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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392 - I

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SECOND WORKSHOP ON MATHEMATICS IN INDUSTRY

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IDENTIFICATION OF LINEAR DYNAMICAL SYSTEMS

M. HAZEWINKEL
CWI
P.O.Box 4079
1009AB Amsterdam

Lectures on identification of linear dynamical systems

Michiel Hazewinkel

CWI

PO Box 4079

1009 AB Amsterdam

1 Introduction to the problem.

We are concerned with dynamical systems of the types

(1.1) $$x(t+1) = Ax(t) + Bu(t) \quad , \quad y(t) = Cx(t)$$ (discrete time, no direct feedthrough)

(1.2) $$\dot{x} = Ax + Bu \;, \quad y = Cx$$ (continuous time, no direct feedthrough)

(1.3) $$x(t+1) = Ax(t) + Bu(t) \;, \quad y(t) = Cx(t) + Du(t)$$ (discrete time and direct feedthrough)

(1.4) $$\dot{x} = Ax + Bu, \quad y = Cx + Du$$ (continuous time and direct feedthrough)

For definiteness I shall concentrate on systems of type (1.1). The considerations below change very little, if at all, for systems of one of the other three types.

In all of the above x is an n-vector (the state vector), u is an m-vector (inputs or controls) and y is a p-vector (outputs or observations), A, B, C are real matrices of the appropriate sizes. In particular they do not depend on time.

The problem we want to address is the following. Suppose we have available time series of data $u(0), u(1), \ldots, u(t)$; $y(0), y(1), \ldots, y(t)$. Suppose also that we have reason to believe that the u's and y's are related by a model like (1.1) (or for lack of any better idea we want to find out if they could be related like that). How

Can one find the A, B, C matrices of (11) from the given data?

The problem is of some importance because very many ~~com~~ phenomena in engineering, chemistry, economics and physics can be modeled by such systems. And once we have identified the system, i.e. calculated A, B, C, all the powerful techniques of linear control theory become available, ~~for~~ to do stabilization, decoupling, Kalman filtering, etc. etc.

Variants of the problem deal with systems like the above with possibly a noise term $Bw(t)$ instead of $Bu(t)$ or with such a noise term added and possible with a noise term $v(t)$ added to $y(t)$ as well (noise corrupted observations). Similar ideas as those outlined below work in those cases.

We shall in any case have to live with the fact that the actually available data are not completely exact.

The first remark is that, as posed, the problem is unsolvable. No amount of input-output data can determine A, B, C uniquely. Indeed let $S \in GL_n(\mathbb{R})$ the group of invertible $n \times n$ real matrices. I claim that the new triple of matrices (SAS^{-1}, SB, CS^{-1}) determines precisely the same input-output relations as (A, B, C). Indeed the relation between y and u is (if we start at $x(0)=0$)

$$(15) \qquad y(n) = CA^{n-1}Bu(0) + CA^{n-2}Bu(1) + \cdots + CABu(n-2) + CBu(n-1)$$

And, clearly, if we substitute SAS^{-1} for A, SB for B, CS^{-1} for C this relation continues to hold.

If we do not assume that the system necessarily starts in $x(0)=0$ the problem becomes the one of identifying A, B, C and the initial state $x(0)$ and exactly the same phenomenon occurs vis à vis the transformation

$$(A, B, C, x(0)) \longmapsto (SAS^{-1}, SB, CS^{-1}, Sx(0))$$

as immediately follows from

$$(16) \qquad y(n) = CA^n x(0) + CA^{n-1}Bu(0) + \dots + CABu(n-2) + CBu(n-1)$$

Under mild restrictions (complete reachability and complete observability) the indeterminacy described above is the only indeterminacy in (A,B,C). ~~Thus~~ Let $L^{co,cr}_{m,n,p}$ be the space of all triples of matrices which are completely reachable (cr) and completely observable (co). Then the best we can do in the way of identification is to find (A,B,C) up to a possible S-transformation. Or in other words identifying the system means finding a particular point of the quotient space

$$(17) \qquad M^{co,cr}_{m,n,p} = L^{co,cr}_{m,n,p} / GL_n(\mathbb{R})$$

of $L^{co,cr}_{m,n,p}$ by the equivalence relation defined by the action of $GL_n(\mathbb{R})$ on $L^{co,cr}_{m,n,p}$ given by $(A,B,C)^S = (SAS^{-1}, SB, CS^{-1})$.

