INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392 - I

SMR/161 - 10

COLLEGE ON
REPRESENTATION THEORY OF LIE GROUPS
(4 November - 6 December 1985)

BASIC NOTIONS FOR
LIE GROUPS II & III

S. GUTT
Universite Libre de Bruxelles
Bruxelles, Belgium

---

1. The Lie algebra of a Lie group.

A. Definitions

<u>Definition 1</u> Let G be a group which has the structure of a smooth n-dimensional manifold. Then G is said to be a <u>Lie group of dimension</u> n if

$$(i)\ G\times G\longrightarrow G\ (g,h)\longrightarrow gh$$

$$(ii)\ G\longrightarrow G : g\to g^{-1}$$

are smooth maps.

<u>Examples</u> (i) A 0-dimensional Lie group is a discrete group.

(ii) $(\mathbb{R}^n,+)$ is a n-dimensional <u>abelian</u> Lie group

(iii) $S^1\subset\mathbb{R}^2=\mathbb{C}=\{e^{i\theta}\mid\theta\in\mathbb{R}\}$ with the multiplication $e^{i\theta}\cdot e^{i\theta'}=e^{i(\theta+\theta')}$ is a 1-dimensional Lie group.

(iv) $T^m=S^1\times\dots\times S^1$ = direct product of m groups S^1 is a m-dimensional abelian Lie group

(v) $GL(n,\mathbb{R})=\{A\in\mathbb{R}^{n^2}\mid\det A\neq 0\}$ is a n^2 dimensional Lie group. Similarly $GL(n,\mathbb{C})=\{A\in\mathbb{C}^{n^2}\mid\det A\neq 0\}$

is a $2m^2$ dimensional Lie group.

(vi) Let $H_m = \mathbb{R}^{2m} \times \mathbb{R} = \mathbb{R}^{2m+1}$; assume we are given a nondegenerate, antisymmetric, bilinear form F on $\mathbb{R}^{2m}$. Then define a group structure by

$$(x, t)(x', t') = (x + x', t + t' + \tfrac{1}{2} F(x, x'))$$

with $x \in \mathbb{R}^{2m}$, $t \in \mathbb{R}$. It is a $(2m+1)$ dimensional Lie group called the <u>Heisenberg group</u>.

<u>Remarks</u> (i) Conditions (i) and (ii) of definition 1 can be replaced by smoothness of the map:

$$G \times G \longrightarrow G \quad (g, h) \longrightarrow g h^{-1}$$

(ii) The map $G \longrightarrow G : g \to g^{-1}$ is a diffeomorphism

(iii) The <u>left translation</u> $L_g : G \to G : h \longrightarrow gh$ and the <u>right translation</u> $R_g : G \to G : h \to hg$ are diffeomorphisms of G. All left translations commute with all right translations. The map $L_g \circ R_{g^{-1}}$ is <u>an automorphism of</u> G and the map $G \longrightarrow \mathrm{Aut}\, G$ (= automorphism group of G) $g \longrightarrow L_g \circ R_{g^{-1}}$ is a group homomorphism.

(iv) The <u>connected component of the identity</u> G^0 of G is a group and an open (and closed) submanifold of G, hence a Lie group.

(v) If G_i $(i = 1, 2)$ are 2 Lie groups, then their <u>direct product</u> $G_1 \times G_2$ is also a Lie group.

B. <u>The Lie algebra of a Lie group.</u>

<u>Definition</u> 1. A vector field $\tilde{X}$ on G is said to be <u>left invariant</u> if, for all $g \in G$, $L_{g*} \tilde{X} = \tilde{X}$. Such a vector field is necessarily a smooth vector field. We shall denote by $\mathcal{G}$ the subspace of $\mathcal{X}(G)$ consisting of left invariant vector fields.

<u>Lemma</u> 1. (i) The map $\mathcal{G} \longrightarrow (TG)_e : \tilde{X} \longrightarrow \tilde{X}_e$ (where e is the neutral element of G) is a linear isomorphism.

(ii) If $\tilde{X}, \tilde{Y} \in \mathcal{G}$, then $[\tilde{X}, \tilde{Y}] \in \mathcal{G}$.

<u>Proof</u> (i) If $\tilde{X}_e = 0$, then $\tilde{X}_g = L_{g*} \tilde{X}_e = 0$ and the map is injective; it is clearly surjective and linear.

(ii) Use proposition (0.E.1) observing that

$$L_{g*}[\tilde X, \tilde Y] = [L_{g*}\tilde X, L_{g*}\tilde Y] = [\tilde X, \tilde Y]$$

Definition 2 The Lie algebra $\mathcal{G}$ ~~of a Lie group~~ G is the subalgebra of $\mathcal{X}(G)$ of left invariant vector fields; as in $\mathcal{X}(G)$ the Lie bracket will be denoted $[\tilde X, \tilde Y]$.

