INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA CONTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392 - I

SMR.201/ 20

SECOND WORKSHOP ON MATHEMATICS IN INDUSTRY

(2 - 27 February 1987)

DENSITY INDUCED FLOW IN POROUS MEDIA

C.J. Van DUIJN
Dept of Mathematics &
Informatics
Delft University of Technology
P.O.Box 356
2600 AJ Delft
NETHERLANDS

Density Induced Flow in Porous Media

C.J. van Duijn

Department of Mathematics and Informatics

Delft University of Technology

P.O. Box 356

2600 AJ Delft

Netherlands.

Introduction.

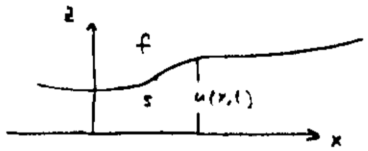
In these lecture notes the phenomenon of density induced flow through porous media is studied. The important applicati of this is the movement of fresh and salt groundwater in coastal aquifers, for which different models are given. In detail we consider the mathematical consequences of each descrip The results are presented in the form of four chapters. Each chapter is the result of joint research with people who were (or still are) part of our group in Delft. The activities of this group are concerned with problems from nonlinear anal (nonlinear partial differential equations and integral equations) with applications to groundwater flow problems.

In deriving the partial differential equations for the flow, w assume throughout that the movement is two dimensional, that the porous medium is homogeneous and that the specific weight γ ($= \rho g$, ρ density) is the only fluid quantity which is allowed to vary.

In Chapter 1 we first give the general description of the flow of a fluid due to variations in its specific weight. This we do in terms of the stream function ψ for which an equation is found of the form

$$-\Delta \psi = \frac{k}{\mu} \frac{\partial \gamma}{\partial x} , \qquad (1.$$

where k denotes the permeability of the porous medium and μ the dynamic viscosity of the fluid.
Next we assume that the fluid consists of fresh and salt water which are separated by an abrupt interface. We further assume that for each time t this interface can be represented in the x,z plane as a function of x:



The specific weight is then given by

$$\gamma = (\gamma_s - \gamma_f)\, H(u(x,t) - z) + \gamma_s \quad , \qquad (2)$$

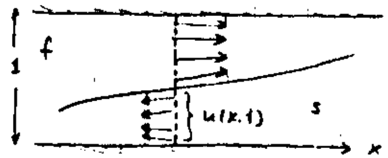
where H denote the Heaviside function.
For the evolution of the interface we obtain the first order equation

$$\varepsilon \frac{\partial u}{\partial t} = \frac{d}{dx} \varphi(x, u(x,t)) \qquad (3)$$

in which ε denotes the porosity of the porous medium.

We solve the system (1) - (3) numerically. The stream function equation is solved by means of the finite element method, while a predictor - corrector method (the socalled $S^{\alpha,\beta}$ scheme) is used for the discretization of the interface motion equation (3).

In Chapter 2 we consider a simplification of the flow problem which is due to Dupuit. That is we assume that the x-component of the discharge field is constant over the height in each fluid:



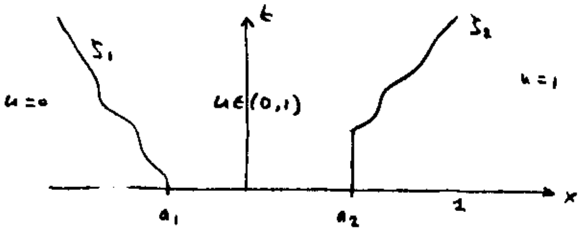
Then a nonlinear diffusion equation arises of the form

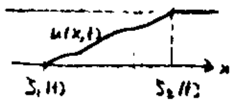
$$\varepsilon \frac{\partial u}{\partial t} = \Gamma \frac{\partial}{\partial x} \left\{ u(1-u) \frac{\partial u/\partial x}{1 + (\partial u/\partial x)^2} \right\} \quad . \qquad (4)$$

This is a doubly degenerated equation: it degenerates at points where $u=0$, $u=1$ and where $u_x = \pm 1$. Therefore we introduce weak solutions and we prove existence, uniqueness and large time behaviour for the Cauchy problem, the Cauchy Dirichlet problem and the Neumann problem. We also give some explicit solutions for equation (4).

In Chapter 3 we continue with equation (4). In particular we prove a number of regularity properties which have a clear physical interpretation. We prove that there exist functions $S_1, S_2 : [0,\infty) \to \mathbb{R}$ such that for all $t \geq 0$ we have

$$u(x,t) = 0 \qquad x \le \zeta_1(t),$$
$$0 < u(x,t) < 1 \qquad \zeta_1(t) < x < \zeta_2(t),$$
$$u(x,t) = 1 \qquad x \ge \zeta_2(t)$$





Further we prove that each function satisfies:

There exists $t_i^* \in [0,\infty)$ such that $\zeta_i(t) = a_i$ for $t \in [0,t_i^*]$, $(-1)^i \zeta_i$ is strictly increasing on (t_i^*,∞) and the functions ζ_i are right differentiable for each $t > 0$ and satisfy

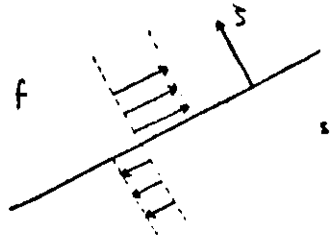
$$\varepsilon D^+ \zeta_1(t) = - \lim_{x \downarrow \zeta_1(t)} \frac{\partial u/\partial x}{1+(\partial u/\partial x)^2} \tag{5}$$

and

$$\varepsilon D^+ \zeta_2(t) = + \lim_{x \uparrow \zeta_2(t)} \frac{\partial u/\partial x}{1+(\partial u/\partial x)^2} \tag{6}$$

The right hand side of (5) denotes the velocity of the salt water in the toe of the interface and the right hand side of (6) denotes the velocity of the fresh water in the top of the interface.

In Chapter 4 the dispersion and diffusion from an originally sharp fresh-salt interface is studied.
Consider the situation where initially a sharp and inclined interface is given which extends in all directions up to infinity.



To simplify the analysis, the conditions at the boundaries of the infinite aquifer are assumed to be such, that the flow is constant and parallel to the original interface plane. Then we can derive a partial differential equation which describes the mixing process of the fresh and salt water due to molecular diffusion and transversal dispersion. It gives the distribution of the specific weight as a function of ζ (coordinate normal to the original interface) and time t. In fact the distribution is given in terms of a similarity solution which depends only on the combination $\zeta/\sqrt{t}$.
A detailed analysis is given about the construction of this similarity solution. The influence of the flow conditions at infinity and the molecular diffusivity is made explicitly.

CHAPTER 1.

THE INTERFACE BETWEEN FRESH AND SALT GROUNDWATER

A NUMERICAL STUDY.

by

J.R.Chan Hong *, C.J.van Duyn *, D.Hilhorst **, J.van Kester ***.

ABSTRACT. In this paper the two dimensional flow of fresh and salt water through a homogeneous aquifer is considered. The two fluids are assumed to be separated by a sharp interface. They differ only in their specific weight. This difference induces a flow in the aquifer which in turn causes a motion of the interface.
We present a mathematical formulation of this problem which consists of a Poisson equation for the stream function coupled to a time evolution equation for the moving interface. The equation for the streamfunction is solved by means of a finite element method while a predictor-corrector method (the $S^{\alpha,\beta}$ scheme) is used for the discretization of the equation for the interface.

* Department of Mathematics, Delft University of Technology, Delft The Netherlands. (The first author is on leave from the Ecole Centrale de Lyon, France).

** CNRS et Laboratoire d'Analyse Numérique, Université de Paris-Sud 91405 Orsay, France.

*** Delft Hydraulics Laboratory, Delft, The Netherlands.

1. Introduction.

In this paper we present a model which describes the two-dimensional motion of fresh and salt water through a horizontal aquifer. The fresh and salt water have different specific weights, denoted by γ_f and γ_s ($\gamma_f < \gamma_s$), respectively. As is common in hydrology (e.g. see Bear [1], and De Josselin de Jong [9]) it is assumed here that the fluids do not mix and are separated by an abrupt interface. The difference in specific weight induces a flow and thus a displacement of the fluids and the corresponding interface. Our interest here is in the evolution in time of this interface. Mathematically, this leads to the following rescaled problem:

$$
P \quad \begin{cases} -\Delta\psi = \dfrac{\partial}{\partial x}\,[H(u(x,t)-z)] & \text{in } \Omega \times \mathbb{R}^+ \\ \psi = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u_t = \dfrac{d}{dx}[\psi(x,u(x,t),t)] & \text{in } I \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in I \end{cases}
$$

where

$$I = (-1,1) \text{ or } I = \mathbb{R}, \quad \Omega = \{(x,z) \in I \times (0,1)\},$$

H is the Heaviside function : $H(s) = 1$ when $s > 0$ and $H(s) = 0$ when $s < 0$, u_0 is a sufficiently smooth function such that $0 \le u_0 \le 1$ and, if $I = \mathbb{R}$, $u_0 - H$ has compact support.
The function ψ denotes the streamfunction of the flow and u represents the height of the fresh-salt water interface, $0 \le u \le 1$. At points where $u = 0$ (bottom of aquifer) only fresh water is present and at points where $u = 1$ (top of aquifer) only salt water is present. A full derivation of Problem P is given in Section 2.
In Section 3, we recall some results of Van Duyn & Hilhorst [6] about a simplified model which arises when a Dupuit assumption with respect to the flow is being made. Then the height u of the interface satisfies a parabolic degenerate equation. Its generalised solutions u(t) have the following convergence property. When $I = (-1,1)$, u(t) converges to the constant $\int_I u_0(x)dx/2$ as $t \to \infty$ and when $I = \mathbb{R}$, u(t) converges to a similarity solution as $t \to \infty$; one of the purposes of this work is to show numerically similar results for solutions of Problem P.

We describe the numerical method in Section 4. The equation for the streamfunction is solved by means of a finite element method. The equation for the interface, when considered apart, is hyperbolic; the $S^{\alpha,\beta}$ scheme introduced by Lerat & Peyret [11] is used for its discretization. An essential problem is to calculate as precisely as possible the x-coordinates $S_1(t)$ and $S_2(t)$ of the points where the interface reaches the bottom $z = 0$ and the top $z = 1$ of the aquifer. We do so by discretizing as well the differential equations for S_1 and S_2 and calculating u only between S_1 and S_2. Similar techniques have been used by DiBenedetto & Hoff [3] and Hoff [8] for the discretization of the porous media equation. The numerical results are presented in Section 5. Part of the computations shown there are obtained by using a fixed triangle distribution throughout the entire flow domain. Tests are performed with and without using the extra equations for S_1 and S_2. The results are virtually the same.
A number of calculations are carried out with using a triangularization of the flow domain which varies in time. This is done with the help of an automatic mesh generator, allowing the mesh to vary at each time step in such a way that the discretised interface coincides with sides of triangles. In this way only values of the streamfunction ψ at mesh points are necessary in the computations. Therefore larger triangles can be used in this case.
The numerical results clearly exhibit the asymptotic behavior as described in Section 3.

Certain one phase problems are treated by related methods; in particular, Rasmussen and Salhani [13] use a Crank-Nicolson procedure to solve the interface equation in a case where it is coupled with the Laplace equation.
Duchon and Robert [4], [5] consider Problem P in $\Omega = \mathbb{R}^2$ and obtain existence and uniqueness results in a scale of Banach spaces.
Chan-Hong and Hilhorst [2] prove the local existence and uniqueness of the solution of problem P in the case that Ω is the strip $\mathbb{R} \times (0,1)$ and $0 < u_0 < 1$.

In the engineering literature, several authors propose finite element models for interface problems in groundwater flow, e.g. Wilson and Da Costa [18] and Verruijt [15]. However their work is based on the Dupuit approximation with respect to the flow. This leads to a different set of differential equations.

Acknowledgement: The authors acknowledge a number of constructive discussions with Ph.Clément about the formulation of the problem. They also wish to thank Ch.H.Bruneau and P.Wilders for pointing out the hyperbolic character of the problem and for suggesting the use of the $S^{\alpha,\beta}$ method. They are indebted to J.Laminie for his technical help and patient explanations. J.R.Chan Hong and D.Hilhorst also wish to thank C.M. Brauner and C.Schmidt-Lainé for useful discussions when this work started.

2. The model.

In this section a derivation of the interface motion problem P is given. We consider first the problem of finding an equation for the streamfunction which describes the flow induced by a given variable density distribution of a fluid in a porous medium. Thereafter the case of a sharp interface is treated and an equation for the interface motion is given.

Throughout this paper, $\Omega \subset \mathbb{R}^2$ denotes either the rectangle $(-1,1) \times (0,1)$ (bounded case) or the strip $\mathbb{R} \times (0,1)$ (unbounded case).

2.1. The streamfunction equation.

Consider the 2-dimensional flow of an incompressible fluid of variable specific weight γ and constant viscosity μ through a homogeneous porous medium with permeability κ. As an example may serve the flow of fresh and salt groundwater near coastal aquifers, see Bear [1], de Josselin de Jong and Van Duyn [10]. Let Ω denote the flow domain of the fluid. The movement of the fluid is governed by the momentum balance equation (Darcy law)

$$\frac{\mu}{\kappa} q + \text{grad } p + \gamma e_z = 0 \qquad \text{in } \Omega \tag{2.1}$$

and the continuity equation (expressing the incompressibility of the fluid)

$$\text{div } q = 0 \tag{2.2}$$

In these equations, q and p denote the velocity field and the pressure, respectively, and e_z denotes the unit vector in the vertical positive z-direction. Since we are interested in a description of the density induced flow only, we consider at the boundary $\partial\Omega$ the no flow condition

$$q \cdot \nu = 0 \qquad \text{on } \partial\Omega \tag{2.3}$$

where ν denotes the outward normal unit vector on $\partial\Omega$.

With respect to the prescribed specific weight distribution γ we make the following hypothesis :

H_γ : $\gamma \in L^2(\Omega) + \{(\gamma_s-\gamma_f) H_x + \gamma_f\}$,

where $H_x : \Omega \to \{0,1\}$ is such that $H_x = 1$ when $x > 0$ and $H_x = 0$ when $x < 0$.

When Ω is bounded $\gamma \in L^2(\Omega)$ and one can immediately deduce from Temam [14, Thm.1.4, p.15] that there exists a unique vector $q \in (L^2(\Omega))^2$ and a function $p \in H^1(\Omega)$, unique up to an additive constant, such that equations (2.1), (2.2) and (2.3) are satisfied. When Ω is unbounded $\gamma \notin L^2(\Omega)$ and we cannot immediately deduce the existence of q and p. However hypothesis H_γ ensures that $\frac{\partial\gamma}{\partial x} \in H^{-1}(\Omega)$. In both the cases where Ω is bounded and unbounded, the following result holds.

Lemma 2.1

Let $q \in (L^2(\Omega))^2$ satisfy (2.2) and (2.3). Then there exists a function $\psi \in H_0^1(\Omega)$ such that $q = \text{curl } \psi$.

Proof. Let $\{q_m\} \in C_0^\infty(\Omega)$ satisfy div $q_m = 0$ and $\lim_{m\to\infty} \|q_m - q\|_{(L^2(\Omega))^2} = 0$. We denote by q_{mx} and q_{mz} the x and the z components of q_m and we set

$$\psi_m(x,z) = -\int_0^z q_{mx}(x,r)dr.$$

Then one can check that curl $\psi_m = q_m$. Obviously $\psi_m = 0$ for $z = 0$. Using the fact that div $q_m = 0$ and $q_m = 0$ on $\partial\Omega$ one can also verify that $\psi_m = 0$ for $z = 1$ and that, in the case where $I = (-1,1)$, $\psi_m = 0$ for $x = \pm 1$. Finally, since, by Poincaré's inequality (see for instance [14,p.4], this yields

$$\|\psi_m\|_{H_0^1(\Omega)} \le C \|q_m\|_{(L^2(\Omega))^2} \le C$$

there exists $\psi \in H_0^1(\Omega)$ and a subsequence $\{\psi_{m_k}\}$ of ψ_m such that $\psi_{m_k} \to \psi$ weakly in $H_0^1(\Omega)$ as $m_k \to \infty$ and $q = \text{curl } \psi$.

Thus if some functions $q \in \{L^2(\Omega)\}^2$ and $p \in H^1(\Omega)$ (or $p \in H^1_{loc}(\bar{\Omega})$ if Ω is unbounded) satisfy the equations (2.1)-(2.3) and if $\psi \in H^1_0(\Omega)$ is such that $q = \text{curl}\ \psi$, then it follows from (2.1) that ψ satisfies

$$\frac{\mu}{\kappa}\,\text{curl}\ \psi + \gamma e_z = -\,\text{grad}\ p \qquad \text{a.e. in } \Omega \tag{2.4}$$

In turn (2.4) implies that

$$\int_\Omega \left(\frac{\mu}{\kappa}\,\text{curl}\ \psi + \gamma e_z\right) \text{curl}\ \phi = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega) \tag{2.5}$$

or equivalently

$$P_\psi \begin{cases} -\Delta\psi = \dfrac{\kappa}{\mu}\dfrac{\partial\gamma}{\partial x} & \text{in } H^{-1}(\Omega) \\ \psi = 0 & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

It follows from standard theory (see for instance Vo Khac Khoan [16, p.201,202] that Problem P_ψ has a unique solution $\psi \in H^1_0(\Omega)$.
Next we show that the solution ψ of Problem P_ψ, in both the bounded and unbounded case, defines a flow field which satisfies the equations (2.1)-(2.3). Since $\psi \in H^1_0(\Omega)$, $q = \text{curl}\ \psi \in \{L^2(\Omega)\}^2$ and satisfies the no-flow condition (2.3). Then from (2.5) we have

$$\int_\Omega \left(\frac{\mu}{\kappa}\,q + \gamma e_z\right) \text{curl}\ \phi = 0 \qquad \text{for all } \phi \in C_0^\infty(\Omega).$$

This implies by Temam [14, Prop. 1.1 and 1.2, p.14] that there exists a function $p \in H^1(\Omega)$ (or $p \in H^1((-N,N) \times (0,1))$ for all $N > 0$ if Ω is unbounded) such that q and p satisfy Darcy's law almost everywhere in Ω. Note that p is uniquely defined up to a constant. Also, since $q = \text{curl}\ \psi$, it follows by direct computation that $\langle q, \text{grad}\ \phi\rangle = 0$ for all $\phi \in C_0^\infty(\Omega)$. Thus div q=0 in the sense of distributions.
Before we proceed to the particular case of an abrupt change in density we note the following. In order to solve numerically problem P_ψ when Ω is unbounded, we shall solve in fact the boundary value problem on $\Omega = \Omega_R := (-R,R) \times (0,1)$ with R large enough. This procedure is justified by the next lemma.

Lemma 2.2.

Let ψ_R be the solution of Problem P_ψ on the domain Ω_R and let $\tilde{\psi}_R$ be the extension of ψ_R on Ω with $\psi_R = 0$ on $\Omega\backslash\Omega_R$. Then

$$\tilde{\psi}_R \to \psi \quad \text{weakly in } H^1_0(\Omega) \text{ as } R \to \infty$$

where ψ is the solution of Problem P_ψ with Ω unbounded.

Proof. By construction, $\tilde{\psi}_R \in H^1_0(\Omega)$. It satisfies

$$\int_\Omega (\text{grad}\ \tilde{\psi}_R)^2 = \int_{\Omega_R} (\text{grad}\ \psi_R)^2 = -\frac{K}{\mu}\int_{\Omega_R} \gamma\,\frac{\partial\psi_R}{\partial x}$$
$$= -\frac{K}{\mu}\int_\Omega \gamma\,\frac{\partial\tilde{\psi}_R}{\partial x} \le \frac{K}{\mu}\left\|\frac{\partial\gamma}{\partial x}\right\|_{H^{-1}(\Omega)} \|\psi_R\|_{H^1_0(\Omega)}$$

This yields $\|\psi_R\|_{H^1_0(\Omega)} \le C$. The result of Lemma 2.2 then easily follows.

Next suppose that there is an abrupt change in specific weight, say from fresh water with γ_f to salt water with γ_s, and that the height of the interface Γ_u separating the fluids can be parametrized in the form $z = u(x)$, where u is a continuous function such that $u(x) = 0$ for x small enough and that $u(x) = 1$ for x large enough (see Fig. 2.1 below).

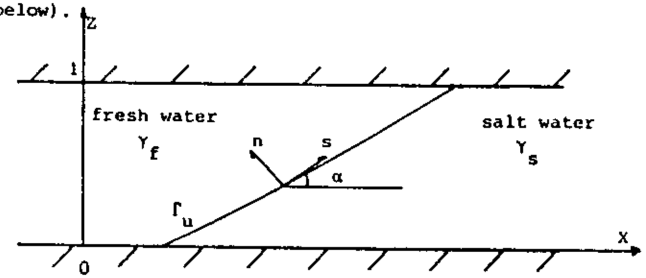


Fig. 2.1. The interface between fresh and salt groundwater in a horizontal aquifer.

Then the specific weight throughout the flow domain is given by

$$\gamma(x,z) = (\gamma_s - \gamma_f)\, H(u(x)-z) + \gamma_f \quad \text{for } (x,z) \in \Omega \tag{2.7}$$

where H denotes the Heaviside function.
With this expression, the equation for the streamfunction becomes

$$-\Delta\psi = \Gamma \frac{\partial}{\partial x} H(u(x)-z) \quad \text{in } \Omega \tag{2.8}$$

where $\Gamma = \frac{\kappa}{\mu}(\gamma_s - \gamma_f)$.

Remark. Since by (2.7), $\gamma \in L^\infty(\Omega)$, it follows from Lions & Magenes [12, Theorem 8.1] that if Ω is bounded, $\psi \in C^{0,\alpha}(\bar{\Omega})$ for all $\alpha \in (0,1)$; furthermore one can show that if Ω is unbounded $\psi \in C^{0,\alpha}(\bar{\Omega}_R)$ for all $R > 0$ and all $\alpha \in (0,1)$.
Let Ω_f denote the domain occupied by the fresh water, Ω_s the domain occupied by the salt water, s the tangential unit vector to Γ_u, n the normal unit vector directed towards the outside of Ω_s and α the angle between the positive x-axis and s, see Fig. 2.1. In the case that the function u is smooth in the neighborhood of each point x where $0 < u(x) < 1$, we show below that the problems for q and ψ can be written as problems on the two subdomains Ω_s and Ω_f, with matching conditions on the interface.

Lemma 2.3.

Let $q \in \{L^2(\Omega)\}^2$ satisfy (2.1), (2.2) and (2.3) and be such that the restrictions q_i (i = f,s) of q to Ω_i are such that $q_i \in \{H^1(\theta)\}^2$ for every open bounded set $\theta \subset \bar{\Omega}_i$. Then q_f and q_s satisfy

$$\begin{cases} \operatorname{div} q_i = 0,\ \operatorname{curl} q_i = 0 & \text{in } \Omega_i,\ i = f,s \\ q_f \cdot n = q_s \cdot n\ ,\ q_f \cdot s - q_s \cdot s = \Gamma \sin\alpha = \Gamma \dfrac{u_x}{\sqrt{1+u_x^2}} & \text{on } \Gamma_u \\ q \cdot n = 0 & \text{on } \partial\Omega \end{cases}$$

Corollary 2.4.

Let $\psi \in H^1_0(\Omega)$ satisfy Problem P_ψ and suppose that the restrictions ψ_i (i = f,s) of ψ to Ω_i are such that $\psi_i \in H^2(C)$ for every open bounded set $\theta \subset \bar{\Omega}_i$. Then ψ_f and ψ_s satisfy

$$\begin{cases} -\Delta\psi_i = 0 & \text{in } \Omega_i \quad i = f,s \\ \psi_f = \psi_s,\ \dfrac{\partial\psi_f}{\partial n} - \dfrac{\partial\psi_s}{\partial n} = -\Gamma\sin\alpha = -\Gamma \dfrac{u_x}{\sqrt{1+u_x^2}} & \text{on } \Gamma_u \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

Proof. Corollary 2.4 follows from Lemma 2.3 by substituting $q = \operatorname{curl}\psi$.

We remark that $\frac{\partial\psi}{\partial n}$ is discontinuous across Γ_u at each point where Γ_u has a non zero slope. This will appear clearly in the numerical results.

Proof of Lemma 2.3. Form $\langle \operatorname{div} q, \phi\rangle = -\int_\Omega q \operatorname{grad} \phi = 0$ for all $\phi \in C_0^\infty(\Omega)$ it follows that div $q_i = 0$ a.e. in Ω_i(i = f,s) and thus that

$$\int_{\Gamma_u} (q_f - q_s) \cdot n\, \phi\, ds = 0 \qquad \text{for all } \phi \in C_0^\infty(\Omega)$$

which implies that $q_f.n = q_s.n$ on Γ_u. Similarly, since curl $q = \frac{\kappa}{\mu}\frac{\partial\gamma}{\partial x}$ in the sense of distributions in Ω, we deduce that curl $q_i = 0$ a.e. in Ω_i (i = f,s) and that

$$\int_{\Gamma_u} (q_f - q_s) \cdot s\, \phi\, ds = \Gamma \int_I \frac{\partial u}{\partial x}\, \phi(x,u(x))dx \ \text{for all } \phi \in C_0^\infty(\Omega)$$

which implies that

$$q_f \cdot s - q_s \cdot s = \Gamma \sin\alpha = \Gamma \frac{u_x}{\sqrt{1+u_x^2}} \ .$$

2.2. The interface motion equation; the equations for S_1 and S_2.

Let u = u(x,t) denote the height of the fresh-salt interface at a certain time t > 0. Then the corresponding velocity field can- in principle-be found by solving equation (2.8) with boundary conditions (2.6). From this the displacement of the interface is calculated with the kinematic condition, which says that the normal displacement velocity of an interface point is equal to the projection on the normal of its vertical displacement velocity u_t time the porosity of the medium, see Bear [1]:

$$q.n = \varepsilon u_t \cos\alpha = \varepsilon \frac{u_t}{\sqrt{1+u_x^2}} \tag{2.9}$$

where the subscripts x and t denote differentiation with respect to these variables. Using q = curl ψ, equation (2.9) becomes

$$\varepsilon u_t = \{\psi(x,u(x,t),t)\}_s \sqrt{1+u_x^2}$$

or

$$\varepsilon u_t = \frac{d}{dx}\{\psi(x,u(x,t),t)\} \tag{2.10}$$

in the region $(x,t) \in I \times \mathbb{R}^+$, where either $I = (-1,1)$ or $I = \mathbb{R}$. Note that the derivation of (2.10) is only formal because we have not enough information about the regularity of u and q . n . Properly rescaled, equations (2.8) and (2.10) together with boundary and initial conditions give problem P.
We remark that in the case $I = (-1,1)$ boundary conditions for u are not needed; this comes from the fact that $\psi = 0$ for $x = \pm 1$ which implies that the lines $x = \pm 1$ are characteritics for equation (2.10).

Based on the nature of the problem, we conjecture that the speed of propagation of the points $S_1(t)$ and $S_2(t)$, where the interface $u(x,t)$ reaches respectively the bottom and the top of the aquifer (see Fig.2.1), is finite. Below we give definitions and differential equations for S_1 and S_2.
For simplicity we suppose that $I = \mathbb{R}$. For any $t \geq 0$, we define $S_1(t)$ and $S_2(t)$ by

$$S_1(t) = \sup\{x \in \mathbb{R} \mid u(s,t) = 0 \text{ for all } s \leq x\},$$

and

$$S_2(t) = \inf\{x \in \mathbb{R} \mid u(s,t) = 1 \text{ for all } s \geq x\}.$$

The differential equation for S_1 is found by observing that the speed at which S_1 travels in the (x,t) plane must be equal to the velocity of the saltwater in the saltwater toe.
We have the formula

$$\varepsilon \dot{S}_1(t) = \lim_{x \downarrow S_1(t)} q_x(x,0,t) = -\lim_{x \downarrow S_1(t)} \frac{\partial\psi}{\partial z}(x,0,t) \tag{2.11}$$

Similarly we have for S_2

$$\varepsilon \dot{S}_2(t) = \lim_{x \uparrow S_2(t)} q_x(x,1,t) = -\lim_{x \uparrow S_2(t)} \frac{\partial\psi}{\partial z}(x,1,t). \tag{2.12}$$

Again by a rescaling one can drop the factor ε in these equations. Finally we observe that equations (2.11) and (2.12) are not a part of the original problem. However they will be used in the numerical algorithm.

3. The Dupuit approximation and some consequences.

In this section we consider a simplified form of Problem P which arises when a Dupuit approximation with respect to the flow is made. In this approximation one assumes that the horizontal component of the velocity field is constant in each fluid and has a jump at the interface.

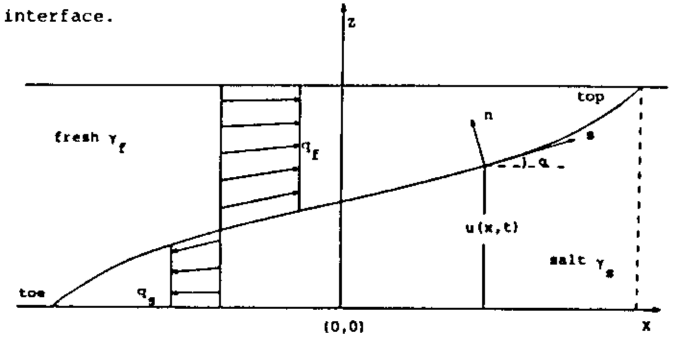


Fig. 2.2 : The Dupuit approximation of the flow.

In hydrology one often uses this simplification, which is assumed to hold when the interface is not too steep . The particular form considered here is due to De Josselin de Jong [9]. It leads to a nonlinear diffusion equation for the function u. We also recall some results of Van Duyn & Hilhorst [6] about the asymptotic behaviour as $t \to \infty$ of solutions of this equation . The numerical simulation of solutions of Problem P shows a similar asymptotic behaviour.