As it turns out $M^{co,cr}_{m,n,p}$ is as a rule quite a complicated space and that is precisely what makes identification a far from trivial problem.

We shall also see below that $M^{co,cr}_{m,n,p}$ is very nice in the sense that it is a smooth differentiable manifold of dimension $nm + np$, noncompact, of finite Ljusternik-Schnirelman category, special as an algebraic variety, ..., and with a whole host of other nice properties.

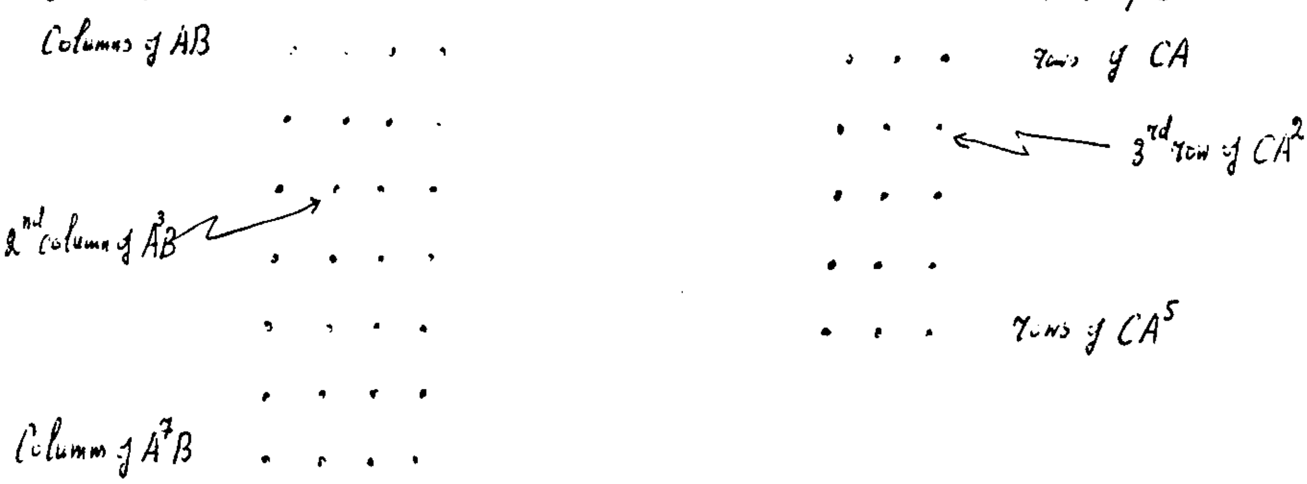
2. Nice selections

Both for identification purposes and other things we need the concept of a nice selection. Given a system AH (1.1), i.e. a triple of matrices $(A,B,C) \in L_{m,n,p}$ the space of all triples of real matrices of sizes $n \times n$, $n \times m$, $p \times n$ respectively, one defines the controllability matrix $R(A,B)$ and the observability matrix $Q(A,C)$ as follows

$$(2.1) \qquad R(A,B) = (B \;\; AB \;\; A^2B \;\cdots\; A^nB) \qquad Q(A,C) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^n \end{pmatrix}$$

Thus $R(A,B)$ consists of the m column vectors of B, then the m column vectors of (the product) AB, ... and is an $n \times (n+1)m$ matrix. Similarly $Q(A,C)$ is an $(n+1)p \times n$ matrix.
The system (A,B,C) is cr if $R(A,B)$ has its maximal rank n and it is co if $Q(A,C)$ has its maximal rank n. These two conditions define the open subspace $L^{co,cr}_{m,n,p}$ of $L_{m,n,p}$.

It is convenient to have a good way of labelling the columns of $R(A,B)$ and the rows of $Q(A,C)$. For the columns of $R(A,B)$ this is done by an $(n+1) \times p$ array as depicted below (on the left) (for the case $m=4$, $n=7$).