Remarks ① Equivalently one can define the Lie algebra $\mathcal{G}$ of G as the tangent space to the neutral element, $(TG)_e$ with the multiplication:

$$(TG)_e \times (TG)_e \longrightarrow (TG)_e : (X, Y) \longrightarrow [\tilde X, \tilde Y]_e \underset{\text{def}}{=} [X, Y]$$

where $\tilde X$ (resp $\tilde Y$) is the unique left invariant vector field, whose value at e is X (resp Y)

② Let $\tilde X$ be a left invariant vector field on G. This vector field is complete. Indeed let $\gamma(t)$ be the unique maximal solution of the Cauchy problem relative to $(\tilde X, e, 0)$. (0,E,5). Assume γ to be defined on $I \subsetneq \mathbb{R}$; let $t_1 \neq 0$ belong to I. The curve $\tilde\gamma$:

$$\tilde\gamma(t) = \begin{cases} \gamma(t) & \text{if } t \in I \\ \gamma(t_1)\gamma(t - t_1) & \text{if } t \in I + t_1 \end{cases}$$

is a solution of the Cauchy problem relative to $(\tilde X, e, 0)$ defined on $I \cup (I + t_1)$. By maximality $I \cup (I + t_1) \subset I$, whatever $t_1 \in I$. Hence a contradiction and $I = \mathbb{R}$. If $g \in G$, the curve $g\gamma(t)$ is a solution of the Cauchy problem relative to $(\tilde X, g, 0)$ and defined on $\mathbb{R}$; hence it is the unique maximal solution of the Cauchy problem relative to $(\tilde X, g, 0)$. This proves completeness.

Let $\psi_t^{\tilde X}$ denote the one parametric group of diffeomorphisms of G associated to $\tilde X$. Then:

$$\psi_t^{\tilde X} = R_{\gamma(t)}$$

where $\gamma(t) = \psi_t^{\tilde X}(e)$. In particular:

$$\gamma(t)\gamma(s) = \gamma(t+s)$$

③ Let $X, Y \in (TG)_e$. Then:

$$[X, Y] = \frac{d}{dt}\left(L_{\gamma(t)*} R^{-1}_{\gamma(t)*} Y\right)_{t=0}$$

Indeed:

$$[X, Y] = [\tilde X, \tilde Y]_e = \frac{d}{dt}\left(\psi^{\tilde X}_{-t*} \tilde Y_{\psi_t^{\tilde X}(e)}\right)(e) = \frac{d}{dt}\left(R^{-1}_{\gamma(t)*} L_{\gamma(t)*} Y\right)(e)$$

using (0,E,6).

④ A Lie algebra $\mathcal{G}$ over a field $\mathbb{K}$ of characteristic $\neq 2$ is a $\mathbb{K}$ vector space with a bilinear map

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G} : (X, Y) \longrightarrow [X, Y]$$

such that (i) $[X,Y] = -[Y,X]$ and (ii) $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$.

<u>Examples</u> (i) The Lie algebra of $(\mathbb{R}^n, +)$ is isomorphic to $\mathbb{R}^n$. The Lie bracket of any 2 elements vanishes.

(ii) The Lie algebra of S^1 is isomorphic to $\mathbb{R}$.

(iii) The tangent space to $GL(m,\mathbb{R})$ at I is isomorphic to $\mathbb{R}^{m^2}$; if $A \in (T\,GL(m,\mathbb{R}))_I$, one has

$$\tilde{A}_B = BA$$

Using the expression of the Lie bracket of vector fields in a coordinate system (0,E,1):

$$[\tilde{A}, \tilde{A}']_I = AA' - A'A$$

and thus $\mathfrak{gl}(m,\mathbb{R}) \underset{\text{not}}{=}$ Lie algebra of $GL(m,\mathbb{R}) = \mathbb{R}^{m^2}$ with a bracket which is the usual commutator of matrices

(iv) The tangent space to H_m at $(0,0)$ is isomorphic to $\mathbb{R}^{2m+1}$; if $(y,a) \in (TH_m)_{(0,0)}$ $(y \in \mathbb{R}^{2m}, a \in \mathbb{R})$:

$$\widetilde{(y,a)}_{(z,t)} = \left(y, a + \tfrac{1}{2} F(z,y)\right)_{(z,t)}$$

and thus, as in example (iii),

$$\left[\widetilde{(y,a)}, \widetilde{(y',a')}\right]_{(0,0)} = F(y,y')$$

C. <u>Homomorphisms</u>

<u>Definition 1</u> A linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ of the Lie algebra $\mathfrak{g}_1$ over $\mathbb{K}$ into the Lie algebra $\mathfrak{g}_2$ over $\mathbb{K}$ is a <u>Lie algebra homomorphism</u> if $\phi([X,Y]) = [\phi(X), \phi(Y)]$. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathfrak{g}_2 = \mathfrak{gl}(m,\mathbb{K}) = \mathrm{End}(\mathbb{K}^m)$, then ϕ is called a <u>representation of $\mathfrak{g}_1$ on</u> $\mathbb{K}^m$.

