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SECOND WORKSHOP ON MATHEMATICS IN INDUSTRY

(2 - 27 February 1987)

GROUNDWATER FLOW

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GROUNDWATER FLOW.

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FEB 1987.

Groundwater Flow:

references.

standard texts.

Hydraulics of Groundwater J. Bear 1979
Mc Graw-Hill U.S.A. (£65.50).
- good mathematical introduction

Groundwater R.A. Freeze, J.A. Cherry 1979
Prentice-Hall U.S.A. (£50.25).
- introduction to subject as a whole.

other sources.

Dynamics of Fluids in Porous Media J. Bear 1972
American Elsevier
- similar to above but on general porous media.

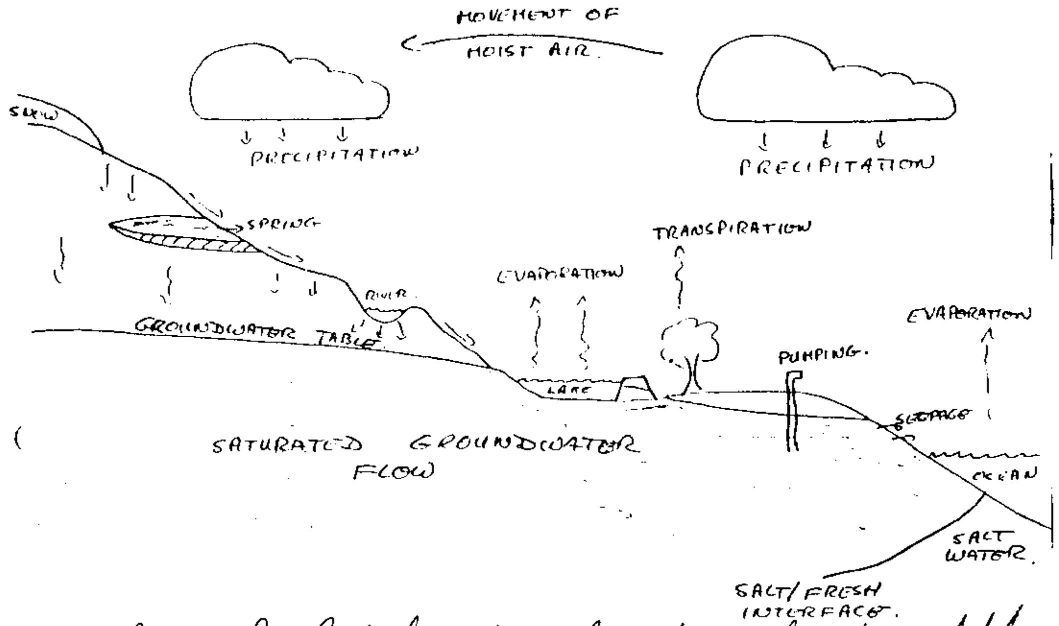
Seepage and Groundwater Flow K.R. Rushton S.C. Redshaw
1979 J. Wiley U.S.A.
- numerical methods and examples.

Flow Phenomena in Porous Media R.A. Greenkorn 1983.
Marcel Dekker U.S.A.
- excellent introduction to oil, groundwater flows.

Fundamentals of Transport Phenomena in Porous Media
ed.- J. Bear H.Y. Corapcioglu
Martinus Nijhoff 1984 USA.
- extensive collection of lectures covering the whole subject area.

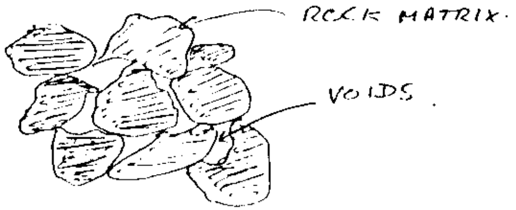
Seepage Hydraulics G. Kovács
American Elsevier USA 1981 (£75.00).
or. Akadémiai Kiadó. HUNGARY 1981 (?)
- comprehensive introduction to subject and different mathematical examples.

INTRODUCTION

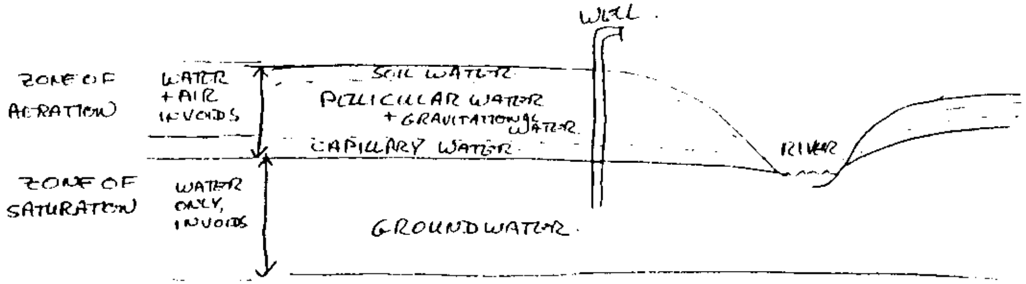


This is the hydrological cycle. it is almost completely closed ie water is conserved.

Once the water enters the ground it exists in the voids between soil particles, sand particles or blocks of rock. The soil sand or rock form a matrix of material and this matrix, with the voids is called a porous media. The interest to groundwater problems is to determine the flow of water within such a porous media.



A typical vertical section in the ground can be described in terms of the distribution of water and the way it is held in the matrix.



SOIL WATER — Seasonal variations in water content. Water movement affected by plant root systems.

PELLICULAR WATER — Non-moving water held to the matrix by hydroscopic and capillary forces.

GRAVITATIONAL WATER — Water passing down through the aeration zone due to gravity and capillary forces.

CAPILLARY WATER — Water held above the water table by capillary forces

WATER TABLE — Surface below which the matrix is saturated

There are four general categories for the rock matrix which indicate the type of groundwater flow that may occur.

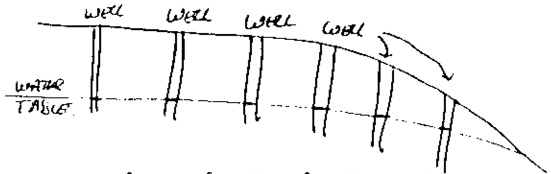
I. <u>AQUIFER</u> — this is a geological formation which contains water and permits significant amounts of water movement eg: sand or gravel or fractured rock. (also known as ground water reservoir or water bearing zone)

II. <u>AQUICLUDE</u> — this is a geological formation which contains water but does not permit significant amounts of water movement (ie: it is impervious) an example is clay.

