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WORKSHOP ON GRADED DIFFERENTIAL GEOMETRY
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GRADED VARIATIONAL THEORY

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GRADED VARIATIONAL THEORY

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Lecture based on work by

D.H. Ruipérez and J. Muñoz Masqué
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First note the main results from my paper "Sheaf representations and Graded manifolds".

Theorem 1 (Mulvey – Bryant)

1. Let A be any $\mathbb{Z}_2$-graded-commutative Gelfand $\mathbb{R}$-algebra. Topologise $\operatorname{Max} A$ with the Zariski topology.
$\exists$ sheaf $\tilde{A}$ of graded-commutative $\mathbb{R}$-algebras over $\tilde{X} = \operatorname{Max} A$ and an isomorphism $C_A : A \longrightarrow \Gamma(\tilde{X}, \tilde{A})$ of graded algebras.
Similarly if M is a graded A-module $\exists$ sheaf $\tilde{M}$ of $\tilde{A}$-modules and a canonical isomorphism $C_M : M \longrightarrow \Gamma(\tilde{X}, \tilde{M})$ of A-modules.

2. If $(X, \mathcal{A})$ is a graded manifold, $A = \Gamma(X, \mathcal{A})$ then A is a $\mathbb{Z}_2$-graded-commutative Gelfand algebra. Denote by $\bar{A}$ the restriction of the above sheaf representation to the set of maximal ideals of codimension 1 in A. Then ~~[illegible]~~ C_A gives an isomorphism (of sheaves of graded algebras)
$\mathcal{A} \xrightarrow{\sim} \bar{A}$.
X compact $\Rightarrow$ the correspondence $M \longrightarrow \tilde{M}$ is an equivalence of categories $A\text{-mod} \longleftrightarrow \mathcal{A}\text{-Mod}$.
X arbitrary $\Rightarrow$ we restrict to an equivalence of categories between the projective/finitely generated A-modules and the locally free $\mathcal{A}$-Modules of finite rank.

Remarks Allows us to work globally or locally and the localization procedure is completely determined. 1 gives a Stone-Cech compactification for graded manifolds and 2 gives a graded version of Swan's theorem.

To make Swan's theorem concrete we obtain the following result (after Kostant, Ruiperez/Munoz Masqué)

Proposition 1 (concrete Swan's theorem)
Let X be a smooth manifold and M a projective/finitely generated $C^\infty(X)$-module. There exists a locally trivial vector-bundle $V(M)$ over X for which the global sections are isomorphic to M.

Proof Localize by thm 1 to obtain $\tilde{M}^*$ a locally free C^∞_X-Module of rank r say.
Write $M^* \simeq C^\infty(X)\cdot M_1 \oplus \cdots \oplus C^\infty(X)\cdot M_r$
(give this meaning on the lines of thm 1).
Consider $S_{C^\infty(X)}(M^*)$ the symmetric algebra of the dual $C^\infty(X)$-module M^*.
If $V(M) = \operatorname{Alg}(S_{C^\infty(X)}(M^*), \mathbb{R})$ with the weak* topology

then $\operatorname{Alg}(S_{C^\infty(X)}(M^*), \mathbb{R}) \longrightarrow \operatorname{Alg}(C^\infty(X), \mathbb{R}) \simeq X$ provides a smooth surjection $\pi : V(M) \longrightarrow X$.
To define the smooth structure of $V(M)$ complete $S(M^*)$ [use coalgebraic completion i.e. weak* closure in $S(M^*)^{\circ *}$, or locally Grothendieck π-product].

locally $S(M^*) \simeq C^\infty(X) \otimes \mathbb{R}[M_1, \ldots M_r]$ so the completion $\overline{S(M^*)} \simeq C^\infty(X \times \mathbb{R}^r)$. This latter is Gelfand providing a (globally defined) smooth structure on $V(M)$ and showing that $\pi : V(M) \longrightarrow X$ is a locally trivial vector bundle of rank r.

Sections of π can be described as $C^\infty(X)$-algebra maps $s : S(M^*) \longrightarrow C^\infty(X)$.
These all arise from elements of
$\operatorname{Hom}_{C^\infty(X)}(M^*, C^\infty(X)) \simeq M^{**} \simeq M$

(since the completion is functorial and does not affect $C^\infty(X)$). /

Remark: The construction of $V(M)$ and its smooth structure was entirely global in contrast to the usual approach involving explicit construction of the fibres.