<u>Definition 2</u> A group homomorphism $\varphi : G_1 \to G_2$ which is smooth is called a <u>Lie homomorphism</u>. If $G_2 = GL(m,\mathbb{K})$, then ϕ is called a <u>representation of G_1 on</u> $\mathbb{K}^m$

<u>Proposition 1.</u> Let $\varphi : G \to H$ be a Lie homomorphism and let $\mathfrak{g}$ (resp $\mathcal{H}$) be the Lie algebra of G (resp H). Then φ_* induces a Lie algebra homomorphism $\mathfrak{g} \to \mathcal{H}$.

<u>Proof</u> Let $\tilde{X}$ be a left invariant vector field on G. Then:

$$\varphi_{*g} \tilde{X} = \varphi_{*g} L_{g*} \tilde{X}_e = (\varphi \circ L_g)_{*e} \tilde{X}_e$$
$$= (L_{\varphi(g)} \circ \varphi)_{*e} \tilde{X}_e = L_{\varphi(g)*e'} (\varphi_{*e} \tilde{X}_e) \quad (e' = \text{neutral element of } H$$

Hence $\tilde{X}$ and $\widetilde{(\varphi_{*e} X_e)}$ are "φ-related". The conclusion follows from (0,E,1).

<u>Definition</u> 3 The automorphism of G corresponding to $L_g \circ R_{g^{-1}}$ induces, by its differential $L_{g*} \circ R_{g^{-1}*}$ a Lie algebra automorphism $\mathfrak{g} \to \mathfrak{g}$ denoted $Ad\, g$. Clearly:

$$Ad(gh) = Ad\, g\; Ad\, h$$

and thus we have an homomorphism $Ad : G \to Aut\, \mathfrak{g}$ (= automorphism group of $\mathfrak{g}$). If we look at Ad as taking its values in $GL(\mathfrak{g})$, one sees that it is a smooth map and thus defines a representation of G on $\mathfrak{g}$, called the <u>adjoint representation</u>

D. <u>Subgroups and subalgebras</u>

<u>Definition</u> 1 Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, if A, B are subsets of $\mathfrak{g}$, we denote by $[A, B]$ the vector space generated by $\{[X, Y] \mid X \in A, Y \in B\}$. A <u>subalgebra</u> B of $\mathfrak{g}$ is a subspace such that $[B, B] \subset B$; an <u>ideal</u> C of $\mathfrak{g}$ is a subspace such that $[C, \mathfrak{g}] \subset C$.

<u>Definition</u> 2 Let H, G be Lie groups; let $i : H \to G$ be a group homomorphism such that (H, i) is a smooth submanifold of G. Then (H, i) is called a <u>Lie subgroup of</u> G

<u>Proposition</u> 1 Let (H, i) be a Lie subgroup of G; let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp H). Then $\mathfrak{h} (= i_* \mathfrak{h})$ is a subalgebra of $\mathfrak{g}$

<u>Proof</u> $i : H \to G$ is a Lie group homomorphism, so i_* is a Lie algebra homomorphism (1.C.1), thus $i_* \mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ $\square$

<u>Proposition</u> 2 Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then there exists a unique connected Lie subgroup (H, i) of G with algebra $\mathfrak{h}$.

<u>Proof</u> . Assume $\dim G = m$, $\dim \mathfrak{h} = p$ $(\leq m)$. Let $\tilde{\mathfrak{h}}$ be the p-dimensional distribution on G defined by $\tilde{\mathfrak{h}}_g = L_{g*} \mathfrak{h}$.

- The distribution $\tilde{\mathfrak{h}}$ is smooth and integrable. Indeed let X_i $(i \leq p)$ be a basis of $\mathfrak{h}$ and $\tilde{X}_i$ $(i \leq p)$ be the corresponding left invariant smooth vector fields on G. As $\tilde{\mathfrak{h}}_g = \langle \tilde{X}_{i_g} ; i \leq p \rangle$ the distribution is smooth. Let X, Y be smooth vector fields on G which belongs to $\tilde{\mathfrak{h}}$; then there exist smooth functions a_i, b_i $(i \leq p)$ on G such that $X = \sum_{i \leq p} a_i \tilde{X}_i$; $Y = \sum_{i \leq p} b_i \tilde{X}_i$. Then

$$[X, Y] = \sum_{i,j} \{ a_i b_j [\tilde{X}_i, \tilde{X}_j] + a_i (\tilde{X}_i(b_j)) \tilde{X}_j - b_j (\tilde{X}_j(a_i)) \tilde{X}_i$$

so $[X, Y]$ belongs to $\tilde{\mathfrak{h}}$ and the distribution is involutive.