For the rows of $Q(A,C)$ an array of size $(n+1)\times p$ is used as is illustrated above on the right for $p=3$ and $n=5$

A nice input selection α_R is now a subset, of size n of the $R(\ ,\ )$ array with the property that if a given element is in α_R than all the elements ~~at~~ straight above it are also in it.

A nice output selection α_Q is a subset of size n of the Q array with the same property. Thus for example if we indicate the elements which are in a selection with crosses then below the selections on the left and on the right are nice and the ~~two~~ three ones in the middle are not nice

$$
\begin{array}{ccc}
\times & \times & \times \\
\cdot & \times & \cdot \\
\cdot & \times & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array}
\qquad
\begin{array}{ccc}
\cdot & \times & \times \\
\times & \times & \times \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array}
\qquad
\begin{array}{ccc}
\times & \cdot & \cdot \\
\times & \cdot & \cdot \\
\times & \cdot & \cdot \\
\times & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array}
\qquad
\begin{array}{ccc}
\cdot & \cdot & \times \\
\cdot & \cdot & \times \\
\cdot & \times & \times \\
\cdot & \cdot & \times \\
\cdot & \cdot & \cdot \\
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\end{array}
\qquad
\begin{array}{ccc}
\times & \cdot & \times \\
\times & \cdot & \cdot \\
\times & \cdot & \cdot \\
\times & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array}
$$

(nice) (gap) (too small) (gaps) (nice)

A nice selection is thus uniquely given by a sequence of numbers in $\mathbb{N}\cup\{0\}$ giving the number of crosses in each column of the array. E.g. ~~by~~ $\alpha=(1,3,1)$ is the leftmost case above.

2.2. Lemma Let $(A,B,C)\in L^{co,cr}_{m,n,p}$ then there is a nice input selection α_R and a nice output selection α_Q such that the matrices $R(A,B)_{\alpha_R}$ and $Q(A,C)_{\alpha_Q}$ are invertible

(– more precisely the column labels

Here if Π is a matrix and α is a subset of its columns ~~(indices)~~, then Π_α denotes the matrix obtained from Π by removing all columns which are not in α. Similarly if β is a subset of its row labels then Π_β is found by removing all rows whose label is not in β.

One way to find an α_R and α_Q is as follows. Go through the columns of $R(A,B)$ in the order in which they appear in $R(A,B)$. Every time you meet one which is linearly independent of all the preceding ones ~~put a cross~~ select it, i.e. put a cross. Because $R(A,B)$ has rank n you will find precisely n crosses this way and it is a mild exercise in linear algebra (cf. [1]) to prove that the resulting selection is nice.
To find a nice output selection do the same with the rows of $Q(A,C)$.
The nice selections thus found are the <u>Kronecker input selection</u> and the <u>Kronecker output selection</u>. I shall denote them $\kappa_R(A,B,C)$, $\kappa_Q(A,B,C)$.

2.3. <u>Example</u>.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 1 & \varepsilon & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q(A,C) = \left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \eta & \varepsilon\eta & 0 & \eta & 0 \\ 1 & \varepsilon & 0 & 1 & 0 \\ 0 & 0 & 2\eta+\varepsilon\eta^2 & 0 & \eta \\ 0 & 0 & 2+\varepsilon\eta & 0 & 1 \\ \cdots & & & & \\ \cdots & & & & \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} \text{rows of } C \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \text{rows of } CA \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \text{rows of } CA^2 \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \text{rows of } CA^3 \\ \\ \\ \end{array}$$

Thus if $\eta \neq 0$ the Kronecker output selection is $K_Q = (4, 1)$, i.e.

$$\begin{array}{ll} \times & \times \\ \times & \\ \times & \\ \times & \end{array}$$