Hence Prop 1 generalises immediately to

<u>Proposition 2</u>
If $(X, \mathcal{A})$ is a graded manifold and $\mathcal{M}$ a locally free $\mathcal{A}$-Module of graded rank (p,q) $\exists$ submersion of graded manifolds $\Phi : (V(M), \hat{S}) \longrightarrow (X, \mathcal{A})$ that is locally trivial with fibre a standard graded manifold $(\mathbb{R}^p, \Lambda \mathbb{R}^q)$.
Moreover $\mathcal{M}$ can be regarded as the sheaf of local 'sections' of Φ.

<u>Proof</u> Let $M = \Gamma(X, \mathcal{M})$ and consider $\overline{S_A(M^*)}$ [the completion of the symmetric algebra of the graded A-dual M^* taken in the graded sense].

Find a locally free C^∞_X-Module of rank p via
$M_{ord} \equiv \left(M / _{N(A).M} \right)_0$ where $N(A) \triangleleft A$ is the nilpotent ideal of A.
Use Prop 1 to turn this into a concrete (ordinary) vector bundle $V(M) \longrightarrow X$ justifying the terminology by $V(M) \equiv Alg(S(M_{ord}^*), \mathbb{R}) \simeq Alg(S(M^*), \mathbb{R})$
The same construction as in Prop 1 gives a sheaf over $V(M)$ which is a graded manifold since
$M^* \overset{loc}{\simeq} A.M_1 \oplus A.M_2 \oplus \cdots \oplus A.M_p \oplus A.\hat{n}_1 \oplus \cdots \oplus A.\hat{n}_q$
(M_i even elts of M^*, $\hat{n}_j$ odd).

$$\text{So} \quad \overline{S(M^*)} \overset{loc}{\simeq} A \otimes \mathbb{R}\widehat{[M_1 \cdots M_p]} \otimes \Lambda \langle \hat{n}_1, \cdots \hat{n}_q \rangle$$
$$\simeq A \otimes C^\infty(\mathbb{R}^p) \otimes \Lambda \mathbb{R}^q .$$

Moreover as before
$M \longleftrightarrow Hom_A(S\widehat{A(M^*)}, A)$. ✓

<u>IMPORTANT REMARK</u>
Construction of M_{ord} involved taking the even part of a <u>graded</u> $C^\infty(X)$-module: this will lead to an important consequence of this approach – the underlying (body) manifold of our graded jet bundles is <u>not</u> the ordinary jet bundle. (Page 26 of [234]).

<u>Example</u>
The graded cotangent bundle constructed by Kostant comes in exactly this way from the A-module

$$\Omega(A) = Hom_A(Der A, A)$$

which will be important to us.

<u>Definition of graded jet bundles</u>
(We describe only the first bundle to avoid technicalities).

If $\mathcal{A}_X, \mathcal{B}_Y$ are GMs let $\Omega(A) = Hom_A(Der A, A)$
$\Omega(B) = Hom_B(Der B, B)$

set $M = Hom_{A \otimes B}(\Omega B, \Omega A)$

(the individual module structures via natural projections $P_1 : A \otimes B \to A$, $P_2 : A \otimes B \to B$).

Prop 2 $\Longrightarrow$ We have a GM total space $\underline{J'(A,B)}$ associated to M which is a locally trivial GM "vector-bundle" over the product GM $(X \times Y, \mathcal{A} \hat{\otimes} \mathcal{B})$.
This is called the first jet bundle of maps from $\mathcal{A}_X$ to $\mathcal{B}_Y$.

<u>Remark</u>: higher order jet bundles difficult to define based on $\Omega_{(k)}(A) = \Delta / \Delta^{k+1}$ for $\Delta = \ker$(diagonal map)

since for $k > 1$ these have non-trivial algebra structures.
The approach goes back to Golubitsky - Guillemin.

<u>Properties</u>
Each GM morphism $P : \mathcal{B}_Y \longrightarrow \mathcal{B}'_Y$ induces
$P_* : J'(A,B) \longrightarrow J'(A,B')$
(a submersion whenever P is a submersion).
Similarly for $A \rightsquigarrow J'(A,B)$ and higher order jet bdles.