By O.F.1 there exists a unique maximal connected integral manifold (H, i) of $\tilde{\mathfrak{h}}$ containing e

• Let us show that $i(H)$ is a subgroup of G. Let $h \in i(H)$; then $(H, L_{h^{-1}} \circ i)$ is a connected integral manifold of $\mathfrak{H}$ containing e, thus $(L_{h^{-1}} \circ i)(H) \subset i(H)$, hence $\forall\, h' \in i(H)$, $h^{-1}h' \in i(H)$ and $i(H)$ is a subgroup of G. (One puts the group structure on H so that i is a group homomorphism so H is a group)

• To show that H is a Lie group, let $\mu : G \times G \to G : (g,h) \to gh^{-1}$. The map $\mu \circ (i \times i) : H \times H \to G$ is smooth and its image is contained in $i(H)$. Hence by (0.F.2) the induced map $H \times H \to H$ is smooth.

This concludes the proof that (H, i) is a connected Lie subgroup of G, with Lie algebra $\mathfrak{H}$.

To prove uniqueness, let (H', i') be another connected Lie subgroup of G with algebra $\mathfrak{H}$; it is also a connected integral manifold of $\mathfrak{H}$ containing e, so $i'(H') \subset i(H)$. Thus, by (0.F.2) the map $\eta\, H' \to H$, so that $i \circ \eta = i'$ which is a group homomorphism, is smooth. Thus H' is a Lie subgroup of H. As H and H' have the same dimension, $\eta(H')$ is an open - and thus closed - subgroup of H. H being connected, $\eta(H') = H$ and η is a Lie group isomorphism. □

__Corollary__ There is a bijective correspondance between connected Lie subgroups of a Lie group G and Lie subalgebras of its Lie algebra $\mathfrak{g}$.

__Proposition 3__ Let H be a subgroup of the Lie group G; assume that (H, i) is a submanifold of G. Then (H, i) is a Lie subgroup of G and the manifold structure on H is unique.

__Remark__. Proposition 3 shows that it is not ambiguous to say that a subset H of a Lie group G is a Lie subgroup of G; it means that H is an abstract subgroup of G and H has a manifold structure (hence a unique one) making (H, i) into a Lie subgroup of G.

• For a __proof__ of proposition 3, see F. Warner: "Foundations of differentiable manifolds and Lie groups". Theorem 3.20, page 95.

__Proposition 4__ Let (H, i) be a Lie subgroup of G. Then i is an embedding if and only if H is a closed subgroup of G.

• For a __proof__ see F. Warner, same ref. as above, theorem 3.21, page 97.

2. The exponential map

A One-parameter subgroups

Definition 1 A one-parameter subgroup of a Lie group G, is a Lie homomorphism $\varphi : \mathbb{R} \to G$ of the additive group of real numbers into G.

Thus φ is a smooth curve in G through e and $\varphi(s) \cdot \varphi(t) = \varphi(s+t)$

Proposition 1 Let G be a Lie group and let $\tilde{X} \in \mathfrak{g}$. Then, there exists a unique one parameter subgroup $\varphi_{\tilde{X}}$ of G such that $L_\varphi = \varphi_{\tilde{X} * (0)}\left(\frac{\partial}{\partial t}\right) = \tilde{X}_e$.

Furthermore the map $\mathfrak{g} \times \mathbb{R} \to G \quad (\tilde{X}, t) \longrightarrow \varphi_{\tilde{X}}(t)$ is smooth.

Proof. By 1.B remark 2, the unique maximal solution of the Cauchy problem relative to $(\tilde{X}, e, 0)$ is a one parameter subgroup of G. Such a subgroup is unique because it must be a solution of the Cauchy problem. Indeed, as $\varphi_{\tilde{X}}(t) \cdot \varphi_{\tilde{X}}(s) = \varphi_{\tilde{X}}(t+s)$, one has

$$\begin{cases} \varphi_{\tilde{X}}(0) = e \\ \varphi_{\tilde{X} *_t}\left(\frac{\partial}{\partial t}\right) = L_{\varphi_{\tilde{X}}(t) *} \; \varphi_{\tilde{X} * (0)}\left(\frac{\partial}{\partial t}\right) = \tilde{X}_{\varphi_{\tilde{X}}(t)} \end{cases}$$

. Consider the manifold $G \times \mathfrak{g}$ and the C^∞ vector field Y on $G \times \mathfrak{g}$:

$$Y_{(g,X)} = \tilde{X}_g \oplus 0_X$$

This vector field is complete, indeed the maximal solution of the Cauchy problem relative to $(Y, (g,X), 0)$ is. $\psi^Y_t(g,X) = (\varphi^{\tilde{X}}_t(g), X)$, defined $\forall t \in \mathbb{R}$.