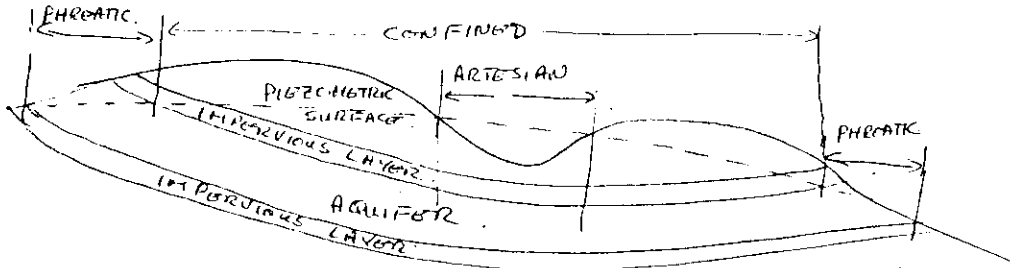
III <u>AQUITARD</u> — this is a geological formation which contains water and is not quite impervious.

VI <u>AQUIFUGE</u> — this is a geological formation which contains no water and is impervious eg: unfractured granite.

There are two types of aquifer usually considered and to delineated then consider drilling a large number a wells into the ground



The surface made by the level of water in each well is called the <u>piezometric surface</u>. If the piezometric surface and the boundary of the saturated zone are the same (as above) then this is called a <u>PHREATIC AQUIFER</u>. Otherwise it is called a <u>CONFINED AQUIFER</u>. Consider for example an aquifer bounded by two impervious layers.



The region where the piezometric surface is above the soil surface is sometimes referred to as an <u>ARTESIAN AQUIFER</u>. A well in this region will freely flow with water.

There are many reasons for studying groundwater movement but here we shall just briefly list two.

1) To determine the feasability and the management necessary for using a particular source of water for drinking etc. Aspects to consider might be

a) How much can be drawn from the source

b) How can the source be refilled

c) How can contamination be avoided

d) How much land movement can be expected.

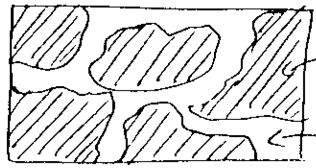
e) How will other nearby sources be affected.

2) To determine the behaviour of various civil engineering structures. Examples might be

a) construction of dams to determine what materials they should be made of

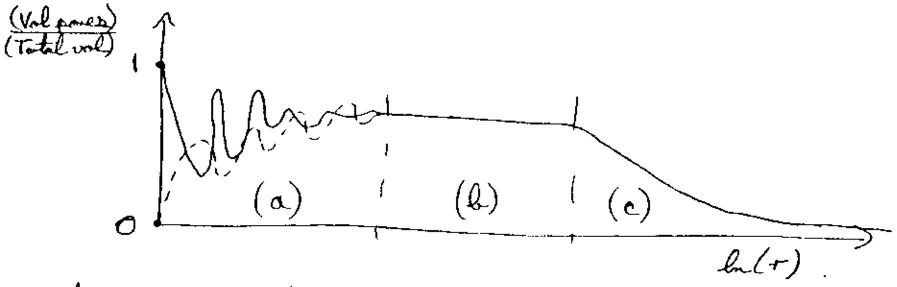
b) installation of drainage to determine how a wet region can be reclaimed.

FLOW IN SATURATED POROUS MEDIA.



On a microscopic level the water flows through the channels between the grains of the matrix and the grains can redistribute themselves due to the applied stresses. For groundwater flows the appropriate Reynolds number, based on pore size is less than 10 in most situations (there are cases, such as in impervious rock with large fractures when larger values occur). The water motion can therefore be modelled using Navier-Stokes equations for a viscous compressible fluid. The inertia terms in the equation are usually also neglected due to the small Reynolds number. The matrix is made of almost incompressible material (compared with water) and redistributes itself due to stresses induced by the water and due to external loads (eg: overlying rock or buildings).

Rather than study the microscopic behaviour of each grain and in each pore, a model is sought which gives the macroscopic behaviour of the porous media. How to get from the microscopic equations to some 'average' equations in a logical manner is a source of much discussion (see references). It depends on finding a Representative Elementary volume (REM) to describe the material. For example given any point in a particular porous media (either in a void or the matrix) consider a sphere of radius, r, about the point and plot (the volume of pores in the sphere)/(the total volume of the sphere).



Here the dotted and solid lines represent 'nearby' points. The curves have 3 regions

(a) where microscopic variations are important and must be accounted for.

(b) where the function is insensitive to microscopic detail

(c) where long range effects, such as changes in rock type, become important.

The equations for macroscopic behaviour are derived to be valid in (b) by averaging over (a) but including variations from (c). The REM are chosen of size (b) therefore.

The quantities of most importance are

$$n = \frac{\text{volume of pores in sphere}}{\text{total volume of sphere}} = \underline{\text{POROSITY}}$$

$$\underset{\sim}{v} = \frac{\text{integral of velocity of water normal to sphere}}{\text{total surface area of sphere}} = \underline{\text{DARCY VELOCITY}}$$

$$\rho = \frac{\text{mass of water in sphere}}{\text{total volume of pores in sphere}} = \underline{\text{AVERAGE WATER DENSITY}}$$

$$p = \frac{\text{pressure in water on surface of sphere}}{\text{total surface of sphere covered by pores}} = \underline{\text{AVERAGE PRESSURE}}$$

It is also useful to introduce the concept of effective porosity, n_e, which is the porosity but excludes pores which are not connected together (hence are excluded from any flow). It should also be noted that the Darcy velocity is not the actual water velocity because it takes no account of pore size.

We start by considering the case when the water is completely stationary. The equations are then

$$\underset{\sim}{v} = 0$$

$$0 = -\underset{\sim}{\nabla} p + \rho(0,0,-g)$$

where $(0,0,-g)$ is a vector pointing in the direction of gravity. The equation of state for water is then, in the simplest case where concentration and temperature differences are negligible,

$$\rho = \rho_0 \exp(\beta(p - p_0))$$

where β is the compressibility of water. Because of the complexity of the matrix redistribution problem and because it approximates the resulting behaviour a similar equation of state is usually considered for the matrix by assuming it is in equilibrium with

$$(1-n) = (1-n_0)\exp(\alpha(p_0 - p))$$

where α is the 'compressibility' of the matrix due to redistribution.

It is common in the literature not to use pressure as a variable but to introduce ϕ, the PIEZOMETRIC HEAD

where $$\phi = \int \frac{dp}{\rho g} + z$$

with z measuring distance up against gravity. The momentum equation is $$0 = \underset{\sim}{\nabla}\phi$$

in the simplest case described above.

The basis of most groundwater analysis is a relationship first derived by Henry Darcy (1856) experimentally. It states that steady flow in a uniform medium with uniform fluid of viscosity, μ, satisfies

$$\mu \underset{\sim}{v} = k(-\underset{\sim}{\nabla} p - \rho g \underset{\sim}{\nabla} z)$$

or $$\underset{\sim}{v} = -\frac{k\rho g}{\mu}\underset{\sim}{\nabla}\phi = -K\underset{\sim}{\nabla}\phi \qquad \text{DARCY'S LAW}$$

Here k, which only depends on matrix properties, is the permeability and K is the hydraulic conductivity. Typical extensions to this law include adding additional terms to account for gradients in

temperature or ion concentration, or more importantly making k a tensor to account for anisotropic structure of the matrix.