Each morphism $f : \mathcal{A}_X \longrightarrow \mathcal{B}_Y$ induces the 1-jet lifting of f:
$j'f : \mathcal{A}_X \hookrightarrow J'(A,B)$ actually a closed immersion.

Given a locally free A-Module of finite rank with corresponding "graded vector bundle" $P : \mathcal{B}_Y \longrightarrow \mathcal{A}_X$ there is a pull-back diagram defining the first <u>jet bundle of sections</u> of P.

$$\begin{tikzcd} \boxed{J'(B/A)} \arrow[r, hook] \arrow[d] & J'(A,B) \arrow[d, "P_*"] \\ \mathcal{A}_X \arrow[r, hook] & J'(A,A) \end{tikzcd}$$

(same for higher orders).

True for <u>all</u> submersions in fact.

[Alternative approach for graded vector bundles only — not arbitrary submersions.
Let $\mathcal{A}_X$ be a GM and $\mathcal{F}, \mathcal{G}$ be $\mathcal{A}_X$-bundles (i.e. locally free / finite rank).
Define $\underline{\mathrm{Diff}(F,G)} \subset \mathrm{Hom}_{\mathbb{R}}(F,G)$ inductively by
$\mathrm{Diff}_0(F,G) = \mathrm{Hom}_A(F,G)$ and
$\mathrm{Diff}_k(F,G) = \{ D \in \mathrm{Hom}_{\mathbb{R}}(F,G) : [D,a] \in \mathrm{Diff}_{k-1}(F,G)\ \forall a \in A \}$
where $[D,a] : f \longmapsto D(a.f) - (-1)^{|D||a|}\, a.D(f)$.
— definition of R. Hermann (1969).

Differential operators localise well and we set
$$\mathcal{J}^k(F) = \mathcal{H}om_{\mathcal{A}}(\mathrm{Diff}_k(F,\mathcal{A}), \mathcal{A})$$
giving a new $\mathcal{A}$-bundle satisfying
$\mathrm{Diff}(F,G) \cong \mathcal{H}om_{\mathcal{A}}(\mathcal{J}^k F, G)$ (see my thesis!)

Return to the first graded jet bundle of sections of some GM submersion $p : \mathcal{B}_Y \to \mathcal{A}_X$ with diagram

$$\begin{tikzcd} \mathcal{J}^1(p) \arrow[r, "p_1"] \arrow[dr, "q"'] & \mathcal{B}_Y \arrow[d, "p"] \\ & \mathcal{A}_X \end{tikzcd}$$

Let $J^1(p)$ be the ring of global sections of $\mathcal{J}^1(p)$ with similar notation for higher orders.

They consider $\mathrm{Der}_A(B)$ = graded derivations of B that vanish on A (p^* is injective)
and construct the fundamental $J^1(p)$-module $\boxed{M = J^1(p) \otimes_B \mathrm{Der}_A(B)}$

Geometrically this gives the sections of the vertical bundle $\mathrm{Vert}(p)$ when it is pulled back over $\mathcal{J}^1(p)$ via p_1:

$$\begin{tikzcd} p_1^* \mathrm{Vert}(p) \arrow[r] \arrow[d] & \mathrm{Vert}(p) \arrow[d] \\ \mathcal{J}^1(p) \arrow[r, "p_1"'] & \mathcal{B}_Y \end{tikzcd}$$

We will generally be doing graded differential calculus with values in the module M — cf Koszul.

Now on $\mathcal{J}^1(p)$ there exists an (M-valued) structure 1-form that characterises those sections of $q = p \cdot p_1$ that are the 1-jet lifts of sections of p

Proposition 3
$\exists\ \theta^1 \in \mathrm{Hom}_{J^1(p)}(\mathrm{Der}\, J^1(p), M)$ — an M-valued 1-form over $\mathcal{J}^1(p)$ s.t.
a section $\bar{\sigma} : \mathcal{A}_X \to \mathcal{J}^1(p)$ of q is the 1-jet lifting of a section $\sigma = p_1 \cdot \bar{\sigma} : \mathcal{A}_X \to \mathcal{B}_Y$ of p iff $\boxed{\bar{\sigma}^* \theta^1 = 0}$.