3.1. The simplified problem.

Our starting point is the rescaled interface motion equation of Problem P

$$u_t = \frac{d}{dx}\{\psi(x,u(x,t),t)\} \tag{3.1}$$

and the observation that

$$\psi(x,u,t) = -\int_0^u q_x(x,z,t)dz \tag{3.2}$$

or

$$\psi(x,u,t) = +\int_u^1 q_x(x,z,t)dz. \tag{3.3}$$

Using the simplifying assumption on the flow, see Fig.2.2, these equations become

$$\psi(x,u,t) = -uq_{s_x}(x,u,t) = (1-u)\, q_{f_x}(x,u,t) \tag{3.4}$$

Further, it follows from Lemma 2.3 , with $\Gamma = 1$, that

$$-(q_{f_x}-q_{s_x})\sin\alpha + (q_{f_z}-q_{s_z})\cos\alpha = 0 \quad \text{on } \Gamma_u \tag{3.5}$$

and

$$(q_{f_x}-q_{s_x})\cos\alpha + (q_{f_z}-q_{s_z})\sin\alpha = \sin\alpha \text{ on } \Gamma_u. \tag{3.6}$$

Eliminating $(q_{f_z}-q_{s_z})$ gives

$$q_{f_x}-q_{s_x} = \frac{\tan\alpha}{1+\tan^2\alpha} = \frac{u_x}{1+u_x^2}\ . \tag{3.7}$$

Combining (3.4) and (3.7) gives

$$q_{f_x} = u\,\frac{u_x}{1+u_x^2} \quad \text{and } q_{s_x} = -(1-u)\frac{u_x}{1+u_x^2} \tag{3.8}$$

and finally

$$\psi(x,u) = u(1-u)\frac{u_x}{1+u_x^2}\ . \tag{3.9}$$

We deduce from (3.1), (3.9) and the fact that $\psi = 0$ on $\partial\Omega$ that u satisfies the Neumann problem

$$N \begin{cases} u_t = \left(u(1-u)\dfrac{u_x}{1+u_x^2}\right)_x & \text{in } I \times \mathbb{R}^+ \\ u(1-u)\dfrac{u_x}{1+u_x^2} = 0 & \text{on } \partial I \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in I \end{cases} \tag{3.10}$$

if $I = (-1,1)$ and the Cauchy problem

$$C \begin{cases} u_t = \left(u(1-u) \dfrac{u_x}{1+u_x^2} \right)_x & \text{in } I \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in I \end{cases}$$

if $I = \mathbb{R}$.

3.2. Asymptotic behavior of the solution.

It follows from [6] that if u_0 satisfies the hypothesis

$$H_0 \begin{cases} u_0 \in W^{1,\infty}(I), \quad 0 \le u_0 \le 1, \quad -1 \le u_0' \le 1 \quad \text{a.e. in } I \quad \text{and if } I = \mathbb{R} \\ u_0 - H \ \text{ has compact support} \end{cases}$$

then Problems N and C have a unique generalized solution u which is continuous in $\overline{I \times \mathbb{R}^+}$ and has the following asymptotic behaviour as $t \to \infty$.

Theorem 3.1.

The solution $u(t,u_0)$ of Problem N converges to $\int_I u_0(x)dx/2$ uniformly in $\bar{I}$ as $t \to \infty$.

Theorem 3.2.

The solution of Problem C satisfies

$$\| u(t,u_0) - f(./g(t)) \|_{L^\infty(\mathbb{R})} \le C/g(t)$$

for all $t > 0$ where C is a positive constant, $g(t) = \phi^{-1}(2t+\phi(g_0))$ with $\phi(s) = s^2/2 + \ln s - 1/2$ and $g_0 \ge 1$ and

$$f(s) = \begin{cases} 0 & \text{if } s < -1/2 \\ s+1/2 & \text{if } -1/2 \le s \le 1/2 \\ 1 & \text{if } s > 1/2. \end{cases}$$

4. The numerical method.

In this section we describe a numerical algorithm for solving Problem P. It is based on an explicit time integration scheme for the initial value problem

$$\begin{cases} u_t = \dfrac{d}{dx} [\psi(x,u(x,t),t)] & \text{in } I \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } I \end{cases} \qquad (4.1)$$

where $I = (-1,1)$ or $I = (-R,R)$ with R large enough. In practice we have in turn to solve the problem for the streamfunction ψ and to proceed with the integration in time.

4.1. Discretization of the problem for ψ.

Let $u^n(x)$ be the interface at time t^n. In order to determine the stream-function ψ, we have to solve the problem

$$P_\psi^n \begin{cases} -\Delta\psi = \dfrac{\partial}{\partial x} \{H(u^n(x)-z)\} & (x,z) \in \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega = I \times (0,1)$. In the case that $I = (-R,R)$ it follows from Lemma 2.2 that if R is large enough, this problem is a good approximation for the problem in the strip $\mathbb{R} \times (0,1)$.

With the purpose of solving Problem P_ψ^n by means of the finite element method, we rewrite it under the form

$$\begin{cases} \text{Find } \psi \in H_0^1(\Omega) \text{ such that} \\ \int_\Omega \text{grad } \psi \text{ grad } v = -\int_\Omega H(u^n(x)-z) \dfrac{\partial v}{\partial x} \quad \text{for all } v \in H_0^1(\Omega) \end{cases} \qquad (4.2)$$

Let T_h be a triangularization of $\bar{\Omega}$. Using the finite element method with piecewise linear basis functions, we obtain the following dis-cretized problem

$$\begin{cases} \text{Find } \psi_h \in V_h \text{ satisfying} \\ \int_\Omega \text{grad } \psi_h \text{ grad } v_h = -\int_\Omega H(u_h^n(x)-z) \dfrac{\partial v_h}{\partial x} \quad \text{for all } v_h \in V_h \end{cases} \qquad (4.3)$$

where

$$V_h = \{v_h \in C(\bar{\Omega}) \mid \forall K \in T_h \quad v_h \text{ is linear on } K \text{ and } v_h = 0 \text{ on } \partial\Omega\}$$

and where

u_h^n is a piecewise linear approximation of u^n.

Let $\{\phi_i\}$, $i = 1,\ldots,N$ be the piecewise linear basis functions (they take value one at one node and vanish at all other nodes). Then Problem (4.3) is equivalent to the linear system

$$\sum_{i=1}^{N} \psi_i \int_\Omega \text{grad } \phi_i \text{ grad } \phi_j = -\int_\Omega H(u_h^n(x)-z) \frac{\partial \phi_j}{\partial x} \qquad (4.4)$$

for all $j = 1,\ldots,N$ where the constants ψ_i, $i = 1,\ldots,N$ are defined by

$$\psi_h(x,z) = \sum_{i=1}^{N} \psi_i \, \phi_i(x,z) \qquad x,z \in \Omega$$

Two variants of the method have been used, with two different triangularizations of $\bar{\Omega}$.

<u>a. The fixed mesh method.</u>

For some of the numerical tests, we have used a fixed uniform mesh, as shown in Figure 4.1 below

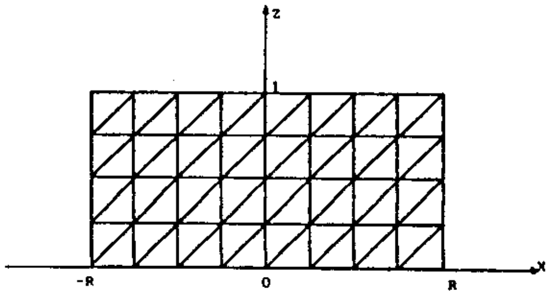


Figure 4.1.

The intervals $[-R,R]$ and $[0,1]$ are regularly subdivided, the mesh points having the coordinates

$$\{(x_i = -R + ih, \; z_j = jk), \; i = 0,\ldots,N, \; j = 0,\ldots,M\}$$

with $h = 2R/N$ and $k = 1/M$. To avoid the technical difficulties induced by the jump of the Heaviside function at the interface we discretize in fact the following problem

$$P_\psi^{\varepsilon,n} \begin{cases} -\Delta\psi = \dfrac{\partial}{\partial x}\{H^\varepsilon(u^n(x)-z)\} & (x,z) \in \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

where

$H^\varepsilon(s) = \phi(s/\varepsilon)$ and $\phi \in C(\mathbb{R})$, ϕ is nondecreasing,
$\lim_{s\to-\infty} \phi(s) = 0$, $\lim_{s\to+\infty} \phi(s) = 1$.

In particular, H^ε tends to H uniformly in compact subsets of $\mathbb{R} \setminus \{0\}$ as $\varepsilon \downarrow 0$. Furthermore one can show that as $\varepsilon \downarrow 0$ the solution ψ_ε of Problem $P_\psi^{\varepsilon,n}$ converges in $H_0^1(\Omega)$ to the solution ψ of Problem P_ψ^n.

The computation of the right-hand side of equation (4.4) requires the evaluation of integrals of the form

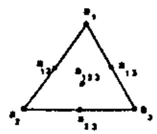
$$\int_K H^\varepsilon(u_h^n(x)-z) \frac{\partial p}{\partial x}$$

where p is one of the three shape functions associated with K. In order to calculate these integrals with sufficient precision, we use the seven points numerical integration formula

$$\int_K f = 2 \text{ meas}(K) \Big\{ \frac{1}{40}(f(a_1) + f(a_2) + f(a_3)) + \frac{9}{40} f(a_{123}) + \frac{1}{15}(f(a_{12}) + f(a_{23}) + f(a_{13})) \Big\}$$

where the points a_i 's are given as in Figure 4.2 below

Figure 4.2



Finally the linear system (4.4) is solved by means of the SSOR-preconditionned conjugate gradient method.

The numerical tests have been performed with $h = k = 10^{-2}$ and the relaxation parameter $\omega = 2/(1+\sin^2(\pi h))$. We have chosen to take

$$H^{\varepsilon}(s) = 1/2\left(\frac{s}{\varepsilon+|s|}+1\right) \text{ with } \varepsilon = 10^{-10}.$$

However, a difficulty is that we have to calculate ψ_h at points of the discretized interface in order to solve the interface equation and that these points are not in general mesh points. Therefore it is necessary to use very small discretization steps h and k. This has motivated us to construct an adaptive mesh in such a way that the discretized interface coincides with sides of triangles of T_h.

b. The adaptive mesh method.

The idea is to generate a new triangularization of Ω at each time step. Let

$$\Omega_{hf}^{n} = \{(x,z) \in \Omega \text{ , } z > u_h^n(x)\}$$

and

$$\Omega_{hs}^{n} = \{(x,z) \in \Omega \text{ , } z < u_h^n(x)\}.$$

We construct triangularizations T_{hf}^{n} and T_{hs}^{n} of Ω_{hf}^{n} and Ω_{hs}^{n} in such a way that the discretized interface coincides with sides of triangles of T_{hf}^{n} and T_{hs}^{n}. In practice the meshes in Ω_{hf}^{n} and in Ω_{hs}^{n} are generated by the mesh generator of the SEPRAN package of the Delft University of Technology. An example is shown in Figure 4.3. We choose a fine mesh in the neighborhood of the numerical interface

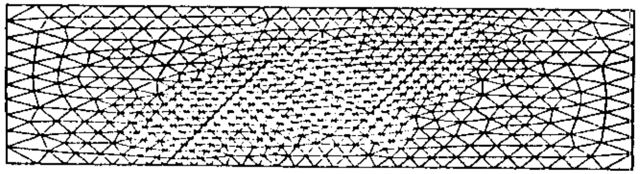


Figure 4.3.

$$\Gamma_h^n = \overline{\Omega_{hf}^{n}} \cap \overline{\Omega_{hs}^{n}}$$

and in particular near the endpoints of Γ_h^n and a coarse mesh further away from Γ_h^n. We remark that if u^n is smooth enough, the right-hand-side of the variational equality in (4.2) may be rewritten as $\int_I (u^n)'(x)v(x,u^n(x))dx$. We set $\Gamma_h^n = \bigcup_{k=1}^{M} \Gamma_k$ where the Γ_k's are sides of triangles. Then another form for the right-hand-side of (4.4) is given by

$$\int_I (u_h^n)'(x)\ \phi_j(x,u_h^n(x))dx = \sum_{k=1}^{M} \int_{\Gamma_k} \frac{(u_h^n)'}{\sqrt{1+((u_h^n)')^2}}\ \phi_j\ d\gamma$$

Here, the linear system (4.4) is solved by means of the LU decomposition.

It will be useful in section 4.2 to be able to calculate as well an approximation of the discharge rate $q = \text{curl}\ \psi$. We set $q_h = \text{curl}\ \psi_h$. Since ψ_h is piecewise linear, q_h is constant on each triangle. On a mesh point outside Γ_h^n, q_h is taken as a weighted average of the discharge rates in the triangles around that point. However, we know from Lemma 2.3 that q is discontinuous accross the interface; thus we must define two approximated discharge rates at mesh points on Γ_h^n corresponding to the discretized fresh water region and to the discretized saltwater region. On a mesh point on Γ_h^n, q_h^f (resp. q_h^s) is taken as a weighted average of the discharge rates on the triangles around that point which lie in Ω_{hf}^{n} (resp. Ω_{hs}^{n}).

4.2. Discretization of the interface equation.

Motivated by the fact that, if ψ is known, the partial differential equation in (4.1) is hyperbolic, we use the $S^{\alpha,\beta}$ explicit scheme of Lerat & Peyret [11] with α,β "optimal" for its discretization. There are two possibilities : either compute u on the whole interval I or use the extra equations (2.11), (2.12) for S_1 and S_2 and calculate u only between S_1 and S_2. When programming on a fixed mesh, we can perform numerical tests in both ways and compare the results. On the other hand we always use the equations for S_1 and S_2 when calculating with an adaptive mesh.

a. The $S^{\alpha,\beta}$ scheme with α,β optimal.

The $S^{\alpha,\beta}$ scheme is a second order explicit scheme. We present it in the case that an adaptive mesh is used. Let u_i^n be the approximation of $u(x_i^n, t^n)$. The function u_h^n that we have introduced above is obtained by linear interpolation; for more details we refer to subsection 4.2.d. below. Further we use the notation $h_i^n = x_{i+1}^n - x_i^n$ and $\Delta t^n = t^{n+1} - t^n$. Then the $S^{\alpha,\beta}$ scheme is given by (see [11] or [17,p.6.8])

$$\tilde{u}_i^n = (1-\beta)u_i^n + \beta u_{i+1}^n + \alpha \frac{\Delta t^n}{h_i^n} \{\psi_h(x_{i+1}^n, u_{i+1}^n, t^n) - \psi_h(x_i^n, u_i^n, t^n)\} \quad (4.5)$$

$$u_i^{n+1} - u_i^n = \frac{\Delta t^n}{\alpha(h_i^n + h_{i-1}^n)} \{(\alpha-\beta)\, \psi_h(x_{i+1}^n, u_{i+1}^n, t^n) + (2\beta-1)\psi_h(x_i^n, u_i^n, t^n) \quad (4.6)$$
$$+ (1-\alpha-\beta)\, \psi_h(x_{i-1}^n, u_{i-1}^n, t^n) + \tilde{\psi}_h(\tilde{x}_i^n, \tilde{u}_i^n, \tilde{t}^n) - \tilde{\psi}_h(\tilde{x}_{i-1}^n, \tilde{u}_{i-1}^n, \tilde{t}^n)\}$$

where $\tilde{x}_i^n = x_i^n + \beta h_i^n$, $\tilde{t}^n = t^n + \alpha(t^{n+1} - t^n)$, the predictor term $\tilde{u}_i^n$ is an approximation of $u(\tilde{x}_i^n, \tilde{t}^n)$ and $\psi_h \in V_h$, $\tilde{\psi}_h \in \tilde{V}_h$ are respectively the solutions of

$$\int_\Omega \operatorname{grad} \psi_h \operatorname{grad} v_h = \int_I u_h^{n\prime}(x)\, v_h(x, u_h^n(x))\,dx \quad \text{for all } v_h \in V_h \quad (4.7)$$

and of

$$\int_\Omega \operatorname{grad} \tilde{\psi}_h \operatorname{grad} \tilde{v}_h = \int_I \tilde{u}_h^{n\prime}(x)\, \tilde{v}_h(x, \tilde{u}_h^n(x))\,dx \quad \text{for all } \tilde{v}_h \in \tilde{V}_h \quad (4.8)$$

where $\tilde{u}_h^n$ and $\tilde{V}_h$ are the analogs of u_h^n and V_h. Our choice of the parameters $\alpha = 1+\sqrt{5}/2$, $\beta = 1/2$ is called "optimal" [11]. The choice $\beta = 1/2$ ensures that the scheme(4.5), (4.6) is space centered; furthermore, when applied to Burger's equation, this choice of the parameters minimizes the dissipative effect of the scheme.

It follows from (4.7), (4.8) that two new meshes must be generated at each time step : one for the calculation of ψ_h at time t^n, the other for the computation of $\tilde{\psi}_h$ at time $\tilde{t}^n$. We remark that also in the case of the fixed mesh, the discretized version of Problem P_ψ^ε is solved on a different grid for the calculation of $\tilde{\psi}_h^\varepsilon$; the z-coordinates of the mesh points remain unchanged while their x-coordinates become $\{(-R+ih/2),\ i = 0,1,3,\ldots,2k+1,\ldots,2N-1,2N\}$.

In order to insure the stability of the $S^{\alpha,\beta}$ scheme, we choose Δt^n such that it satisfies the Courant-Friedrichs-Lewy (CFL) condition

$$C^n \Delta t^n / h_{max}^n \leq 1 \quad (4.9)$$

where $h_{max}^n = \max_i \{h_i^n\}$ and where C^n is an approximation of the maximum of $\left|\frac{\partial \psi_h}{\partial z}\right|$ on both sides on Γ_h^n . Since $q_{hx} = -\frac{\partial \psi_h}{\partial z}$, we calculate C^n as

$$C^n = \max_{(x_i^n, u_i^n) \in \Gamma_h^n} [\max(|q_{hx}^f(x_i^n, u_i^n)|, |q_{hx}^s(x_i^n, u_i^n)|)].$$

In the case of the fixed mesh, we calculate C^n as

$$C^n = \max_{\substack{0 \leq i \leq N \\ 0 \leq j \leq M-1}} \left| \frac{\psi_h^\varepsilon(x_i, z_{j+1}) - \psi_h^\varepsilon(x_i, z_j)}{k} \right|$$

b. Boundary conditions.

We have already remarked that since the lines $x = \pm R$ are characteristics of the differential equation (4.1), no boundary conditions are necessary to define the analytical problem. However, we need numerical boundary conditions; we obtain them by approximating the equations on the characteristics

$$u_t(\pm R, t) = \psi_x(\pm R, u(\pm R, t), t) \quad (4.10)$$

In order to find a suitable scheme, close enough to the $S^{\alpha,\beta}$ scheme used to approximate u in the interior of the domain we rewrite (4.6) as

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{1}{2\alpha}\left[2\alpha\,\frac{\psi_{i+1}^n-\psi_{i-1}^n}{2\Delta x} + \beta\Delta x\,\frac{-\psi_{i+1}^n+2\psi_i^n-\psi_{i-1}^n}{\Delta x^2} - \frac{\psi_i^n-\psi_{i-1}^n}{\Delta x} + \frac{\tilde\psi_i^n-\tilde\psi_{i-1}^n}{\Delta x}\right]$$

in which we formally let Δx and Δt tend to zero. We obtain the equation

$$u_t = \frac{1}{2\alpha}\left[(2\alpha-1)\,\frac{\partial\psi}{\partial x} + \frac{\partial\tilde\psi}{\partial x}\right]$$

which is completely formal but motivates us to discretize (4.10) by means of the following analog of the second - order Runge - Kutta scheme (see for instance Hildebrand [7,p.146]).

$$\tilde u_0 = u_0^n + \alpha\Delta t^n\,\frac{\psi_h(x_1^n,u_1^n,t^n)}{h_0^n}$$

$$u_0^{n+1} = u_0^n + \Delta t^n\left[\left(1-\frac{1}{2\alpha}\right)\frac{\psi_h(x_1^n,u_1^n,t^n)}{h_0^n} + \frac{1}{2\alpha}\,\frac{\tilde\psi_h(\tilde x_1^n,\tilde u_1^n,\tilde t^n)}{\tilde h_0^n}\right].$$

in the point $x_0 = -R$ and similar formulas in the point $x_N = R$.

These boundary conditions are necessary as soon as Γ_h^n intersects the lines $x = \pm R$. In the case of the fixed mesh and when Γ_h^n intersects the lines $z = 0$ and $z = 1$, we either use them or calculate u_h^n only between the approximated values of S_1 and S_2; in the case of the adaptive grid, we always compute u_h^n only between the approximated values of S_1 and S_2.

c. Computation of S_1 and S_2.

Let S_1^n and S_2^n be the approximated values of $S_1(t^n)$ and $S_2(t^n)$ We assume that $S_1^n > -R$ and that $S_2^n < R$ and define k_1^n and k_2^n by

$$k_1^n = \min\{i, x_i^n > s_1^n\} \text{ and } k_2^n = \max\{i,\ x_i^n < s_2^n\}\ .$$

The second order Runge-Kutta method that we also use to compute u at the boundaries yields

$$\tilde S_1^n = S_1^n - \alpha\Delta t^n\,\frac{\psi_h(x_{k_1^n}^n,\ u_{k_1^n}^n,\ t^n)}{u_{k_1^n}^n}$$

$$\tilde S_2^n = S_2^n + \alpha\Delta t^n\,\frac{\psi_h(x_{k_2^n}^n,\ u_{k_2^n}^n,\ t^n)}{1-u_{k_2^n}^n}\ .$$

and

$$S_1^{n+1} = S_1^n - \Delta t^n\left[\left(1-\frac{1}{2\alpha}\right)\frac{\psi_h(x_{k_1^n}^n,\ u_{k_1^n}^n,t^n)}{u_{k_1^n}^n} + \frac{1}{2\alpha}\,\frac{\tilde\psi_h(\tilde x_{\tilde k_1^n}^n,\tilde u_{\tilde k_1^n}^n,\tilde t^n)}{\tilde u_{\tilde k_1^n}^n}\right]$$

$$S_2^{n+1} = S_2^n + \Delta t^n\left[\left(1-\frac{1}{2\alpha}\right)\frac{\psi_h(x_{k_2^n}^n,\ u_{k_2^n}^n,\ t^n)}{1-u_{k_2^n}^n} + \frac{1}{2\alpha}\,\frac{\tilde\psi_h(\tilde x_{\tilde k_2^n}^n,\ \tilde u_{\tilde k_2^n}^n,\tilde t^n)}{1-\tilde u_{\tilde k_2^n}^n}\right]$$

where

$$\tilde k_1^n = \min\{i\,|\ \tilde x_i^n > \tilde S_1^n\} \text{ and } \tilde k_2^n = \max\{i\,|\tilde x_i^n < \tilde S_2^n\}.$$

We do not impose any linearized stability condition on Δt^n since it would be less constraining than the CFL condition (4.9).

d. Some practical details about the computation of u_h^n.

In the case of the fixed mesh we set

$$u_h^{n+1}(x) = 0 \qquad \text{for all } x \le S_1^{n+1}$$

$$u_h^{n+1}(x) = 1 \qquad \text{for all } x \ge S_2^{n+1} \tag{4.11}$$

and use formulas (4.5), (4.6) to calculate u_i^{n+1} for $i \in \{k_1^{n+1},\dots,k_2^{n+1}\}$. We then define u_h^{n+1} on (S_1^{n+1},S_2^{n+1}) by linear interpolation. An example is shown in Figure 4.1.

In the case of the adaptive mesh, we define u_h^{n+1} by (4.11) outside of (S_1^{n+1},S_2^{n+1}) and calculate values $\bar{u}_{\cdot}^{n+1}$ in previously computed mesh points by using formulas (4.6), (4.7). This yields a first discretised interface at time t^{n+1}.

$$(S_1^{n+1},0),(S_1^n,\bar{u}^{n+1}_{k_1^n-1}),\ (x^n_{k_1^n},\bar{u}^{n+1}_{k_1^n}),\ \dots,(x^n_{k_2^n},\bar{u}^{n+1}_{k_2^n}),(S_2^n,\bar{u}^{n+1}_{k_2^n+1}),(S_2^{n+1},1) \tag{4.12}$$

Then we make a new distribution $\{x_i^{n+1}\}$ of the interval (S_1^{n+1},S_2^{n+1}) based on a new distribution of coordinates p along the interface and use linear interpolation to compute $\{u_i^{n+1}\}$.

5. Numerical results.

5.1. The finite time results

As a test case we have chosen the solution of Problem P with the initial condition

$$u_0(x) = \begin{cases} 0 & -R \le x < -1/2 \\ 2x+1 & -1/2 \le x < 1/6 \\ -x+1/2 & -1/6 \le x < 1/6 \\ 2x & 1/6 \le x < 1/2 \\ 1 & 1/2 \le x \le R \end{cases}$$

Figures 5.1 and 5.2 represent the streamfunction for several values of the height z and the velocity field in the fresh water region, corresponding to the initial interface $u_0(x)$. We observe jumps in $\frac{\partial\psi}{\partial x}$ at u_0; this corresponds to the matching conditions stated in Corollary 2.4 and the shear flow described in [9]. Moreover the interface is a line of local extrema of $\psi(.,z)$ for all $z \in [0,1]$ and of $\psi(x,.)$ for all $x \in [-1,1]$. Due to this behaviour the space steps must be taken small (at least in a neighborhood of the interface) in order to obtain a reasonable approximation of the stream function along Γ_u. When using the fixed mesh, we take $h = 1/50$, $k = 1/50$. With the adaptive mesh, the triangles generated near the interface are smaller than those in the far field (see Figure 4.3).

The computations of Figures 5.3 and 5.4 have been performed in the domain $\Omega = (-1,1) \times (0,1)$ with the fixed mesh procedure. The solution u with initial condition u_0 is computed by means of the $S^{\alpha\beta}$ scheme with α, β optimal while use is made of the discretized equations for $S_1(t)$ and $S_2(t)$. We have taken the time step

$$\Delta t^n = 1/2\ \Delta t^n_{CFL} \quad \text{where } \Delta t^n_{CFL} = h^n_{max}/C^n$$

Computations were also done with $\Delta t^n = \Delta t^n_{CFL}$. Then one observes small jumps in the velocities of the toe $\dot{S}_1$ and the top $\dot{S}_2$ which seem to occur at values of t where the difference $|S_1(t^{n+1}) - S_1(t^n)|$ or $|S_2(t^{n+1}) - S_2(t^n)|$ is equal to h; this difference then has the maximum value allowed by the

CFL condition. This has motivated us to perform the numerical computations with $\Delta t^n = \Delta t^n_{CFL}/2$.

As stated in section 4.2.b, the interface can also be computed without use of the differential equations for S_1 and S_2. The resulting curves differ very little form those shown in Figure 5.4.

When the automatic mesh generator is used, fewer grid points are necessary. Consequently larger time steps are allowed and the program requires less computing time. When comparing the results with those obtained by means of the fixed mesh (Figures 5.3, 5.4) we find that the differences in the position of the free boundaries $S_1(t)$ and $S_2(t)$ are less than 0.02.

Finally we observe that the discrete analog of the conservation of mass equation $\int_I u(t) = \int_I u_0$ is satisfied by the numerical solution.

5.2 The large time behaviour

In order to verify the convergence to a constant for large t we performed a numerical test with the following initial condition

$$u_{0N}(x) = 1/2(\tanh(x) + 1) \qquad -1 \le x \le 1.$$

The interface curves drawn in Figure 5.5 have been obtained with the fixed mesh. In this case the values of Δt^n_{CFL} are much larger than the space step h which leads us to choose $\Delta t^n = h$.

We remark that the Dupuit approximation is verified when the slope of the interface is sufficiently small. Figure 5.6 shows the streamfunction at different cross sections perpendicular to the x-axis, for the interface u_{0N}. We observe the ψ is linear in z on each subdomain which means that the component q_x of the velocity is constant in each fluid.

Figure 5.7 shows the long time behaviour of the solution u of problem P with initial condition u_0 in the case of a strip. The computations were performed in the domain $\Omega_R = (-3,3) \times (0,1)$ with the adaptive mesh. One observes that for large t the interface behaves as a rotating line.

Next, we compare the computed solution with a similarity solution u_s of the simplified problem C of Section 3, namely the similarity solution with initial value $u_s(0) = f$; then $g_0 = 1$. The points $S_1(0)$ and $S_2(0)$ are the same for both initial conditions u_0 and f; moreover, we have that $\int_I (u_0 - f)dx = 0$.

It follows from [6] that the curves $S_{1s}(t)$ and $S_{2s}(t)$ corresponding to the similarity solution are given by

$$S_{1s}(t) = -\frac{g(t)}{2} \quad \text{and} \quad S_{2s}(t) = \frac{g(t)}{2}$$

where g is the function defined in Theorem 3.2. In Figure 5.8, the values of S_2 and S_{2s} are given at several times.

t	$S_2(t)$ (computed values)	$S_{2s}(t)$
1.428	1.26	1.13
2.770	1.69	1.56
4.570	2.14	2.03

Figure 5.8

REFERENCES

[1] BEAR, J., Dynamics of Fluids in Porous Media, Elsevier, New York, 1975.

[2] CHAN HONG, J.R. & D. HILHORST, in preparation.

[3] DIBENEDETTO, E. & D. HOFF, An interface tracking algorithm for the porous media equation, Trans. Amer. Math. Soc. 284 (1984) 463-500

[4] DUCHON, J. & R.ROBERT, Estimation d'opérateurs intégraux du type de Cauchy dans les échelles d'Ovsjannikov et application, Annales Inst. Fourier, 36 (1986) 83-95.