Darcy's law can be derived by averaging techniques if the flow in the pores is very small (Reynold's number $= \frac{|\underset{\sim}{v}|\, d\, \rho}{\mu}$ much less than one for d a typical pore diameter) so that pressure gradients in each pore balance viscous dissipation.

An unsteady version of Darcy's Law to account for some inertia is

$$\frac{\rho\eta}{\mu}\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} = -K\underset{\sim}{\nabla}\phi$$

where η is the channel conductance or the effective crossectional area of the pores. To this equation another equation describing conservation of mass is added

$$\frac{\partial(\rho n)}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho\underset{\sim}{v}) = 0$$

For groundwater flow where density and porosity are assumed to be only functions of pressure as above this can be well approximated by.

$$(\alpha(1-n) + \beta n)\frac{\partial p}{\partial t} + \underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0$$

where $(\alpha(1-n)+\beta n)$ is called the specific volume storativity which usually has to be inferred from experiments.

If this system of equations is non-dimensionalized with $\hat{\rho}$, $\hat{k}$ typical densities and permeability and with

$$p = (\hat{\rho} g L)\bar{p} \qquad \underset{\sim}{x} = (L)\underset{\sim}{\bar{x}} \qquad \underset{\sim}{v} = (\hat{k}\hat{\rho} g)\underset{\sim}{\bar{v}} \qquad t = \left(\frac{L}{\hat{k}\hat{\rho} g}\right)\bar{t}$$

so that $x = O(10^3 m)$, $|\underset{\sim}{v}| = O(10\text{–}10^{-8}\ m/s)$ and $t = O(10^2\text{–}10^4 s)$

then

$$\left(\frac{\hat{\rho}\eta\hat{K}g}{\mu L}\right)\frac{\partial\bar{v}}{\partial\bar{t}} + \bar{v} = -\bar{k}(\nabla\bar{p} - \bar{\rho}\underset{\sim}{\nabla}\bar{z})$$

$$(\beta\hat{\rho}gL)\left(\frac{\alpha}{\beta}(1-n)+n\right)\frac{\partial\bar{p}}{\partial\bar{t}} + \underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0$$

where

$$\frac{\hat{\rho}^2\eta\hat{K}g}{\mu L} = O(10^{-1}\text{–}10^{-20})$$

$$\beta\hat{\rho}gL = O(10^{-4}\text{–}10^{-6})$$

$$\alpha/\beta = O(1)$$

Hence the dynamic behaviour in the equations is typically very quick and equilibrium equations are usually employed. One case where this is not true is when determining how much water can be pumped out of a confined aquifer when the specific volume storativity can be crucially important.

The equilibrium model usually employed is therefore

$$\underset{\sim}{v} = -K(\nabla p - \rho g\underset{\sim}{\nabla}z)$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0$$

or $$\underset{\sim}{\nabla}\cdot(K(\underset{\sim}{\nabla}p - \rho g\underset{\sim}{\nabla}z)) = 0$$

For mathematical simplicity the case of constant density and conductance is often studied when

$$\nabla^2 p = 0, \quad \underset{\sim}{v} = -K(\underset{\sim}{\nabla}p - \rho g\nabla z)$$

or $$\nabla^2\phi = 0, \quad \underset{\sim}{v} = -K\underset{\sim}{\nabla}\phi$$

FLOW IN UNSATURATED POROUS MEDIA

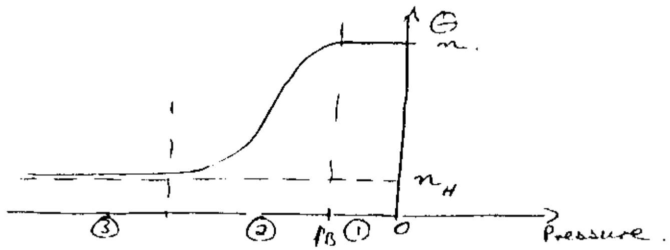
In the region above the water table in a phreatic aquifer the water is held in the pores by surface tension and by hydroscopic action ie electrical adhesion to the matrix. Within the matrix there may also be air in the pores, and models exist for both water and air motion, but most models neglect air movement. This avoids the problems when air pockets form in the pores. Within this region effects such as concentration or temperature gradients can create substantial flows but again these are usually neglected.

The conventional method of accomodating unsaturated porous media into the model is to introduce an additional variable, Θ, the moisture content of the media. By definition this variable satisfies

$$0 \le \Theta(\underset{\sim}{x}, t) \le n$$

The unsaturated region is then defined as that part of the flow where the average water pressure is less than atmospheric $(p < 0)$, due to the surface tension effects. A number of sweeping assumptions are then made in order to derive a simple model of the flow. These assumptions are based on some of the following experimental evidence.

If a completely saturated sample of a porous media is slowly drained then the pressure, due to surface tension, can be measured and plotted against moisure content to create a retention curve for the material.



The curve has three fairly distinct parts

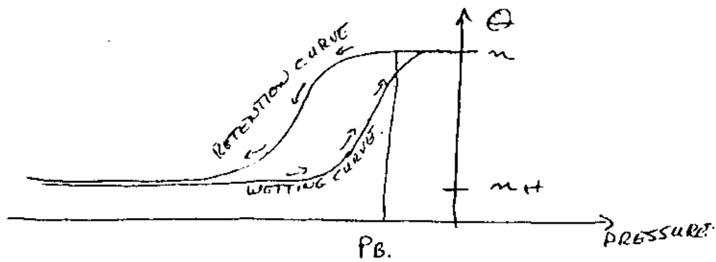
1 – the pores are still full of water and the pressure is higher than the pressure drop due to surface tension in the largest pore (this is called the bubbling pressure p_B).

2 – the pores contain air and water

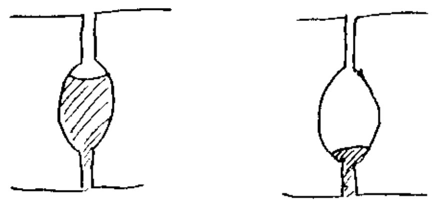
3 – only pellicular water (held by hydroscopic action) remains in the matrix. Here n_H is the effective porosity minus the volume fraction taken by pellicular water.

It is found that region ② shrinks in size as the spread in pore size is reduced. (eg: well graded sand has a small region ②) The retention curve represents the maximum water content which can occur at a particular pressure.

If the same sample is subsequently slowly wetted and the same quantities are measured they create a wetting curve.

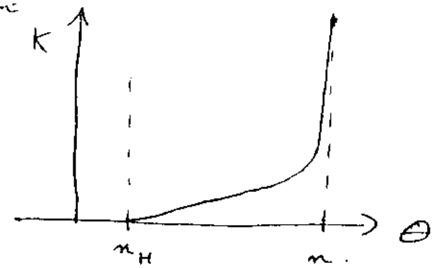


This hysteresis is commonly explained by the 'rain drop' phenomena, that an advancing water front has a larger contact angle that a retreating front, or by the 'ink bottle' phenomena, as shown below.