Proof The exact sequence of $J^1(p)$-modules
$$0 \to M \rightleftarrows J^1(p) \otimes_B \mathrm{Der}\, B \to J^1(p) \otimes_A \mathrm{Der}\, A \to 0$$
splits at the second place and p_1 induces
$p_{1,*} : \mathrm{Der}\, J^1(p) \to J^1(p) \otimes_B \mathrm{Der}\, B$ etc.

We also need a structure form on the second graded jet-bundle $\mathcal{J}^2(p)$. The proof is similar:

Prop 4
on $\mathcal{J}^2(p)$ $\exists$ 1-form θ^2 (valued in $J^2(p) \otimes_{J^1(p)} \mathrm{Der}_A J^2(p)$ — i.e. $p_{2,1}^*.\mathrm{Vert}(q)$ where $p_{2,1} : \mathcal{J}^2(p) \to \mathcal{J}^1(p)$)

A section $\bar{\sigma} : \mathcal{A}_X \to \mathcal{J}^2(p)$ is the 2-jet lift of a section $\sigma = (p_1 \cdot p_{2,1}) \cdot \bar{\sigma} : \mathcal{A}_X \to \mathcal{B}_Y$ of p iff $\bar{\sigma}^* \theta^2 = 0$.

Now a graded derivation law ("connection") for M is a map of $J^1(p)$-modules $\nabla : \mathrm{Der}\, J^1(p) \to \mathrm{End}_{\mathbb{R}}(M)$ s.t.
$$D^\nabla (j.m) = D(j).m + (-1)^{|D||j|}\, j.D^\nabla m \quad (\text{homogeneous } j \in J^1(p),\ D \in \mathrm{Der}\, J^1(p),\ m \in M)$$

Then we can take, for example, the exterior derivative $d\theta^1$ of the M-valued 1-form θ^1 w.r.t. some ∇.

[Choose one from a graded derivation law for the B-module $\mathrm{Der}\, B$ with "vanishing vertical torsion"
$D^\nabla D' = (-1)^{|D||D'|} D'^\nabla D$ — this produces a connection for $\mathrm{Vert}(p)$ hence its pull-back — M].

Define a graded infinitesimal contact transformation on $\mathcal{J}^1(p)$ as a vector field $\Xi \in \mathrm{Der}\, J^1(p)$ s.t.
$$\mathcal{L}_\Xi \theta^1 = h \circ \theta^1 \quad \text{for some } h \in \mathrm{End}_{J^1(p)}(M)$$
(Lie derivative may be taken with respect to any derivation law on M, the definition being independent of a particular choice).

__Graded variational theory__

$L \in \mathcal{J}^1(P)$ is a __graded Lagrangian__ based on P.

A __volume form__ ω on $\mathcal{A}_X$ is a closed graded m-form such that $\omega^\sim$ becomes a volume form on X.
(e.g. use a section $C^\infty_X \hookrightarrow \mathcal{A}$ coming from Batchelor's theorem to lift a volume form on X to $\mathcal{A}$).
- we will denote by ω also the lifted form on $\mathcal{J}^1(P)$ via q.

Then describe __$L.\omega$__ as a __graded Lagrangian density__ (it is an m-form on $\mathcal{J}^1(P)$).

[Set $\Omega(B/A) = \mathrm{Der}_A(B)^*$ (dual as B-module)]
$\exists$ exact sequence of $\mathcal{J}^1(P)$-modules:

$$0 \longrightarrow \Omega(B) \otimes_B \mathcal{J}^1(P) \longrightarrow \Omega(\mathcal{J}^1(P)) \xrightarrow{\ \gamma\ } \mathrm{Der}A \otimes_A M^* \longrightarrow 0$$

Let $\boxed{F = -\gamma(dL)}$ vector-field on $\mathcal{A}$ with values now in M^*.

The __graded Legendre form__ $\Omega = F \lrcorner\, \omega$
an $(m-1)$ form on $\mathcal{A}$ valued in M^*. ↖ interior product

The __graded Poincaré-Cartan form__ is

$$\textcircled{H} = \theta^1 \wedge \Omega - L.\omega$$

↖ duality $M \otimes M^* \to \mathcal{J}^1(P)$

an m-form on $\mathcal{J}^1(P)$: "Hamiltonian Formulation of $L.\omega$".