The map $\psi \; \mathbb{R} \times G \times \mathfrak{g} \to G \times \mathfrak{g} \quad (t, g, X) \to \psi^Y_t(g,X)$ is C^∞. (q.o.E)

Let $p : G \times \mathfrak{g} \to G$ be the canonical projection. Then $p \circ \psi|_{\mathbb{R} \times \{e\} \times \mathfrak{g}} : \mathbb{R} \times \mathfrak{g} \to G$ is C^∞ □

B. Exponential map.

Definition 1 Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and let $\tilde{X} \in \mathfrak{g}$. Let $\varphi_{\tilde{X}}$ be the unique one parameter group of G such that $L_\varphi = \tilde{X}_e$. Then the exponential map, $\exp : \mathfrak{g} \longrightarrow G : \tilde{X}$

$\to \varphi_{\tilde{X}}(1) \underset{not}{=} \exp X \quad$ (where $X \underset{not}{=} \tilde{X}_e$)

Lemma 1. $\exp tX = \varphi_{\tilde{X}}(t)$

Proof Consider the curve $s \longrightarrow \varphi_{\tilde{X}}(st) \underset{not}{=} \psi(s)$, it is such that $\psi(0) = e$ and:

$$\psi_{*u}\left(\frac{\partial}{\partial s}\right) = t\, \varphi_{\tilde{X} * tu}\left(\frac{\partial}{\partial s}\right) = t\, \tilde{X}_{\psi(u)} = \widetilde{(tX)}_{\psi(u)}$$

Hence $\psi(s)$ is the unique maximal solution of the Cauchy problem relative to $(\widetilde{tX}, e, 0)$ and thus

$$\varphi_{t\tilde{X}}(u) = \varphi_{\tilde{X}}(tu)$$

which proves the point.

Corollary (i) $\exp tX \cdot \exp t'X = \exp(t+t')X$

(ii) $(\exp tX)^{-1} = \exp -tX$

Proposition 1 The exponential map $\mathfrak{g} \to G$ is smooth and there exists a neighborhood ω of 0 in $\mathfrak{g}$ and a neighborhood ω' of e in G such that $\exp : \omega \to \omega'$ is a diffeomorphism. Notice also

Proof The first part is proven in (2 A 1).

that:

$$\exp_{*0} X = \frac{d}{dt}(\exp tX)_{t=0} = X$$

and thus $\exp_{*0} = id$ (if one identifies $\mathfrak{g}$ and $(TG)_e$).

<u>Proposition</u> 2 (for explanation of notations see 2.C). Let $\Omega = \{X \in \mathfrak{g} \mid$ each eigenvalue of ad X has a module smaller than $\pi\}$. Then Ω is an open neighborhood of 0 in $\mathfrak{g}$ containing the center of $\mathfrak{g}$, which is stable by Ad G. If G is simply connected then exp is a diffeomorphism of Ω onto its image.
For a <u>proof</u> see Varadarajan "Lie groups, Lie algebras and their representation" p. 112 and 113, thm 2.14.6.

<u>Definition</u> 2. If U is a neighborhood of e in G such that that there exists a neighborhood V of 0 in $\mathfrak{g}$ such that $\exp : V \longrightarrow U$ is a diffeomorphism, $(U, \exp^{-1})$ is a chart of G called a <u>logarithmic</u> (or <u>canonical</u>) chart. If one chooses a basis $X_1, \dots X_m$ of $\mathfrak{g}$ $(= (TG)_e)$ one has

$$\exp^{-1} : U \longrightarrow \mathbb{R}^m : g = \exp\left(\sum_i x_i X_i\right) \longrightarrow (x_1, \dots x_m)$$

<u>Example</u>. Let G $GL(m, \mathbb{R})$; then if $S \in \mathfrak{gl}(m, \mathbb{R})$

$$\exp tS = e^{tS} = \sum_{k=0}^{\infty} \frac{t^k S^k}{k!}$$

as can be seen by solving the differential equation

$$\frac{d\gamma}{dt}(t) = \gamma(t) S$$

C. The Campbell - Baker - Hausdorff formula

This theorem gives the formula for multiplication in a Lie group in terms of the logarithmic coordinates on the Lie group.