Both diagrams have the same pressure due to surface tension but very different water content. In a porous media there is a spread in pore size and this accounts for the large difference between wetting and draining a sample. All the points on the graph between the two curves can be reached by changing from draining to wetting at different times. The major assumption made in most models is that either only wetting or only draining occurs and therefore a unique curve of Θ vs p can be taken.

The flow through the unsaturated zone is usually taken to be driven by gradients in the surface tension pressure between different parts of the porous media. The enable Darcy's law to be continued into the unsaturated zone but with a different hydraulic conductivity, K. This quantity can most easily be measured as a function of Θ. For example if the saturated sample is slowly drained then:



The use of the previous curve with Θ vs p enable a function $K(p)$ to be approximated. A typical form for this is

$$K(p) = K_0 \qquad p > p_B$$

$$K(p) = K_0 \cdot (p/p_B)^m \qquad p \leq p_B$$

where $K(\Theta) = B(\Theta - n_H)^3$ with K_0, m, B constant.

The model equations are then

$$\frac{\partial \Theta(p)}{\partial t} - \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$$

$$\underset{\sim}{v} = -K(p)\,\underset{\sim}{\nabla}(p + \rho g z)$$

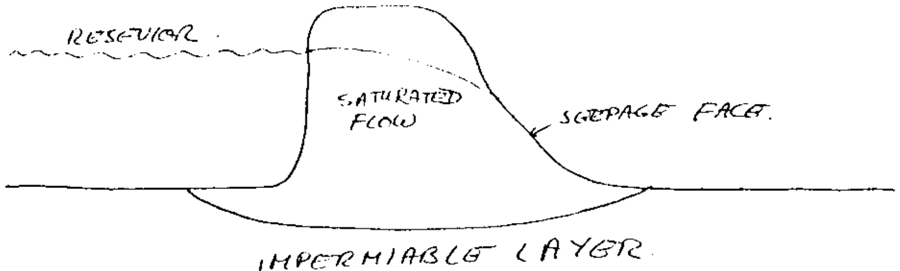
or

$$\frac{\partial \Theta(p)}{\partial t} + \underset{\sim}{\nabla} \cdot \left(K(p)\,\underset{\sim}{\nabla}(p + \rho g z)\right) = 0$$

This is a non-linear diffusion equation and has been studied by many researchers, in this area they typically consider the problem of wetting due to rain on the soil surface. It is interesting to note that if the problem is initially posed with $0 \leq \Theta \leq n_H$ in some region then the equation is a singular parabolic problem since $K(p) = 0$ for $0 \leq \Theta \leq n_H$.

SEEPAGE FLOW IN A DAM

One of the classical groundwater problems is to predict the water flow in a dam which acts to contain a reservoir.



The two pieces of information of interest are the flux of water lost through the dam and the water pressure distribution (this affects the stability of the dam and may indicate different materials are needed in different parts of the dam). The simplest model of the flow assumes the following

1 – negligible capillary water is present so pellicular and groundwater can be considered with a sharp interface between.

2 – air in the unsaturated region is at atmosphere

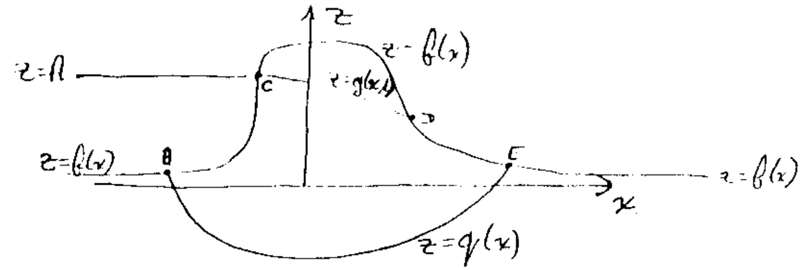
3 – water running off downstream is of negligible depth

4 – there is no fluid motion in the reservoir

5 – the matrix properties are uniform throughout the dam

6 – compressibility of water and matrix are negligible.

Mathematically we get:



solve $\nabla^2 p = 0$ in CDEB

with a) hydrostatic pressure in the reservoir

on BC $p = p_A + \rho g (A - f(x))$ p_A = atmosphere pressure

b) very thin layer of water downstream and flow is outward

on DE $p = p_A$, $\frac{\partial p}{\partial x}\frac{\partial f}{\partial x} - \frac{\partial p}{\partial z} - \rho g \geq 0$

c) no water flows into the impermeable layer.

on BE $\frac{\partial p}{\partial x}\frac{\partial q}{\partial x} - \frac{\partial p}{\partial z} - \rho g = 0$

d) water table is within the dam with atmosphere pressure and it moves to ensure mass is conserved.

on CD $g(x,t) < f(x)$

$$p = p_A$$

$$\frac{\partial g}{\partial t} - \frac{K}{n_H}\left(\frac{\partial p}{\partial x}\frac{\partial g}{\partial x} - \frac{\partial p}{\partial z} - \rho g\right) = 0$$

with some initial conditions concerning the position of $g(x,t)$

To simplify the notation of the problem it can be non-dimensionalised with

$$z = L\bar{z} \qquad x = l\,\bar{x} \qquad A = La$$

$$f(x) = LF(\bar{x}) \qquad g(x,t) = LG(\bar{x},\bar{t}) \qquad q(x) = LQ(\bar{x})$$

$$p = p_A + \rho g L(\bar{\phi} - \bar{z})$$, $\bar{\phi}$ is a non-dimensional piezometric head.

$$t = \frac{L\,n_H}{K\rho g}\,\bar{t}$$

to give

$$\nabla^2\bar{\phi} = 0 \quad \text{in } CDEB$$

on BC $\quad \bar{\phi} = a \quad$ on $\bar{z} = F(\bar{x})$

on DE $\quad \bar{\phi} = \bar{z} \quad$ on $\bar{z} = F(\bar{x})$

and $\quad \left(\frac{\partial\bar{\phi}}{\partial\bar{x}}\right)^2 - \frac{\partial\bar{\phi}}{\partial\bar{z}} > 0 \quad \left(\text{by using } \frac{\partial F}{\partial\bar{x}} = \frac{\partial\bar{\phi}}{\partial\bar{x}}\right)$

on BE $\quad \left(\frac{\partial\bar{\phi}}{\partial\bar{x}}\right)^2 - \frac{\partial\bar{\phi}}{\partial\bar{z}} = 0 \quad$ on $\bar{z} = Q(\bar{x})$

on CD $\quad G(\bar{x},\bar{t}) < F(\bar{x})$

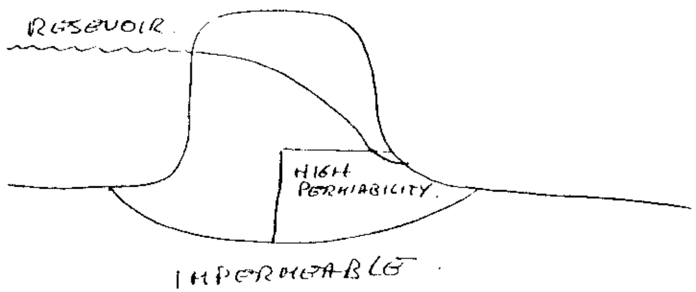
$$\bar{\phi} = \bar{z} \qquad \text{on } \bar{z} = G(x,t)$$

and $\quad \frac{\partial G}{\partial\bar{t}} - \left(\left(\frac{\partial\bar{\phi}}{\partial\bar{x}}\right)^2 - \frac{\partial\bar{\phi}}{\partial\bar{z}}\right) = 0$