__Proposition__
$\exists\ \delta_\nabla : \mathrm{Der}A \otimes_A M^* \longrightarrow \Omega(\mathcal{J}^1(P)/\mathcal{A})$ such that

$\delta_\nabla . \gamma : \Omega(\mathcal{J}^1(P)) \to \Omega(\mathcal{J}^1(P)/\mathcal{A})$ is the natural projection (written $\nu \to \bar{\nu}$) dual to $\mathrm{Der}_A \mathcal{J}^1(P) \hookrightarrow \mathrm{Der}\,\mathcal{J}^1(P)$.

Moreover letting $f = \overline{dL} + \delta_\nabla(F)$ $\in M^*$
we obtain

$$\underline{dL \equiv -F \lrcorner\, d\theta^1 + \theta^1 \wedge f + \alpha} \quad (\text{some } \alpha \in \Omega A \otimes_A \mathcal{J}^1(P))$$

__Proof__ consider the diagram (with exact rows and columns)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega A \otimes_A \mathcal{J}^1(P) & = & \Omega A \otimes_A \mathcal{J}^1(P) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega B \otimes_B \mathcal{J}^1(P) & \longrightarrow & \Omega \mathcal{J}^1(P) & \xrightarrow{\gamma} & \mathrm{Der}A \otimes_A M^* & \longrightarrow & 0 \\
 & & \downarrow \uparrow \bar{\theta}^1 & & \downarrow \text{bar} & & \| & & \\
0 & \longrightarrow & M^* & \longrightarrow & \Omega(\mathcal{J}^1(P)/A) & \underset{\delta_\nabla}{\leftrightarrows} & \mathrm{Der}A \otimes_A M^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

First column is dual of $0 \to M \to \mathcal{J}^1(P) \otimes_B \mathrm{Der}B \to \mathcal{J}^1(P) \otimes_A \mathrm{Der}A \to 0$ split by existence of θ^1.

δ_∇ splits the bottom row and exists because the vanishing vertical torsion condition on $\nabla \Rightarrow$
$\forall\ \xi \in \mathrm{Der}_A \mathcal{J}^1(P) \otimes_B \Omega(B/A)$ $\quad \xi \lrcorner\, d\theta^1 \equiv 0$

We obtain

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \theta^1 \wedge f & \longrightarrow & dL & \longrightarrow & -F & \longrightarrow & 0 \\
 & & \uparrow \bar{\theta}^1 & & \downarrow \text{bar} & & \| & & \\
0 & \longrightarrow & f & \longrightarrow & \overline{dL} & \underset{\delta_\nabla}{\leftrightarrows} & -F & \longrightarrow & 0
\end{array}$$

/

__IMPORTANT REMARK__
At this point we have to pull everything back to the second graded jet bundle. Why ??

Consider the dual map: $\delta_\nabla^* : \mathrm{Der}_A \mathcal{J}^1(P) \to M \otimes_A \Omega(A)$

Write $\theta^{2,\nabla} = (1 \otimes \delta_\nabla^*)(\theta^2)$
↖ structure 1-form.
i.e. $\theta^{2,\nabla}$ is a 1-form valued in $M \otimes_A \Omega A$ over $\mathcal{J}^2(P)$.

__Theorem!__
$\exists_1$ m-form ξ valued in $\mathrm{Der}A \otimes_A M^*$ over $\mathcal{A}_X$ s.t.

$$d\textcircled{H} = -\theta^1 \wedge (d\Omega + \omega \otimes f) + \theta^{2,\nabla} \,\bar{\wedge}\, \xi + \alpha$$

↖ dual connection ↖ duality

for $\alpha \in \Lambda^{m+1} \Omega A \otimes_A \mathcal{J}^2(P)$

Moreover ξ __does not depend on the derivation law ∇.__

__Proof__
Local computation unfortunately.

Definition A section s of the GM submersion $p: B_Y \to A_X$ is said to be CRITICAL for the variational problem given by $L\cdot\omega$ if for all graded infinitesimal contact transformations $\bar{D}$ in $\mathcal{J}^1(p)$ [with compact support on A_X]:

$$\int_X [(j^1 s)^* \mathcal{L}_{\bar D}(L\cdot\omega)]^\sim = 0$$

Theorem 2

i) s is critical $\Leftrightarrow$ for all (any) derivation laws ∇, both $[(j^1 s)^* \xi^\nabla]^\sim = 0$ and $[(j^1 s)^* \xi]^\sim = 0$

where $\xi^\nabla = d\Omega + \omega \otimes F$

ii) s is critical $\Leftrightarrow$ $[(j^1 s)^* \; \bar D \lrcorner d\Theta]^\sim = 0$ (Hamiltonian version).