2.4 Exercise Show that if $\eta = 0$ $K_Q = (1, 4)$. Calculate K_R for $\varepsilon \neq 0$ and for $\varepsilon = 0$

3 Recovering (A, B) from $R(A,B)$

As a preliminary exercise and a necessary technique in identification let us describe how for a completely reachable pair of matrices (A, B) one can recover the matrices (A, B) from the reachability matrix $R(A,B) = (B\ AB \cdots A^n B)$. Let α be a nice selection such that $R(A,B)_\alpha = S$ is invertible. First consider $(S^{-1}AS, S^{-1}B)$. Note that

$$(3.1) \qquad R(S^{-1}AS, S^{-1}B) = S^{-1} R(A,B)$$

and that for any selection (of columns) α

$$(3.2) \qquad (S^{-1} R(A,B))_\alpha = S^{-1}(R(A,B)_\alpha)$$

Thus for our particular S and α we have

$$(3.3) \qquad R(S^{-1}AS, S^{-1}B)_\alpha = S^{-1} R(A,B)_\alpha = I_n$$

Thus $R(S^{-1}AS, S^{-1}B)_\alpha$ is the identity matrix and that means that we can read of the columns of $S^{-1}AS$ and $S^{-1}B$ directly from $R(S^{-1}AS, S^{-1}B)$. Let us illustrate that in the case of the selection on the left below

$$\begin{matrix} \times & \cdot & \times & \times & \times \\ \cdot & \cdot & \times & \times & \cdot \\ \cdot & \cdot & \times & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \qquad\qquad \begin{matrix} e_1 & s_2 & e_2 & e_3 & e_4 \\ s_1 & \cdot & e_5 & e_6 & s_5 \\ \cdot & \cdot & e_7 & s_4 & \cdot \\ \cdot & \cdot & s_3 & \cdot & \cdot \end{matrix}$$

Denote with $s_1, .., s_5$ the columns of $R(S^{-1}AS, S^{-1}B)$ whose labels are directly below the crosses of the nice selection, cf above on the right. We also know that $R(S^{-1}AS, S^{-1}B)_\alpha = I_n$, so at the crosses there are the various unit vectors, cf. also above on the right. Thus

$$S^{-1}B = (e_1 \ s_2 \ e_2 \ e_3 \ e_4)$$

$$S^{-1}AS = (s_1 \ e_5 \ e_6 \ s_5 \ e_7 \ s_4 \ s_3)$$

and knowing $S = R(A,B)_\alpha$ we also have found the matrices A and B. (Of course B was no problem anyway as it consists of the first m columns of $R(A,B)$).

Similarly one can of course recover A, C from $Q(A,C)$ in the completely observable case.

Incidentally note that one may need all of $R(A,B)$ and $Q(A,C)$ to do this calculation including the blocks A^nB and CA^n respectively. This is the reason for making $R(A,B)$ and $Q(A,C)$ one block longer than is customary in other contexts e.g. the definition of CR and Co

4. The Hankel matrix

The Hankel matrix of a system Σ_n ~~such as~~ $\Sigma = (A,B,C)$ such as (1.1) is defined by

$$(4.1) \qquad H(A,B,C) = Q(A,C)\,R(A,B)$$

Note that $H(SAS^{-1}, SB, CS^{-1}) = H(A,B,C)$. Thus the entries of $H(A,B,C)$ are invariant for the action of $GL_n(\mathbb{R})$ on $L_{m,n,p}$ (cf. below). They are also in a sense the only invariants, cf. also below.

The matrix $H(A,B,C)$ is much more directly related to the input-output behaviour (and hence our data) than the matrices (A,B,C). Indeed

$$(4.2) \qquad H(A,B,C) = \begin{pmatrix} CB & CAB & \cdots & CA^{n}B \\ CAB & CA^{2}B & & CA^{n+1}B \\ \vdots & \vdots & & \vdots \\ CA^{n}B & CA^{n+1}B & \cdots & CA^{2n}B \end{pmatrix}$$

and thus if $x(0) = 0$, one has, ~~cf. because~~ for the input sequence $u(0), u(1), \ldots, u(n), 0, 0, \ldots$

$$(4.3) \qquad \begin{pmatrix} y(n+1) \\ \vdots \\ y(2n+1) \end{pmatrix} = H(A,B,C) \begin{pmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(0) \end{pmatrix}$$

In any case the blocks occurring in the block Hankel matrix are precisely the ~~blocks or~~ matrices occurring in the input-output formula (1.5). Note also that the columns of $H(A,B,C)$ can be labelled in the same way as the columns of $R(A,B)$ and that the rows of $H(A,B,C)$ can be labelled as the rows of $Q(A,C)$

5. Conceptual description of an identification algorithm

The conceptual background of the identification algorithm to be described in section 6 below is now as follows.