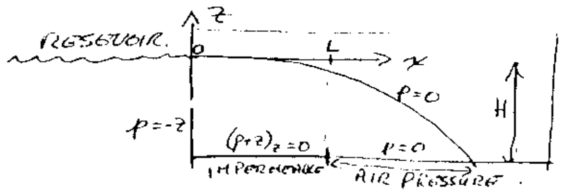
A solution is sought with $\bar{\phi}$ continuous and $|\nabla\bar{\phi}|$ bounded if possible (note $|\nabla\bar{\phi}|$ is the velocity of the flow). A local analysis can be done of the points where singularities might occur. The technique is to use complex variables eg. $\bar{\phi}+i\psi$ or $p+i\Psi$ and a local variable $Z = (x-x_0)+i(y-y_0)$. From this it is found that $|\nabla\bar{\phi}|$ is bounded everywhere but near D $g(x)$ has a singular curvature in steady state, $x - x_0 \sim (y-y_0)\ln(|y-y_0|)$. If a lower reservoir is considered $|\nabla\bar{\phi}|$ becomes unbounded where the seepage face meets the

reservoir surface. Similar unbounded behaviour can occur if capillary water is considered.

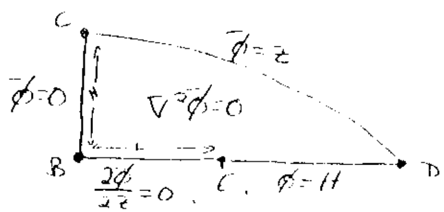
In any physical dam it is inadvisable to have a seepage face, if possible, because the water movement makes the soil unstable. A common way of avoiding this problem is to introduce material of higher permeability near the 'toe' of the dam to remove the seepage water through a drain system.



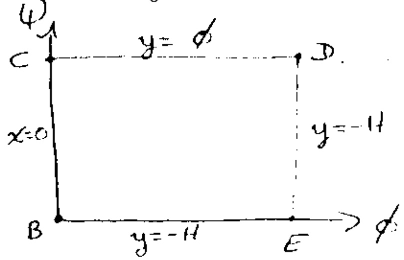
The extreme example of this problem can be solved exactly.



by changing to $\bar{\phi}$



and then changing to the $\phi + i\psi$ plane where



This problem can be solved by a series method and the solution found. It has the form

$$x + iy = -i + \sum_{n=0}^{\infty} \frac{8 \sin\left(\left(\frac{2n+1}{2}\right)\pi(\phi + i\psi)\right)}{\pi^2 (2n+1)^2 \sinh\left(\left(\frac{2n+1}{2}\right)\pi Q\right)}$$

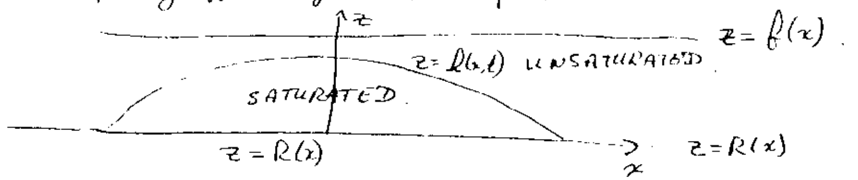
where Q, the flux through the dam is given by.

$$L = \sum_{n=0}^{\infty} \frac{8(-1)^n}{\pi^2 (2n+1)^2 \sinh\left(\left(\frac{2n+1}{2}\right)\pi Q\right)}$$

MODELLING OF REGIONAL GROUNDWATER FLOW

The previous section dealt with the details of water motion in a saturated aquifer including the free surface effects. If the behaviour of a large aquifer needs modelling to determine how it may be utilized as a resource the computational effort needed to solve the problem becomes prohibitive. The 3-Dimensional time dependent problem is beyond most computer resources and there are few techniques for measuring all the necessary physical information needed to set up the model. An approximation is therefore made, which makes the problem tractable, and which gives excellent approximate solutions. This is based on the fact that most aquifers are formed by pervious rock surrounded by impervious layers or free surfaces and that their thickness is a much smaller dimension than their horizontal extent. This allows vertical velocities to be taken to be much smaller than horizontal velocities and for pressure to be essentially hydrostatic. This is conventially called the 'Hydraulic Approach' or 'Dupuit Approximation' after Dupuit (1863) who first used it to calculate discharge rates from aquifers. As will be seen this reduces the dimensionality of the problem by one.

Consider, for example a phreatic aquifer in an isotropic matrix with capillary effects neglected. The problem is



for $R(x) < z < h(x,t)$ $\quad \nabla \cdot \left(K(x,z) \nabla (p + \rho g z) \right) = 0$

on $z = R(x)$ $\quad K(x,z)\left\{ \frac{\partial p}{\partial x}\frac{\partial R}{\partial x} - \frac{\partial p}{\partial z} - \rho g \right\} = 0$

on $z = h(x,t)$, $\quad h(x,t) < f(x)$

$$p = p_A$$

$$\frac{\partial h}{\partial t} - \frac{K(x,z)}{n_H}\left\{ \frac{\partial p}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial p}{\partial z} - \rho g \right\} = 0$$

+ boundary conditions in x and initial conditions

Non dimensionalize by putting

$$p = p_A + \rho g l \bar{p} \qquad x = L\bar{x} \quad z = l\bar{z}$$

$$f = l\bar{f} \qquad h = l\bar{h} \qquad R = l\bar{R} \qquad K = k\bar{K}$$

$$\text{and} \quad t = \left(\frac{L n_H}{\varepsilon k \rho g} \right) \bar{t}$$

Where l is a typical thickness of the aquifer and L is a typical length so that, by assumption

$$l/L = \varepsilon \ll 1$$

The time has been non-dimensionalized with the time for water to travel a distance L under a pressure gradient of $l(\varepsilon)$.