<u>Lemma</u> 1 If $g(t)$ is any C^∞ curve in G through e representing $X \in TG_e$ (i.e $g_{*0}\left(\frac{\partial}{\partial t}\right) = X$) then $[X, Y] = \frac{d}{dt} L_{g(t)*} R_{(g(t))^{-1}*} Y$ for all $Y \in TG$

<u>Proof</u>. • If $g(t)$ represents $X \in TG_e$ and $g'(t)$ represents X' then $g(t) \cdot g'(t)$ represents $X + X'$

indeed $\frac{d}{dt} f(g(t) g'(t))\big|_{t=0} = \frac{d}{dt} f(g(t))\big|_{t=0} + \frac{d}{dt} f(g'(t))\big|_{t=0}$

• In particular $(g(t))^{-1}$ represents $-X$

• We have $\left(\frac{d}{dt} L_{g(t)*} R_{(g(t))^{-1}*} Y\right)(f) = \frac{\partial}{\partial t} \frac{\partial}{\partial s} f\left(g(t) h(s) (g(t))^{-1}\right)\big|_{t=s=0}$

if $h(s)$ represents Y, and f is a smooth function on G.

. On the other hand, as $(\tilde{Y} f)(g) = \frac{d}{ds} f(g h(s))\big|_{s=0}$, $(-\tilde{X} f)(h) = \frac{d}{dt} f(h g(t)^{-1})$

$$[X, Y] f = [\tilde{X}, \tilde{Y}]_e f = \tilde{X}_e(\tilde{Y} f) - \tilde{Y}_e(\tilde{X} f)$$

$$= \frac{\partial}{\partial t} \frac{\partial}{\partial s} f(g(t) h(s))\big|_{s=t=0} + \frac{\partial}{\partial t} \frac{\partial}{\partial s} f(h(s) g(t)^{-1})\big|_{t=s=0}$$

□

<u>Definition</u> Let $\mathfrak{g}$ be a Lie algebra, $X \in \mathfrak{g}$. We define ad X as the linear transformation of $\mathfrak{g}$ defined by $\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ $Y \to [X, Y]$

<u>Remark</u> By Jacobi's identity, the map $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ $X \to \operatorname{ad} X$ is a Lie algebra homomorphism

Lemma 2 Let $\mathfrak{g}$ be the Lie algebra of a Lie group G and let $X, Y \in \mathfrak{g}$. Then

$$\mathrm{Ad}(\exp X) \cdot Y = (\mathrm{Exp}(\mathrm{ad}\, X)) \cdot Y$$

where Exp is the exponential of an endomorphism i.e.

$$\mathrm{Exp}(\mathrm{ad}\, X) = \mathrm{Id} + \mathrm{ad}\, X + \frac{(\mathrm{ad}\, X)^2}{2!} + \cdots + \frac{(\mathrm{ad}\, X)^n}{n!} + \cdots$$

Proof. Let $\mu(t) = \mathrm{Ad}(\exp tX) \cdot Y$; it is a C^∞ curve in $\mathfrak{g} \simeq TG_e$ through Y. As $\mathrm{Ad}(h) = L_{h*} \circ R_{h^{-1}*} : TG_e \simeq \mathfrak{g} \to TG_e \simeq \mathfrak{g}$, and as $\exp tX$ is a C^∞ curve in G through e representing X, we know by lemma 1 that $[X, Y] = \frac{d}{dt}(\mathrm{Ad} \exp tX\, Y)\big|_{t=0}$

Hence as $\mu(t + t_0) = \mathrm{Ad}(\exp tX)\, \mu(t_0)$ we have

$$\frac{d}{dt}\mu(t)\Big|_s = [X, \mu(s)] \qquad \mu(0) = Y$$

This differential equation in $\mathfrak{g}$ has the solution

$$\mu(t) = Y + t\, \mathrm{ad}\, X\, Y + \cdots + \frac{t^n (\mathrm{ad}\, X)^n}{n!} Y$$

we obtain the result for $t = 1$ □

Remark In particular we have seen that

$$\mathrm{Ad} : G \to Gl(\mathfrak{g}) \quad h \mapsto \mathrm{Ad}\, h \quad \text{is a Lie group}$$

homomorphism so

$$\mathrm{Ad}_{*_e} : \mathfrak{g} \simeq TG_e \to \mathrm{End}(\mathfrak{g}) \quad \text{is a Lie algebra homomorphism}$$

and $\mathrm{Ad}_* = \mathrm{ad}$

Proposition Let $Y(t)$ be a C^∞ curve in the Lie algebra $\mathfrak{g}$ of a Lie group G.
Let $\varphi(z) = \frac{e^z - 1}{z} = 1 + \frac{1}{2!} z + \frac{1}{3!} z^2 + \cdots$