Call ξ^∇, ξ the graded global Euler-Lagrange operators.

CRUCIAL REMARK ξ^∇ depends on ∇, ξ does not

ξ^∇ gives expected Euler-Lagrange equations.

ξ gives completely 'new' constraint equations.

FACT For a naïve supergravity theory with Lagrangian $R^2 + T^2$ ξ produces the supergravity constraints on curvature/torsion found by Wess-Zumino!

However local equations (from ξ^∇) are independent of the derivation law.

Proof of thm 2 via

Lemma 1

$$\mathcal{L}_{\bar D}(L\cdot\omega) = -(\bar D \lrcorner \theta^1)\wedge(d\Omega + \omega\otimes F) + (\bar D \lrcorner \theta^{2,\nabla})\bar\wedge \xi$$
$$- d(\bar D \lrcorner \Theta) - \theta^1 \wedge (\mathcal{L}_{\bar D}\Omega - \bar D \lrcorner (d\Omega + \omega\otimes F) + g\circ\Omega)$$
$$- \theta^{2,\nabla} \bar\wedge (\bar D \lrcorner \xi) + \beta$$

$\beta \in \Lambda^m RA \otimes_A \mathcal{J}^2(p)$

with $[(j^2 s)^* \beta]^\sim = 0$ and

Proof from $\Theta = \theta^1 \wedge \Omega - L\cdot\omega$ and $\mathcal{L}_{\bar D}\alpha = \bar D \lrcorner d\alpha + d(\bar D \lrcorner \alpha)$ use theorem 1 etc.

Proof (thm 2)

$$\int_X [j^1 s^* \mathcal{L}_{\bar D}(L\cdot\omega)]^\sim = -\int_X (j^1 s)^*(\bar D \lrcorner \theta^1)^\sim \wedge (j^1 s^* \xi^\nabla)^\sim$$
$$+ \int_X (j^2 s)^* (\bar D \lrcorner \theta^{2,\nabla})^\sim \wedge (j^1 s)^* \xi^\sim$$

giving sufficiency (i). Converse by local computation. (ii) is similar.

APPENDIX

Local descriptions

Let $(r_1 \cdots r_m, s_1 - s_n)$ be local coords for (X, A)
$(\rho_1 \cdots \rho_{m'}, \sigma_1 \cdots \sigma_{n'})$ be local coords for (Y, B).

if $D_{\alpha,\beta} = (\partial/\partial r_i)^{\alpha_i}(\partial/\partial s_\beta)$ basis for $\mathrm{Der}A$

Then

1) $\{r_i, s_j, \rho_h, \sigma_c, D_{\alpha,\beta} \otimes d\rho_h, D_{\alpha,\beta} \otimes d\sigma_c\}$ are local coordinates for $\mathcal{J}^1(A, B)$

2) If $f : (X, A) \longrightarrow (Y, B)$ is a GM morphism and $f^* : B \to A$ the induced map we have a lift $j^1 f : (X, A) \hookrightarrow \mathcal{J}^1(A, B)$ and

$(j^1 f)^* : \mathcal{J}^1(A, B) \longrightarrow\!\!\!\!\rightarrow A$ is described by

$r_i \to r_i, \quad s_j \to s_j, \quad \rho_h \to f^*\rho_h, \quad \sigma_c \to f^*\sigma_c$

$D_{\alpha,\beta} \otimes d\rho_h \longrightarrow \frac{1}{\alpha!}(\partial/\partial r)^\alpha(\partial/\partial s_\beta)(f^*\rho_h)$

$D_{\alpha,\beta} \otimes d s_c \longrightarrow \frac{1}{\alpha!}(\partial/\partial r)^\alpha(\partial/\partial s)_\beta (f^*\sigma_c)$

latter two are coefficients in the Taylor's series.

Consider submersion $p : (Y, B) \longrightarrow (X, A)$

if $(x_1, \cdots x_m, s_1 \cdots s_n)$ are coords for A
$(x_1' \cdots x_m', y_1' \cdots y_n', s_1' \cdots s_n', \sigma_1 \cdots \sigma_{n'})$ coords for B.