This gives the problem

$$\varepsilon^2 \frac{\partial}{\partial x}\left(\bar{K}(\bar{x},\bar{z}) \frac{\partial \bar{p}}{\partial x} \right) + \frac{\partial}{\partial \bar{z}}\left(\bar{K}(\bar{x},\bar{z})\left(\frac{\partial \bar{p}}{\partial \bar{z}} + 1 \right) \right) = 0$$

on $\bar{R}(\bar{x}) = \bar{z}$ $\quad \bar{K}(\bar{x},\bar{z})\left(\varepsilon^2 \frac{\partial \bar{p}}{\partial \bar{x}}\frac{\partial \bar{R}}{\partial \bar{x}} - \frac{\partial \bar{p}}{\partial \bar{z}} - 1 \right) = 0$

on $\bar{h}(\bar{x},\bar{t}) = \bar{z}$ $\quad \bar{h}(\bar{x},t) < f(\bar{x})$

$$\bar{p} = 0$$

$$\varepsilon^2 \frac{\partial \bar{h}}{\partial t} - K(\bar{x},\bar{z})\left\{ \varepsilon^2 \frac{\partial \bar{p}}{\partial \bar{x}}\frac{\partial \bar{h}}{\partial \bar{x}} - \frac{\partial \bar{p}}{\partial \bar{z}} - 1 \right\} = 0$$

+ boundary and initial conditions.

An asymptotic expansion is now sought in the limit $\varepsilon \to 0$.

We assume

$$\bar{p}(\bar{x},\bar{t};\varepsilon) \sim p_0(\bar{x},\bar{t}) + \varepsilon^2 p_1(\bar{x},\bar{t}) + \cdots$$

$$\bar{h}(\bar{x},\bar{t};\varepsilon) \sim h_0(\bar{x},\bar{t}) + \varepsilon^2 h_1(\bar{x},\bar{t}) + \cdots$$

Hence the $O(1)$ problem is

$$\frac{\partial}{\partial \bar{z}}\left(\bar{K}(\bar{x},\bar{z})\left(\frac{\partial p_0}{\partial \bar{z}} + 1 \right) \right) = 0.$$

on $\bar{R}(\bar{x}) = \bar{z}$ $\quad \bar{K}(\bar{x},\bar{z})\left(-\frac{\partial p_0}{\partial \bar{z}} - 1 \right) = 0.$

on $h_0(\bar{x},\bar{t}) = \bar{z}$ $\quad h_0 < f(\bar{x})$

$$p_0 = 0$$

$$\frac{\partial p_0}{\partial \bar{z}} + 1 = 0$$

which has the solution $p_0(\bar{x},\bar{t}) = h_0(\bar{x},\bar{t}) - \bar{z})$ ie. the pressure is hydrostatic. If the $O(\varepsilon^2)$ problem is then set up.

$$\frac{\partial}{\partial \bar{x}}\left(\bar{K}(\bar{x},\bar{z}) \frac{\partial h_0}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{z}}\left(\bar{K}(\bar{x},\bar{z}) \frac{\partial p_1}{\partial \bar{z}} \right) = 0$$

on $\bar{R}(\bar{x}) = \bar{z}$ $\quad \frac{\partial p_1}{\partial \bar{z}} = \frac{\partial h_0}{\partial \bar{x}}\frac{\partial \bar{R}}{\partial \bar{x}}$

on $h_0(\bar x,t)=\bar z$, $h_1(\bar x,t)=0$ $\quad h_0<b(\bar x)$.

$$p_1=0$$

$$\frac{\partial p_1}{\partial \bar z} = \left(\frac{\partial h_0}{\partial \bar x}\right)^2 - h_1\frac{\partial^2 p_0}{\partial \bar z^2} - \frac{\partial h_0}{\partial t}\Big/K(\bar x, h_0)$$

Integrating the equation over and imposing the condition at $\bar R(\bar x)=\bar z$ gives

$$\int_{\bar R(\bar x)}^{\bar z} \frac{\partial}{\partial \bar x}\left(\bar K(\bar x,\bar z)\frac{\partial h_0}{\partial \bar x}\right)dz + \bar K(\bar x,\bar z)\frac{\partial p_1}{\partial \bar z} - \bar K(\bar x,\bar R(\bar x))\frac{\partial h_0}{\partial \bar x}\frac{\partial \bar R}{\partial \bar x} = 0$$

If this is then evaluated at $\bar z = h$ this gives

$$\int_{\bar R(\bar x)}^{h_0(\bar x,t)} \frac{\partial}{\partial\bar x}\left(\bar K(\bar x,\bar z)\frac{\partial h_0}{\partial \bar x}\right)dz + \bar K(\bar x,h_0)\left(\frac{\partial h_0}{\partial \bar x}\right)^2 - \frac{\partial h_0}{\partial t} - \bar K(\bar x,\bar R)\frac{\partial h_0}{\partial\bar x}\frac{\partial \bar R}{\partial \bar x}=0$$

or

$$\frac{\partial}{\partial \bar x}\left\{\int_{\bar R(\bar x)}^{h_0(\bar x,t)}\bar K(\bar x,\bar z)\,dz\,\frac{\partial h_0}{\partial\bar x}\right\}=\frac{\partial h_0}{\partial t}$$

This is an equation for $h_0(\bar x,t)$ which must be solved with appropriate boundary and initial data. The equivalent equation for an aquifer confined between $\bar T(\bar x)$ and $\bar R(\bar x)$ is

$$\frac{\partial}{\partial \bar x}\left\{\int_{\bar R(\bar x)}^{\bar T(\bar x)} K(\bar x,\bar z)\,dz\,\frac{\partial p_0}{\partial\bar x}\right\}=0$$

The quantities

$$\int_{\bar R(\bar x)}^{h_0(\bar x,t)} \bar K(\bar x,\bar z)\,dz = T(\bar x,\bar z,h_0)$$

and

$$\int_{\bar R(\bar x)}^{\bar T(\bar x)} \bar K(\bar x,\bar z)\,dz = T(\bar x,\bar z)$$

are called the nondimensional transmissivity of the aquifer. These equations do not contain z as an independent variable and are therefore easier to study. It is also possible to infer the transmissivity of an aquifer from physical pumping data. For a physical problem sources and sinks (due, for instance, to rain infiltration or pumping) must be included and in deep confined aquifers 'compressional' effects of water and rock can be important.

Because the equation for a phreatic aquifer is non-linear analytical methods of solution are limited and numerical methods must be carefully constructed. For the confined aquifer the problem is linear and if $\bar R$, $\bar T$ and $\bar K$ are constant then Laplace's equation appears and complex variable techniques apply. Two common approximations for phreatic aquifers are firstly to assume an average $\hat T(\bar x,\bar z)$ can be taken independent of h_0, or secondly to consider the steady state with $\bar R$, $\bar K$ constant when Laplace's equation for h_0^2 is relevant.

The equation for h_0 in a phreatic aquifer is parabolic except for points where $h_0=\bar R(x)$ when the problem is singular. Behaviour of such singular parabolic equations is of great interest to mathematicians. At the singular point a classical solution is appropriate and instead a weak solution is sought which ensures the fluid velocity is bounded near the singular point. In practice this implies the singular point travels at the velocity of the fluid velocity at that point. A particular interesting aspect of such equations is that for parabolic equations

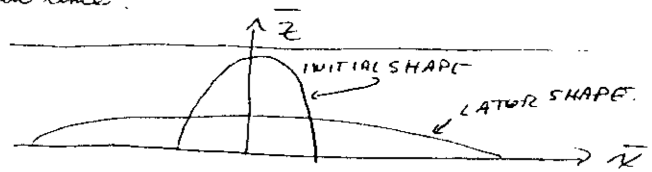
$$\text{eg:} \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with initial data with compact support the support is lost for all later times. However the singular parabolic problem

$$\text{eg} \quad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right) \qquad n > 0$$

retains its support for all time.