[5] DUCHON, J. & R.ROBERT, Perturbation quasi-différentielle d'un semi-groupe régularisant dans une échelle d'espaces de Banach, C.R. Acad. Sc. Paris 301 (1985) 561-564.

[6] VAN DUYN, C.J. & D.HILHORST, On a doubly degenerate parabolic equation in hydrology, to appear in J.Nonlinear Analysis TMA 1987.

[7] HILDEBRAND, F.B., Finite-Difference Equations and Simulations, Prentice-Hall, Englewood Cliffs 1968.

[8] HOFF, D., A scheme for computing solutions and interface curves for a doubly degenerate parabolic equation, SIAM J.Num. Anal. 22 (1985)687-712.

[9] DE JOSSELIN DE JONG, G., The simultaneous flow of fresh and salt water in aquifers of large horizontal extension determined by shear flow and vortex theory, in Proc. Euromech. 143, A. Verruit & F.B.J. Barends eds., Balkema Rotterdam, 1981.

[10] DE JOSSELIN DE JONG, G. & C.J.VAN DUYN, Transverse dispersion from an originally sharp fresh-salt interface caused by shear flow, J. of Hydrology 84 (1986), 55-79.

[11] LERAT, A. & R. PEYRET, Sur le choix de schémas aux différences finies du second ordre fournissant des profils de choc sans oscillation, C.R. Acad. Sc. Paris 277 (1973) 363-366.

[12] LIONS, J.L. & E. MAGENES, Problemi ai limiti non omogenei III, Ann. Scuola Norm. Sup. Pisa 15 (1961) 41-101.

[13] RASMUSSEN, H. & D.SALHANI, Unsteady porous flow with a free surface, IMA J. Appl. Math. 27 (1981) 307-318.

[14] TEMAM, R., Navier-Stokes Equation, Theory and Numerical Analysis, North Holland Amsterdam, 1979.

[15] VERRUIT, A., A finite element model for interface problems in ground-water flow in Microcomputers in Engineering Applications, B.A. Schrefler ed., John Wiley 1985.

[16] VO KHAC KHOAN, Distributions, Analyse de Fourier, Opérateurs aux Dérivées Partielles, Tome II, Vuibert, Paris 1972.

[17] WILDERS, P., Minimization of dispersion in difference methods for hyperbolic conservation laws, Doctoral Thesis, University of Amsterdam 1983.

[18] WILSON, J.L. & A.DACOSTA, Finite element simulation of a salt water-fresh water interface with indirect toe tracking, Water Resour. Res. 18 (1982) 1069-1080.

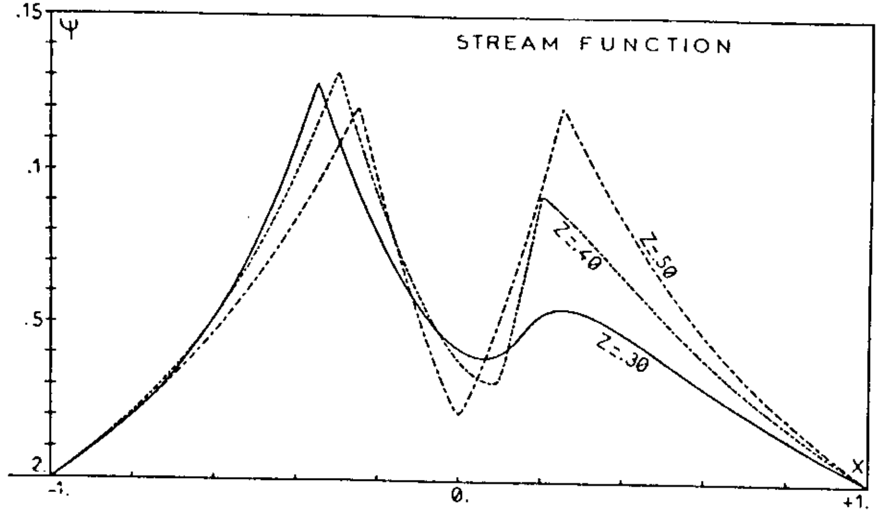


Figure 5.1 Stream Function for the interface u_o

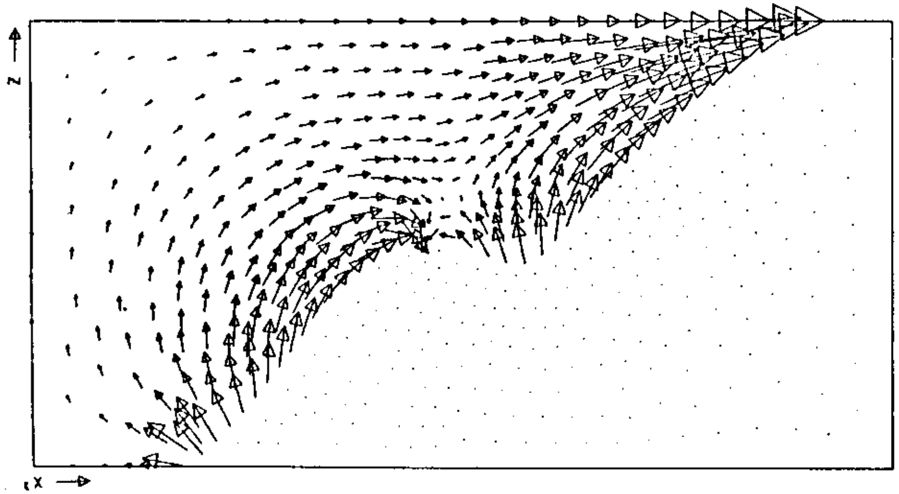


Figure 5.2 Velocity field of the fresh water region for the interface u_o

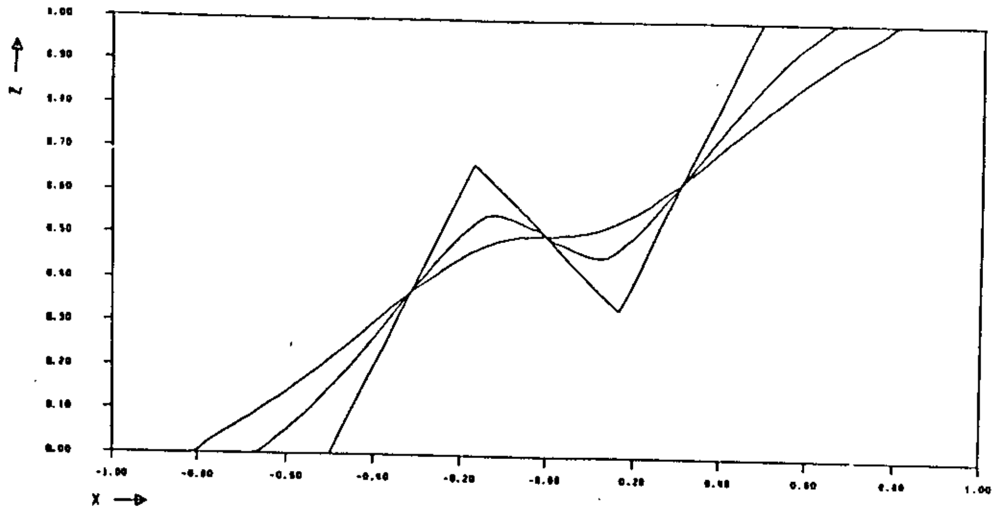


Figure 5.3 Interface computed between S_1 and S_2 at t=0.204 and t=0.455 α =2.118 $\beta=\frac{1}{2}$ $\Delta t^n = \frac{1}{2}\Delta t^n_{cfl}$

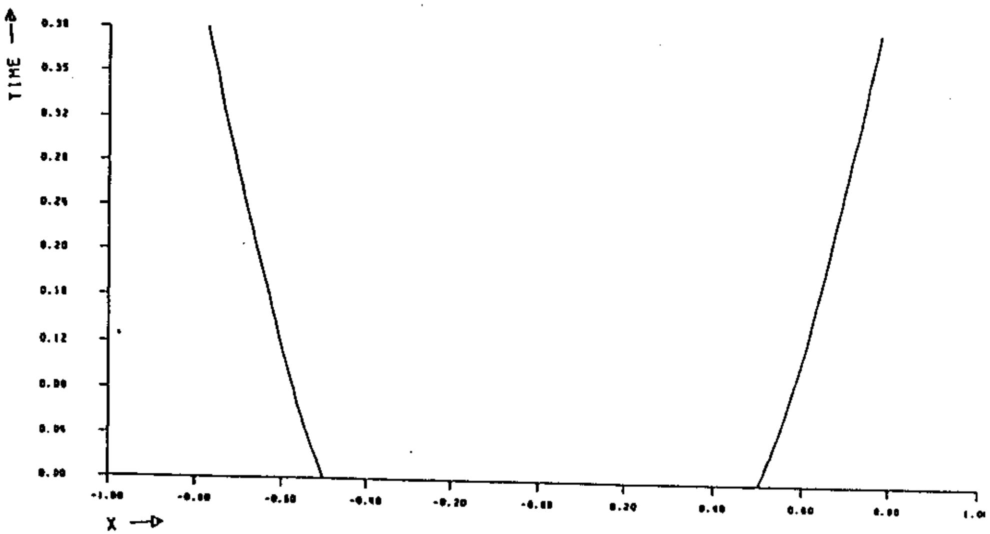


Figure 5.4 Evolution of the free boundaries S_1 and S_2 α= 2.118 $\beta=\frac{1}{2}$ $\Delta t^n = \frac{1}{2}\Delta t^n_{cfl}$

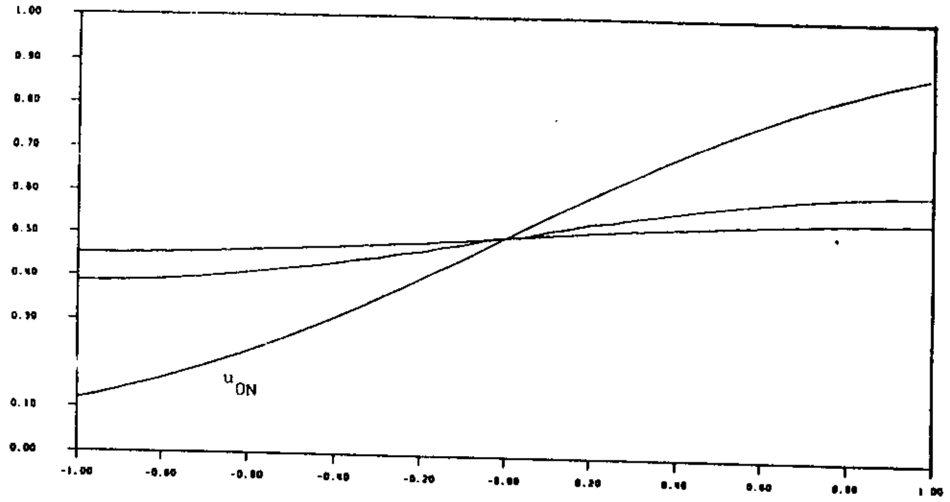


Figure 5.5 Interface computed at t = 2.5 and t = 4.25

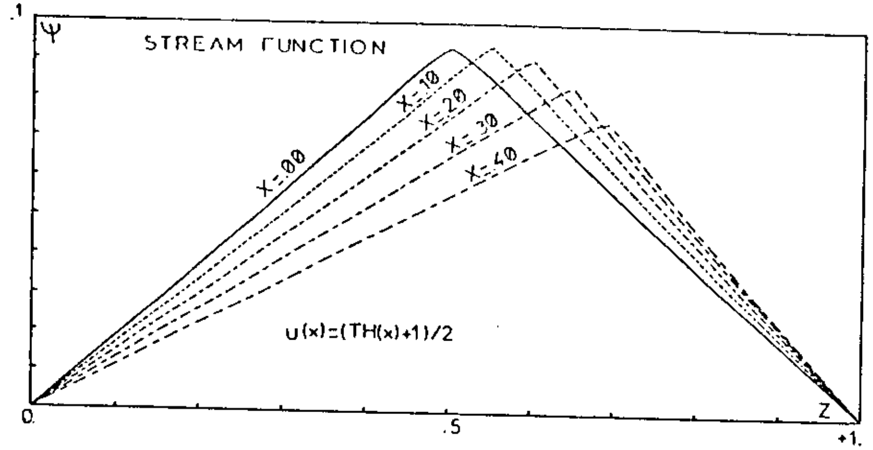


Figure 5.6 Justification for the Dupuit approximation.

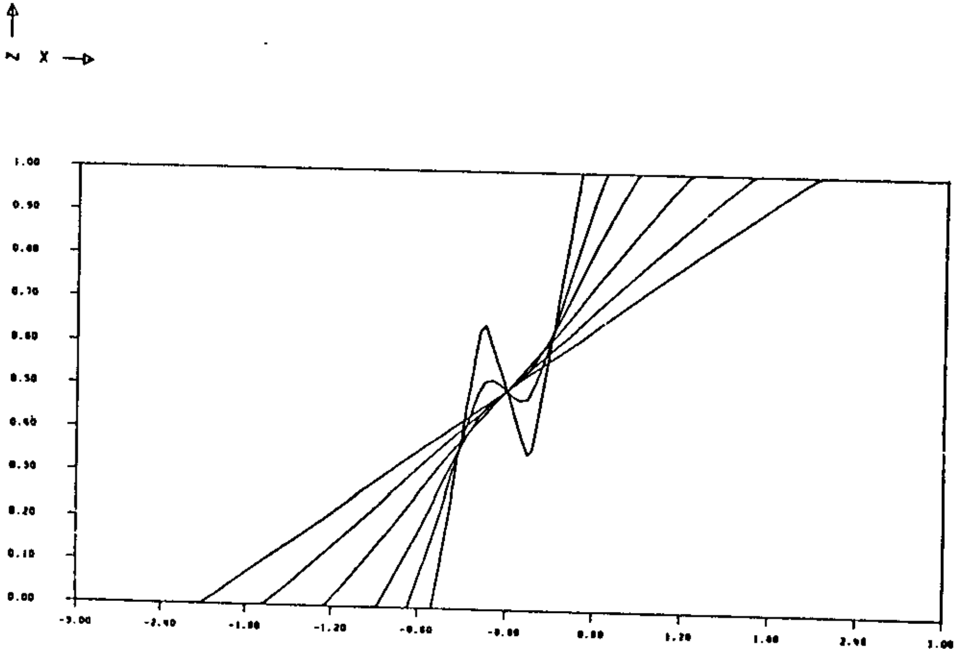


Figure 5.7 Convergence to a similarity solution. The interface is computed at t=0.24, t=0.6, t=1.43, t=2.77, t=4.57 with the mesh generator.

CHAPTER 2

On a doubly nonlinear diffusion equation in hydrology.

C.J.van Duyn
Department of Mathematics and Informatics
Delft University of Technology, Delft, The Netherlands
and
D.Hilhorst *
CNRS, Laboratoire d'Analyse Numérique
Université Paris-Sud, Orsay, France

1. Introduction.

In this paper we consider the nonlinear partial differential equation

(1.1) $$u_t = (D(u)\phi(u_x))_x$$

where the functions D and ϕ satisfy the hypotheses

H_ϕ : $\phi \in C^1([-1,1]) \cap C^2((-1,1))$, $\phi(0) = 0$, $\phi'(-1) = \phi'(1) = 0$,
$\phi' > 0$ on $(-1,1)$.

and

H_D : $D \in C^1([0,1]) \cap C^2((0,1))$, $D > 0$ on $(0,1)$, $D(0) = D(1) = 0$,
$D'' \le 0$ on $(0,1)$.

The main difficulty in studying equation (1.1) is that it has two kinds of degeneracies, namely one in points where $u = 0$ or 1 and one in points where $u_x = 1$ or -1. An equation of type (1.1) arises in the theory of hydrology, with $D(s) = s(1-s)$ and $\phi(s) = s/(1+s^2)$. We give its derivation in section 2.

* This author's research was supported in part by funds of the Delft University of Technology, The Netherlands.

We are interested in the following three problems related to equation (1.1) : the Neumann problem on $(-1,1)$ with the natural boundary conditions $D(u)\phi(u_x)(\pm 1,t) = 0$ for $t > 0$, the Cauchy problem and a related Cauchy Dirichlet problem on $(0,\infty)$ with the boundary condition $u(0,t) = A$ for $t > 0$, $A \in (0,1)$. For each problem we assume that the initial function u_0 is Lipschitz continuous and such that $0 \le u_0 \le 1$ and $-1 \le u_0' \le 1$ a.e. in the corresponding domain. For the precise assumptions we refer to section 3.

In section 4, we show that solutions of the three problems satisfy a contraction property in L^1. It then follows immediately that each problem has at most one solution.

Considering related uniformly parabolic problems and using the monotony of the function ϕ , we prove that there exists a solution of each problem (for the Cauchy-Dirichlet problem under some extra assumptions on the data). This is done in section 5.

In section 6 we study the large time behaviour. In the case of the Neumann problem, we show with the help of a suitably chosen Lyapunov functional that the solution converges to a constant as $t \to \infty$. For the Cauchy problem, we give conditions on the initial function under which the solution converges to a similarity solution as $t \to \infty$ in the case that $D(u) = u(1-u)$. Finally, we show by means of a method based on the comparison principle that the solution of the Cauchy-Dirichlet problem converges to the unique stationary solution as $t \to \infty$.

Other doubly degenerate problems, with differential equations of the form

(1.2) $$u_t = (\psi((\Phi(u))_x))_x$$

have been considered by several authors : in the case of the Cauchy problem, Kalashnikov [15,16] gives a method for studying the existence of a solution and proves some properties related to the support of u. Bamberger [4] constructs a solution of the Dirichlet problem and remarks that it has at most one solution such that $u_t \in L^1$. For the study of the semi-group solution we refer to Benilan & Crandall [6] and Cortazar [7].

Atkinson & Bouillet [2,3] study similarity solutions for the Cauchy-Dirichlet problem and give a comparison principle for solutions satisfying $u_t \in L^1$. We remark that the methods used in this paper to study the large time behaviour of solutions could also be applied to solutions of the differential equation (1.2).

Acknowledgement. The authors are very grateful to M.Bertsch for the many discussions which have been at the origin of this paper. They wish to thank Ph. Clément for pointing out the contraction property, R.Kersner for showing the articles of Kalashnikov and L.A.Peletier for many inspiring comments.

2. The physical derivation of the problem.

Consider the two-dimensional flow of fresh and salt groundwater in a homogeneous coastal aquifer, which is vertically confined (with height H) and horizontally extended. The fresh and salt groudwater have a different specific weight, γ_f and γ_s respectively. In addition to external factors, the difference in specific weight induces a flow and thus a movement of both fluids.

It is common practice in hydrology to assume that the fluids are separated by a sharp *interface*, e.g. see Bear [5]. Adopting this assumption, it is then sufficient to know the evolution of this fresh-salt interface in order to determine the movement of the fluids.

In this section, a derivation of a differential equation is given which describes the fresh-salt interface as a function of position and time. The analysis is based on the work of de Josselin de Jong [14], further references are given there.

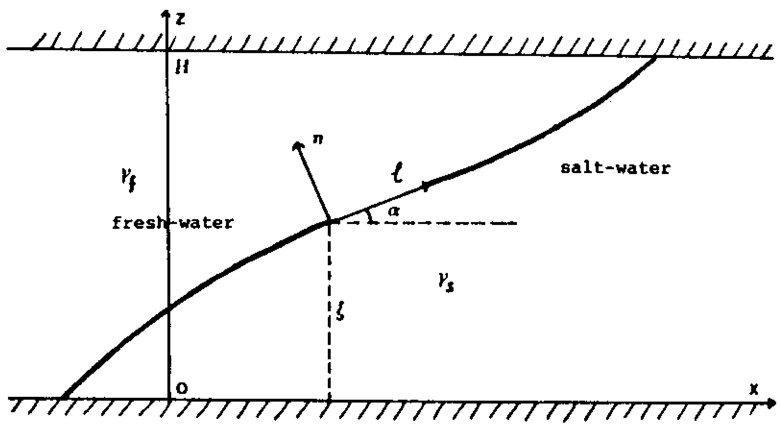


Fig.1. The distribution of fresh and salt water in an aquifer.

Let the flow take place in the xz-plane. The height of the interface is denoted by $\zeta(x,t)$: when $0 \le z < \zeta(x,t)$ only salt water is present, when $\zeta(x,t) < z \le H$ only fresh water is present. Here t denotes the time.

Further, let α denote the angle of the tangent at the interface with the horizontal and let n and ℓ denote the local orthogonal coordinates, normal and tangential to the interface (see fig.1). In both fluids, the specific discharge $\underline{q}$, the pressure p and specific weight γ are related through Darcy's law as :

(2.1) $$\frac{\mu}{k} q_i + \text{grad } p_i + \gamma_i \underline{e}_z = 0 , \quad i = f,s,$$

where μ is the dynamic viscosity of the fluids (which is assumed here to be the same for both fluids), k is the intrinsic permeability of the porous medium and $\underline{e}_z$ is the unit vector in the positive z-direction.
If the fluids are incompressible, the following *continuity* condition is required at the interface :

(2.2) $$q_{f_n} - q_{s_n} = 0 \quad \text{at } z = \xi.$$

At the interface, the fluids must also be in *equilibrium*. This means that the pressure on either side of the interface must be equal : $p_f - p_s = 0$ along the interface. This implies that

(2.3) $$\frac{\partial p_f}{\partial \ell} - \frac{\partial p_s}{\partial \ell} = 0 \quad \text{at } z = \xi.$$

Equations (2.2) and (2.3), written out in x and z coordinates become

(2.4) $$(q_{f_x} - q_{s_x}) \sin\alpha - (q_{f_z} - q_{s_z}) \cos\alpha = 0$$

and

(2.5) $$\left(\frac{\partial p_f}{\partial x} - \frac{\partial p_s}{\partial x}\right)\cos\alpha - \left(\frac{\partial p_f}{\partial z} - \frac{\partial p_s}{\partial z}\right)\sin\alpha = 0 \quad .$$

Substituting Darcy's law in (2.5) yields

(2.6) $$(q_{f_x} - q_{s_x}) \frac{\mu}{k} \cos\alpha + (q_{f_z} - q_{s_z}) \frac{\mu}{k} \sin\alpha = \sin\alpha \, (\gamma_s - \gamma_f) \quad .$$

Then from (2.4) and (2.6), the unknown $(q_{f_x} - q_{s_x})$ can be solved:

(2.7) $$q_{f_x} - q_{s_x} = \Gamma \frac{\tan\alpha}{1+\tan^2\alpha} \quad \text{at } z = \xi,$$

where $\Gamma = \frac{k}{\mu}(\gamma_s - \gamma_f)$. Here (2.7) represents the x-component of the discontinuity which occurs in the tangential component of $q_f - q_s$ at the interface : this is the *shear flow* observed by de Josselin de Jong.

The total fresh water discharge through the aquifer in the positive x-direction is given by

(2.8) $$Q_{f_x}(x,t) = \int_{\xi(x,t)}^{H} q_{f_x}(x,z,t)dz \quad .$$

The corresponding expression for the saltwater is

$$Q_{s_x}(x,t) = \int_{0}^{\xi(x,t)} q_{s_x}(x,z,t)dz.$$

When the aquifer is confined in the sense that $q_{f_z} = q_{s_z} = 0$ when $z = 0$ or $z = H$ then the following continuity equations hold:

(2.9) $$\frac{\partial Q_{f_x}}{\partial x} = n \frac{\partial \xi}{\partial t}$$

and

$$\frac{\partial Q_{s_x}}{\partial x} = -n \frac{\partial \xi}{\partial t}$$

where n denotes the porosity of the medium. Consequently, the total discharge

(2.10) $$Q = Q_{f_x} + Q_{s_x} ,$$

does not depend on x : it is considered here as a given constant.

Next a simplification is being made which is related to the *Dupuit-approximation* in hydrology : it is assumed here that the horizontal components of the specific discharges q_{f_x} and q_{s_x} are constant over the height of the aquifer. Thus

$$q_{f_x}(x,z,t) = q_{f_x}(x,\xi,t) \qquad \text{for } \xi \le z \le H \tag{2.11}$$

and

$$q_{s_x}(x,z,t) = q_{s_x}(x,\xi,t) \qquad \text{for } 0 \le z \le \xi \ . \tag{2.12}$$

Strictly speaking, this simplification is only valid when the interface is rather flat : thus for large angles α we expect this model to break down.

The total discharge can now be written as

$$Q = Q_{f_x} + Q_{s_x} = q_{f_x}(x,\xi,t)(H-\xi) + q_{s_x}(x,\xi,t)\xi \ . \tag{2.13}$$

From equations (2.7) and (2.13) the unknowns $q_{f_x}(x,\xi,t)$ and $q_{s_x}(x,\xi,t)$ can be solved : for q_{f_x} one finds,

$$q_{f_x}(x,\xi,t) = Q + \Gamma\xi\frac{\tan\alpha}{1+\tan^2\alpha} \ . \tag{2.14}$$

Finally, expression (2.14) is substituted into equation (2.8) and the result into equation (2.9). This gives the partial differential equation

$$n\xi_t = \{(H-\xi)Q + \Gamma(H-\xi)\xi\frac{\xi_x}{1+\xi_x^2}\}_x \ , \tag{2.15}$$

where the subscripts t and x denote differentiation with respect to these variables and $\tan\alpha = \xi_x$ is used.

Setting $n = H = \Gamma = 1$, $Q = -\lambda$ and $\xi(x,t) = u(x,t)$, (2.15) becomes

$$u_t = (u(1-u)\frac{u_x}{1+u_x^2})_x + \lambda u_x \ . \tag{2.16}$$

Observe that in the case of the Cauchy problem, the term λu_x can be eliminated. We set $\bar{x} = x + \lambda t$ and $u(x,t) = \bar{u}(\bar{x},t)$. Then since $u_t = +\lambda\bar{u}_x + \bar{u}_t$, we have that

$$\bar{u}_t = \left(\bar{u}(1-\bar{u})\frac{\bar{u}_x}{1+\bar{u}_x^2}\right)_x \ .$$

In this paper, we study the following problems : the Cauchy problem

$$C\begin{cases} u_t = \left(u(1-u)\dfrac{u_x}{1+u_x^2}\right)_x & \text{for } (x,t) \in \mathbb{R}\times\mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

the Neumann problem which is interesting in its own right and which is useful for understanding Problem C

$$N\begin{cases} u_t = \left(u(1-u)\dfrac{u_x}{1+u_x^2}\right)_x & \text{for } (x,t) \in (-1,1)\times\mathbb{R}^+ \\ u(1-u)\dfrac{u_x}{1+u_x^2} = 0 & \text{for } (x,t) \in \{-1,1\}\times\mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in (-1,1), \end{cases}$$

and the Cauchy Dirichlet problem

$$\text{CD}\quad\begin{cases} u_t = \left(u(1-u)\dfrac{u_x}{1+u_x^2}\right)_x + \lambda u_x & \text{for } (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(0,t) = A & t \in \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in \mathbb{R}^+ \;, \end{cases}$$

with $\lambda \geq 0$ and $A \in (0,1)$.

Instead of studying these three problems with this specific differential equation, we consider the more general case where the nonlinear term has been replaced by $(D(u)\phi(u_x))_x$, where D and ϕ are given real functions such that D defined on the interval $[0,1]$ and ϕ on $[-1,1]$ satisfy the hypotheses H_D and H_ϕ.

3. Definitions.

Let us first give a definition of a solution of the three problems and state for each of these problems the precise hypotheses on the initial function u_0

The Neumann problem

$$\text{N}\quad\begin{cases} u_t = (D(u)\phi(u_x))_x & (x,t) \in (-1,1) \times \mathbb{R}^+ \\ D(u)\,\phi(u_x) = 0 & (x,t) \in \{-1,1\} \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in (-1,1) \;, \end{cases}$$

where u_0 satisfies the hypothesis

H_{0N} : $u_0 \in W^{1,\infty}(-1,1)$, $0 \leq u_0 \leq 1$, $-1 \leq u_0' \leq 1$ a.e.

Definition 3.1. We say that u is a weak solution of Problem N if it satisfies for every $T > 0$

(i) $u \in L^\infty(0,T;W^{1,\infty}(-1,1)), u_t \in L^2(Q_{NT})$ with $Q_{NT} := (-1,1)\times(0,T)$;

(ii) $0 \leq u \leq 1$, $-1 \leq u_x \leq 1$ a.e. in Q_{NT};

(iii) $u(.,0) = u_0(.)$;

(iv) $\iint_{Q_{NT}} \{u_t\,\psi + D(u)\phi(u_x)\,\psi_x\} = 0$ for all $\psi \in L^2(0,T;H^1(-1,1))$.

The Cauchy problem

$$\text{C}\quad\begin{cases} u_t = (D(u)\phi(u_x))_x & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in \mathbb{R} \;, \end{cases}$$

where u_0 satisfies

H_{0C} : $u_0 \in W^{1,\infty}(\mathbb{R})$, $0 \leq u_0 \leq 1$, $-1 \leq u_0' \leq 1$ a.e. and $u_0 - H \in L^1(\mathbb{R})$ where H denotes the Heaviside function: $H(x) = 1$ when $x>0$ and $H(x) = 0$ when $x \leq 0$.

Definition 3.2. We say that u is a weak solution of Problem C if it satisfies for every $T > 0$

(i) $u \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}))$, $u_t \in L^2((-R,R) \times (0,T))$ for all $R > 0$;

(ii) $0 \le u \le 1$, $-1 \le u_x \le 1$ a.e. in Q_{CT} where $Q_{CT} := \mathbb{R} \times (0,T)$;

(iii) $u(.,0) = u_0(.)$;

(iv) $\iint_{Q_{CT}} \{u_t \psi + D(u)\phi(u_x)\psi_x\} = 0$ for all $\psi \in L^2(0,T; H^1(\mathbb{R}))$

such that ψ vanishes for large $|x|$.