(i.e. $p^* : A \hookrightarrow B$ is $p^*(x_i) = x_i'$ etc.)

Then $p_* : \mathrm{Der}B \to B \otimes_A \mathrm{Der}A$ is given by

$p_*(\partial/\partial x_i') = 1 \otimes \partial/\partial x_i, \quad p_*(\partial/\partial y_i) = 0$

$p_*(\partial/\partial s_h') = 1 \otimes \partial/\partial s_h, \quad p_*(\partial/\partial \sigma_c) = 0$

Then a local system for $\mathcal{J}^1(P)$ is given by

$(x_i, y_h, s_j, \sigma_k, \; p_{hi} = \partial/\partial x_i \otimes dy_h, \; \bar{p}_{hj} = \partial/\partial s_j \otimes dy_h,$
$q_{ki} = \partial/\partial x_i \otimes d\sigma_k, \; q_{kj} = \partial/\partial s_j \otimes d\sigma_k)$

Description of θ'

$$\theta' = \theta_h \otimes \partial/\partial y_h + \bar{\theta}_k \otimes \partial/\partial \sigma_k \in \mathrm{Der}_A B$$

where $\theta_h, \bar{\theta}_k$ are 1-forms over $\mathcal{J}^1(P)$ given by

$\theta_h = dy_h - dx_i\, p_{hi} - ds_j\, \bar{p}_{hj}$

$\bar{\theta}_k = d\sigma_k - dx_i\, q_{ki} - ds_j\, q_{kj}$

Description of F, Ω

$$F = \left(-\frac{\partial L}{\partial y_i^h}\,\partial/\partial x_i + \frac{\partial L^\tau}{\partial y_j^h}\,\partial/\partial s_j\right) \otimes dy_h$$

$$+ \left(\frac{\partial L^\tau}{\partial \sigma_i^k}\,\partial/\partial x_i - \frac{\partial L}{\partial \sigma_j^k}\,\partial/\partial s_j\right) \otimes d\sigma_k$$

$$\Omega = -\frac{\partial L}{\partial y_i^h}\,\omega_i \otimes dy_h + \frac{\partial L^\tau}{\partial \sigma_i^k}\,\omega_i \otimes d\sigma_k$$

Description of ζ

$$\zeta = \left(-\omega\frac{\partial L^\tau}{\partial y_j^h} - ds_j \wedge \omega_i \frac{\partial L^\tau}{\partial y_i^h}\right) \otimes \left(\frac{\partial}{\partial s_j} \otimes dy_h\right)$$

$$+ \left(-\omega\frac{\partial L}{\partial \sigma_j^k} + ds_j \wedge \omega_i \frac{\partial L}{\partial \sigma_i^k}\right) \otimes \left(\frac{\partial}{\partial s_j} \otimes d\sigma_k\right)$$

<u>The Euler-Lagrange equations are</u>

$$\left[(j^1 s)^* \frac{\partial L_0}{\partial y_r} - \partial/\partial x_i \left((j^1 s)^* \frac{\partial L_0}{\partial y^r_i}\right)\right]^\sim = 0 \qquad (r = 1, \ldots p)$$

$$\left[(j^1 s)^* \frac{\partial L_1}{\partial \sigma_L} - \partial/\partial x_i \left((j^1 s)^* \frac{\partial L_1}{\partial \sigma^L_i}\right)\right]^\sim = 0 \qquad (L = 1, \ldots q)$$

$$\left[(j^1 s)^* \frac{\partial L_1}{\partial y^h_j}\right]^\sim = 0 \qquad (h = 1 \ldots p, \; J = 1 \ldots n)$$

$$\left[(j^1 s)^* \frac{\partial L_0}{\partial \sigma^K_R}\right]^\sim = 0 \qquad (K = 1 \ldots q, \; R = 1 \ldots n)$$

$B_y \longrightarrow A_x$ coords for B (adapted) $(x_1 \ldots x_m, y_1 \ldots y_n$
$s_1' \ldots s_n', \sigma_1 \ldots \sigma_m)$

<u>and</u>

$$\left.\begin{aligned} y^h_J &= \partial/\partial s_J \otimes dy_h \quad \text{etc} \\ \sigma^K_i &= \partial/\partial x_i \otimes d\sigma_K \end{aligned}\right\} \text{local coordinates on } J^1(P)$$