As an example of this behaviour consider a region of uniform porous rock bounded by a flat lower surface, $\bar{K}$ is constant and $\bar{R} = 0$. Determine how a blob of water in the media will spread in time.



$$\text{so} \quad \frac{\partial}{\partial x}\left(K h \frac{\partial h}{\partial x}\right) = K n_H \frac{\partial h}{\partial t} \quad \text{(in dimensional terms)}$$

and we will seek a similarity solution to this problem so we expect $\int_{-\infty}^{\infty} h(x,t)\,dx = M n_H$ where M is the total volume of water in the blob.

We assume a solution to exist of the form

$$h(x,t) = t^{\alpha} f(x/t^{\beta}) = t^{\alpha} f(\eta)$$

where α and β are constants to be chosen and $\eta = x/t^{\beta}$. The integral condition becomes

$$\int_{-\infty}^{\infty} t^{\alpha} f(x/t^{\beta})\,dx = M n_H$$

or $t^{\alpha+\beta} \int_{-\infty}^{\infty} f(\eta)\,d\eta = M n_H$ so that if this is true $\forall t$ we require $\alpha = -\beta$. Putting $h = t^{-\beta} f(x/t^{\beta})$ in the equation gives

$$t^{2\beta} \frac{\partial}{\partial \eta}\left(f \frac{\partial f}{\partial \eta}\right) = t^{-\beta-1}(-\beta) n_H \left(f + \eta \frac{\partial f}{\partial \eta}\right)$$

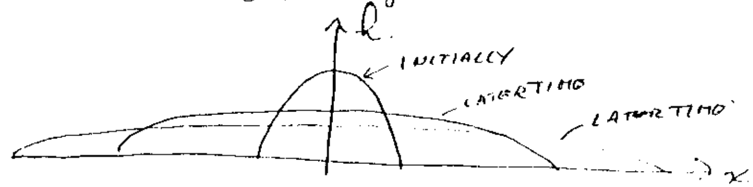
and since this holds $\forall t$ we require $\beta = -1/3$, $\alpha = 1/3$.

A solution can then be found by integrating the equation to give

$$f(\eta) = \frac{n_H}{6} t^{-1/3}(A^2 - \eta^2) \qquad |\eta| \le A$$

$$= 0 \qquad |\eta| \ge A$$

where $A = \left(\frac{9M}{2}\right)^{1/3}$. The condition at the singular points $|\eta| = A$ has been satisfied. Graphically the solution is



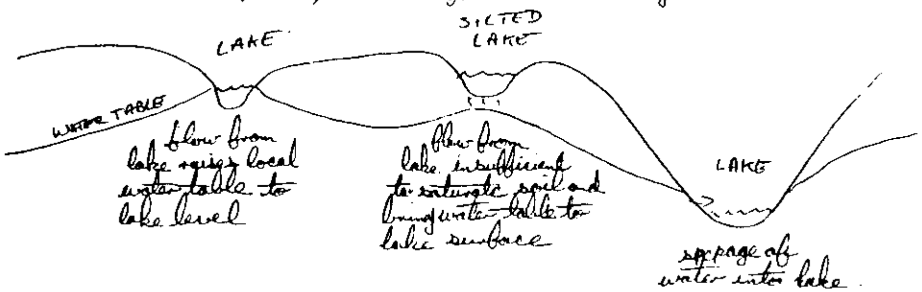
When modelling regional groundwater flow it is necessary to consider various types of sources and sinks within the model region. The main effects to be accounted for are

1) rainfall
2) rivers and lakes
3) leakage into other aquifers
4) wells.

1) Rainfall – The way in which rainwater gets into the groundwater system depends crucially on the permeability of the top layer of soil and requires analysis of the soil water layer (note: in urban areas there tends to be no infiltration of water into the ground due to the extensive use of drains). The water that gets into the soil water layer is affected by surface tension, gravity, evaporation and absorption by plants (transpiration). Estimates of the water lost by evaporation and transpiration tend to be fairly unreliable and depend on air temperature, humidity, type of vegetation and other effects. The water that gets through the soil layer falls under gravity and surface tension to the water table. It can be represented in the equations as a source term on the phreatic

surface over the area of interest with, perhaps, seasonal fluctuations. A confined aquifer has no rainfall source term.

2) Rivers and Lakes – As with rainfall the permeability of the soil layer lining the river bed or lake can substantially alter the amount of water exchanged with an aquifer. In particular a lake which seeps into an aquifer may develop a layer of silt on its bed which greatly reduces the amount of infiltration. Consider a phreatic aquifer, then a typical case may be as below.



For a confined aquifer exchange with lakes and rivers only occurs if they penetrate into the aquifer through the impermeable layer. The simplest exchange model is called the 'leaky stream' model and assumes the lake is silted so

$$\text{flux} \propto (H_s - h) \qquad \text{for } h > H_d$$
$$\text{or} \quad \text{flux} \propto (H_s - H_d) \qquad \text{for } h < H_d$$

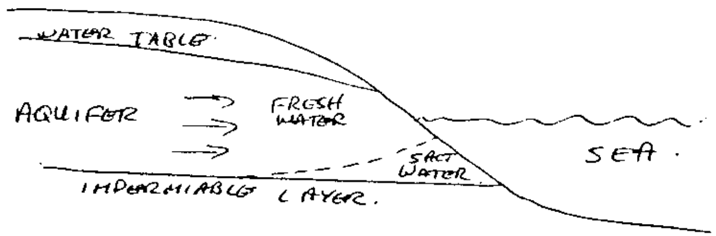
where h is the local water table, H_d is the stream bed, and H_s is the stream height. It should be noted that in a practical problem equations relating stream and lake depth to run-off and seepage rate will be required.

3) <u>Leakage into other aquifers</u> — This usually takes place through a semipervious layer (an aquitard). In such a layer the flow is also normal to the interface, due to the large change in permeability. Hence if the semipervious layer is thin we can take $\text{flow} \propto (p_1 - p_2)$ where p_1 is the pressure in aquifer 1 and p_2 in aquifer 2. This requires some knowledge about both aquifers, such as setting up equations in both.

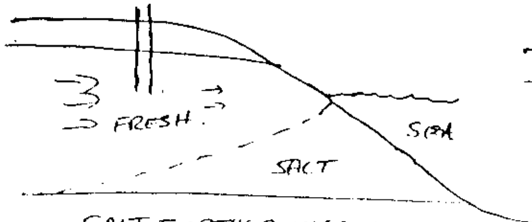
4) <u>Wells</u> — These can be pumping water into or out of the aquifer. If such a well penetrates the aquifer it may be possible to approximate it as a point source or sink in the equations. However, locally there may be 3 dimensional effects of importance in which case the Dupuit approximation may be invalid or the fluid velocities may get so large that Darcy's becomes inappropriate.