The Cauchy-Dirichlet problem

$$CD \begin{cases} u_t = (D(u)\phi(u_x))_x + \lambda u_x & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^- \\ u(0,t) = A & t \in \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in \mathbb{R}^+ , \end{cases}$$

where the constants λ and A are such that $\lambda \ge 0$ and $0 < A < 1$ and where u_0 satisfies the hypothesis

H_{0D} : $u_0 \in W^{1,\infty}(0,\infty) \cap L^1(0,\infty)$, $0 \le u_0 \le 1$, $-1 \le u_0' \le 1$ a.e., $u_0(0) = A$.

Definition 3.3. We say that u is a weak solution of Problem CD if it satisfies for every $T > 0$

(i) $u - A \in L^\infty(0,T; V)$ where $V := \{v \in W^{1,\infty}(\mathbb{R}^+), v(0) = 0\}$ and $u_t \in L^2((0,R) \times (0,T))$ for all $R > 0$;

(ii) $0 \le u \le 1$, $-1 \le u_x \le 1$ a.e. in Q_{DT} where $Q_{DT} := \mathbb{R}^+ \times (0,T)$;

(iii) $u(.,0) = u_0(.)$;

(iv) $\iint_{Q_{DT}} \{u_t \psi + (D(u)\phi(u_x) + \lambda u)\psi_x\} = 0$ for all $\psi \in L^2(0,T; H_0^1(0,\infty))$

such that ψ vanishes for large x.

We remark that if u is a solution of any of the three problems then $u(t) \in W^{1,\infty}(\Omega)$ for all $t > 0$, where Ω denotes either $(-1,1)$ or $\mathbb{R}$ or $\mathbb{R}^+$. This is a consequence of a result given by Temam [19, Lemma 1.4 p.263].

4. Contraction property and uniqueness of the solution.

In this section we show how solutions of each problem satisfy a contraction property in L^1. The uniqueness of the weak solution follows immediately.

Lemma 4.1. Let u be a solution of any of the three problems. Then $D(u)\phi(u_x)(t) \in C(\bar\Omega)$ for a.e. $t>0$ where Ω denotes either $(-1,1)$ or $\mathbb{R}$ or $\mathbb{R}^+$.

Proof. We prove Lemma 4.1 in the case of Problem N. By Definition 3.1 $u_t \in L^2(-1,1)$ and $u_t = (D(u)\phi(u_x))_x$ for a.e. $t > 0$. Thus

$$(D(u)\phi(u_x))_x \in L^2(-1,1) \quad \text{for a.e. } t > 0$$

and consequently

$$D(u)\phi(u_x) \in C([-1,1]) \quad \text{for a.e. } t > 0 . \qquad \square$$

Remark 4.2. Let t be such that $D(u)\phi(u_x)(t) \in C(\bar\Omega)$. Then $u_x(t)$ is continuous as a function of x in every point x such that $u(x,t) \in (0,1)$.

Theorem 4.3. Let u and v be solutions of Problem N with initial conditions u_0 and v_0 respectively. Then

$$\|u(t) - v(t)\|_{L^1(-1,1)} \le \|u_0 - v_0\|_{L^1(-1,1)} \quad \text{for all } t > 0 .$$

Proof. Let W denote either u or v. By Definition 3.1 W satisfies for a.e. $t > 0$

$$(4.1) \quad \begin{cases} W_t \in L^2(-1,1) \text{ and } W_t = (D(W)\phi(W_x))_x \\ \text{and} \\ D(W)\phi(W_x)(\pm 1,t) = 0 . \end{cases}$$

Multiplying by sgn(u-v) the difference of the equations for u and v yields

$$(4.2) \quad \int_{-1}^{1} (u-v)_t \, \text{sgn}(u-v) = \int_{-1}^{1} \{D(u)\phi(u_x) - D(v)\phi(v_x)\}_x \, \text{sgn}(u-v)$$

for a.e. $t > 0$

where sgn $s = -1$ if $s < 0$, 0 if $s = 0$ and 1 if $s > 0$.

Next we use the following lemma, given for instance by Crandall & Pierre [8].

Lemma 4.4. Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. If $w \in W^{1,1}(0,T;L^1(-1,1))$, then $G(w) \in W^{1,1}(0,T;L^1(-1,1))$ and $\frac{d}{dt} G(w) = G'(w) \frac{dw}{dt}$ a.e.

It follow from Lemma 4.4 that

$$(u-v)_t \operatorname{sgn}(u-v) = |u-v|_t \quad \text{a.e.}$$

so that (4.2) implies that

(4.3) $$\frac{d}{dt} \|u-v\|_{L^1(-1,1)} = \int_{-1}^{1} \{D(u)\phi(u_x) - D(v)\phi(v_x)\}_x \operatorname{sgn}(u-v)$$

for a.e. $t > 0$.

We show below that the right hand side of (4.3) is nonpositive for a.e. $t > 0$. This corresponds to the accretivity in $L^1(-1,1)$ of the operator $Au = -(D(u)\phi(u'))'$ when defined on a suitable domain.

Let t be such that (4.1) holds ; since $u(t)$ and $v(t)$ are Lipschitz continuous the open interval $(-1,1) \setminus \{x | u(x,t) - v(x,t) = 0\}$ is the union of open intervals where either $u-v > 0$ or $u-v < 0$. Since the proofs for both kinds of intervals are similar, we only consider the intervals where $u-v > 0$. In order to simplify the notations, we omit the variable t in what follows.

(i) if $[a,b] \subset (-1,1)$ is such that $u-v > 0$ on (a,b) and $u = v$ in a and in b then

(4.4) $$\int_a^b (D(u)\phi(u_x) - D(v)\phi(v_x))_x \operatorname{sgn}(u-v)$$
$$= \{D(u)(\phi(u_x) - \phi(v_x))\}(b) - \{D(u)(\phi(u_x) - \phi(v_x))\}(a) .$$

Then if $u(b) = 0$ or 1 the first term on the right-hand side of (4.4) is equal to zero and if $0 < u(b) < 1$ it follows from Remark 4.2 that $u_x(b)$ and $v_x(b)$ are well defined; then $u_x(b) \le v_x(b)$ and this term is nonpositive; similarly one can see that the second term on the right-hand-side of (4.4) is also nonpositive.

(ii) if $(-1,c] \subset (-1,1)$ is such that $u - v > 0$ in $(-1,c)$ and $u(c) = v(c)$ then, in view of the boundary condition,

$$\int_{-1}^{c} \left(D(u)\phi(u_x) - D(v)\phi(v_x)\right)_x \operatorname{sgn}(u - v) = \left\{D(u)(\phi(u_x) - \phi(v_x))\right\}(c)$$

which, similarly as in the case (i), is nonpositive.

Finally, one finds that

$$\frac{d}{dt}\|u - v\|_{L^1(-1,1)} \le 0 \qquad \text{for a.e. } t > 0.$$

and thus

$$\|u(t) - v(t)\|_{L^1(-1,1)} \le \|u_0 - v_0\|_{L^1(-1,1)} \qquad \text{for all } t > 0. \qquad \square$$

Corollary 4.5. The solution of Problem N is unique.

In what follows, we prove similar properties for the problems C and CD.

Lemma 4.6. Let u be a solution of Problem C.

Then $u(t) - H \in L^1(\mathbb{R})$ for all $t>0$.

Proof. We show below that $\int_{-\infty}^{0} u(t) < \infty$. The proof that $\int_{0}^{+\infty} (1-u(t)) < \infty$ is similar. It follows from Definition 3.2 (iv) that u satisfies

$$\int_{\mathbb{R}} u(t)\psi = \int_{\mathbb{R}} u_0\psi + \int_0^t \int_{\mathbb{R}} \Big(D(u)\,\phi(u_x)\Big)_x \psi$$

for all $\psi \in H^1(\mathbb{R})$ with compact support and all $t > 0$. Let $R > 0$ be arbitrary. The characteristic function $\chi_{[-R,0]}$ of the interval $[-R,0]$ can be constructed as the limit in $L^2(\mathbb{R})$ of H^1 functions with compact support. Thus

$$\int_{\mathbb{R}} u(t)\chi_{[-R,0]} = \int_{\mathbb{R}} u_0\chi_{[-R,0]} + \int_0^t \int_{\mathbb{R}} \Big(D(u)\phi(u_x)\Big)_x \chi_{[-R,0]}$$

which implies that

$$\int_{\mathbb{R}} u(t)\chi_{[-R,0]} \le \int_{-\infty}^{0} u_0 + \int_0^t D(u)\phi(u_x)\Big]_{x=-R}^{x=0} .$$

Finally applying Lebesgue's monotone convergence theorem one finds

$$\int_{-\infty}^{0} u(t) \le \int_{-\infty}^{0} u_0 + Ct.$$

□

Corollary 4.7. Let u be a solution of Problem C. Then, for all $t > 0$, $u(x,t) \to 0$ as $x \to -\infty$, $u(x,t) \to 1$ as $x \to +\infty$.

Proof. Corollary 4.7 follows from Lemma 4.6 together with the fact that u(t) is Lipschitz continuous for all $t > 0$. □

We are now in a position to show that the solution u of Problem C defines a contraction in $L^1(\mathbb{R})$.

Theorem 4.8. Let u and v be solutions of Problem C with initial functions u_0 and v_0 respectively. Then

$$\|u(t) - v(t)\|_{L^1(\mathbb{R})} \le \|u_0 - v_0\|_{L^1(\mathbb{R})} \quad \text{for all } t > 0.$$

Corollary 4.9. The solution of Problem C is unique.

Proof of Theorem 4.8. Let $R > 1$ be arbitrary. Then, for a.e. $t > 0$

$$\frac{d}{dt}\|u-v\|_{L^1(-R,R)} = \int_{-R}^{R} \big(D(u)\phi(u_x) - D(v)\phi(v_x)\big)_x \operatorname{sgn}(u-v) .$$

Using the proof of Theorem 4.3 we deduce that for all t

$$\int_{-\infty}^{+\infty} |u(t) - v(t)|\chi_{[-R,R]} \le \int_0^t \Big\{ \big(|D(u)\phi(u_x)| + |D(v)\phi(v_x)|\big)(R,t) + \tag{4.5}$$
$$+ \big(|D(u)\phi(u_x)| + |D(v)\phi(v_x)|\big)(-R,t)\Big\}\,dt + \|u_0 - v_0\|_{L^1(\mathbb{R})} .$$

Let us denote by f_R the integrand in the first term at the right-hand-side of (4.5). Since by Corollary 4.7, f_R tends to zero as $R \to \infty$ for a.e. $\tau \in (0,t)$ and since $\|f_R\|_{L^1(0,t)} \le C$, it follows from the dominated convergence theorem that $\int_0^t f_R$ tends to zero as $R \to \infty$, which completes the proof. □

Similar results hold for the Cauchy-Dirichlet Problem CD, namely that

(i) $u(t) \in L^1(\mathbb{R}^+)$ for all $t > 0$;

(ii) $u(x,t) \to 0$ as $x \to +\infty$ for all $t > 0$;

(iii) If u and v are solutions of Problem CD with initial functions u_0 and v_0, then

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^+)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^+)} \quad \text{for all } t > 0 .$$

Since the proof of properties (i) - (iii) is very similar to the one given above, we omit it here.

5. Existence of solution.

In this section, we adapt a proof of Kalashnikov [17] in order to show that there exists a solution of each of the three problems (in the case of Problem CD under an extra assumption relating D,ϕ,λ and A). We first consider the Cauchy and the Neumann problem; we then show how one can modify the proof in order to obtain the existence result for the Cauchy Dirichlet problem as well.

We consider the following problems, with $n \in \mathbb{N}$ large enough

$$P_n \begin{cases} u_t = \left(D_n(u)\phi_n(u_x)\right)_x & \text{in } Q_{nT} := (-n,n) \times (0,T) \\ u_x(-n,t) = 0 \quad u_x(n,t) = 0 & \text{for } t \in (0,T] \\ u(x,0) = u_{0n}(x) & \text{for } x \in (-n,n) \end{cases} ,$$

where

$$D_n \in C^2(\mathbb{R}) \text{ is such that } D_n(s) = D(s) \text{ for } s \in \left[\frac{1}{n}, 1-\frac{1}{n}\right]$$

$$\text{and } \frac{1}{2} \inf_{[\frac{1}{n},1-\frac{1}{n}]} D \le D_n \le \sup_{[0,1]} D \quad \text{on } \mathbb{R} ,$$

and

$$\phi_n \in C^2(\mathbb{R}) \text{ is such that } \phi_n(s) = \phi(s) \text{ for } s \in [-1+\tfrac{1}{n}, 1-\tfrac{1}{n}]$$

$$\text{and } \frac{1}{2} \inf_{[-1+\frac{1}{n},1-\frac{1}{n}]} \phi' \le \phi_n' \le \sup_{[-1,1]} \phi' ,$$

and where

H_{0n} : $u_{0n} \in C^\infty(\mathbb{R})$, $\frac{1}{n} \le u_{0n} \le 1 - \frac{1}{n}$, $|u_{0n}'| \le 1 - \frac{1}{n}$, $u_{0n}'(x) = 0$ for $|x| \ge n$, u_{0n} converges uniformly to u_0 on all compact subsets of $\mathbb{R}$ as $n \to \infty$.

We show in the appendix that given u_0 satisfying the hypothesis H_{0C}, one can construct a sequence of functions $\{u_{0n}\}$ satisfying H_{0n}. First we prove a comparison principle.

Lemma 5.1. Let u_1 and $u_2 \in C^{2,1}(\bar{Q}_{nT})$ be two solutions of Problem P_n with corresponding initial functions $u_{01} \le u_{02}$. Then $u_1(t) \le u_2(t)$ for every $t \ge 0$.

Proof. The function $z := u_1 - u_2$ satisfies the problem

$$\begin{cases} z_t = \Big(D_n(u_1)A_n(x,t)z_x\Big)_x + \Big(B_n(x,t)\phi_n(u_{2x})z\Big)_x & \text{in } Q_{nT} \\ z_x(-n,t) = 0 \qquad z_x(n,t) = 0 & \text{for } t \in (0,T] \\ z(x,0) = u_{01}(x) - u_{02}(x) & \text{for } x \in (-n,n) \end{cases}$$

where

$$A_n(x,t) = \int_0^1 \phi_n'(\Theta u_{1x}(x,t) + (1-\Theta)\, u_{2x}(x,t))d\Theta$$

and

$$B_n(x,t) = \int_0^1 D_n'(\Theta u_1(x,t) + (1-\Theta)\, u_2(x,t))d\Theta \quad .$$

Since $z(0) \le 0$, it is a consequence from the standard maximum principle that $z(t) \le 0$ for all $t \in (0,T]$. □

Lemma 5.2. Problem P_n has a unique classical solution $u_n \in C^{2+\alpha}(\bar{Q}_{nT})$ for each $\alpha \in (0,1)$. Furthermore we have that $\frac{1}{n} \le u_n \le 1 - \frac{1}{n}$ and $-1 + \frac{1}{n} \le u_{nx} \le 1 - \frac{1}{n}$ in $\bar{Q}_{nT}$.

Proof. The existence and uniqueness of the solution of Problem P_n follows from [18, Theorem 7.4 p.491 and a remark at the end of Section 7 p.492]. Also we remark that both $\frac{1}{n}$ and $1 - \frac{1}{n}$ satisfy problem P_n which, by Lemma 5.1 implies that $\frac{1}{n} \le u_n \le 1 - \frac{1}{n}$. Next we show that $|u_{nx}| \le 1 - \frac{1}{n}$. We set $w = u_{nx}$. Using the linear theory (see for instance in [12, p.72]) we deduce that $w \in C^{2+\alpha}(Q_{nT})$. Thus $w \in C(\bar{Q}_{nT}) \cap C^{2,1}(Q_{nT})$. Differentiating the differential equation in P_n yields

$$\begin{aligned} w_t &= D(u_n)\phi_n'(w)w_{xx} + \Big(D(u_n)\phi_n''(w)w_x + 2D'(u_n)w\phi_n'(w) \\ &\quad + D'(u_n)\phi_n(w)\Big)w_x + D''(u_n)\phi_n(w)w^2 && \text{in } Q_{nT} \\ w(-n,t) &= 0 \quad w(n,t) = 0 && \text{for } t \in (0,T] \\ w(x,0) &= u_{0n}'(x) && \text{for } x \in (-n,n) \ . \end{aligned}$$

In order to simplify the notation, we rewrite the equation above as

$$w_t = a(x,t)w_{xx} + b(x,t)w_x - c(x,t)w$$

where a,b,c are continuous functions and $a > 0$, $c \ge 0$.
The function $w - 1 + \frac{1}{n}$ satisfies

$$\begin{aligned} a(x,t)(w-1+\tfrac{1}{n})_{xx} + b(x,t)(w-1+\tfrac{1}{n})_x - c(x,t)(w-1+\tfrac{1}{n}) - (w-1+\tfrac{1}{n})_t \\ \text{in } Q_{nT} \\ w(-n,t) - 1 + \tfrac{1}{n} \le 0 \qquad w(n,t) - 1 + \tfrac{1}{n} \le 0 \qquad \text{for } t \in (0,T] \\ w(x,0) - 1 + \tfrac{1}{n} \le 0 \qquad \text{for } x \in (-n,n) \ . \end{aligned}$$

Thus by the maximum principle $w - 1 + \frac{1}{n} \le 0$, that is $u_{nx} \le 1 - \frac{1}{n}$.
The bound $u_{nx} \ge -1 + \frac{1}{n}$ follows in the same way. □

Lemma 5.3. $\int_0^T \int_{-R}^{R} u_{nt}^2 \le C(R,T)$ for all $R \le n-2$

where the constant $C(R,T)$ does not depend on n.

Proof. Let $m,n \in \mathbb{N}$ such that $0 < m < n$. Set

$$\zeta_m(x) = \begin{cases} 1 & x \le m-1 \\ m - x & m-1 \le x \le m \\ 0 & x \ge m \\ \zeta_m(-x) & x \le 0 \end{cases} \quad .$$

We multiply the differential equation for u_n by $\zeta_m^2 u_{nt}$ to obtain the equality

$$\iint_{Q_{nT}} u_{nt}^2 \zeta_m^2 = \iint_{Q_{nT}} \Big(D(u_n)\phi(u_{nx})\Big)_x u_{nt}\, \zeta_m^2$$

that is, after integration by parts

$$\iint_{Q_{nT}} u_{nt}^2 \zeta_m^2 = -\iint_{Q_{nT}} D(u_n)\phi(u_{nx})\, u_{nx\,t}\, \zeta_m^2 - 2\iint_{Q_{nT}} D(u_n)\phi(u_{nx})u_{nt}\, \zeta_m \zeta_m' .$$

Thus

$$\iint_{Q_{nT}} u_{nt}^2 \zeta_m^2 = -\iint_{Q_{nT}} D(u_n)\frac{\partial}{\partial t}F(u_{nx})\zeta_m^2 - 2\iint_{Q_{nT}} D(u_n)\phi(u_{nx})u_{nt}\, \zeta_m \zeta_m'$$

where the positive function F is defined by $F(s) = \int_0^s \phi(\tau)d\tau$.
Thus

$$\iint_{Q_{nT}} u_{nt}^2 \zeta_m^2 + \int_{-n}^{n} D(u_n(T))F(u_{nx}(T))\zeta_m^2$$

$$\le \int_{-n}^{n} D(u_{0n})F(u_{0n}')\zeta_m^2 + \Big\{2\sup_{[0,1]} D \sup_{[-1,1]}\phi\Big(\iint_{Q_{nT}} \zeta_m'^2\Big)^{1/2} + \sup_{[0,1]} D' \sup_{[-1,1]} F \Big(\iint_{Q_{nT}} \zeta_m^2\Big)^{1/2}\Big\}\Big(\iint_{Q_{nT}} u_{nt}^2\zeta_m^2\Big)^{1/2}$$

which we rewrite as

$$\iint_{Q_{nT}} u_{nt}^2 \zeta_m^2 \le C_1 m + C_2\sqrt{mT}\Big(\iint_{Q_{nT}} u_{nt}^2 \zeta_m^2\Big)^{1/2} .$$

Finally, we find that

$$\int_0^T \int_{-m+1}^{m-1} u_{nt}^2 \le C(m,T)$$

which concludes the proof of Lemma 5.3. □

Theorem 5.4. There exists a solution of Problem C.

Proof. Fix $R > 0$. Since $1/n \le u_n \le 1 - 1/n$ and since $|u_{nx}| \le 1 - 1/n$, it follows from Gilding [13] that

$$|u_n(x,t') - u_n(x,t)| \le C|t - t'|^{1/2}$$

for all $r > R$ and for all (x,t), $(x,t') \in \bar{Q}_{RT} := [-R,R] \times [0,T]$. Here the constant C depends on R but does not depend on n. The set $\{u_n\}_{r > R}$ is bounded and equicontinuous in Q_{RT}. Thus there exists a continuous function u_R and a subsequence $\{u_{nk}\}$, with $n_k > R$ such that u_{nk} converges uniformly to u_R in $\bar{Q}_{RT}$ as $n_k \to \infty$. Then by a diagonal process there exists a function $u \in C(\bar{Q}_{CT})$ and a converging subsequence $\{u_j\}$ such that u_j converges to u as $j \to \infty$, pointwise on $\bar{Q}_{CT}$ and uniformly on all bounded subsets of $\bar{Q}_{CT}$. Also it follows from the estimates above that $u_{jx} \rightharpoonup u_x$ and $u_{jt} \rightharpoonup u_t$ weakly in $L^2(Q_{RT})$ for all $R > 0$ as $j \to \infty$. Thus u satisfies conditions (i), (ii), (iii) of Definition 3.2. In what follows we check that u also satisfies (iv). The function u_j satisfies for j sufficiently large:

$$\text{(5.1)} \qquad \iint_{Q_{CT}} \Big\{u_{jt}\psi + D(u_j)\phi(u_{jx})\ \psi_x\Big\} = 0$$

for all $\psi \in V := \{v \in L^2(0,T\ ;\ H^1(\mathbb{R}))$ such that $v = 0$ for large $|x|\}$. Since $\|\phi(u_{jx})\|_{L^\infty(Q_{CT})} \le \sup_{[-1,1]}|\phi|$, there exists $\chi \in L^\infty(Q_{CT})$ and a subsequence of $\{u_j\}$, that we denote again by $\{u_j\}$ such that

$$\phi(u_{jx}) \rightharpoonup \chi \text{ weakly in } L^2(Q_{RT}) ,$$

for any $R > 0$ as $j \to \infty$. Letting now $j \to \infty$ in (5.1) yields

$$\text{(5.2)} \qquad \iint_{Q_{CT}} \{u_t\psi + D(u)\chi\ \psi_x\} = 0 \quad \text{for all } \psi \in V$$

Next we show that .

(5.3) $$\iint_{Q_{CT}} D(u)(\chi - \phi(u_x))\,\psi_x = 0 \text{ for all } \psi \in V \quad .$$

We first write an inequality which is based on the monotonicity of the function ϕ and which involves the functions u_j. This was also done by Kalashnikov [17], for example.

Let us extend ϕ on $\mathbb{R}$ such that $\phi(s) = \phi(1)$ for $s > 1$ and $\phi(s) = \phi(-1)$ for $s < -1$ and let v be such that $v - u \in V$. We have, with $m < n$, $Q_{mT} := (-m,m) \times (0,T)$ and $\delta \in (0,1]$

$$\iint_{Q_{CT}} D(u)(\phi(u_{jx}) - \phi(v_x))\left(\zeta_m^2(u_j - v)\right)_x = \iint_{Q_{mT}} \zeta_m^2\, D(u)(\phi(u_{jx}) - \phi(v_x))(u_{jx} - v_x)$$

$$+ 2\iint_{Q_{mT}} \zeta_m\, \zeta_m'\, D(u)(\phi(u_{jx}) - \phi(v_x))(u_j - v)$$

$$\geq \iint_{Q_{mT}} \zeta_m^2\, D(u)(\phi(u_{jx}) - \phi(v_x))(u_{jx} - v_x)$$

$$- \delta \iint_{Q_{mT}} \zeta_m^2\, D(u)(\phi(u_{jx}) - \phi(v_x))^2 - \frac{1}{\delta}\iint_{Q_{mT}} D(u)(u_j - v)^2$$

$$\geq (1 - \delta \sup_{[-1,1]} \phi') \iint_{Q_{mT}} \zeta_m^2\, D(u)(\phi(u_{jx}) - \phi(v_x))(u_{jx} - v_x) - \frac{1}{\delta}\iint_{Q} D(u)(u_j - v)^2 .$$

Setting $\delta = 1/\sup_{[-1,1]} \phi'$ yields

(5.4) $$\iint_{Q_{mT}} D(u)\Big(\phi(u_{jx}) - \phi(v_x)\Big)\Big(\zeta_m^2(u_j - v)\Big)_x \geq -C\iint_{Q_{mT}} D(u)(u_j - v)^2 .$$

Since u_n satisfies the differential equation in Problem P_n, we also have

(5.5) $$\iint_{Q_{mT}} D(u)\phi(u_{jx})(\zeta_m^2\, u_j)_x$$

$$= \iint_{Q_{mT}} (D(u) - D(u_j))\phi(u_{jx})(\zeta_m^2\, u_j)_x - \iint_{Q_{mT}} u_{jt}\, \zeta_m^2\, u_j \quad .$$

Letting $j \to \infty$ while keeping m fixed in (5.4) and (5.5) yields

$$-\iint_{Q_{mT}} u_t u\, \zeta_m^2 - \iint_{Q_{mT}} D(u)\,\chi\,(\zeta_m^2\, v)_x$$

(5.6) $$-\iint_{Q_{mT}} D(u)\phi(v_x)((\zeta_m^2\, u)_x - (\zeta_m^2\, v)_x) \geq -C\iint_{Q_{mT}} D(u)(u - v)^2 .$$

Replacing ψ by $\zeta_m^2\, u$ in (5.2) yields

(5.7) $$\iint_{Q_{mT}} (u_t(\zeta_m^2\, u) + D(u)\,\chi\,(\zeta_m^2\, u)_x) = 0 \quad .$$

Next we add (5.6) and (5.7) to obtain

$$\iint_{Q_{mT}} D(u)(\chi - \phi(v_x))((\zeta_m^2\, u)_x - (\zeta_m^2\, v)_x) \geq -C\iint_{Q_{mT}} D(u)(u - v)^2 .$$

We set $v = u - \mu\xi$ with $\mu > 0$, $\xi \in L^2(0,T\,;\,H^1(\mathbb{R}))$ with $\xi(x) = 0$ for $|x| \geq m - 1$. Then

$$\iint_{Q_{CT}} D(u)(\chi - \phi(u_x - \mu\xi_x))\,\mu\xi_x \geq -\mu^2 C\iint_{Q_{DT}} D(u)\,\xi^2 \quad .$$

Dividing by μ and letting $\mu \downarrow 0$ yields

$$\iint_{Q_{CT}} D(u)(\chi - \phi(u_x))\,\xi_x \geq 0$$

which in turn implies (5.3). □

In section 4 we proved that $u(t)-H \in L^1(\mathbb{R})$. We show below

some extra invariance properties of Problem C, namely that if $u_0' \in L^2(\mathbb{R})$), $u_x(t) \in L^2(\mathbb{R})$ for all $t > 0$ and that $u_t \in L^2(Q_{CT})$. We suppose that the approximating functions u_{0n} are such that

$\limsup_{n\to\infty} \int_{-n}^{n} u_{0n}'^2 \le \int_{\mathbb{R}} u_0'^2$ (it follows from the appendix that the construction of such functions is possible)

Lemma 5.5. Let $u_0' \in L^2(\mathbb{R})$. Then $\int_{\mathbb{R}} u_x^2(t) \le \int_{\mathbb{R}} u_0'^2$ for all $t > 0$.

Proof. We multiply the differential equation

$$u_{nxt} = \left(D(u_n)\phi(u_{nx})\right)_{xx}$$

by u_{nx} and integrate on $Q_{nt} := (-n,n) \times (0,t)$ to obtain

$$\iint_{Q_{nt}} u_{nxt}\, u_{nx} = \iint_{Q_{nt}} \left(D(u_n)\phi(u_{nx})\right)_{xx} u_{nx},$$

that is

(5.8)
$$\int_{-n}^{n} u_{nx}^2(t) - \int_{-n}^{n} u_{0n}'^2 = -2 \iint_{Q_{nt}} \left(D(u_n)\phi(u_{nx})\right)_x u_{nxx}$$
$$= -2 \iint_{Q_{nt}} D'(u_n)\phi(u_{nx})u_{nx}\, u_{nxx} - 2 \iint_{Q_{nt}} D(u_n)\phi'(u_{nx})\, u_{nxx}^2 .$$

Next we define the monotone function $\Phi(s) := \int_0^s \phi(\tau)\tau\, d\tau$. It follows from (5.8) that

$$\int_{-n}^{n} u_{nx}^2(t) - \int_{-n}^{n} u_{0n}'^2 \le -2 \iint_{Q_{nt}} D'(u_n) \frac{\partial}{\partial x} \Phi(u_{nx})$$
$$\le 2 \iint_{Q_{nt}} D''(u_n)\Phi(u_{nx})u_{nx} \le 0.$$

Thus

$$\int_{-n}^{n} u_{nx}^2(t) \le \int_{-n}^{n} u_{0n}'^2 .$$

Since the L^2-norm is w.l.s.c., this implies that

$$\int_{-R}^{R} u_x^2(t) \le \int_{\mathbb{R}} u_0'^2 \quad \text{for all } R > 0$$

and finally that $\|u_x^2(t)\|_{L^2(\mathbb{R})} \le \|u_0'\|_{L^2(\mathbb{R})}$. □

Lemma 5.6. Let $u_0' \in L^2(\mathbb{R})$. Then $u_t \in L^2(Q_{CT})$.