(i) calculate $H(A,B,C)$ from the input output data. It is to be expected that this can be done provided enough data are available. Indeed observe that the columns of the CA^iB appear as outputs in response to input sequences of the form $(e_k, 0, 0, 0, \ldots)$ and also $(0, 0, \ldots, 0, e_k, 0, 0, \ldots)$. Thus roughly speaking if there are enough input sequences available to recover enough of these special ones ~~all the CA^iB~~ as linear combinations all the CA^iB are known

(ii) calculate a nice suitable input selection α_R and a nice output selection α_Q (we are assuming $c\tau$ and co for the moment). This can be done from $H(A,B,C)$ because the columns of $H(A,B,C)$ are labelled just as the columns of $R(A,B)$ and because $H(A,B,C) = Q(A,C)\,R(A,B)$ ~~so that~~. Indeed because $Q(A,C)$ is a full rank matrix the linear dependency relations between the columns of $R(A,B)$ ~~..~~ are exactly the same as the linear dependency relations between the columns of $H(A,B,C)$. Similarly the ~~dep~~ linear dependency relations between the rows of $H(A,B,C)$ are the same as those of the rows of $Q(A,C)$.

(iii) Find some A,B,C such that their block Hankel matrix is precisely the given one. Here there is choice. Because if (A,B,C) works then so does (SAS^{-1}, SB, CS^{-1}) for every $S \in Gl_n(\mathbb{R})$. We shall choose that

particular unique triple (A, B, C) with the additional property that $Q(A,C)_{\alpha_Q} = I_n$.
(Exercise: prove that there is indeed exactly one such, cf [2] for a proof)
Now if $Q(A,C)_{\alpha_Q} = I_n$ we have

$$H(A,B,C)_{\alpha_Q} = (Q(A,C)\, R(A,B))_{\alpha_Q} = Q(A,C)_{\alpha_Q} R(A,B) = R(A,B)$$

So we know the $R(A,B)$ of this particular triple and that means that we can calculate A and B by the preliminary work of section 3 above. Finally the first p rows of $H(A,B,C)$ form the matrix

$$C\,R(A,B)$$

Now we know $R(A,B)$ and hence $R(A,B)_{\alpha_R}$ which is invertible. So C results from

$$(C\,R(A,B))_{\alpha_R} = C\,R(A,B)_{\alpha_R}$$

because both the left hand side and $R(A,B)_{\alpha_R}$ are known and the latter is invertible.

The attentive reader may have noticed that it seems that I have cheated a little bit in that I seem to have assumed that I know n, the dimension of the system. Actually this is part of (ii) above in that $n = \mathrm{rk}\, H(A,B,C)$ and that its determination is more or less directly linked with finding an α_R and α_C. For both one considers larger and larger Hankel block matrices

$$(CB), \quad \begin{pmatrix} CB & CAB \\ CAB & CA^2B \end{pmatrix}, \quad \begin{pmatrix} CB & CAB & CA^2B \\ CAB & CA^2B & CA^3B \\ CA^2B & CA^3B & CA^4B \end{pmatrix}, \ldots$$

and one determines when the rank stabilizes. This particular conceptual method of recovering A,B,C from H(A,B,C) comes from [2] where also more details can be found.

In practice the algorithm goes not quite this way, largely because there are shortcuts particularly if one is only interested in the future input-output behaviour of the system.

Also one cannot always assume that $x(0)=0$. If A is stable this can be more or less safely assumed by throwing away ~~the~~ a suitable initial chunk of the observed time series. But if A is not stable this can ~~be~~ not be done and also it may be a serious waste of data to do so.

However, to understand the algorithm the sketch above should help.