AQUIFERS NEAR THE COAST.

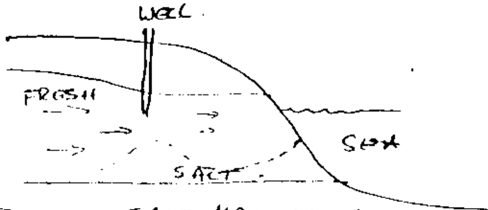
In the previous sections it has been assumed that the density differences in the water are very small and can be neglected. This is usually a reasonable assumption except when considering an aquifer that extends into the sea. In such cases the flow of water from the land must act against the heavier salt water in the sea which pushes inland within the aquifer.



Predicting the behaviour of this salt water is very important as any pumping from wells near the coast will reduce the outflow of fresh water thereby allowing saltwater further inland. Also the act of pumping reduces the pressure locally and may cause an upwelling of the heavier salt water. In either case the pumped water may become contaminated and undrinkable.



SALT FURTHER INLAND



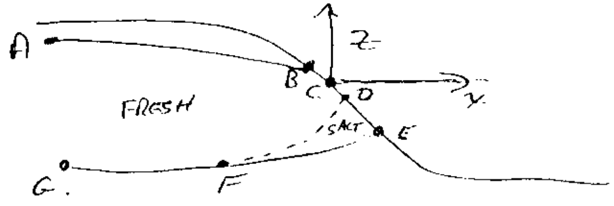
SALT UPWELLING

In practice there is no sharp interface between salt water and fresh because of dispersion within the porous media. In the case of a coastal aquifer this dispersion can be greatly enhanced because of the periodic flows created by tidal variations in the sea level. There are additional problems in that the salt gradients can generate flow independently of gravitational effects.

In order to proceed however, the interface between salt and fresh water is assumed to be sharp for the flow problem and the dispersion problem is usually solved subsequently to determine how sharp the interface actually is.

As an example of how such a problem can be put into a mathematical form consider the flow given at the beginning of this section and we shall set up equations to describe the steady state flow. In such a case the fresh water flows from

some inland source and the salt water remains stationary.



Hence in DEF the water pressure is hydrostatic with

$$p = p_A - \rho_s g z \qquad \text{where } \rho_s \text{ is the density of saltwater}$$

We take $z = f(x)$ as the soil surface, $z = h(x)$ as the saturated/unsaturated interface AB, $z = I(x)$ as the fresh/salt interface DF, and $z = R(x)$ as the lower impassable bottom.

We have $\nabla^2 p = 0$ in ABCDFG

with on $z = h(x)$ AB
$$p = p_A$$
$$R(x) < h(x) < f(x)$$
$$\frac{\partial p}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial p}{\partial z} - \rho g = 0$$

on $z = f(x)$ BC $\quad p = p_A$

on $z = f(x)$ CD $\quad p = p_A - \rho_s g z$

on $z = I(x)$ DF
$$R(x) < I(x) < f(x)$$
$$p = p_A - \rho_s g z$$
$$\frac{\partial p}{\partial x}\frac{\partial I}{\partial x} - \frac{\partial p}{\partial z} - \rho g = 0.$$

on $z = R(x)$ GF
$$\frac{\partial p}{\partial x}\frac{\partial R}{\partial x} - \frac{\partial p}{\partial z} - \rho g = 0.$$

plus a condition near AG to describe the upstream flow.

Nondimensionalizing with

$$x = L\bar{x} \quad z = l\bar{z}, \; f = l\bar{f}, \; h = l\bar{h}, \; R = L\bar{R} \quad I = L\bar{I}$$

$$p = p_A + \rho g l(\bar{\phi} - \bar{z})$$

gives $$\varepsilon^2 \frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{z}^2} = 0 \qquad \text{in } ABCDFG.$$

on $\bar{z} = \bar{h}$ AB
$$\bar{\phi} = \bar{h}$$
$$\bar{R} < \bar{h} < \bar{f}$$
$$\varepsilon^2 \frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{\partial\bar{h}}{\partial\bar{x}} - \frac{\partial\bar{\phi}}{\partial\bar{z}} = 0$$

on $\bar{z} = \bar{f}$ BC $\quad \bar{\phi} = \bar{f}$

on $\bar{z} = \bar{f}$ CD $\quad \bar{\phi} = \left(\frac{\rho_s - \rho}{\rho}\right)(-\bar{f})$

on $\bar{z} = \bar{I}$ DF
$$R < I < f.$$
$$\bar{\phi} = \left(\frac{\rho_s - \rho}{\rho}\right)(-\bar{I})$$
$$\varepsilon^2 \frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{\partial\bar{I}}{\partial\bar{x}} - \frac{\partial\bar{\phi}}{\partial\bar{z}} = 0$$

on $\bar{z} = \bar{R}$ GF $\quad \varepsilon^2 \frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{\partial\bar{R}}{\partial\bar{x}} - \frac{\partial\bar{\phi}}{\partial\bar{z}} = 0$

This problem has two parameters $\varepsilon = l/L$ and $\frac{\rho_s - \rho}{\rho} = \delta$, the specific density difference, which for sea water is approximately 0.025. If the Dupuit approximation, $\varepsilon \to 0$, is made along with assumption $\delta = O(1)$ then an approximation called the Ghyben–Herzberg is generated. The equations are

$$\frac{\partial^2\phi_0}{\partial\bar{z}^2} = 0 \quad \text{so that} \quad \phi_0 = A(x)z + B(x).$$

$\bar{z} = \bar{h}$ AB $\quad \phi_0 = h_0$, $\quad \frac{\partial\phi_0}{\partial\bar{z}} = 0$

$\bar{z} = \bar{f}$ BC $\quad \phi_0 = \bar{f}$

$\bar{z} = \bar{f}$ CD $\quad \phi_0 = -\delta\bar{f}$

$\bar{z} = \bar{I}_0$ DF $\quad \phi_0 = -\delta I_0$, $\quad \frac{\partial\phi_0}{\partial\bar{z}} = 0.$

$\bar{z} = \bar{R}$ GF $\quad \frac{\partial\phi_0}{\partial\bar{z}} = 0$

Hence
$$x < B \qquad \phi_0 = h_0$$
$$B < x < C \qquad \phi_0 = \bar{f}$$
$$C < x < D \qquad \phi_0 = -\delta\bar{f}$$

$$F < x < B \qquad I_0 = -\frac{1}{\delta}h_0.$$
$$B < x < C \qquad I_0 = -\bar{f}$$
$$C < x < D \qquad I_0 = \bar{f}$$

It follows therefore that $C \equiv D$ and the fresh water does not extend under the sea to this approximation. This solution is not valid close to D and there are some interesting asymptotic analysis questions concerning the behaviour there.