Proof. The proof is close to that of Lemma 5.3. Here one multiplies the equation for u_n by u_{nt}. □

Remark 5.7. If one assumes that $u_0' \in L^2(\mathbb{R})$, it is not necessary to use the cut-off function ζ_m in the proof of Theorem 5.4.

Theorem 5.8 There exists a solution of Problem N.

Proof. The proof is very similar to that of Theorem 5.4. It uses the same auxiliary problem P_n, but now on the fixed domain Q_{NT}. □

We shall now study the existence of a solution of Problem CD. An essential problem here is to find a lower bound on $u_x(0,t)$, which is obtained by considering a suitable lower solution for the problem. Therefore, we first study the corresponding stationary problem

$$S_\lambda \begin{cases} \left(D(y)\phi(y')\right)' + \lambda y' = 0 & \text{in } \mathbb{R}^+ \\ y(0) = A \ ; \ y(\infty) = 0 & \text{if } \lambda > 0 \end{cases}$$

Definition 5.9. A function y_λ is said to be a weak solution of Problem S_λ if it satisfies

(i) $y_\lambda \in H^1(0,R)$ for all $R > 0$

(ii) $0 \le y_\lambda \le 1$, $-1 \le y_\lambda' \le 1$ a.e. in $(0,\infty)$

(iii) $y_\lambda(0) = A$; $y_\lambda(\infty) = 0$ if $\lambda > 0$

(iv) $\int_{\mathbb{R}^+} (D(y_\lambda)\phi(y_\lambda') + \lambda y_\lambda)\psi' = 0$ for all $\psi \in H_0^1(\mathbb{R}^+)$ such that ψ vanishes for large x.

Remark 5.10. If y_λ is a weak solution of Problem S_λ, then y_λ satisfies the differential equation a.e.

Lemma 5.11. Let y_λ be a weak solution of Problem S_λ for $\lambda > 0$. On the set where y_λ is positive, it satisfies:
(i) y_λ' is continuous, (ii) y_λ is strictly decreasing, (iii) y_λ is convex.

Proof. If y_λ is a weak solution of S_λ for $\lambda > 0$, then it satisfies

$$\begin{cases} D(y_\lambda)\phi(y_\lambda') + \lambda y_\lambda = 0 & \text{in } \mathbb{R}^+ \\ y_\lambda(0) = A \quad y_\lambda(\infty) = 0 & . \end{cases}$$

Thus

(5.9) $$\phi(y_\lambda') = -\lambda \frac{y_\lambda}{D(y_\lambda)}$$

at points where $0 < y_\lambda < 1$, which implies that y_λ' is continuous and strictly negative in those points. Next, we show that y_λ is convex in a neighborhood of each point where it is positive. We define $d(x) = -\frac{x}{D(x)}$ on $(0,1)$. Since $d'(x) = -\frac{D(x) - xD'(x)}{D^2(x)}$, it follows from the concavity of D that d is nonincreasing on $(0,1)$. Let $0 < x_1 < x_2$ be such that $y_\lambda(x_1), y_\lambda(x_2) \in (0,A)$. Then

$$y_\lambda(x_2) < y_\lambda(x_1)$$

and thus

$$\frac{-y_\lambda(x_2)}{D(y_\lambda(x_2))} \ge \frac{-y_\lambda(x_1)}{D(y_\lambda(x_1))}$$

which yields

$$\phi(y_\lambda'(x_2)) \ge \phi(y_\lambda'(x_1))$$

and finally

$$y_\lambda'(x_2) \ge y_\lambda'(x_1). \qquad \square$$

Lemma 5.12. Suppose $\lambda > 0$. Problem S_λ has a unique weak solution if and only if $\lambda \le \lambda_{max} := \frac{D(A)}{A}(-\phi(-1))$.

Proof. It follows from (5.9) that S_λ has no solution if the condition $\lambda \le \frac{D(A)}{A}(-\phi(-1))$ is not satisfied. Next we supppose that this condition holds and construct a solution y_λ which will turn out to be the unique solution of Problem S_λ.
We deduce from Lemma 5.11 that if y_λ is a solution, there exists $L_\lambda \in (0,\infty]$ such that y_λ is positive and strictly decreasing on $(0,L_\lambda)$ and that $y_\lambda(L_\lambda) = 0$. In order to calculate L_λ and y_λ on the interval $(0,L_\lambda)$ we take as new unknown on that interval the inverse function $x := \sigma(y_\lambda)$. It comes to solve

$$\begin{cases} \lambda = -\frac{D(y_\lambda)}{y_\lambda}\phi\left(\frac{1}{\sigma'(y_\lambda)}\right) & \text{on } (0,A) \\ \sigma(A) = 0 & . \end{cases}$$

Thus

$$\sigma(y_\lambda) = -\int_{y_\lambda}^{A} \frac{ds}{\phi^{-1}(-\lambda s/D(s))} \tag{5.10}$$

and

$$L_\lambda = \sigma(0) = -\int_{0}^{A} \frac{ds}{\phi^{-1}(-\lambda s/D(s))} \tag{5.11}$$

so that L_λ, which by (5.11) has a finite value, and the function y_λ on the interval $(0,L_\lambda)$ are uniquely determined. Note that $y_\lambda(x) = 0$ for all $x > L_\lambda$ since otherwise there would be a point $\tilde{x}$ such that $0 < y_\lambda(\tilde{x}) < 1$ and $y'_\lambda(\tilde{x}) > 0$ which contradicts (5.9). □

Corollary 5.13. Let $\tilde{\delta} \in [0,L_\lambda]$ be such that $y' > -1$ on $(\tilde{\delta},L_\lambda)$. Then $y_\lambda \in C^2((\tilde{\delta},L_\lambda))$.

Proof. Corollary 5.13 follows from the fact that the elliptic equation in Problem S_λ is non degenerate in $(\tilde{\delta},L_\lambda)$. □

We remark that if $\lambda < \lambda_{max}$, then $y'_\lambda(0) > -1$ and $y_\lambda \in C^2([0,L_\lambda))$.

Lemma 5.14. If $\lambda = 0$, the unique solution of Problem S_λ is $y_0 = A$.

Proof. Integrating the differential equation in S_λ yields

$$\begin{cases} D(y_0)\phi(y'_0) = C & \text{in } \mathbb{R}^+ \\ y_0(0) = A & . \end{cases}$$

If $C = 0$, then $y_0 = A$. We claim that C must be equal to zero. Suppose not and let $C > 0$. Then $y_0 \in C^1([0,+\infty))$, $0 < y_0 < 1$ and $y'_0 > 0$. Since y_0 is increasing and bounded from above, it tends to a constant as $x \to \infty$. Hence there exists a subsequence $\{x_n\}$ such that $y'_0(x_n) \to 0$ as $x_n \to \infty$. This is in contradiction with the differential equation. Similarly, one can show that C cannot be negative. □

Lemma 5.15. (i) y_λ is decreasing in λ.

(ii) Let $\lambda_n := \lambda_{max} - 1/n$; as $n \to \infty$, y_{λ_n} converges to $y_{\lambda_{max}}$ uniformly on compact subsets of $\overline{\mathbb{R}^+}$.

Proof. Property (i) follows from (5.10). As $n \to \infty$, $y_{\lambda n}$ decreases to a limit $\tilde{y}$, which satisfies Properties (i), (ii) and (iii) of Definition 5.9. In order to show that $\tilde{y} = y_{\lambda max}$, one also has to show that $\tilde{y}$ satisfies the integral relation (iv) in Definition 5.9. This is done in a similar way as in the proof of Theorem 5.4. □

We are now in a position to prove that there exists a solution of Problem CD, however with some extra assumptions on λ, A and u_0.

$$H_{CD}\begin{cases} 0 \le \lambda \le \lambda_{max} = \frac{D(A)}{A} \quad (-\phi(-1))\,;\ u_0 \le A \text{ on } \mathbb{R}^+\,; \\ \text{if } \lambda = 0,\ A \le \bar{s} := \sup\ \{s \in (0,1) \text{ such that } D'(s) = 0\}\,; \\ \text{if } \lambda > 0,\ u_0 \ge y_{\lambda_{max}} \quad \text{on } \mathbb{R}^+. \end{cases}$$

Next we consider the regularized problems, with $n \in \mathbb{N}$ large enough

$$CD_n\begin{cases} u_t = \left(D_n(u)\phi_n\,(u_x)\right)_x + \tilde{\lambda}_n u_x \text{ in } Q_T^n := (0,n) \times (0,T) \\ u(0,t) = A \quad u_x(n,t) = 0 \qquad \text{for } t \in (0,T] \\ u(x,0) = u_{0n}(x) \qquad \text{for } x \in (0,n) \end{cases}$$

where $\tilde{\lambda}_n := \min\,(\lambda,\lambda_n)$ and where u_0 satisfies

H_0^n : $u_{0n} \in C^\infty(\mathbb{R}^+)$, $u_{0n}(x) = A$ for x in a neighborhood of zero, $u_{0n} \le A$, $u'_{0n}(x) = 0$ for $x \ge n$, u_{0n} converges uniformly to u_0 on compact subsets of $\mathbb{R}^+$ as $n \to \infty$. If $\lambda > 0$, then $u_{0n} \ge \max\{A(1+y'_{\lambda_n}(0)),\ y_{\lambda_n}\}$ and $|u'_{0n}| \le -y'_{\lambda_n}(0)$; if $\lambda = 0$, then $u_{0n} \ge 1/n$ and $|u'_{0n}| \le 1-1/n$.

We show in the appendix that, given an initial function u_0 which satisfies H_{0D} and H_{CD}, one can construct a sequence $\{u_{0n}\}$ which satisfies the above properties.

As in the case of Problem P_n, one can show that a comparison principle holds and that Problem CD_n has a unique solution u_n which is such that $A(1+y'_{\lambda_n}(0)) \le u_n \le A$ and $u_{nx} \le -y'_{\lambda_n}(0)$ if $\lambda > 0$ (resp. $1/n \le u_n \le A$ and $u_{nx} \le 1-1/n$ if $\lambda = 0$). In order to show that $u_{nx} \ge y'_{\lambda_n}(0)$ if $\lambda > 0$ (resp. $u_{nx} \ge -1 + \frac{1}{n}$ if $\lambda = 0$) a lower bound for $u_{nx}(0,.)$ is necessary.

Lemma 5.16. Suppose that $\lambda > 0$. Then $u_n \ge y_{\lambda_n}$ in $\overline{Q_T^n}$, which implies that $u_{nx}(0,t) \ge y'_{\lambda_n}(0)$ for all $t \in [0,T]$.

Proof. Assume that n is large enough so that $L_{\lambda_n} < n$. Since $u_n \ge 0$ in Q_T^n, we have that $u_n \ge y_{\lambda_n}$ in $[L_{\lambda_n}, n] \times [0,T]$. Also, since y_{λ_n} satisfies

$$\begin{aligned} &(D(y_{\lambda_n})\phi(y'_{\lambda_n}))' + \tilde{\lambda}_n y'_{\lambda_n} = (\tilde{\lambda}_n - \lambda_n)y'_{\lambda_n} \ge 0 && \text{for } x \in (0,L_{\lambda_n}) \\ &y_{\lambda_n}(0) = u_n(0,t) = A \qquad y_{\lambda_n}(L_{\lambda_n}) = 0 \le u_n(L_{\lambda_n},t) && \text{for } t \in (0,T] \\ &y_{\lambda_n} \le u_{0n} && \text{for } x \in (0,L_{\lambda_n}) \end{aligned}$$

it follows from a comparison principle argument that $u_n \ge y_{\lambda_n}$ on $[0,L_{\lambda_n}] \times [0,T]$. □

Lemma 5.17. If $\lambda = 0$, $u_n \ge \underline{y} := \max\,(A-(1-1/n)x,\ 0)$ in $\overline{Q_T^n}$.

Proof. Again suppose that n is large enough so that $n > A/(1-1/n)$. Obviously $u_n \ge \underline{y}$ for $x \in [A/(1-1/n),n]$. It remains to show that

$$(D(\underline{y})\phi(\underline{y}'))' \ge 0 \text{ for } x < A/(1-1/n)$$

that is

$$D'(\underline{y})\phi(1/n-1)(1/n-1) \ge 0$$

which follows from the assumption that $A \le \bar{s}$. □

Theorem 5.18. There exists a solution of Problem CD.

Proof. From Lemma 5.16 (resp. Lemma 5.17 if $\lambda = 0$), it follows that $u_{nx} \ge y'_{\lambda_n}(0)$ (resp $u_{nx} \ge 1/n-1$ if $\lambda = 0$). In particular, $|u_{nx}| < 1$ on Q_T^n, thus there exists a sequence $\{n_k\}$ and a function $u \in C(\bar{Q}_{DT})$ such that u_{n_k} tends to u as $n_k \to \infty$ uniformly on compact subsets of $\overline{Q_{DT}}$. It follows at once that u satisfies condition (i)-(iii) of the definition of a solution of Problem CD (Definition 3.3). The proof that u also satisfies the integral condition (iv) of Definition 3.3 is quite similar to that of Theorem 5.4. However it is convenient to consider the function $\bar{u} := u - y_\lambda$ as the new unknown function, in order to have a homogeneous boundary condition in the point $x = 0$. □

Note that the following results hold .

Theorem 5.19. Let u be the solution of any of the three problems. Then $u \in C(\bar{Q})$ where Q denotes either $(-1,1) \times \mathbb{R}^+$ or $\mathbb{R} \times \mathbb{R}^+$ or $\mathbb{R}^+ \times \mathbb{R}^+$.

Theorem 5.20. (Comparison principle).

(i) Let u_1 and u_2 be the solutions of any of the three problems with initial functions $u_{01} \le u_{02}$. Then $u_1(t) \le u_2(t)$ for all $t > 0$.

(ii) Let $\lambda \in [\lambda_2, \lambda_1] \subset [0, \lambda_{max}]$ and let u be the solution of Problem CD. Then if $y_{\lambda_1} \le u_0 \le y_{\lambda_2}$, $y_{\lambda_1} \le u(t) \le y_{\lambda_2}$ for all $t > 0$.

Finally we remark that the hypothesis $D'' \le 0$ is necessary to obtain the uniform bounds on u_{nx} in the proof of Lemma 5.2.

6. The large time behavior

6.1. The case of the Neumann problem. Convergence to a constant.

In this subsection, we show that the solution of Problem N converges to a constant as $t \to \infty$; we adapt a proof of Alikakos & Rostamian [1] and Cafermos [9] based on the use of a suitable Lyapunov functional . We first give a result in the case that u is bounded away from 0 and 1. We denote by $u(t,u_0)$ the solution of Problem N with initial function u_0.

Theorem 6.1. Let $\delta \le u_0 \le 1 - \delta$ for some $\delta \in (0,1/2)$. Then there exist constants $K > 0$ and $\sigma = \sigma(\delta) > 0$ such that

$$\left\| u(t,u_0) - \frac{1}{2}\int_{-1}^{1} u_0 \right\|_{L^\infty(-1,1)} \le Ke^{-\sigma t} \qquad t \ge 0.$$

Proof. We first consider the solution u_n of the problem N_n

$$N_n \begin{cases} u_t = \left(D(u)\phi(u_x)\right)_x & \text{in } Q_{NT} \\ u_x(-1,t) = 0 \quad u_x(1,t) = 0 & \text{for } t \in (0,T] \\ u(x,0) = u_{0n}(x) & \text{for } x \in (-1,1) \end{cases} ,$$

where u_{0n} satisfies

$u_{0n} \in C^\infty([-1,1])$, $\delta \le u_{0n} \le 1-\delta$, $|u'_{0n}| \le 1 - 1/n$,
$u'_{0n}(-1) = u'_{0n}(1) = 0$, u_{0n} converges to u_0 as $n \to \infty$
uniformly in $[-1,1]$.

Using the methods developped in section 5, one can show that the solution u_n of Problem N_n converges to the solution u of Problem N uniformly in $\overline{Q_{NT}}$.

Let $v_n = u_n - \frac{1}{2}\int_{-1}^{1} u_{0n}$. Then v_n satisfies the problem

$$v_t = \left(D(u_n)\phi(v_x)\right)_x \qquad \text{in } Q_{NT}$$

$$v_x(-1,t) = 0 \qquad v_x(1,t) = 0 \qquad \text{for } t \in (0,T]$$

$$v(x,0) = v_{0n}(x) := u_{0n}(x) - \frac{1}{2}\int_{-1}^{1} u_{0n} \qquad \text{for } x \in (-1,1) \;.$$

and is such that $\int_{-1}^{1} v_n(t) = 0$ for all $t \in [0,T]$. We multiply by v_n the equation for v_n and integrate by parts. This yields

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} v_n^2 = -\int_{-1}^{1} D(u_n)\phi(v_{nx})v_{nx} \;.$$

Now since $\phi' \geq 0$ with $\phi'(0) > 0$, there exists $\mu > 0$ such that $|\phi(s)| \geq \mu|s|$ for $s \in [-1,1]$. Thus

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} v_n^2 \leq -\mu \inf_{[\delta,1-\delta]} D \int_{-1}^{1} v_{nx}^2 \leq -\frac{\mu}{4} \inf_{[\delta,1-\delta]} D \int_{-1}^{1} v_n^2$$

which implies that

$$\int_{-1}^{1} v_n^2 \leq \left(\int_{-1}^{1} v_{0n}^2\right) e^{-\frac{\mu}{2}\inf_{[\delta,1-\delta]} Dt} \;.$$

Letting $n \to \infty$, we obtain

(6.1) $$\|v(t)\|_{L^2(-1,1)} \leq K_1 e^{-\sigma_1 t}$$

where $\sigma_1 = \frac{\mu}{4} \inf_{[\delta,1-\delta]} D$. Next, observe that since $v(t)$ is Lipschitz continuous with respect to the space variable, it satisfies the inequality

$$\frac{1}{2}\|v(t)\|^2_{L^\infty(-1,1)} \leq \int_{-1}^{1}|v(t)| \leq \sqrt{2}\,\|v(t)\|_{L^2(-1,1)}$$

which combined with (6.1) yields

$$\|v(t)\|_{L^\infty(-1,1)} \leq 2^{3/4}\sqrt{K_1}\, e^{-\sigma_1 t/2} \;. \qquad \square$$

Theorem 6.2. Let $\delta \leq u_0 \leq 1$ for some $\delta > 0$. When $t \to \infty$, $u(t,u_0)$ converges to the constant $\frac{1}{2}\int_{-1}^{1} u_0$ uniformly on $[-1,1]$.

Proof. Since $\{u(t),\ t \geq 0\}$ is precompact in $C([-1,1])$, there exists a sequence $\{t_n\}$ and a function $q \in C([-1,1])$ such that

$$u(t_n) \to q \text{ as } t_n \to \infty \text{ uniformly on } [-1,1] \;.$$

In particular $u(t_n)$ converges to q in $L^1(-1,1)$. Defining the ω-limit set of u_0 by

$$\omega(u_0) = \{w \in L^1(-1,1) : \text{there exists a sequence } t_n \to \infty \text{ such that } u(t_n) \to w \text{ in } L^1(-1,1) \text{ as } t_n \to \infty\}$$

we conclude that $\omega(u_0)$ is not empty. Define $V : L^1(-1,1) \to \overline{\mathbb{R}}$ by

$$V(v) = \begin{cases} -\operatorname*{ess\,inf}_{x \in (-1,1)} v(x) & \text{if } -\operatorname*{ess\,inf}_{x \in (-1,1)} v(x) < +\infty \\ +\infty & \text{otherwise.} \end{cases}$$

Since $u(t_0) \geq \operatorname*{ess\,inf}_{x \in (-1,1)} u(t_0)$, we have that, for $t \geq t_0$, $u(t) \geq \operatorname*{ess\,inf}_{x\,(-1,1)} u(t_0)$. This follows from the comparison principle and the fact that constants are solutions of Problem N. Thus

$$V(u(t)) \leq V(u(t_0)) \quad \text{for all } t, t_0 \text{ such that } t \geq t_0$$

which shows that V is a Lyapunov functional for Problem N. Since this functional is lower semi-continuous in $L^1(-1,1)$ and since the orbits are Lyapunov stable (because u satisfies a contraction property in $L^1(-1,1)$) , it follows from DaFermos [9, Proposition 4.1] that V is constant on $\omega(u_0)$, say $V = -W$. Next, we show that for any $w_0 \in \omega(u_0)$

(6.2) $$w_0(x) = W \qquad \text{for all } x \in (-1,1).$$

Since

$$\int_{-1}^{1} w_0 = \int_{-1}^{1} u_0 < 2$$

and since $w_0 \le 1$, it follows that $W < 1$. Now suppose that (6.2) is not true. Then for sufficiently small $\mu \in (0,1-W)$ the set

$$\Omega_\mu = \{x \in (-1,1) : w_0(x) \ge W + \mu\}$$

has a positive measure. Define

$$\underline{w}_0(x) = \begin{cases} w_0(x) & \text{if } w_0(x) < W + \mu \\ W + \mu & \text{if } w_0(x) \ge W + \mu \end{cases}$$

and let $\underline{w}$ and w be the solutions of Problem N with intial values $\underline{w}_0$ and w_0 respectively. Since $\delta \le \underline{w}_0 \le W + \mu < 1$, we have that $\delta \le \underline{w} \le W + \mu < 1$. Thus, by Theorem 6.1, $\underline{w}(t)$ converges to $\frac{1}{2}\int_{-1}^{1} \underline{w}_0$ as $t \to \infty$, uniformly on $[-1,1]$.

Hence for given $\eta > 0$, there exists $T(\eta,\mu)$ such that

$$\underline{w}(t) \ge \frac{1}{2}\int_{-1}^{1} \underline{w}_0 - \eta \text{ for } t \ge T(\eta,\mu) .$$

Since $\underline{w}_0 \le w_0$, the comparison principle implies that $\underline{w} \le w$. Therefore for $t \ge T(\eta,\mu)$

$$w(t) \ge \frac{1}{2}\int_{-1}^{1} \underline{w}_0 - \eta = \frac{1}{2}\int_{(-1,1)\setminus\Omega_\mu} \underline{w}_0 + \frac{1}{2}\int_{\Omega_\mu} \underline{w}_0 - \eta.$$

Thus

$$w(t) \ge \frac{1}{2}(2 - |\Omega_\mu|)W + \frac{1}{2}|\Omega_\mu|(W + \mu) - \eta = W - \eta + \frac{|\Omega_\mu|}{2}\mu.$$

For fixed μ , we choose η sufficiently small so that

$$w(t) \ge W + \nu \quad \text{for some } \nu > 0.$$

Then

$$V(w(t)) < -W, \text{ for t sufficiently large,}$$

which is a contradiction. □

Theorem 6.3. When $t \to \infty$, $u(t,u_0)$ converges to the constant $\frac{1}{2}\int_{-1}^{1} u_0$ uniformly on $[-1,1]$.

Proof. We now take as the Lyapunov functional

$$\bar{V}(v) = \begin{cases} \operatorname*{ess\,sup}_{x \in (-1,1)} v(x) & \text{if } \operatorname*{ess\,sup}_{x \in (-1,1)} v(x) < +\infty \\ +\infty & \text{otherwise} . \end{cases}$$

Then $\bar{V}$ is constant on $\omega(u_0)$, say $\bar{V} = \bar{W}$. The reasoning then follows as in the proof of Theorem 6.2. The auxiliary function is now defined as

$$\bar{w}_0(x) = \begin{cases} w_0(x) & \text{if } w_0(x) > \bar{W} - \bar{\mu} \\ \bar{W} - \bar{\mu} & \text{if } w_0(x) \le \bar{W} - \bar{\mu} \end{cases}$$

with $\bar{\mu} \in (0,\bar{W})$. Then $\bar{w}_0 > 0$ and by Theorem 6.2 the solution $\bar{w}$ of Problem N with initial function $\bar{w}_0$ converges to $\frac{1}{2}\int_{-1}^{1} \bar{w}_0$ as $t \to \infty$, which in turn implies the contradiction

$$\bar{V}(w(t)) < \bar{W}$$

for t sufficiently large. □

Corollary 6.4. There exists $t_0 > 0$, $K > 0$ and $\sigma = \sigma(t_0) > 0$ such that

$$\left\| u(t,u_0) - \frac{1}{2}\int_{-1}^{1} u_0 \right\|_{L^\infty(-1,1)} \le Ke^{-\sigma t} \text{ for all } t \ge t_0.$$

Proof. Corollary 6.4 follows from the uniform convergence of $u(t)$ to $\frac{1}{2}\int_{-1}^{1} u_0 \in (0,1)$ as $t \to \infty$. □

6.2. The Cauchy problem in the case that $D(u) = u(1 - u)$. Convergence to similarity solutions.

In this section, we first construct a class of similarity solutions and then give a convergence theorem.

Following de Josselin de Jong [14] and van Duyn [10], we look for a similarity solution of Problem C of the form

(6.3) $$u_s(x,t) = f(\eta) = \begin{cases} 0 & \text{if } \eta < -\frac{1}{2} \\ \frac{1}{2} + \eta & \text{if } -\frac{1}{2} \le \eta \le \frac{1}{2} \\ 1 & \text{if } \eta > \frac{1}{2} \end{cases}$$

with $\eta = x/g(t)$ where the function g is still unknown and has to be determined. Substituting (6.3) in equation (1.1) with $D(s) = s(1-s)$, we formally deduce that g must satisfy the differential equation

(6.4) $$g'(t) = 2\phi\left(\frac{1}{g(t)}\right)$$

which we solve below together with the initial condition

$$g(0) = g_0 \ge 1 .$$

Note that $1/g_0$ corresponds to the slope of the initial value $u_s(x,0)$ for $x \in (-g_0/2, g_0/2)$.
We set

$$\Phi(\tau) = \int_1^\tau \frac{ds}{\phi(\frac{1}{s})} = \int_{1/\tau}^{1} \frac{du}{\phi(u)u^2} \quad \text{for } \tau \ge 1.$$

Remark that since $\phi'(0) > 0$, $\Phi(+\infty) = +\infty$. Thus the function Φ, which is strictly increasing, maps $[1,\infty)$ on to $[0,\infty)$. Integrating the differential equation (6.4) yields

$$\Phi(g(t)) = 2t + \Phi(g_0)$$

and thus

$$g(t) = \Phi^{-1}(2t + \Phi(g_0)).$$

The function u_s is such that

$$u_s(x,t) = 0 \qquad \text{for } x \le S_f(t)$$
$$0 < u_s(x,t) < 1 \qquad \text{for } S_f(t) < x < S_s(t)$$
$$u_s(x,t) = 1 \qquad \text{for } x \ge S_s(t)$$

where

$$S_f(t) = -g(t)/2 \quad \text{and } S_s(t) = g(t)/2$$

and the velocity of the two fronts is given by

$$S_f'(t) = -\phi\left(\frac{1}{g(t)}\right) \text{ and } S_s'(t) = \phi\left(\frac{1}{g(t)}\right).$$

It remains to show that u_s is a weak solution of Problem C. It is immediate that u_s satisfies properties (i) and (ii) of Definition 3.2. Since $\left(u_s(1-u_s)\phi(u_{sx})\right)_x \in L^2(Q_{CT})$ and since u_s satisfies equation (1.1) a.e., it easily follows that u_s satisfies the integral equation (iv) of Definition 3.2.

Next, we give a convergence theorem which extends a result of van Duyn [11] in the case that $\phi(s) = s$.

Theorem 6.5. Suppose that $D(u) = u(1-u)$. Let u_0 be such that $u_0(x) = 0$ for $x < x_1$ and $u_0(x) = 1$ for $x > x_2$ with $-\infty < x_1 < x_2 < +\infty$. Then there exists $C > 0$ such that

$$\| u(t,u_0) - f(./g(t)) \|_{L^\infty(\mathbb{R})} \le C/g(t) \qquad \text{for all } t \ge 0.$$

Proof. We choose $g_0 = 1$; then $g(t) = \Phi^{-1}(2t)$. In view of the hypothesis on u_0, there exists $d > 0$ such that

$$f(x-d) \le u_0(x) \le f(x+d) \qquad \text{for } x \in \mathbb{R}.$$

Then by the comparison theorem

$$f((x-d)/g(t)) \le u(x,t) \le f((x+d)/g(t)) \qquad \text{for all } (x,t) \in \bar{Q}_{CT},$$

which implies that

$$|u(x,t) - f(x/g(t))| \le |f((x+d)/g(t)) - f((x-d)/g(t))| \le 2d/g(t)$$

for all $x \in \mathbb{R}$ and $0 \le t \le T < \infty$. □

6.3 The Cauchy-Dirichlet problem : convergence to the stationary solution.

In what follows, we show that the solution of Problem CD stabilizes as $t \to \infty$. The idea of considering sets of the form $\mathbb{R}^+ \times (t, t+\tau)$ was suggested to us by M.Bertsch.

Theorem 6.6. (i) If $\lambda > 0$ and if u_0 satisfies the hypothesis H_{CD} and is such that $u_0 \le y_{\bar\lambda}$ for some $\bar\lambda \in (0,\lambda]$, then the solution $u(t,u_0)$ of Problem CD converges to the stationary solution y_λ as $t \to \infty$, uniformly on $\overline{\mathbb{R}^+}$.

(ii) If $\lambda = 0$ and u_0 satisfies H_{CD}, $u(t,u_0)$ converges to A as $t \to \infty$, uniformly on compact subsets of $\overline{\mathbb{R}^+}$.

Proof. (i) It follows from the comparison theorem 5.20 that

$$y_{\lambda_{max}} \le u(t,y_{\lambda_{max}}) \le u(t,u_0) \le u(t,y_{\bar\lambda}) \le y_{\bar\lambda} \quad \text{for all } t \ge 0.$$

The proof will be completed if we show that $u(t,y_{\lambda_{max}})$ and $u(t,y_{\bar\lambda})$ converge to the stationary solution y_λ as $t \to \infty$. Since both proofs are similar, we only show the convergence result for $u(t,y_{\lambda_{max}})$.

