx^n)$ is dual to the basis $\left(\frac{\partial}{\partial x^1}\Big|_x, \ldots\ldots, \frac{\partial}{\partial x^n}\Big|_x\right)$ of $T_x M$, and each p-form may be expressed locally.

A p-form is a smooth map β from $M \longrightarrow \Lambda^p T^*M$ such that $\beta(x)$ is a linear map from $\Lambda^p T_x M$ into $\mathbb{R}$ each x. Then with $f_*: TM \longrightarrow TN$ the differential, we have the pull-back $f^*: \Omega^p(N) \longrightarrow \Omega^p(M)$ defined by if $\beta \in \Omega^p(N)$,

$$(f^*\beta)(x)(X_1, \ldots, X_p) = \beta f(x)(f_* X_1, \ldots\ldots, f_* X_p), \; p \geq 1$$

$X_1, \ldots, X_p \in \mathcal{X}(M)$. Also define for

$$h \in \Omega^0(N) \quad , \quad f^* h = h \circ f \in \Omega^0(M)$$

The exterior derivative $d: \Lambda^p(M) \longrightarrow \Lambda^{p+1}(M)$ satisfies (i) $df(\xi) = \xi f$, $\xi \in \mathfrak{X}(M)$ (ii) $d(\beta\wedge\gamma) = d\beta\wedge\gamma + (-1)^p\beta\wedge d\gamma$ (iii) $d^2=0$

d is unique satisfying (i), (ii), (iii).

Now $d\Lambda^{p-1}(M) \subseteq \operatorname{Ker} d$ and define $Z^p(M) = \operatorname{Ker} d$ the closed p-forms, $B^p(M) = d\Lambda^{p-1}(M)$ the exact forms. The de Rham cohomology $H^p_{dR}(M, \mathbb{R}) = Z^p(M)/B^p(M)$ only depends on the topology of M.

G a Lie group of dimension n. There are n linearly independent left invariant forms $\omega_1, \ldots, \omega_n$ (Maurer-Cartan forms) for take dual basis in $\mathfrak{g}^*$. Any left invariant p-form is obtained from these eg basis for 2-forms is

$\omega_1\wedge\omega_2, \ldots, \omega_{n-1}\wedge\omega_n$ dimension is $\binom{n}{p}$.

Haar measure: A Lie group G is orientable for: let ω be a non-zero ~~alt~~ n form on $\mathfrak{g}$ ($n = \dim G$) ie an element of $\Lambda^n\mathfrak{g}^*$; then we get an n-form on G by

$$\omega_g(\xi_1(g), \ldots, \xi_n(g)) = \omega(\xi_1, \ldots, \xi_n) \quad g\in G,\ \xi_1,\ldots,\xi_n \in \mathfrak{g}.$$

Moreover ω is left invariant ie $l_g^*\omega = \omega$ for all $g\in G$;

($f: M \longrightarrow N$, tangent map $f_*: TM \longrightarrow TN$ and $f^*: \Lambda^p(N) \longrightarrow \Lambda^p(M)$ called the pull-back ie if $\beta \in \Lambda^p(N)$,

$$(f^*\beta)(x)(X_1(x), \ldots, X_p(x)) = \beta(f(x))(f_*X_1(x), \ldots, f_*X_p(x)),\quad X_1,\ldots,X_p \in \mathfrak{X}(M), x\in M)$$

$$l_g^*\omega(h)(\xi_1(h), \ldots, \xi_n(h)) = \omega(gh)(l_{g*}\xi_1(h), \ldots, l_{g*}\xi_n(h))$$
$$= \omega(gh)(\xi_1(gh), \ldots, \xi_n(gh)) = \omega(\xi_1, \ldots, \xi_n).$$

If ω_1 is an n-form on G then $\omega_1 = c\omega$, c a fn on G and if ω_1 is left invariant then c is a constant. Let r_g, $g\in G$ be the right translations on G; then

$$r_g^*\omega = \delta(g)^{-1}\omega \text{ for a homomorphism } \delta: G \longrightarrow \mathbb{R}_*, \quad \delta(g) = \det(\mathrm{Ad}\, g)$$

Write $\int_G f\omega = \int_G f(x)\,dx$, dx the Haar measure, $f \in C_c(G)$. Then the change of variables formula ($\int_M f\phi^*\omega = \int_M (f\circ\phi^{-1})\omega$, ϕ a diffeo on M) implies that

$$\int_G f(xg)\,dx = \int_G f(x)\,|\delta(g)|\,dx$$

$$\int_G f(x^{-1})\,dx = \int_G f(x)\,|\delta(x)|^{-1}\,dx$$

G is called unimodular if $|\delta(g)| = 1$ for all $g \in G$ (if G connected $\delta(G) = 1$) ie ω is both left and right invariant. eg if G/G', G' the derived group, is compact then G is unimodular (any continuous homomorphism from compact G into $\mathbb{R}_*$ is trivial; if $\delta(h) \neq 1$, can assume $|\delta(h)| > 1$ but $|\delta(h^n)|$ is unbounded) thus a compact Lie group is unimodular and so is G connected, semi-simple. The group of rigid motions of $\mathbb{R}^3$; the semi-direct product of the translations in $\mathbb{R}^3$ and $SO(3)$, $\begin{pmatrix} A & x \\ 0 & 1 \end{pmatrix}$, is not unimodular, nor is the affine group of $\mathbb{R}$, $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, (show that the space of left invariant 1-forms is not equal to the space of right invariant 1-forms).

$\dim G = n$, n linearly independent left invariant 1-forms (Maurer-Cartan forms) $\omega_1, \ldots\ldots, \omega_n$. The exterior products of k of these give the left invariant k-forms, dim is $\binom{n}{k}$.

Matrix group G. $\quad G \xrightarrow{\quad X \quad} GL(n, \mathbb{R})$.

dX is a matrix of 1-forms on G; $\Omega = X^{-1}dX$ is a matrix of left invariant 1-forms [illegible] spans the space of left invariant 1-forms sim $dX \cdot X^{-1}$ for right invariant 1-forms.