INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392 - I

SMR.201/ 12

SECOND WORKSHOP ON MATHEMATICS IN INDUSTRY

(2 - 27 February 1987)

CONTROLLED INVARIANT AND CONTROLLABILITY SUBSPACES

H.W. KNOBLOCH
Mathematisches Institut
der Universitat
Am Hubland
87 Wurzburg
FEDERAL REPUBLIC GERMANY

1. Controlled invariant and controllability subspaces

Linear control system: Input/output behavior described by a state space model

$$\left.\begin{array}{ll} \dot{x} = Ax + Bu & \text{(Input/state behaviour)} \\ y = Cx & \text{(State/output behaviour)} \end{array}\right\} (1)$$

$Im(M) = \{Mx,\ x \text{ arbitrary}\}$, $ker(M) = \{x : Mx = 0\}$

<u>Definition 1</u> (i) $Im(B, AB, A^2B, \ldots) =: R_0$ is called the maximal controllable subspace of (1). (ii) The maximal controllable subspace of a system of the form

$$\dot{x} = (A - BF)x + BGu \qquad (2)$$

is called a controllable subspace of (1) and denoted by R.

<u>Lemma 1</u>. Let R_0 be the maximal controllable subspace of (1). Then: $x(0) \in R_0$ implies $x(t) \in R_0$ for all t, if $x(t)$ is solution of $\dot{x} = Ax + Bu(t)$, for any (continuous) control function $u(t)$.

<u>Definition 2</u>. A linear subspace $\mathcal{V}$ of the state space is called controlled invariant with respect to (1) if: To any $x_0 \in \mathcal{V}$ there exists a control function $u(t)$ such that the solution $x(t)$ of the initial value problem

$$\dot{x} = Ax + Bu(t),\ x(0) = x_0$$

satisfies $x(t) \in \mathcal{V}$ for all t.

<u>Remark</u>: Any controllable subspace is controlled invariant.

<u>Theorem 1</u>. $\mathcal{V}$ is controlled invariant iff either one of the following two conditions hold.

(i) $\exists$ matrix F such that $(A - BF)\mathcal{V} \subseteq \mathcal{V}$

(ii) $A\mathcal{V} \subseteq \mathcal{V} + Im(B)$.

<u>Proof</u>. (i) $\Rightarrow$ controlled invariance: $(A - BF)\mathcal{V} \subseteq \mathcal{V} \Rightarrow$ $e^{(A-BF)t}\mathcal{V} \subseteq \mathcal{V}$ for all t

Controlled invariance implies (ii): $x(t) \in \mathcal{V}$ for all $t \Rightarrow$ $\dot{x}(t) = Ax(t) + Bu(t) \in \mathcal{V}$ for all t, in particular $Ax_0 + Bu(0) \in \mathcal{V}$.

<u>Theorem 2</u>. Every linear subspace of the state space contains a maximal controlled invariant subspace. Every controlled invariant subspace contains a maximal controllable subspace.

Standard notation: $\mathcal{V}^*$ = maximal controlled invariant subspace contained in $\{x : Cx=0\}$. $\mathcal{R}^*$ = maximal controllable subspace contained in $\mathcal{V}^*$.

$x_0 \in \mathcal{V}^*$ iff there exists a solution of $\dot{x} = Ax + Bu(t)$ for some control function $u(t)$ satisfying

$$x(0)=x_0, \quad Cx(t)=0 \text{ for all } t.$$

Theorem 3. $\mathcal{R}^*$ is the maximal controllable subspace of a system $\dot{x} = (A-BF)x + BGu$ where F, G satisfy

$$(A-BF)\mathcal{V}^* \subseteq \mathcal{V}^*, \quad \mathrm{Im}(BG) = \mathcal{V} \cap \mathrm{Im}(B). \qquad (3)$$

Conversely if F, G are such that (3) holds, then the maximal controllable subspace of the system (2) is equal to the maximal controllable subspace contained in $\mathcal{V}^*$.

If a new coordinate system is introduced in the state space such that

$x=(x_1,x_2,x_3)$, $\mathcal{R}^* = \{(x_1,0,0)\}$, $\mathcal{V}^* = \{(x_1,x_2,0)\}$

$\mathrm{Im}\,B = \mathcal{B}' \oplus \mathcal{B}''$; $\mathcal{B}' \subseteq \mathcal{R}^*$, $\mathcal{B}'' \subseteq \{(0,0,x_3)\}$

then the system equation can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^{\cdot} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \\ B_2 \end{pmatrix} u, \quad y = (0,0,C_3)x = C_3 x_3$$

Theorem 4 (Input/output normal form of a linear system). The given system (1) is equivalent to a system of the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \\ 0 & B_{32} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$y = (0,0,C_3)x$$

Hence its input/output behavior is completely described by the subsystem

$$\dot{x}_3 = A_{33}x_3 + B_{32}u_2, \quad y = C_3 x_3$$

which has this property: The largest controlled invariant subspace contained in $\{x_3 : C_3x_3 = 0\}$ is $[0]$.

Furthermore: $\dot{x}_1 = A_{11}x_1 + B_{11}u_1$ is controllable

Definition 3. $\Sigma(s) := \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix}$, s being a scalar variable, is called the Rosenbrock-matrix of the linear system $\dot{x} = Ax + Bu$. Normal rank of the Rosenbrock-matrix: Rank, considered as a matrix of rational functions. s Transmission zero if rank $\Sigma(s) <$ normal rank.

Theorem 5. (i) Equivalent systems have the same transmission zeros. (ii) If the system is written in the normal form of Theorem 4 then the transmission zeros become the eigenvalues of the matrix A_{22}.

For a single input/single output system which is observable and controllable the transmission zeros are the zeros of the transfer function $H(s)$.

Theorem 6. Assume that the system is stabilizable. Then there exists a unique maximal subspace $\mathcal{V}^-$ of $\mathcal{V}^*$ having this property: $\exists$ a matrix F such that

$$\sigma(A-BF) \subset \mathbb{C}^-, \quad (A-BF)\mathcal{V}^- \subseteq \mathcal{V}^-.$$