Using again the comparison principle we deduce that

$$u(\tau, y_{\lambda_{max}}) \le u(t+\tau, y_{\lambda_{max}}) \quad \text{for all } t,\ \tau \ge 0$$

and thus that $u(\cdot, y_{\lambda_{max}})$ is nondecreasing. Since furthermore $u(\cdot, y_{\lambda_{max}}) \le A$, there exists a function $q \in C^{0,1}(\overline{\mathbb{R}^+})$ such that

$$u(t, y_{\lambda_{max}}) \to q \text{ as } t \to \infty$$

uniformly on compact subsets of $\overline{\mathbb{R}^+}$. It remains to show that $q = y_\lambda$. Obviously $q(x) = 0$ for large x and q satisfies properties (i),(ii),(iii) of Definition (5.9). Next, we show that q also satisfies the integral relation (iv). In order to have a homogeneous boundary condition in 0, it is convenient to make the change of functions

$$\bar{u} = u - y_\lambda \text{ and } \bar{q} = q - y_\lambda .$$

Then $\bar{u}$ satisfies the differential equation

(6.5) $$\bar{u}_t = \left(D(\bar{u} + y_\lambda)\,\phi\,(\bar{u}_x + y_\lambda')\right)_x + \lambda(\bar{u}_x + y_\lambda')$$

for $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$. Set

$$w^{(t)}(x,s) = \bar{u}(x, s+t).$$

Then $w^{(t)}$ satisfies the differential equation (6.5) as well as the boundary condition $w^{(t)}(0,\cdot) = 0$. Let $\tau > 0$ be given. Then

(6.6) $$\int_{\mathbb{R}^+} w^{(t)}(\tau)\psi(\tau) - \int_{\mathbb{R}^+} w^{(t)}(0)\psi(0) =$$

$$= \int_0^\tau \int_{\mathbb{R}^+} \left\{ w^{(t)}\psi_t - (D(w^{(t)} + y_\lambda)\phi(w_x^{(t)} + y_\lambda') + \lambda(w^{(t)} + y_\lambda))\ \psi_x \right\}$$

for all $\psi \in L^2(0,\tau; H_0^1(\mathbb{R}^+))$ such that $\psi_t \in L^2(\mathbb{R}^+ \times (0,\tau))$.
Note that

(i) $w^{(t)} \to \bar{q}$ as $t \to \infty$, uniformly in $\overline{\mathbb{R}^+}$

(ii) there exists a function $\bar{\chi} \in L^\infty((0,\infty) \times (0,\tau))$ and a sequence $\{t_n\}$ such that

$$\phi(w_x^{(t_n)} + y_\lambda') \rightharpoonup \bar{\chi} \text{ weakly in } L^2(0,\tau;\ L^2(0,\infty))$$

Letting $t \to \infty$ in (6.6) and setting $\psi = \psi(x)$ yields

(6.7) $$\int_0^\tau \int_{\mathbb{R}^+} (D(\bar{q} + y_\lambda)\ \bar{\chi} + \lambda(\bar{q} + y_\lambda))\psi' = 0 \quad \text{for all } \psi \in H_0^1(\mathbb{R}^+).$$

We show below that

(6.8) $$\int_0^\tau \int_{\mathbb{R}^+} D(\bar{q} + y_\lambda)(\bar{\chi} - \phi(\bar{q}' + y_\lambda'))\psi' = 0 \text{ for all } \psi \in H_0^1(\mathbb{R}^+).$$

Let $v \in L^2(0,\tau; H_0^1(\mathbb{R}^+))$. Since ϕ is monotone, we have that

(6.9) $$\int_0^\tau \int_{\mathbb{R}^+} D(\bar{q} + y_\lambda)(\phi(w_x^{(t_n)} + y_\lambda') - \phi(v_x + y_\lambda'))(w_x^{(t_n)} - v_x) \ge 0$$

and since $w^{(t_n)}$ satisfies (6.6), we also have, putting $\psi = w^{(t_n)}$

(6.10) $$\int_0^\tau \int_{\mathbb{R}^+} (D(\bar{q} + y_\lambda)\ \phi(w_x^{(t_n)} + y_\lambda') + \lambda(w^{(t_n)} + y_\lambda))w_x^{(t_n)} =$$

$$= \int_0^\tau \int_{\mathbb{R}^+} \{D(\bar q + y_\lambda) - D(w^{(t_n)} + y_\lambda)\}\phi(w_x^{(t_n)} + y_\lambda')w_x^{(t_n)}$$

$$-\frac{1}{2}\int_{\mathbb{R}^+} (w^{(t_n)}(\tau))^2 + \frac{1}{2}\int_{\mathbb{R}^+} (w^{(t_n)}(0))^2 .$$

Letting t_n tend to infinity in (6.9) and (6.10) yields

(6.11) $$-\int_0^\tau \int_{\mathbb{R}^+} \{D(\bar q + y_\lambda)\ \bar\chi\ v_x + \lambda(\bar q + y_\lambda)\bar q'\}$$

$$-\int_0^\tau \int_{\mathbb{R}^+} D(\bar q + y_\lambda)\phi(v_x + y_\lambda')(\bar q' - v_x) \geq 0.$$

Replacing ψ by $\bar q$ in (6.7) gives

(6.12) $$\int_0^\tau \int_{\mathbb{R}^+} \{D(\bar q + y_\lambda)\ \bar\chi + \lambda(\bar q + y_\lambda)\}\bar q' = 0 .$$

Next we add (6.11) and (6.12) to obtain

$$\int_0^\tau \int_{\mathbb{R}^+} D(\bar q + y_\lambda)(\bar\chi - \phi(v_x + y_\lambda'))(\bar q' - v_x) \geq 0 .$$

We set $v = \bar q - \mu\xi$ with $\mu > 0$ and $\xi \in H_0^1(\mathbb{R}^+)$.
Then

$$\int_0^\tau \int_{\mathbb{R}^+} D(\bar q + y_\lambda)(\bar\chi - \phi(\bar q' + y_\lambda' - \mu\xi')\ \mu\xi' \geq 0$$

Dividing by μ and letting $\mu \downarrow 0$, we obtain

$$\int_0^\tau \int_{\mathbb{R}^+} D(\bar q + y_\lambda)(\bar\chi - \phi(\bar q' + y_\lambda')\xi' \geq 0 \text{ for all } \xi \in H_0^1(\mathbb{R}^+)$$

which in turn implies (6.8). Combining (6.7) and (6.8) we obtain

$$\int_0^\tau \int_{\mathbb{R}^+} (D(\bar q + y_\lambda)\phi(\bar q' + y_\lambda') + \lambda(\bar q + y_\lambda))\psi' = 0 \text{ for all } \psi \in H_0^1(\mathbb{R}^+)$$

from which we deduce that $q = \bar q + y_\lambda$ satisfies the integral relation (iv) of Definition 5.9. Thus $q = y_\lambda$ and $\bar q = 0$.

(ii) The proof of (ii) is quite similar to that of (i). However, since we do not suppose here that u has compact support, one has to use cut-off functions in several formulas. □

A P P E N D I X

We prove below two approximation lemmas

Lemma A1. Let u_0 satisfies H_{0C}. Then there exists $\{u_{0n}\}$ satisfying H_{0n}. If $u_0' \in L^2(\mathbb{R})$, $\{u_{0n}\}$ satisfies in addition

$$\limsup_{n\to\infty} \int_{-n}^{n} u_{0n}'^2 \le \int_{\mathbb{R}} u_0'^2 .$$

Proof. Set

$$\tilde{u}_{0n} = (1 - 2/n)u_0 + 1/n$$

and

$$\hat{u}_{0n}(x) = \begin{cases} \max(\frac{1}{n}, (1-\frac{1}{n})(x+n/4) + \tilde{u}_{0n}(-n/4)) & \text{if } x \le -n/4 , \\ \tilde{u}_{0n}(x) & \text{if } |x| \le r/4 , \\ \min(1-\frac{1}{n}, (1-\frac{1}{n})(x-n/4) + \tilde{u}_{0n}(n/4)) & \text{if } x \ge n/4 . \end{cases}$$

Using the function

$$\rho(x) = \begin{cases} 0 & \text{if } |x| \ge 1 , \\ C\exp(1/(|x|^2 - 1)) & \text{if } |x| < 1 , \end{cases}$$

where C is a constant such that $\int_{\mathbb{R}} \rho(x)dx = 1$, we define the sequence

$$u_{0n}(x) = n \int_{\mathbb{R}} \rho(n(x-y))\hat{u}_{0n}(y)dy .$$

Then one can check that if n is large enough, u_{0n} satisfies the hypothesis H_{0n}. Also if $u_0' \in L^2(\mathbb{R})$, then

$$\int_{-n}^{+n} u_{0n}'^2 \le \int_{-n}^{n} \hat{u}_{0n}'^2 \le \|u_0'\|^2_{L^2(\mathbb{R})} + C(n)$$

where $\lim_{n\to\infty} C(n) = 0$. □

Lemma A2. Let u_0 satisfy H_{0D} and H_{CD}. Then there exists $\{u_{0n}\}$ satisfying H_0^n.

Proof. (i) The case $\lambda = 0$. Set

$$\tilde{u}_{0n} = (1 - 1/(An))(u_0 - A) + A$$

and

$$\hat{u}_{0n}(x) = \begin{cases} A & \text{if } x \le 1/n \\ \max(-(1-1/n)(x-1/n) + A, \tilde{u}_{0n}(x)) & \text{if } 1/n < x < n/4 \\ \max(-(1-1/n)(x-n/4) + \tilde{u}_{0n}(n/4), 1/n) & \text{if } x \ge n/4 \end{cases}$$

where we assume that n is large enough so that $\hat{u}_{0n} = \tilde{u}_{0n}$ in an interval of positive measure. Let

$$u_{0n}(x) = 2n \int_{\mathbb{R}} \rho(2n(x-y))\hat{u}_{0n}(y)dy \qquad \text{for } x \ge 0$$

Then one can check that u_{0n} satisfies the hypothesis H_0^n for n sufficiently large.

(ii) The case $\lambda > 0$. We first construct an approximation of $y_{\lambda_{max}}$. We set

$$\tilde{y}_n = \max(y_{\lambda_n}, A(1 + y'_{\lambda_n}(0)))$$

and

$$\hat{y}_n(x) = \begin{cases} A & \text{if } x < 1/n \\ \tilde{y}_n(x - \frac{1}{n}) & \text{if } x \ge 1/n. \end{cases}$$

Let $\{\varepsilon_n\}$ be such that $\varepsilon_n > 0$ and $\varepsilon_n \downarrow 0$ as $n \to \infty$ and let

$$y_n(x) = \frac{1}{\varepsilon_n} \int_{\mathbb{R}} \rho\left(\frac{x-z}{\varepsilon_n}\right) \hat{y}_n(z)dz \qquad \text{for } x \geq 0$$

Then one can check that for n large enough and for ε_n small enough y_n satisfies the hypothesis H_0^n in the case that $u_0 = y_{\lambda max}$.

In the general case that $u_0 \geq y_{\lambda max}$ on $\overline{\mathbb{R}^+}$, we set

$$\tilde{u}_{0n}(x) = \begin{cases} \max(y_{\lambda_n}(x), -y'_{\lambda_n}(0)(u_0(x)-A)+A) & \text{if } x \leq n/4 \\ \max(A(1+y'_{\lambda_n}(0)), y'_{\lambda_n}(0)(x-n/4)+\tilde{u}_{0n}(n/4)) & \text{if } x > n/4 \end{cases}$$

and

$$\hat{u}_{0n}(x) = \begin{cases} A & \text{if } x < 1/n \\ \tilde{u}_{0n}(x-1/n) & \text{if } x \geq 1/n \end{cases}$$

Again one can check that the function u_{0n}, defined by

$$u_{0n}(x) = \frac{1}{\varepsilon_n} \int_{\mathbb{R}} \rho\left(\frac{x-y}{\varepsilon_n}\right) \hat{u}_{0n}(y)dy \qquad \text{for } x \geq 0$$

which is such that $u_{0n} \geq y_{\lambda_n}$, satisfies the hypothesis H_0^n for n large enough and ε_n small enough.

References.

[1] ALIKAKOS, N.D. & R.ROSTAMIAN, Large time behavior of solutions of Neumann boundary value problem for the porous medium equation, Indiana Univ.Math.J.30 (1981) 749-785.

[2] ATKINSON,C. & J.E.BOUILLET, Some qualitative properties of solutions of a generalized diffusion equation, Math. Proc. Cambridge Philos. Soc.86 (1979) 495-510.

[3] ATKINSON,C. & J.E.BOUILLET, A generalized diffusion equation: radial symmetries and comparison theorems, J.Math.Anal.Appl. 95 (1983) 37-68.

[4] BAMBERGER,A., Etude d'une equation doublement non linéaire, J.Funct.Anal. 24 (1977) 148-155.

[5] BEAR,J., Dynamics of Fluids in Porous Media, American Elsevier Publishing Company 1972.

[6] BENILAN,Ph. & M.G. CRANDALL, Regularizing effects of homogeneous evolution equations, MRC Report 2076, University of Wisconsin.

[7] CORTAZAR,C., The application of dissipative operators to nonlinear diffusion equations, J.Diff.Eq.47 (1983)1-23

[8] CRANDALL,M.G.& M.PIERRE, Regularizing effects for $u_t + A\phi(u) = 0$ in L^1, J.Funct.Anal.45 (1982) 194-212.

[9] DAFERMOS, C.M., Asymptotic behaviour of solutions of evolution equations, in : Nonlinear Evolution Equations, M.G.Crandall ed.,Academic Press 1978.

[10] VAN DUYN ,C.J., Some fundamental properties of the simultaneous flow of fresh and salt groundwater in horizontally extended aquifers, in : Proceedings of Euromech 143, A.Verruijt & F.B.J.Barends eds., Balkema, Rotterdam 1981.

[11] VAN DUYN, C.J., On the diffusion of immiscible fluids in porous media, SIAM J.Math.Anal. 10 (1979) 486-496.

[12] FRIEDMAN, A., Partial Differential Equations of Parabolic Type, Prentice Hall, Englewood Cliffs, N.J., 1964.

[13] GILDING, B.J., Hölder continuity of solutions of parabolic equations, J.London Math.Soc. 13 (1976) 103-106.

[14] DE JOSSELIN DE JONG, G., The simultaneous flow of fresh and salt water in aquifers of large horizontal extension determined by shear flow and vortex theory, in :Proceedings of Euromech 143, A.Verruijt & F.B.J.Barends eds., Balkema, Rotterdam 1981.

[15] KALASHNIKOV, A.S., On some nonlinear equation arising in the theory of nonstaticnary filtration, Trudy Seminara im I.G.Petrovskovo 4 (1978) 137-146.

[16] KALASHNIKOV, A.S., The Cauchy problem for some degenerate second order parabolic equations with general nonlinearities, Trudy Seminara im I.G.Petrovskovo 6 (1981) 83-96.

[17] KALASHNIKOV, A.S., Cauchy's problem in classes of increasing functions for certain quasilinear degenerate parabolic equations of second order, Differential'nye Uravneniya 9 (1973) 682-691.

[18] LADYŽENSKAJA, O.A., V.A. SOLONNIKOV & N.N. URAL'CEVA, Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs Volume 23, Providence R.I.: American Mathematical Society 1968.

[19] TEMAM, R., Navier- Stokes Equations, North Holland 1979.

CHAPTER 3

REGULARITY PROPERTIES OF A DOUBLY DEGENERATE EQUATION IN HYDROLOGY

C.J. van Duijn   Zhang Hongfei*

Department of Mathematics & Informatics
Delft University of Technology
Delft, The Netherlands

1. INTRODUCTION

In this paper we study the regularity of solutions of the initial value problem

$$
(C) \quad \begin{cases} u_t = (D(u)\phi(u_x))_x & \text{in } Q = \mathbb{R} \times \mathbb{R}^+, & (1.1) \\ u(\cdot,0) = u_0(\cdot) & \text{on } \mathbb{R}, & (1.2) \end{cases}
$$

where t and x denote, respectively, a time and a space coordinate

*) This author is on leave from the Department of Applied Mathematics, Tsinghua University, Beijing, People's Republic of China. His research was sponsored by funds of the Dutch Ministry of Education and Sciences.

and where the subscripts t and x denote partial differentiation with respect to these variables. Further

$D: [0,1] \to [0,\infty)$ satisfies

$$D(s) > 0 \text{ for } s \in (0,1) \quad \text{and} \quad D(0) = D(1) = 0 \tag{1.3}$$

and

$\phi: [-1,1] \to \mathbb{R}$ satisfies

$$\phi'(s) > 0 \text{ for } s \in (-1,1) \quad \text{and} \quad \phi'(-1) = \phi'(+1) = 0. \tag{1.4}$$

Observe that equation (1.1) is only of parabolic type at points where the solution $u \in (0,1)$ and its derivative $u_x \in (-1,1)$. Since $D(s)$ vanishes at $s = 0,1$ and $\phi'(s)$ at $s = \pm 1$, the equation degenerates at points where $u = 0,1$ or $u_x = \pm 1$.

An example of a differential equation which has the form of (1.1) with D and ϕ satisfying (1.3) and (1.4), respectively, is found in a model from hydrology. In this model one studies the evolution in time of the interface between fresh and salt groundwater under certain simplifying assumptions on the flow field. This model was first derived by de Josselin de Jong [15]. In Section 2 we give a brief description of its background and of its main features. Here, we mention that it leads to an equation with

$$D(s) = s(1-s) \quad \text{and} \quad \phi(s) = s/(1+s^2). \tag{1.5}$$

Problem C was first studied by Van Duijn & Hilhorst [9]. They introduced a class of solutions and they showed that if D,ϕ and u_0 satisfy the hypotheses:

HD1: (i) $D \in C^1[0,1] \cap C^2(0,1)$;

(ii) $D > 0$, $D'' \leq 0$ on $(0,1)$ and $D(0) = D(1) = 0$;

Hϕ1: (i) $\phi \in C^1[-1,1] \cap C^2(-1,1)$;

(ii) $\phi' > 0$ on $(-1,1)$ with $\phi'(\pm 1) = 0$ and $\phi(0) = 0$;

Hu_0: (i) $u_0 \in W^{1,\infty}(\mathbb{R})$;

(ii) $0 \leq u_0 \leq 1$ and $-1 \leq u_0' \leq 1$ a.e. on $\mathbb{R}$;

(iii) $u_0 - H \in L^1(\mathbb{R})$, where H denotes the Heaviside function: $H(s) = 1$ when $s > 0$ and $H(s) = 0$ when $s < 0$;

then Problem C has a unique solution u on Q.

In Section 3 we recall some of their results for later use. In Section 4 we introduce subsolution and supersolution for Problem C and we prove a comparison principle.

To prove regularity properties for solutions of Problem C, we have to impose additional assumptions on the functions D and ϕ. These are

HD2: (i) There exist $c,d \in \mathbb{R}^+$ such that $-d \leq D'' \leq -c = \sup_{(0,1)} D''$ on $(0,1)$;

(ii) $D \in C^{3+\alpha}(0,1)$ for some $\alpha \in (0,1)$ and $D''' \cdot D' \geq 0$ on $(0,1)$.

and

Hϕ2: (i) $s\phi''(s) \leq 0$ and $\phi(s) = -\phi(-s)$ for $s \in (-1,1)$;

(ii) $\phi \in C^{3+\alpha}(-1,1)$ for some $\alpha \in (0,1)$.

Observe that the functions D and ϕ, given by (1.5), satisfy all the imposed conditions.

In Section 5 we use HD2 and Hϕ2 to obtain a pointwise bound on the derivative of u with respect to x, which is sharp in the case where $D(s) = s(1-s)$. It follows from this bound that for all $t > 0$

$$u_x(\cdot,t) \in (-1,1) \quad \text{a.e. on } \mathbb{R}$$

which means that the degeneracy in the derivative vanishes instantaneously for $t > 0$. This in turn implies that the solutions are classical solutions in the region where $u \in (0,1)$.

The rest of the paper is devoted to the study of the free boundaries induced by the degeneracy of the equation at $u = 0$ and $u = 1$. To avoid unnecessary complications we assume that there exist real numbers $-\infty < a_1 < a_2 < \infty$ such that

$$\begin{aligned} &u_0 = 0 && \text{on} && (-\infty,a_1] \\ &u_0 \in (0,1) && \text{on} && (a_1,a_2) \\ &u_0 = 1 && \text{on} && [a_2,+\infty). \end{aligned} \tag{1.6}$$

Let

$$M(u) = \{(x,t) \in Q \mid u(x,t) \in (0,1)\}.$$

In Section 6 we follow Knerr [16] to show that there exist functions $\zeta_1,\zeta_2\colon [0,\infty) \to \mathbb{R}$ such that the set M(u) can be written as

$$M(u) = \{(x,t) \in Q \mid \zeta_1(t) < x < \zeta_2(t),\ t > 0\}. \tag{1.7}$$

Further, we show that each function satisfies:

There exists $t_i^* \in [0,\infty)$ such that $\zeta_i(t) = a_i$ for $t \in [0,t_i^*]$, $(-1)^i\zeta_i$ is nondecreasing on $[0,\infty)$ with $\lim_{t\to\infty} (-1)^i\zeta_i(t) = +\infty$ and $\zeta_i \in C^{0,1}[0,\infty)$.

In what follows we refer to ζ_1 and ζ_2 as the interfaces of Problem C. They separate the regions where $u = 0$, $u \in (0,1)$ and $u = 1$.

Next, let

$$v_1 := -\frac{D(u)}{u}\phi(u_x) \quad \text{and} \quad v_2 := \frac{D(u)}{1-u}\phi(u_x) \text{ on } M(u). \tag{1.8}$$

In terms of the hydrological model, v_1 represents the x component of the salt water velocity and v_2 the x component of the fresh water velocity, see (2.16) and (2.17) after suitable rescaling. Based on this model, we expect that the speed in the x,t plane of the interfaces is equal to the velocity of the fluids in the corresponding points, see also (2.18) and (2.19). Thus at the interfaces ζ_i one expects for $t > 0$ the differential equations

$$\zeta_i'(t) = \lim_{\substack{x \to \zeta_i(t) \\ (x,t) \in N(u)}} v_i(x,t) \qquad i = 1,2. \qquad (1.9)$$

In order to establish these equations, we first derive an Aronson-Benilan [2] type estimate for $(\phi(u_x))_x$, near the interfaces. As in the case of the porous medium equation and other related degenerate equations, this appears to be the crucial estimate for the regularity theory, e.g. see Vazquez [20] or Esteban & Vazquez [10].

For solutions of Problem C we prove in Section 7 first an interior estimate on M(u). Let $(x_0,t_0) \in M(u)$ such that $u(x_0,t_0) \in (\delta,1-\delta)$ for some $\delta \in (0,1/2)$ and let $|u_0'| \leq 1 - \varepsilon$ a.e. on $\mathbb{R}$ for some $\varepsilon > 0$ and $u_0 \in H^2_{loc}(a_1,a_2)$. Then there exists a constant $K > 0$, depending on δ and ε, such that

$$|(\phi(u_x))_x(x_0,t_0)| \leq Kt_0^{-1/2}. \qquad (1.10)$$

After that we prove one-sided bounds near the interfaces. For this we need

$$\sigma = \sup\left\{r \in (0,1) \,\middle|\, 2\frac{s\phi''(s)}{\phi'(s)} + 3 > 0 \text{ for all } s \in (0,r)\right\} \qquad (1.11)$$

and the sets

$$N_1 = \left\{(x,t) \in Q \,\middle|\, -\infty < x < \zeta_1(t) + s_0/4,\ t > 0\right\} \qquad (1.12)$$

and

$$N_2 = \left\{(x,t) \in Q \,\middle|\, \zeta_2(t) - (1 - s_0)/4 < x < \infty,\ t > 0\right\} \qquad (1.13)$$

where s_0 is the zero of D'. The result is the following. Let $|u_0'| \leq \nu < \sigma$ a.e. on $\mathbb{R}$ and let $u_0 \in H^2_{loc}(a_1,a_2)$. Then there exist constants $K_i > 0$, $i = 1,2$, such that

$$(\phi(u_x))_x \geq -K_1(t^{-1} + t^{-1/2}) \quad \text{in } \mathcal{D}'(N_1) \qquad (1.14)$$

and

$$(\phi(u_x))_x \leq +K_2(t^{-1} + t^{-1/2}) \quad \text{in } \mathcal{D}'(N_2). \qquad (1.15)$$

In establishing the interior estimate we use an energy type inequality, also given by Hoff [14], combined with a Bernstein estimate. The inequalities (1.14) and (1.15) follow from a maximum principle argument.

We observe here that for the special case $\phi(s) = s/(1 + s^2)$, the constant σ from (1.11) is given by

$\sigma = 0.505 \ldots$.

From estimates (1.14) and (1.15) it follows that for every $t > 0$

$$\phi(u_x)(x,t) + K_1 (t^{-1} + t^{-1/2})\, x$$

is nonincreasing when $x \downarrow \zeta_1(t)$ and

$$\phi(u_x)(x,t) - K_2 (t^{-1} + t^{-1/2})\, x$$

is nonincreasing when $x \uparrow \zeta_2(t)$. Since both expressions are bounded, this means that $\lim_{\substack{x\to\zeta_i(t)\\(x,t)\in M(u)}} \phi(u_x)(x,t)$ exists and consequently,

$$\lim_{\substack{x\to\zeta_i(t)\\(x,t)\in M(u)}} u_x(x,t) = u_x(\zeta_i(t),t) \text{ exists.}$$

In Section 8, we then show that the interfaces are right differentiable for each $t > 0$ and that

$$D^+\zeta_i(t) = -D'(i-1)\,\phi(u_x(\zeta_i(t),t)), \tag{1.16}$$

for $i = 1,2$ and for all $t > 0$. We also show that $(-1)^i\zeta_i(t)$ is strictly increasing for $t > t_i^*$.
The differentiability of the interfaces is much more involved, see e.g. Caffarelli & Friedman [6] and Aronson, Caffarelli & Kamin [3]. We leave it for a future paper.

2. THE MODEL

Consider the two-dimensional flow of a fluid of variable density and constant viscosity through a homogeneous porous medium. Let the flow take place in the x,z plane, with x horizontal and z vertical and pointing upwards. If the flow is incompressible, then the underlying equations are, see Bear [5]:

Momentum balance (Darcy law)

$$\frac{\mu}{\kappa} q + \text{grad } p + \gamma e_z = 0, \tag{2.1}$$

and

$$\text{div } q = 0. \tag{2.2}$$

Here q is the specific discharge, p the pressure, γ the specific weight and μ the dynamic viscosity of the fluid. Further, κ denotes the intrinsic permeability of the porous medium and e_z is the unit vector in positive z-direction. Introducing the stream function ψ:

$$q = \text{curl } \psi = \left(-\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial x}\right),$$

and taking the curl of (2.1), one finds the equation

$$-\Delta\psi = \frac{\kappa}{\mu}\frac{\partial\gamma}{\partial x}, \qquad (2.3)$$

see also Chan-Hong *et al.* [7], where these manipulations are made precise.

In our model we consider the specific case where the flow takes place in a coastal aquifer which is vertically confined and horizontally extended and where the fluid consists of fresh and salt water, separated by an abrupt transition - an interface. Our object is to derive an equation for this fresh-salt interface, which has the form of equation (1.1). We follow the derivation given in [7] and the original work of de Josselin de Jong [15].

Suppose that for each time t the interface can be parametrized by the horizontal space coordinate x, see Fig. 1, and that its

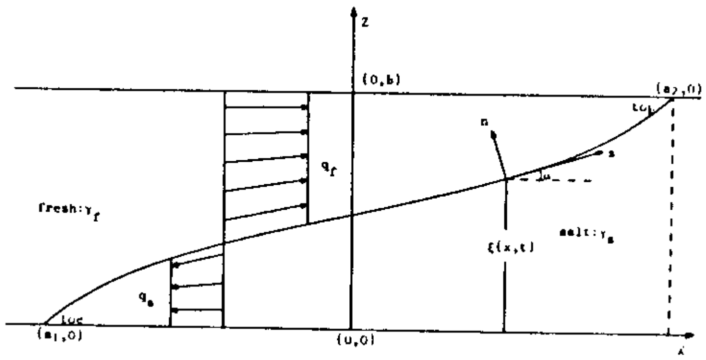


Fig. 1. Example of a fresh/salt distribution in a horizontal aquifer.

height with respect to the z = 0 plane is given by $\xi = \xi(x,t)$. In terms of ξ, the specific weight becomes

$$\gamma(x,z,t) = (\gamma_s - \gamma_f)H(\xi(x,t) - z) + \gamma_f \qquad (2.4)$$

where γ_s and γ_f are constants ($\gamma_s > \gamma_f$) denoting the specific weight of the salt and the fresh water and where H is the Heaviside function.

Inserting (2.4) into (2.3) results in the following jump conditions at the interface (see [7]):

$$[q_n] = \left[\frac{\partial\psi}{\partial s}\right] = 0, \qquad (2.5)$$

$$[q_s] = -\left[\frac{\partial\psi}{\partial n}\right] = \Gamma\cdot\sin\alpha, \quad \Gamma = \frac{\kappa}{\mu}(\gamma_s - \gamma_f). \qquad (2.6)$$

Here s and n are local orthogonal coordinates at the interface, s tangential and n normal, and α denotes the angle of the tangent at the interface with the horizontal.