Then

$$\frac{d}{dt} \exp Y(t)\Big|_t = L_{\exp Y(t) *}\ \varphi(-\mathrm{ad}\, Y(t))\ (Y'(t))$$

where the lefthand side represents the element of $TG_{\exp Y(t)}$ associated to the curve $s \to \exp Y(t+s)$ (cf notations given in 0.C) and $Y'(t)$ is the element of $T\mathfrak{g}_{Y(t)} \simeq \mathfrak{g}$ associated to the curve $s \to Y(t+s)$.

In other words, the differential of the exponential map is given by:

$$\exp_{*X} = L_{\exp X *} \circ \frac{1 - e^{-\mathrm{ad}\, X}}{\mathrm{ad}\, X} \qquad \text{for all } X \in \mathfrak{g}$$

where $\mathfrak{g}$ is identified with the tangent space $T\mathfrak{g}_X$.

Proof. • Consider $g(s,t) = \exp s Y(t)$

Let $\alpha(s,t) = L_{g^{-1}(s,t)*} \frac{\partial}{\partial s} g(s,t) = Y(t)$ where, as above, $\frac{\partial}{\partial s} g(s,t)$ is the element of $TG_{g(s,t)}$ associated with the curve $u \to g(s+u, t)$.

Let $\beta(s,t) = L_{g^{-1}(s,t)*} \frac{\partial}{\partial t} g(s,t) = L_{\exp -sY(t)*} \frac{\partial}{\partial t} \exp s Y(t)$.

Then $\frac{\partial \beta}{\partial s} - \frac{\partial \alpha}{\partial t} + [\alpha, \beta] = 0$ (in the vector space $\mathfrak{g} \simeq TG_e$)

Indeed a curve representing $\alpha(s,t)$ is given by $u \to \exp u Y(t)$, one representing β is $v \to \exp -s Y(t) \exp s Y(t+v)$; hence

$$[\alpha(s,t), \beta(s,t)](f) = \frac{\partial}{\partial u}\frac{\partial}{\partial v} f(\exp u Y(t) \exp -sY(t) \exp sY(t+v) \exp -uY(t))\Big|_{u=v=0}$$

$$\left(-\frac{\partial \beta}{\partial s}(s,t)\right)(f) = \frac{\partial}{\partial u}\frac{\partial}{\partial v} f(\exp(-s+u)Y(t) \exp(s-u)Y(t+v))\Big|_{u=v=0}$$

$$\left(\frac{\partial \alpha}{\partial t}\right)(s,t)(f) = \frac{\partial}{\partial u}\frac{\partial}{\partial v} f(\exp u Y(t+v))\Big|_{u=v=0}$$

So, if one fixes t, set $Y = Y(t)$ and $Y' = \frac{d}{dt} Y(t)\big|_t$

$$\frac{d\beta}{ds} = -\mathrm{ad}\, Y\, \beta + Y' \quad \text{and} \quad \beta(0) = 0$$

This equation may be solved explicitely in terms of an everywhere convergent power series in s

$$\beta(s,t) = sY' + \frac{s^2}{2}(-\operatorname{ad}Y)Y' + \cdots + \frac{s^n}{n!}(-\operatorname{ad}Y)^{n-1}Y' + \cdots$$

Putting $s=1$ one gets

$$L_{\exp -Y *}\ \frac{d}{dt}\exp Y(t)\Big|_t = \left(\frac{e^{-\operatorname{ad}Y}-1}{-\operatorname{ad}Y}\right)(Y') \qquad \square.$$

Theorem (Campbell - Baker - Hausdorff) Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Then, there exists a neighborhood N of 0 in $\mathfrak{g}$ such that

$$\exp X \cdot \exp Y = \exp \mu(X,Y) \qquad \text{for all } X,Y\in N$$

with $\mu(X,Y) = X + \int_0^1 \psi\left[(\operatorname{Exp}\operatorname{ad}X)(\operatorname{Exp} t\operatorname{ad}Y)\right]Y\,dt$

where $\psi(z) = \frac{z\log z}{z-1}$, which is analytic near $z=1$.