From (2.5) and (2.6) we find for the jump in the x-component of the specific discharge at the interface

$$[q_x] = q_{f_x}(x,\xi,t) - q_{s_x}(x,\xi,t) = \Gamma\frac{\tan\alpha}{1+\tan^2\alpha}. \qquad (2.7)$$

If the aquifer has a constant thickness B and if it is bounded below and above by an impervious boundary, then the total fresh and salt water discharge through the aquifer in the positive x-direction is given by

$$Q_{f_x}(x,t) = \int_{\xi(x,t)}^{B} q_{f_x}(x,z,t)dz, \tag{2.8}$$

and

$$Q_{s_x}(x,t) = \int_{0}^{\xi(x,t)} q_{s_x}(x,z,t)dz. \tag{2.9}$$

Also the following fresh and salt water continuity equations hold:

$$\frac{\partial Q_{f_x}}{\partial x} = \varepsilon \frac{\partial \xi}{\partial t}, \tag{2.10}$$

and

$$\frac{\partial Q_{s_x}}{\partial x} = -\varepsilon \frac{\partial \xi}{\partial t}, \tag{2.11}$$

when ε denotes the porosity of the porous material.

Consequently, the total discharge

$$Q = Q_{f_x} + Q_{s_x} \tag{2.12}$$

is constant in space and is considered here as a given quantity.

Next we make an assumption with respect to the discharge field, which in hydrology is called the *Dupuit-approximation*, see also

Bear [5]. We assume that the horizontal component of the specific discharge is constant over the height in each fluid such that

$$q_{f_x}(x,z,t) \approx q_{f_x}(x,\xi,t), \qquad \xi < z < B \tag{2.13}$$

and

$$q_{s_x}(x,z,t) \approx q_{s_x}(x,\xi,t), \qquad 0 < z < \xi. \tag{2.14}$$

These assumptions are generally satisfied for sufficiently flat interfaces, see [7], where numerical computations were carried out. Substituting these simplifications into (2.8) and (2.9), gives for (2.12)

$$Q = q_{f_x}(x,\xi,t)(B - \xi) + q_{s_x}\xi. \tag{2.15}$$

Next the unknowns q_{f_x} and q_{s_x} are solved from (2.7) and (2.15). This gives

$$q_{f_x} = \frac{Q}{B} + \frac{\Gamma}{B}\xi \frac{\tan\alpha}{1 + \tan^2\alpha} \tag{2.16}$$

and

$$q_{s_x} = \frac{Q}{3} - \frac{\Gamma}{B}(B - \xi) \frac{\tan\alpha}{1 + \tan^2\alpha}. \tag{2.17}$$

These equations give the discharges in both fluids in terms of the interface. From (2.16) we find for the velocity of the fresh water particles in the top of the interface:

$$v_f = \frac{q_{f_x}}{\varepsilon} = \frac{Q}{\varepsilon B} + \frac{\Gamma}{\varepsilon} \frac{\tan\alpha}{1+\tan^2\alpha}. \tag{2.18}$$

Similarly, for the salt water particles in the toe we find

$$v_s = \frac{q_{s_x}}{\varepsilon} = \frac{Q}{\varepsilon B} - \frac{\Gamma}{\varepsilon} \frac{\tan\alpha}{1+\tan^2\alpha}. \tag{2.19}$$

To obtain a differential equation for ξ, we combine (2.8), (2.10) and (2.16) to obtain

$$\varepsilon \frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x}\left\{(B-\xi)Q + \Gamma(B-\xi)\xi \frac{\partial\xi/\partial x}{1+(\partial\xi/\partial x)^2}\right\}.$$

Finally we apply the rescaling

$$x := \left(x - \frac{1}{\varepsilon}\int_0^t Q(s)ds\right)/B; \quad t := \Gamma t/\varepsilon; \quad u := \xi/B,$$

which results in the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{u(1-u)\frac{\partial u/\partial x}{1+(\partial u/\partial x)^2}\right\}.$$

3. PREVIOUS RESULTS

In this section we recall some results from [9] with respect to Problem C.

Definition 3.1. A function $u: \bar{Q} \to \mathbb{R}$ is called a solution of Problem C, if it satisfies

(i) $u \in L^\infty((0,\infty);W^{1,\infty}(\mathbb{R}))$, $u_t \in L^2((-R,R)\times(0,T))$ for all $R,T > 0$;

(ii) $u \in [0,1]$, $u_x \in [-1,1]$ a.e. in Q;

(iii) $u(\cdot,0) = u_0(\cdot)$;

(iv) $u_t - (D(u)\phi(u_x))_x = 0$ a.e. in Q.

Theorem 3.2. Suppose HD1, Hϕ1 and Hu_0 are satisfied. Then Problem C has a unique solution with the following properties:

(i) $u(t) - H \in L^1(\mathbb{R})$ for all $t > 0$;

(ii) $u \in C(\bar{Q})$;

(iii) Let u_1 and u_2 be two solutions of Problem C with initial functions u_{01} and u_{02}, respectively. Then $\|u_1(t) - u_2(t)\|_{L^1(\mathbb{R})} \leq \|u_{01} - u_{02}\|_{L^1(\mathbb{R})}$ for all $t > 0$;

(iv) Let u_1 and u_2 be two solutions of Problem C with initial functions $u_{01} \leq u_{01}$. Then $u_1(t) \leq u_2(t)$ for all $t > 0$.

The existence part of this theorem is proved by parabolic regularization: given a u_0 which satisfies Hu_0, one constructs a sequence $\{u_{0n}\}_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R})$ such that for sufficiently large $n \in \mathbb{N}$

$$u_{0n} \in (\frac{1}{n}, 1 - \frac{1}{n}), \qquad u'_{0n} \in (-1 + \frac{1}{n}, 1 - \frac{1}{n}) \text{ on } \mathbb{R},$$

$$u'_{0n}(x) = 0 \quad \text{for } |x| \geq n,$$

and $u_{0n} \to u_0$ uniformly on compact subsets of $\mathbb{R}$.
Then for $n \in \mathbb{N}$ (sufficiently large) and for any $T > 0$ consider the problem

$$(C_n) \begin{cases} u_t = (D(u)\phi(u_x))_x & \text{in } Q_T^n := (-n,n)\times(0,T], \\ u_x(-n,t) = u_x(n,t) = 0 & \text{for } 0 < t \leq T, \\ u(x,0) = u_{0n}(x) & \text{for } -n \leq x \leq n. \end{cases}$$

Problem C_n has a unique classical solution $u_n \in C^{2+\alpha}(\bar{Q}_T^n)$, for each $\alpha \in (0,1)$, and $u_n \in (\frac{1}{n}, 1 - \frac{1}{n})$ and $u_{nx} \in (-1 + \frac{1}{n}, 1 - \frac{1}{n})$. A solution u of Problem C is now obtained as the limit of a subsequence of $\{u_n\}$. The convergence is uniform on compact subsets of Q. The uniqueness part follows from the contraction property (iii), which in turn follows from the accretivity in $L^1(\mathbb{R})$ of the operator $Au := -(D(u)\phi(u'))'$ when defined on a suitable domain.

For the special case when $D(s) = s(1 - s)$ a self-similar solution of Problem C can be constructed. It has the form

$$u_s(x,t) = s(\eta) \quad \text{and} \quad \eta = xf(t), \tag{3.1}$$

where the function $s: \mathbb{R} \to [0,1]$ is given by

$$s(\eta) = \begin{cases} 0 & -\infty < \eta < -\frac{1}{2}, \\ \frac{1}{2} + \eta & -\frac{1}{2} \leq \eta \leq \frac{1}{2}, \\ 1 & \frac{1}{2} < \eta < \infty, \end{cases}$$

and where the function $f: [0,\infty) \to \mathbb{R}$ is determined by the problem

$$(S) \begin{cases} f'(t) = -2\phi(f(t))f^2(t) & t > 0, \\ f(0) = f_0. \end{cases}$$

A solution f of Problem S, with $f_0 \in [-1,1]$, induces a solution u_s of Problem C which has the form of a rotating line: this line rotates clockwise when $f_0 \in (0,1]$ and counter clockwise when $f_0 \in [-1,0)$. The initial slope is given by f_0.

It was shown that a solution of Problem C, with u_0 satisfying (1.6), converges to a self-similar solution with $f_0 > 0$. Also an estimate for the rate of convergence was given.

4. SUB- AND SUPERSOLUTIONS

Definition 4.1. A function u is called a sub-(super)solution of Problem C on a domain $S_T = \mathbb{R}\times(0,T)$ if there exists a $T \in (0,\infty]$ such that

(i) $u \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}))$;

(ii) $u \in [0,1]$, $u_x \in [-1,1]$ a.e. in S_T;

(iii) u_t, $(D(u)\phi(u_x))_x \in L^2((-R,R)\times(0,T'))$ for all $R > 0$ and for all finite $0 < T' \leq T$;

(iv) $D(u(\cdot,t)) \in L^1(\mathbb{R})$ for all $t \in (0,T)$;

(v) $u_t - (D(u)\phi(u_x))_x \leq (\geq)\ 0$ a.e. in S_T and $u(\cdot,0) \leq (\geq)$ $u_0(\cdot)$ on $\mathbb{R}$.

Remark 4.2. In [9] it was proven that if u is any solution of Problem C with D and ϕ bounded functions and $u_0 - H \in L^1(\mathbb{R})$, then $u(t) - H \in L^1(\mathbb{R})$ for all $t \geq 0$. This implies $D(u(\cdot,t)) \in L^1(\mathbb{R})$ for all $t \geq 0$. Therefore under these conditions, a solution u is both a subsolution and a supersolution of Problem C on $S_\infty = Q$.

Theorem 4.3. (Comparison principle). Let $D \in C[0,1]$ satisfy (1.3) and let $\phi \in C[-1,1]$ be strictly increasing. Further let $\underline{u}$ and $\bar{u}$ be respectively a subsolution and a supersolution of Problem C on S_T for some $T \in (0,\infty]$. Then $\underline{u} \leq \bar{u}$ in S_T.

Proof. From (v) of Definition 4.1 we have

$$(\underline{u} - \bar{u})_t \leq (D(\underline{u})\phi(\underline{u}_x))_x - (D(\bar{u})\phi(\bar{u}_x))_x \quad \text{a.e. in } S_T.$$

Multiplying both sides by $H(\underline{u} - \bar{u})$, where H is the Heaviside function, and integrating with respect to x over (-R,R) for every $R > 0$ gives

$$\int_{-R}^{R} (\underline{u} - \bar{u})_t H(\underline{u} - \bar{u}) \leq \int_{-R}^{+R} \{D(\underline{u})\phi(\underline{u}_x) - D(\bar{u})\phi(\bar{u}_x)\}_x\, H(\underline{u} - \bar{u})$$

for a.e. $t \in (0,T)$.

Arguing exactly as in [9], we obtain from this inequality

$$\int_{-R}^{R} (\underline{u} - \bar{u})_t H(\underline{u} - \bar{u}) \leq \{|D(\underline{u})\phi(\underline{u}_x)| + |D(\bar{u})\phi(\bar{u}_x)|\}\ (R,t)$$
$$+ \{|D(\underline{u})\phi(\underline{u}_x)| + |D(\bar{u})\phi(\bar{u}_x)|\}\ (-R,t) \qquad (4.1)$$

for a.e. $t \in (0,T)$.

Since $(\underline{u} - \bar{u}) \in W^{1,1}((0,T'); L^1(-R,R))$ for any finite $T' \leq T$ we have

$$H(\underline{u} - \bar{u}) \frac{d}{dt} (\underline{u} - \bar{u}) = \frac{d}{dt} (\underline{u} - \bar{u})^+ \text{ on } (0,T)$$

with $(s)^+ = \max(s,0)$, see Crandall & Pierre [8, Lemma a.1]. Therefore integrating (4.1) in t and using $(\underline{u}(\cdot,0) - \bar{u}(\cdot,0))^+ = 0$

gives

$$\int_{-R}^{+R} (\underline{u} - \bar{u})^+(t) \leq \int_0^t \{f(R,s) + f(-R,s)\}ds, \text{ for all } t \in (0,T) \quad (4.2)$$

where $f(\pm R,s)$ denote the expressions in the right hand side of (4.1). Because $D(\underline{u}(t))$, $D(\bar{u}(t)) \in L^1(\mathbb{R})$ for all $t \geq 0$ and because ϕ is bounded, $f(\pm R,s)$ tend to zero as $R \to \infty$ pointwise in $s \in (0,t)$. Moreover $f(\pm R,\cdot)$ are bounded on $(0,t)$ because D and ϕ are bounded. Thus we can use the dominated convergence theorem to obtain

$$\int_{-\infty}^{+\infty} (\underline{u} - \bar{u})^+(t) \leq 0, \text{ for all } t \in (0,T)$$

and consequently $\underline{u} \leq \bar{u}$ in S_T.

Remark 4.4. Let (x_0,t_0), $(x_0,t_1) \in Q$ with $0 < t_0 < t_1 < \infty$ and consider in Q the regions $S^- = (-\infty,x_0) \times (t_0,t_1)$ and $S^+ = (x_0,\infty) \times (t_0,t_1)$. On S^- and S^+ we can also define subsolutions and supersolutions. This is done similar to Definition 4.1: one replaces S_T by S^- or S^+, $\mathbb{R}$ by $(-\infty,x_0)$ or (x_0,∞) and one adds the condition $\underline{u}(x_0,t) \leq u(x_0,t)$ or $\bar{u}(x_0,t) \geq u(x_0,t))$ on (t_0,t_1). For S^- the comparison principle is now the following: Let D and ϕ satisfy the hypotheses of Theorem 4.2 and let $\underline{u}$ and $\bar{u}$ be respectively a subsolution and a supersolution of Problem C on S^-. Then $\underline{u} \leq \bar{u}$ in S^-. A similar result holds for S^+.

5. GRADIENT BOUND

Consider the auxiliary problem

$$(\tilde{S}) \quad \begin{cases} f'(t) = -c\phi(f(t))f^2(t) & t > 0, \\ f(0) = f_0, \end{cases}$$

where $f_0 \in (0,1]$ and where $c > 0$ is the constant defined in HD2. For the case $D(s) = s(1 - s)$ we have $c = 2$, which gives Problem S. The solution f then denotes the slope of the self-similar solution. By standard methods, Problem $\tilde{S}$ has a solution f which is positive and strictly decreasing such that $f(t) \searrow 0$ as $t \to \infty$. To obtain a rate of convergence, observe that the assumptions on ϕ imply

$$as \leq \phi(s) \leq bs, \qquad s \in [0,1] \quad (5.1)$$

with $a = \phi(1)$ and $b = \sup_{s\in[0,1]} \phi'(s)$.

Therefore

$$-bc \leq \frac{f'(t)}{f^3(t)} \leq -ac \quad \text{for} \quad t > 0,$$

which gives

$$\frac{1}{2bct + 1/f_0^2} \leq f^2(t) \leq \frac{1}{2act + 1/f_0^2} \quad \text{for } t > 0. \quad (5.2)$$

Lemma 5.1. Let u be the solution of Problem C in which D, ϕ and u_0 are such that the hypotheses HD1-2, Hϕ1-2 and Hu_0 are satisfied. Then

$$|u_x(\cdot,t)| \leq f(t) \quad \text{a.e. on } \mathbb{R}$$

for all $t \geq 0$. Here f denotes the solution of Problem $\hat{S}$ with $f_0 = \sup_{\mathbb{R}} |u_0'|$.

Proof. We prove here the upper bound. The proof for the lower bound is similar since ϕ is odd. Let $n \in \mathbb{N}$, sufficiently large, and let u_n be the classical solution of Problem C_n. The smoothness of D and ϕ implies $u_n \in C^{2+\alpha}(\bar{Q}_T^n) \cap C^{4,2}(Q_T^n)$, $\alpha \in (0,1)$. Therefore the function $w = u_{nx}$ satisfies

$$\begin{cases} \mathcal{L}(w) := Aw_{xx} + Bw_x + D''(u_n)\phi(w)w^2 - w_t = 0 & \text{in } Q_T^n \\ w(\pm n,t) = 0 & \text{for } 0 < t < T \\ w(x,0) = u_{0n}'(x) & \text{for } -n < x < n \end{cases} \tag{5.3}$$

where

$$A = D(u_n)\phi'(u_{nx}),$$

and

$$B = D(u_n)\phi''(u_{nx})u_{nxx} + 2D'(u_n)u_{nx}\phi'(u_{nx}) + D'(u_n)\phi(u_{nx}).$$

Let f be as in the lemma. Then

$$\mathcal{L}(f) = D''(u_n)\phi(f)f^2 - f' = (D''(u_n) + c)\phi(f)f^2 \leq 0 \quad \text{in } Q_T^n.$$

The approximating sequence $\{u_{0n}\}$ can be constructed such that $|u_{0n}'| \leq \sup_{\mathbb{R}} |u_0'| = f_0$. Therefore we have

$$w - f \leq 0 \quad \text{on } \partial Q_T^n$$

where ∂Q_T^n denotes the parabolic boundary of Q_T^n.
A standard maximum principle argument [19] gives $w - f \leq 0$ in $\bar{Q}_T^n$. Thus

$$u_{nx}(x,t) \leq f(t) \quad \text{in } \bar{Q}_T^n$$

for each $n \in \mathbb{N}$, sufficiently large, which implies the desired inequality.

Remark 5.2. Suppose $D(s) = s(1 - s)$. Then Problem C has a self-similar solution u_s which satisfies $u_{s_x}(\cdot,t) = f(t)$, $t > 0$, when $u_s \in (0,1)$. Thus the gradient bound is sharp in this case.

Remark 5.3. Lemma 5.1 shows that the fresh/salt interface rotates and becomes flatter in time. This corresponds to what we physically may expect of the flow problem. In particular the Dupuit-approximation (see (2.13) and (2.14)) holds when the interface is sufficiently flat. Thus the mathematical model gives a good description of the physical problem when t increases.

Since f is a strictly decreasing function, we conclude from Lemma 5.1 that the degeneracy at points where $u_x = \pm 1$ vanishes instantaneously. Therefore we expect the solution to be classical at points where $u \in (0,1)$. We make this the content of the following theorem.

Theorem 5.4. Let u be the solution of Problem C in which D, ϕ and u_0 are such that the hypotheses HD1-2, Hϕ1-2 and Hu_0 are satisfied. Then

$$u \in C^{4,2}(M(u)),$$

where $M(u) = \{(x,t) \in Q \mid u(x,t) \in (0,1)\}$.

Proof. Let $(x_0,t_0) \in M(u)$. Then there exists a $\delta \in (0,1)$, sufficiently small, and a neighbourhood N_0 of (x_0,t_0) such that $u \in (\delta,1-\delta)$ on N_0. Since the approximating sequence of classical solutions, denoted again by $\{u_n\}$, converges uniformly to u on N_0, we have $u_n \in (\delta/2,1-\delta/2)$ on N_0 for $n \in \mathbb{N}$ sufficiently large. By the proof of Lemma 5.1, we also have $|u_{nx}| \leq f < 1-\delta'$ on N_0 for $n \in \mathbb{N}$ large and $\delta' > 0$ sufficiently small. Therefore there exists a $\delta'' > 0$ such that

$$\delta'' < D(u_n)\phi'(u_{nx}) < 1/\delta'' \quad \text{on } N_0 \qquad (5.4)$$

for all $n \in \mathbb{N}$ large enough.

Now each function $w_n := u_{nx}$ satisfies equation (5.3) which we write here in the divergence form

$$\frac{\partial w_n}{\partial t} = \frac{\partial}{\partial x}\left\{D(u_n)\phi'(u_{nx})\frac{\partial w_n}{\partial x} + D'(u_n)\phi(u_{nx})w_n\right\} \quad \text{on } Q_T^n.$$

Then using (5.4), it follows from Ladyzenskaja *et al.* [17, Theorem 10.1] or Aronson & Serrin [4] that there exists a neighbourhood $N_1 \subset N_0$ of (x_0,t_0) for which

$$u_{nx} \in C^{\beta}(\bar{N}_1) \quad \text{for some } \beta \in (0,1)$$

where β and $\|u_{nx}\|_{C^{\beta}(N_1)}$ are estimated independent of n.

Next we consider equation (1.1), written as

$$u_t = D(u_n)\phi'(u_{nx})u_{xx} + D'(u_n)\phi(u_{nx})u_x \quad \text{in } N_1.$$

The coefficients in this equation are Hölder continuous on N_1, uniformly with respect to $n \in \mathbb{N}$. Then it follows from the linear theory, Friedman [11], that there exists a neighbourhood $N_2 \subset N_1$ of (x_0,t_0) on which for all $n \in \mathbb{N}$, sufficiently large,

$$u_n \in C^{2+\beta'}(\bar{N}_2) \qquad \text{for some } \beta' \in (0,1),$$

where $\|u_n\|_{C^{2+\beta'}(N_2)}$ is bounded uniformly in n. Hence the limiting solution u belongs to $C^{2+\beta'}(\bar{N}_2)$. Since $D \in C^{3+\alpha}(0,1)$ and $\phi \in C^{3+\alpha}(-1,1)$, with $\alpha \in (0,1)$, a standard bootstrap argument completes the proof.

The gradient estimate can be used to construct sub- and supersolutions for Problem C. Below we give a sub- and supersolution, which reduce to a translated self-similar solution in the case $D(s) = s(1-s)$. For $x \in \mathbb{R}$, let

$$[x]_0^1 = \begin{cases} 1 & x \geq 1 \\ x & 0 < x < 1. \\ 0 & x \leq 0 \end{cases} \tag{5.5}$$

Proposition 5.5. Let HD1-2, Hϕ1-2, Hu_0 and (1.6) be satisfied and let $f: [0,\infty) \to \mathbb{R}$ denote the solution of Problem $\hat{S}$ with $\sup_{\mathbb{R}} |u_0'| \leq f_0 \leq 1$. Then $\bar{s}: \bar{Q} \to \mathbb{R}$ given by

$$\bar{s}(x,t;\bar{x}) = \left[f(t)\left(x - \bar{x} + \frac{D'(0)}{cf(t)}\right)\right]_0^1, \qquad (x,t) \in \bar{Q}$$

is a supersolution of Problem C on Q for all $\bar{x} \leq a_1 + \frac{D'(0)}{cf_0}$.

Similarly, $\underline{s}: \bar{Q} \to \mathbb{R}$ given by

$$\underline{s}(x,t;\underline{x}) = \left[f(t)\left(x - \underline{x} + \frac{D'(1) + c}{cf(t)}\right)\right]_0^1, \qquad (x,t) \in \bar{Q}$$

is a subsolution of Problem C on Q for all $\underline{x} \geq a_2 + \frac{D'(1)}{cf_0}$.

Proof. We only show Definition 4.1 (v) for $\bar{s}$. Let $\bar{s} \in (0,1)$. Then $L(\bar{s}) := (D(\bar{s})\phi(\bar{s}_x))_x - \bar{s}_t = D'(\bar{s})\phi(f)f - \bar{s}_t$. Using $D'' \leq -c$, which implies $D'(s) \leq D'(0) - cs$ for $s \in (0,1)$, and writing

$$\bar{s}_t = \frac{f'}{f}\bar{s} - \frac{f'}{f}\frac{D'(0)}{c},$$

gives

$$L(\bar{s}) \leq \left(\frac{D'(0)}{cf} - \frac{\bar{s}}{f}\right)(c\phi(f)f^2 + f') = 0$$

because f satisfies Problem $\hat{S}$.

6. THE INTERFACES

In this section we characterize the set

$$M(u) = \{(x,t) \in Q \mid u(x,t) \in (0,1),\ u \text{ a solution of Problem C}\}.$$

Throughout this section we assume that HD1-2, Hϕ1-2, Hu_0 and (1.6) are satisfied.

Proposition 6.1. The set $M(u) \cap \mathbb{R} \times \{t\}$ is bounded, open and connected for all $t \geq 0$.

Proof. The boundedness is a direct consequence of Proposition 5.5 and the comparison principle. It follows that

$$u(x,t) = 0 \quad \text{when} \quad x \leq a_1 + \frac{D'(0)}{cf_0} - \frac{D'(0)}{cf(t)}, \quad t > 0 \tag{6.1}$$

and

$$u(x,t) = 1 \quad \text{when} \quad x \geq a_2 + \frac{D'(1)}{cf_0} - \frac{D'(1)}{cf(t)}, \quad t > 0. \tag{6.2}$$

The set $M(u)$ is open because $u \in C(\bar{Q})$. The connectedness follows from a level curve argument from Knerr [16].

We are now in a position to define the interfaces.

Definition 6.2. The functions $\zeta_i : [0,\infty) \to \mathbb{R}$ $(i = 1,2)$, defined according to

$$\zeta_1(t) = \inf \{x \in \mathbb{R} \mid u(x,t) > 0\}, \qquad t \geq 0$$

$$\zeta_2(t) = \sup \{x \in \mathbb{R} \mid u(x,t) < 1\}, \qquad t \geq 0$$

are called the interfaces of Problem C.

The next proposition shows that ζ_1 is nonincreasing and ζ_2 is nondecreasing.

Proposition 6.2. Let $(x_0,t_0) \in M(u)$. Then $(x_0,t) \in M(u)$ for all $t \geq t_0$.

Proof. We prove here that if $u(x_0,t_0) > 0$, then $u(x_0,t) > 0$ for all $t \geq t_0$ and for convenience we take $(x_0,t_0) = (0,0)$.
Consider the function $\rho : \bar{Q} \to \mathbb{R}$ defined by

$$\rho(x,t) = \left(\frac{1}{2a} e^{-kt}(a^2 - x^2)\right)^+ \qquad (x,t) \in \bar{Q}, \tag{6.3}$$

where a and k are positive constants to be chosen later. Observe that $|\rho_x| \leq 1$ a.e. on Q. We choose a sufficiently small so that

(i) $\rho(\cdot,0) \leq u(\cdot,0)$ on $\mathbb{R}$, where we use the continuity of u,

(ii) $D'(\rho) > 0$ on Q, where we use $D'(0) > 0$.

Then $L(\rho) := (D(\rho)\phi(\rho_x))_x - \rho_t \geq D(\rho)\phi'(\rho_x)\rho_{xx} - \rho_t$ when $\rho > 0$.
Substituting (6.3) gives

$$L(\rho) \geq \rho\left(k - \frac{D(\rho)}{a\rho}\phi'(\rho_x)e^{-kt}\right) \geq 0,$$

for k sufficiently large. Thus ρ is a subsolution on Q. Consequently

$$u(0,t) \geq \frac{a}{2} e^{-kt} > 0 \qquad \text{for all } t > 0.$$

Proposition 6.3. $\zeta_i \in C^{0,1}[0,\infty)$ and $\zeta_i(0) = a_i$ for $i = 1,2$.

Proof. By the monotonicity of the functions ζ_i we only have to consider a lower bound for $\zeta_1(t) - \zeta_1(t_0)$ and an upperbound for $\zeta_2(t) - \zeta_2(t_0)$ for all $t \geq t_0 \geq 0$.

This we do with the functions $\bar{s}$ and $\underline{s}$ from Proposition 5.5. Let $t_0 \geq 0$ and let $Q_{t_0} = \mathbb{R} \times (t_0,\infty)$. Choose

$$\bar{x} = \zeta_1(t_0) + \frac{D'(0)}{cf(t_0)} \quad \text{and} \quad \underline{x} = \zeta_2(t_0) + \frac{D'(1)}{cf(t_0)}.$$

As in Proposition 5.5

$$\underline{s}(x,t;\underline{x}) \leq u(x,t) \leq \bar{s}(x,t;\bar{x}) \qquad \text{for all } (x,t) \in \bar{Q}_{t_0}.$$

The upperbound implies

$$u(x,t) = 0 \quad \text{for} \quad x \leq \bar{x} - \frac{D'(0)}{cf(t)}, \quad t \geq t_0$$

and thus

$$\zeta_1(t) \geq \bar{x} - \frac{D'(0)}{cf(t)} \qquad \text{for all } t \geq t_0.$$

Therefore

$$-\frac{D'(0)}{c}\left\{\frac{1}{f(t)} - \frac{1}{f(t_0)}\right\} \leq \zeta_1(t) - \zeta_1(t_0) \leq 0. \tag{6.4}$$

To estimate the left hand side, consider $F: [0,\infty) \to \mathbb{R}$ defined by

$$F(t) = \frac{1}{f(t)} - c\phi(1)t \qquad\qquad t \geq 0.$$

Clearly F is differentiable and

$$F'(t) = -\frac{f'(t)}{f^2(t)} - c\phi(1) = c\{\phi(f(t)) - \phi(1)\} \leq 0.$$

Thus

$$F(t) \leq F(t_0) \qquad\qquad \text{for all } t \geq t_0.$$

Using this in (6.4) gives

$$-D'(0)\phi(1)(t - t_0) \leq \zeta_1(t) - \zeta_1(t_0) \leq 0. \tag{6.5}$$

In a similar way, we find for the second interface

$$0 \leq \zeta_2(t) - \zeta_2(t_0) \leq -D'(1)\phi(1)(t - t_0). \tag{6.6}$$

Finally, inequalities (6.5) and (6.6) imply $\zeta_i(0) = a_i$, $i = 1,2$.

Next we show

Proposition 6.4. $\lim_{t\to\infty} \zeta_1(t) = -\infty$ and $\lim_{t\to\infty} \zeta_2(t) = +\infty$.

Proof. We prove here only the assertion for ζ_1. From Proposition 5.5 it follows that the interface near zero of the subsolution $\underline{s}$ is given by $\underline{x}(t) = \underline{x} - (D'(1) + c)/cf(t)$. Thus if

$$D'(1) + c > 0, \tag{6.7}$$

then $\underline{x}(t)$ decreases in t and $\lim_{t\to\infty} \underline{x}(t) = -\infty$. Since $\zeta_1(t) \leq \underline{x}(t)$, the proof is completed for this case. If (6.7) is not satisfied*, we cannot use Proposition 5.5 to prove the result. Instead we use

Claim 6.5. The function $\alpha: \bar{Q} \to \mathbb{R}$ defined by

$$\alpha(x,t;x_0) = \frac{1}{2\lambda^m(t)}\left(1 - \frac{(x - x_0)^2}{\lambda^2(t)}\right)^+, \qquad (x,t) \in \bar{Q}$$

with

$$\lambda(t) = (1 + kt)^{\frac{1}{m+2}}, \qquad t \geq 0$$

and with m and k positive constants given by (6.10), is a subsolution of Problem C on Q for $x_0 \in \mathbb{R}$ large enough.