Remark $\log z = \sum_{m\geq 1}(-1)^{m+1}\frac{(z-1)^m}{m}$ near $z=1$ and thus

$$\psi(z) = \sum_{m\geq 1}\left(\frac{(-1)^m}{m+1} + \frac{(-1)^{m+1}}{m}\right)(z-1)^m + 1$$

So $\psi\left[(\operatorname{Exp}\operatorname{ad}X)(\operatorname{Exp} t\operatorname{ad}Y)\right] = \psi\left(1 + \operatorname{ad}X + t\operatorname{ad}Y + t\operatorname{ad}X\operatorname{ad}Y + \tfrac12\operatorname{ad}^2X + \tfrac12\operatorname{ad}^2Y + \cdots\right)$

$$= 1 + \tfrac12\left(\operatorname{ad}X + t\operatorname{ad}Y + t\operatorname{ad}X\operatorname{ad}Y + \tfrac12\operatorname{ad}^2X + \tfrac12\operatorname{ad}^2Y\right) + \left(-\tfrac16\right)\left(\operatorname{ad}^2X + t^2\operatorname{ad}^2Y + t\operatorname{ad}X\operatorname{ad}Y + t\operatorname{ad}Y\operatorname{ad}X\right) + \cdots$$

And the first few terms are given by

$$\exp X\ \exp Y = \exp\left(X + Y + \tfrac12[X,Y] + \tfrac1{12}[X,[X,Y]] + \tfrac1{12}[Y,[Y,X]] + \cdots\right)$$

To see a complete expression (by induction) of $\mu(X,Y) = \sum_{n\geq 0} C_n(X,Y)$ where the C_n's are polynomials of degree n on $\mathfrak{g}$, see (for instance) Varadarajan page 115

Example For $G = Gl(m,\mathbb{R})$ $\exp A = 1 + A + \frac{A^2}{2} + \cdots$ and

$$\exp A\cdot\exp B = 1 + A + B + AB + \frac{A^2}{2} + \frac{B^2}{2} + \frac{A^3}{6} + \frac{B^3}{6} + \frac{AB^2}{2} + \frac{A^2B}{2} + \cdots$$

$$= 1 + A + B + \tfrac12(AB+BA) + \tfrac12[A,B] + \cdots = \exp\left(A + B + \tfrac12[A,B] + \cdots\right)$$

Proof. Let U be a neighborhood of e in which the logarithm exists, and V a neighborhood of e such that $VV\subset U$. Let N be a spherical neighborhood of 0 in $\mathfrak{g}$ such that $\exp N\subset V$.

Fix $X,Y\in N\subset\mathfrak{g}=TG_e$. Set $\Gamma(t) = \log(\exp X\cdot\exp tY)$ $\quad 0\leq t\leq 1$

For each $Z\in\mathfrak{g}$ $\quad \operatorname{Ad}\exp\Gamma(t).Z = \operatorname{Ad}\exp X\cdot\operatorname{Ad}\exp tY.Z$

Thus, by lemma 2 $\quad \operatorname{Ad}\exp\Gamma(t)Z = (\operatorname{Exp}\operatorname{ad}X).(\operatorname{Exp} t\operatorname{ad}Y)Z$

Assume X and Y are small enough so that

$$\left|\operatorname{Exp}\operatorname{ad}X.\operatorname{Exp} t\operatorname{ad}Y - 1\right| < 1 \qquad \text{for } 0\leq t\leq 1$$

Then, as matrices $\operatorname{ad}\Gamma(t) = \operatorname{Log}\left[(\operatorname{Exp}\operatorname{ad}X.\operatorname{Exp}\operatorname{ad}Y)\right]$

Differentiating the relation $\exp\Gamma(t) = \exp X.\exp tY$ we have

$$\frac{d}{dt}\exp\Gamma(t)\Big|_t = \ell_{\exp X *}\frac{d}{dt}\exp tY\Big|_t$$

$$= L_{\exp X *}\,\ell_{\exp tY *}Y = L_{\exp\Gamma(t) *}Y$$

Hence, by the proposition above

$$Y = \varphi(-\operatorname{ad}\Gamma(t))\,(\Gamma'(t))$$

Where $\varphi(z) = \frac{e^z-1}{z}$. If $|z-1|<1$ $\quad \varphi(-\log z) = \frac{z-1}{z\log z}$

Thus $\psi(z)\cdot\varphi(-\log z) = 1$

And

$$\Gamma'(t) = \psi\left[(\operatorname{Exp}\operatorname{ad}X.\operatorname{Exp} t\operatorname{ad}Y)\right].Y$$

We integrate from 0 to 1, recalling that $\Gamma(0) = \log\exp X = X$ $\quad\square$.

Remark: This formula shows that in logarithmic coordinates around e, the multiplication in G is analytic, not merely C^∞. If (U,φ) is a logarithmic coordinate system, we cover G with the translate $(\ell_g U, \varphi\circ L_{g^{-1}})$. Then G becomes an analytic manifold (i.e the change of coordinates in overlapping coordinate systems is analytic); multiplication and inverse are analytic maps