If this claim holds, then clearly, $\zeta_1(t) \leq x_0 - \lambda(t)$, which implies the desired result.

*) For example let D be given by $D(s) = -\frac{1}{12}as^4 + \frac{1}{6}as^3 + -\frac{1}{2}(c + \frac{a}{4})s^2 + \frac{1}{2}(\frac{1}{12}a + c)s$ for $s \in [0,1]$, with $a, c \in \mathbb{R}^+$. Then D satisfies HDI-2 and $D'(1) + c = \frac{1}{2}c - \frac{1}{24}a$. Thus, when $a > 12c$ (6.7) is not satisfied.

Proof of the claim. We only check here property (v) of Definition 4.1. The others are trivially satisfied. To estimate $L(\alpha)$, L defined in the proof of Proposition 6.2, in points where $\alpha > 0$ we use $D'(\alpha) \geq D'(0) - d\alpha$ and

$$\alpha_t = -(m + 2)\frac{\lambda'}{\lambda}\alpha + \frac{\lambda'}{\lambda^{m+1}}.$$

This gives

$$L(\alpha) \geq -\frac{1}{\lambda^{m+2}}D(\alpha)\phi'(\alpha_x) + (D'(0) - d\alpha)\phi(\alpha_x)\alpha_x + \\ + (m + 2)\frac{\lambda'}{\lambda}\alpha - \frac{\lambda'}{\lambda^{m+1}}. \tag{6.8}$$

Next we use (5.1), $D(\alpha) \leq D'(0)\alpha$ and

$$\alpha_x^2 = \frac{1}{\lambda^{2m+2}}(1 - 2\lambda^m\alpha)$$

in (6.8) to find

$$L(\alpha) \geq \alpha\left\{(m + 2)\lambda^{m+1}\lambda' - D'(0)b - 2D'(0)a - \frac{db}{\lambda^m}\right\}\frac{1}{\lambda^{m+2}} \\ + \left\{D'(0)\frac{a}{\lambda^{m+1}} - \lambda'\right\}\frac{1}{\lambda^{m+1}}. \tag{6.9}$$

The choice of λ implies

$$\lambda \geq 1 \quad \text{and} \quad \lambda^{m+1}\lambda' = k/(m + 2).$$

Thus if we choose

$$k = D'(0)b + 2D'(0)a + db \quad \text{and} \quad m = \frac{k}{aD'(0)} - 2, \qquad (6.10)$$

then we obtain $L(\alpha) \geq 0$ a.e. on Q.

As in the case of the porous medium equation, Problem C also has waiting time solutions.

Proposition 6.6. Suppose there exist $A, \ell > 0$ such that

$$u_0(x) \leq A(x - a_1)^2 \qquad \text{for } x \leq a_1 + \ell.$$

Then there exists $t_1^* > 0$ such that $\zeta_1(t) = a_1$ for $0 \leq t \leq t_1^*$. Similarly if

$$u_0(x) \geq 1 - A(x - a_2)^2 \qquad \text{for } x \geq a_2 - \ell,$$

then there exists $t_2^* > 0$ such that $\zeta_2(t) = a_2$ for $0 \leq t \leq t_2^*$.

Proof. Again we only give the proof for ζ_1. We construct a local supersolution in a neighbourhood of $(a_1,0)$ which has a stagnant interface. For $\delta \in (0,\ell)$ and $\tau > 0$, consider in Q the halfstrip

$$S_\tau^\delta = (-\infty, a_1 + \delta) \times (0, \tau/2).$$

On S_τ^δ we consider the function

$$\rho(x,t) = \begin{cases} 2A(x - a_1)^2 \frac{\tau}{\tau - t} & \text{for } (x,t) \in [a_1, a_1 + \delta] \times [0, \tau/2], \\ 0 & \text{for } (x,t) \in (-\infty, a_1) \times [0, \tau/2]. \end{cases}$$

Clearly $u_0(\cdot) \leq \rho(\cdot, 0)$ on $(-\infty, a_1 + \delta)$.

One easily verifies that if

$$0 < \delta \leq \min\left\{\frac{1}{2\sqrt{A}}, \frac{1}{8A}\right\}, \qquad (6.11)$$

then $0 \leq \rho \leq 1$ and $0 \leq \rho_x \leq 1$ on S_τ^δ.

By the continuity of u and the first assumption of the proposition, there exists a $\tau_0 > 0$ such that

$$u(a_1 + \delta, \cdot) < \rho(a_1 + \delta, \cdot) \qquad \text{on } (0, \tau_0/2).$$

Next we estimate $L(\rho)$ on S_τ^δ

$$\begin{aligned} L(\rho) &= D(\rho)\phi'(\rho_x)4A \frac{\tau}{\tau - t} + D'(\rho)\phi(\rho_x)\,\rho_x - \rho_t \\ &\leq D'(0)\rho b\ 4A \frac{\tau}{\tau - t} + D'(0)b\rho_x^2 - \rho_t \\ &\leq \rho \frac{\tau}{\tau - t}\left\{12AbD'(0) - \frac{1}{\tau}\right\}. \end{aligned}$$

Therefore $L(\rho) \leq 0$ on S_τ^δ if $\tau \leq \frac{1}{12AbD'(0)}$.

Thus if we take δ satisfying (6.11) and

$$0 < \tau \leq \min\left\{\frac{1}{12AbD'(0)}, \tau_0\right\}$$

then ρ is a supersolution of Problem C on S_τ^δ. The proposition now follows from the comparison principle, see Remark 4.4.

7. BOUNDS FOR $(\phi(u_x))_x$

In this section we derive bounds for $(\phi(u_x))_x$ on Q. Based on the form of the self-similar solution (3.1) (the rotating line) we expect only one-sided bounds across the interfaces. A lower bound near ζ_1 (where $u = 0$) and an upperbound near ζ_2 (when $u = 1$). At interior points of M(u) we construct a two-sided bound.

Throughout this section $K = K(*,\ldots,*)$ is a positive constant depending only on the quantities appearing in the parentheses. However, since nearly all constants in this section depend on the structure of D and ϕ, this dependence is not explicitly indicated.

7.1. Interior estimate

We first give some notation. Let $-\infty < a < a' < b' < b < \infty$ and $0 \leq \tau < \tau' < T < \infty$ and let $R' \subset R \subset Q$ denote the rectangles

$R = (a,b) \times (\tau,T]$ and $R' = (a',b') \times (\tau',T]$. Further let $\xi \in C^\infty(\bar{R})$ such that $0 \leq \xi \leq 1$ on $\bar{R}$, $\xi = 1$ on $\bar{R}'$ and $\xi = 0$ in a neighbourhood of the parabolic boundary of R. Then $|\xi_x|$ and $|\xi_{xx}|$ are bounded in terms of $a' - a$ and $b - b'$ and $|\xi_t|$ is bounded in terms of $\tau' - \tau$: i.e. $|\xi_t| \leq K/(\tau' - \tau)$ for some $K > 0$.

Lemma 7.1. Let R, R' and ξ be defined as above and let $w \in C^{2,1}(R) \cap L^2((\tau,T);H^1(a,b))$ satisfy

$$w_t = Aw_{xx} + B \qquad \text{in } R,$$

where $A \in L^\infty(R)$ such that $\mu \leq A \leq \mu^{-1}$ on R for some $\mu > 0$ and where $B \in L^2(R)$. Then

$$\max_{[a',b']} |w(T)| + \|w_x(T)\|_{(a',b')} + \|w_{xx}\|_{R'} \leq K\left\{\|w\|_R \frac{1}{\tau' - \tau} + \|B\xi\|_R + \|w_x\|_R + \|w\|_R\right\}$$

where $\|\cdot\|$ denotes the L^2-norm. If $w(\tau) \in H^1_{loc}(a,b)$ and if we do not truncate in time, then

$$\max_{[a',b']} |w(T)| + \|w_x(T)\|_{(a',b')} + \|w_{xx}\|_{R'} \leq K\left\{\|(w\xi)_x(\tau)\|_{(a,b)} + \|B\xi\|_R + \|w_x\|_R + \|w\|_R\right\}.$$

In both inequalities $K = K(\mu, a'-a, b-b')$.

Proof. Since the proof is elementary it is omitted here, see [14].

To obtain the interior estimate we choose to work with the function

$$v = D(u)\phi(u_x). \tag{7.1}$$

The inequalities from Lemma 7.1 are the basic tools in the proof of the following result.

Lemma 7.2. Let u be a solution of Problem C in which D, ϕ and u_0 satisfy the hypotheses imposed in Section 6. In addition suppose that $u_0 \in H^2_{loc}(a_1,a_2)$ and $|u_0'| \leq 1 - \varepsilon$ a.e. on $\mathbb{R}$ for some $\varepsilon > 0$. Let $t_0 > 0$ and let $I \subset (\zeta_1(t_0),\zeta_2(t_0))$ denote an interval on which $\delta < u(\cdot,t_0) < 1 - \delta$ for some $\delta > 0$. Then

$$|v(x_1,t_0) - v(x_2,t_0)| \leq K|x_1 - x_2|^{1/2}$$

for all $x_1,x_2 \in I$. Here $K = K(\varepsilon,\delta,u_0)$.

Proof. In [9] it is shown that there exists a constant $L > 0$ such that the solution u of Problem C satisfies

$$|u(x_1,t_1) - u(x_2,t_2)| \leq L\left\{|x_1 - x_2| + |t_1 - t_2|^{1/2}\right\} \tag{7.2}$$

for all $(x_1,t_1),(x_2,t_2) \in \bar{Q}$.
Now let (a,b) denote the interval I. Then it follows from inequality

(7.2) that there exist real numbers $a_1 < a < b < b_1$ and $0 \leq \tau < t_0$ such that $\delta/2 < u < 1 - \delta/2$ in the rectangle $R_1 = (a_1,b_1) \times (\tau,t_0]$. In fact we choose $\tau = (t_0 - \Delta)^+$ with $\Delta = \delta^2/16L^2$. Inside R_1 we construct the rectangle $R_2 = (a_2,b_2) \times (\tau,t_0]$, with $a_2 = (a_1 + a)/2$ and $b_2 = (b_1 + b)/2$, and the rectangle $R = (a,b) \times ((\tau + t_0)/2,t_0]$. In R_1, u satisfies

$$u_t = D(u)\phi'(u_x)u_{xx} + D'(u)\phi(u_x)u_x.$$

Since $D(u)\phi'(u_x) \geq \min\{D(\delta/2),D(1 - \delta/2)\}\ \phi'(1 - \varepsilon)$, an application of Lemma 7.1, without truncating ξ in time, gives $\|u_{xx}\|_{R_2} \leq K$ with $K = K(\varepsilon,\delta)$.

In R_2, v satisfies

$$v_t = D(u)\phi'(u_x)v_{xx} + D'(u)\phi(u_x)\{D(u)\phi'(u_x)u_{xx} + D'(u)\phi(u_x)u_x\}.$$

Again we apply Lemma 7.2. For $t_0 > \Delta$, we truncate in time. This gives the bound

$$\|v_x(t_0)\|_I \leq K(\varepsilon,\delta). \tag{7.3}$$

For $t_0 \leq \Delta$ we do not truncate in time. This gives the bound

$$\|v_x(t_0)\|_I \leq K(\varepsilon,\delta,u_0). \tag{7.4}$$

From the estimates (7.3) and (7.4) the desired inequality immediately follows.

Lemma 7.3. Let u be a solution of Problem C in which D, ϕ and u_0 satisfy the hypotheses from Lemma 7.2. Let $R \subset M(u)$ be a bounded open rectangle on which $\delta < u < 1 - \delta$ for some $\delta > 0$. Then

$$|v(x_1,t_1) - v(x_2,t_2)| \leq K \left\{|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4}\right\}$$

for all $(x_1,t_1),(x_2,t_2) \in \bar{R}$. Here $K = K(\varepsilon,\delta,u_0)$.

Proof. Let the rectangle R be given by $(a,b) \times (\bar{t}_1,\bar{t}_2)$ with $0 \leq \bar{t}_1 < \bar{t}_2 < \infty$. Since $u \in (\delta, 1 - \delta)$ on R and $|u_x| \leq 1$, we have that $u \in (\delta/2, 1 - \delta/2)$ on the larger rectangle $R_L = (a - \delta/2, b + \delta/2) \times (\bar{t}_1,\bar{t}_2)$. On R_L, v satisfies the equation

$$v_t = D(u)\phi'(u_x)v_{xx} + D'(u)\phi(u_x)v_x. \tag{7.5}$$

From Lemma 7.2, v is Hölder continuous in x with exponent 1/2 on R_L. Then we use a result of Gilding [12] to obtain the Hölder continuity of v in t with exponent 1/4 on the smaller rectangle R.

We are now in a position to prove the interior estimate.

Theorem 7.4. Let u be a solution of Problem C in which D, ϕ and u_0 satisfy the hypotheses imposed in Section 6. Further let $u_0 \in H^2_{loc}(a_1,a_2)$ and $|u_0'| \leq 1 - \varepsilon$ a.e. on $\mathbb{R}$ for some $\varepsilon > 0$. If $(x_0,t_0) \in M(u)$ such that $\delta < u(x_0,t_0) < 1 - \delta$ for some $\delta \in (0,1/2)$, then

$$|v_x(x_0,t_0)| \leq K\, t_0^{-1/2}.$$

Here $K = K(\varepsilon,\delta,u_0)$.

Proof. The proof of this theorem consists of three steps.

Step 1: Construction of rectangle. Let $(x_0,t_0) \in M(u)$ and $u(x_0,t_0) \in (\delta, 1 - \delta)$. We construct a rectangle $R = (a,b) \times (\tau,t_0)$ in $M(u)$, with the point (x_0,t_0) centered at the top, on which $u \in (\delta/2, 1 - \delta/2)$ and on which the difference between two values of v is bounded by a small given constant. We choose

$$b - a = \min\{\delta/2, (\sigma/2K)^2\} \tag{7.6}$$

and

$$\Delta = \min\{(\delta/4L)^2, (\sigma/2K)^4\} \tag{7.7}$$

and let $\tau = (t_0 - \Delta)^+$. Here L denotes the constant from inequality (7.2) which implies $u \in (\delta/2, 1 - \delta/2)$ on R. Further, K denotes the constant from Lemma 7.3 applied to the rectangle with $b - a = \delta/2$

and $\Delta = \delta^2/16L^2$. This implies that $|v(x_1,t_1) - v(x_2,t_2)| < \sigma$ for all $(x_1,t_1),(x_2,t_2) \in R$. Finally σ is a positive constant which satisfies the inequality

$$\max_{[0,1-\varepsilon]} |\phi''|\sigma \leq (\phi'(1-\varepsilon))^2 \frac{\sqrt{2\pi}}{8} e^{1/2} \min\{D(\delta/2), D(1-\delta/2)\} \tag{7.8}$$

When $\max_{[0,1-\varepsilon]} |\phi''| = 0$, we take σ sufficiently large so that $b - a = \delta/2$ and $\Delta = \delta^2/4L^2$.

Step 2: Bernstein technique (see also Aronson [1] and Herrero & Vazquez [13]). First rewrite the equation for v as

$$v_t = D(u)\phi'(u_x)v_{xx} + \frac{D'(u)}{D(u)} vv_x \qquad \text{in } R. \tag{7.9}$$

Then set $v = g(w)$, where g is a smooth and strictly increasing function and let $p = w_x$. For p we find the equation

$$p_t - D(u)\phi'(u_x)p_{xx} = c_1pp_x + c_2p_x + c_3p^3 + c_4p^2 + c_5p \tag{7.10}$$

where

$$c_1 = \frac{\phi''}{\phi'} g' + 2D\phi' \frac{g''}{g'};$$

$$c_2 = D'\phi'u_x - \frac{\phi''}{\phi'} \phi D'u_x + \frac{D'}{D} g;$$

$$c_3 = D\phi' \left(\frac{g''}{g'}\right)' + \frac{\phi''}{\phi'} g'';$$

$$c_4 = -D' \frac{\phi''}{\phi'} \phi \frac{g''}{g'} u_x + D'\phi' \frac{g''}{g'} u_x + \frac{D'}{D} g';$$

$$c_5 = \left(\frac{D'}{D}\right)' g\, u_x.$$

In R we consider the smaller rectangle $R' = (a',b') \times (\tau',t_0]$, with $a' = (a + x_0)/2$, $b' = (b + x_0)/2$ and $\tau' = (t_0 + \tau)/2$. Let $\xi \in C^\infty(\bar{R})$ such that $0 \leq \xi \leq 1$ on $\bar{R}$, $\xi = 1$ on $\bar{R}'$ and $\xi = 0$ in a neighbourhood of the lower and lateral boundary of R. Then $|\xi_t| \leq K(\frac{1}{t_0} + 1)$ for some $K > 0$. Finally set $z = \xi^2p^2$. At the point where z attains its maximum we have

$$z_x = 0 \quad \text{and} \quad z_t - D(u)\phi'(u_x)z_{xx} \geq 0.$$

Combining this with equation (7.10) gives the inequality

$$-c_3\xi^2p^4 \leq \{c_4\xi - c_1\xi_x\}\, \xi p^3 + \left\{\xi\xi_t + \left(2\xi_x^2 - \xi^2\xi_{xx}\right)D\phi' - c_2\xi\xi_x + 2c_5\xi^2\right\} p^2. \tag{7.11}$$

Next we want to choose a function g such that the coefficient c_3 is bounded above by a negative constant.

Step 3: Choice of g. We distinguish two cases.

(i) $u_x \neq 0$ on $\bar{R}$. Suppose $u_x > 0$ on $\bar{R}$. Consequently also $v > 0$ on $\bar{R}$. We choose $g(w) = \frac{N}{2} w(w + 1)$ with $N = \max v$. Then $w \in (0,1]$, $g'' = N > 0$ and $\left(\frac{\phi''}{\phi'}\right)' \leq -\frac{4}{9}$. Since $\phi'' < 0$ when $u_x > 0$, we find for c_3

$$c_3 \leqslant -\frac{4}{9} D(u)\phi'(u_x) \leqslant -\frac{4}{9} \min \{D(\delta/2), D(1-\delta/2)\} \phi'(1-\varepsilon).$$

Using this inequality in (7.11) gives

$$|\xi p| \leqslant K\sqrt{\frac{1}{t_0}+1} \qquad \text{on } R, \tag{7.12}$$

where $K = K(\varepsilon,\delta,u_0)$. Then

$$|v_x(x_0,t_0)| \leqslant \sup_R |\xi g'(w)p| \leqslant \frac{3}{2} NK\sqrt{\frac{1}{t_0}+1} \tag{7.13}$$

where

$$N = \max_R D(u)\phi(u_x) \leqslant \max_{[0,1]} D\,\phi'(0)f(\tau). \tag{7.14}$$

To finish the proof, we substitute estimate (5.2) into (7.14) and combine the resulting inequality with (7.13).
When $u_x < 0$ on $\bar{R}$, we choose $g(w) = \frac{N}{2} w(1-w)$ and use similar arguments as above to obtain the desired inequality.

(ii) $u_x = 0$ at some point of $\bar{R}$. Consequently, $v = 0$ at some point of $\bar{R}$ and, by the choice of R, $|v| \leqslant \sigma$ on $\bar{R}$. Here σ is a positive constant which satisfies inequality (7.8). Now we choose $g(w) = 2N\,\mathrm{erfc}(w)$, where again $N = \max_R |v|$. Clearly $N \leqslant \sigma$. Further $\left(\frac{g''}{g'}\right)' = -2$ and $|g''| \leqslant \frac{8\sigma}{\sqrt{2\pi}} e^{-1/2}$. The choice of σ implies

$$c_3 \leqslant -D(u)\phi'(u_x) \leqslant -\min \{D(\delta/2), D(1-\delta/2)\} \phi'(1-\varepsilon).$$

One then proceeds as in case (i) to complete the theorem.

Corollary 7.5. $|(\phi(u_x))_x(x_0,t_0)| \leqslant K\, t_0^{-1/2}$, with $K = K(\varepsilon,\delta,u_0)$.

Proof. Use $v_x = D(u)(\phi(u_x))_x + D'(u)u_x\phi(u_x)$, Lemma 5.1 and inequality (5.2).

Remark 7.6. If $u_0' \geqslant 0$ a.e. on $\mathbb{R}$, then $u_x \geqslant 0$ a.e. on Q. In fact one can prove that $u_x > 0$ on $M(u)$. Therefore in this case we do not have to use Lemma 7.3 to construct the rectangle R. A straightforward application of the Bernstein method, with the function g from step 3 (i) in the proof of Theorem 7.4, suffices for our purpose.

7.2. Estimates near the interfaces

We start with a result concerning the positivity of u and $1-u$ in $M(u)$. To make this clear, we first introduce some notation. Below, u denotes a solution of Problem C.
For any $t \geqslant 0$ and $\mu \in (0,1)$, let

$$\underline{\chi}_\mu(t) = \min \{x \in (\zeta_1(t),\zeta_2(t)) \mid u(x,t) = \mu\}, \tag{7.15}$$

$$\bar{\chi}_\mu(t) = \max \{x \in (\zeta_1(t),\zeta_2(t)) \mid u(x,t) = \mu\}, \tag{7.16}$$

and

$$\underline{f}_\mu(t) = \min\{u(x,t) \mid x \in [\chi_\mu(t),\zeta_2(t)]\}, \tag{7.17}$$
$$\bar{f}_\mu(t) = \max\{u(x,t) \mid x \in [\zeta_1(t),\bar{\chi}_\mu(t)]\}. \tag{7.18}$$

Since $u \in C(\bar{Q})$, these expressions are well-defined.

Lemma 7.7. Let u be a solution of Problem C in which D, ϕ and u_0 satisfy the hypothesis imposed in Section 6. For any $\mu \in (0,1)$

$\underline{f}_\mu(\cdot)$ is nondecreasing on $\mathbb{R}^+$,

and

$\bar{f}_\mu(\cdot)$ is nonincreasing on $\mathbb{R}^+$.

Here $\underline{f}_\mu$ and $\bar{f}_\mu$ are defined by (7.17) and (7.18).

Proof. We prove here only the first assertion and we argue by contradiction. Thus suppose there exist $t_2 > t_1 \geq 0$ such that $\underline{f}_\mu(t_2) < \underline{f}_\mu(t_1)$. Then by Sard's theorem [18], there exists $\mu' \in (\underline{f}_\mu(t_2),\underline{f}_\mu(t_1))$ which is a noncritical value of u. Let

$$x' = \min\{x \in [\chi_\mu(t_2),\zeta_2(t_2)] \mid u(x,t_2) = \underline{f}_\mu(t_2)\}.$$

Since u is continuous and since

$$u(x',t_2) = \underline{f}_\mu(t_2) < \mu' < \underline{f}_\mu(t_1) \leq \mu = u(\chi_\mu(t_2),t_2),$$

there exists $x_2' \in (\chi_\mu(t_2),x')$ such that $u(x_2',t_2) = \mu'$. Then we use [16, Lemma 5.4 and Remark 5.6] to find that there exists a level curve $x = \ell'(t)$, $\ell' \in C[t_1,t_2]$, on which $u = \mu'$. Since $\mu' < \underline{f}_\mu(t_1)$, $\ell'(t_1) < \chi_\mu(t_1)$. Next let $x'' = (\ell'(t_1) + \chi_\mu(t_1))/2$ and let $\mu^* = \max\{u(x,t_1) \mid x \in [\zeta_1(t_1),x'']\}$. Clearly $\mu' \leq \mu^* < \mu$. Then choose again a $\mu'' \in (\mu^*,\mu)$ which is a noncritical value of u. Let $x_2'' \in (\zeta_1(t_2),\chi_\mu(t_2))$ such that $u(x_2'',t_2) = \mu''$. Again there exists a level curve $x = \ell''(t)$, $\ell'' \in C[t_1,t_2]$ on which $u = \mu^*$. Clearly $\ell''(t_1) > x''$. Hence the curves ℓ' and ℓ'' intersect. This gives a contradiction. Therefore $\underline{f}_\mu(t_2) \geq \underline{f}_\mu(t_1)$.

We are now in a position to prove the one-sided bounds for $(\phi(u_x))_x$ across the interfaces.

Theorem 7.8. Let u be a solution of Problem C in which D, ϕ and u_0 satisfy the hypotheses of Theorem 7.4. In addition, let $|u_0'| \leq \nu$ a.e. on $\mathbb{R}$ for some $\nu \in (0,\sigma)$, where σ is the constant given by (1.11). Then

$$(\phi(u_x))_x \geq -K_1(t^{-1} + t^{-1/2}) \quad \text{in} \quad \mathcal{D}'(N_1)$$

and

$$(\phi(u_x))_x \leq K_2(t^{-1} + t^{-1/2}) \quad \text{in} \quad \mathcal{D}'(N_2)$$

where $K_i = K_i(\nu,u_0)$ and where $N_1,N_2 \subset Q$ are neighbourhoods of the interfaces given by (1.12) and (1.13).

Proof. Again we only prove here the first assertion. It is convenient to approximate the solution of Problem C by solutions of a sequence of initial value problems. Let T be a fixed, but arbitrary positive constant. Then, for each $n \in \mathbb{N}$, consider

$$(C_n')\quad \begin{cases} u_t = (D(u)\phi(u_x))_x & \text{in } S_T = \mathbb{R}\times(0,T], \\ u(\cdot,0) = u_{0n}(\cdot) & \text{on } \mathbb{R}. \end{cases}$$

Here $\{u_{0n}\}_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R})$, is chosen such that each u_{0n} has bounded derivatives of all orders, $u_{0n} \in (\frac{1}{n}, 1-\frac{1}{n})$ and $|u_{0n}'| \leq \nu$ on $\mathbb{R}$ and $u_{0n} \to u_0$ as $n \to \infty$ uniformly on $\mathbb{R}$. Moreover, the functions u_{0n} satisfy: if I denotes an interval on which $u_{0n} \in (\delta, 1-\delta)$ for some $\delta \in (0,1/2)$, then $\|u_{0n}''\|_I$ is bounded uniformly in $n \in \mathbb{N}$.

Problem C_n' can be considered as nondegenerate. Therefore we apply standard methods (see [17, Theorem 8.1]) and find that there exists a unique classical solution $u_n \in C^{4+\alpha,2+\alpha/2}(\bar{S}_T)$ such that $u_n \in (\frac{1}{n}, 1-\frac{1}{n})$ and $|u_{nx}| \leq \nu$ on $\bar{S}_T$. Moreover, each solution u_n also satisfies the interior estimate: if $(x_0,t_0) \in S_T$ such that $u_n \in (\delta, 1-\delta)$ for some $\delta \in (0,1/2)$, then

$$|(\phi(u_{nx}))_x(x_0,t_0)| \leq K\, t_0^{-1/2} \tag{7.19}$$

where $K = K(\delta,\nu,u_0)$, independent of n.

Finally, the sequence of solutions $\{u_n\}$ convergences pointwise to the solution of Problem C on Q. This convergence is uniform on compact subsets of $\bar{Q}$.

Let

$$x_0 = \chi_{s_0/2}(0) \qquad \text{where } D'(s_0) = 0.$$

Then $u(x_0,0) = s_0/2$ and by (7.2) there exists $\Delta = s_0^2/16L^2$ such that $u(x_0,t) \in (s_0/4, 3s_0/4)$ for $t \in [0,\Delta]$. Let

$$S_1 = \{(x,t) \in Q \mid -\infty < x < x_0,\ 0 < t \leq \Delta\}.$$

Since $u_n \to u$ uniformly on compact subsets of $\bar{Q}$ and since $u_{0n} \to u_0$ uniformly on $\mathbb{R}$, we have for sufficiently large $n \in \mathbb{N}$ that $u_n \leq 7s_0/8$ on ∂S_1, the parabolic boundary of S_1. A maximum principle argument then gives

$$u_n \leq \frac{7}{8}s_0 \quad \text{on } \bar{S}_1, \tag{7.20}$$

for all $n \in \mathbb{N}$ sufficiently large. Moreover for large $n \in \mathbb{N}$ and for all $t \in (0,\Delta]$, $u_n(x_0,t) \in (s_0/8, 7s_0/8)$ and thus

$$|(\phi(u_{nx})_x(x_0,t)| \leq K\, t^{-1/2}. \tag{7.21}$$

If we apply the proof of Theorem 7.4, without truncation in time, to Problem C_n' we obtain (see also Corollary 7.5)

$$|(\phi(u_{nx}))_x| \leq K(n) \qquad \text{in } \bar{S}_T \tag{7.22}$$

where K(n) are positive constants satisfying $K(n) \to \infty$ as $n \to \infty$. Define the comparison function

$$z(x,t) = -\min\left\{K(n), \bar{K}(t^{-1} + t^{-1/2})\right\} \qquad (x,t) \in \bar{S}_T. \tag{7.23}$$

where n is chosen sufficiently large, so that (7.20) and (7.21) hold, and where $\bar{K} \geq K$ is a constant to be chosen later (here K is the constant from (7.21)). Then

$$(\phi(u_{nx}))_x \geq z \qquad \text{on } \partial S_1. \tag{7.24}$$

Further introduce the operator

$$\mathcal{L}(w) := Aw_{xx} + Bw_x + D'(u)\left(2\,\frac{u_x\phi''(u_x)}{\phi'(u_x)} + 3\right)w^2$$

$$+ D''(u)\left\{2\phi(u_x)u_x + \phi'(u_x)u_x^2 + \frac{\phi''(u_x)\phi(u_x)u_x^2}{\phi'(u_x)}\right\} w - w_t,$$

where $A = D(u)\phi'(u_x)$

$B = D(u)\phi''(u_x)u_{xx} + 3D'(u)\phi'(u_x)u_x + D'(u)\phi(u_x)$

and where in the coefficients u denotes u_n: the subscript n is left

out for convenience.

Then

$$\mathcal{L}((\phi(u_{nx}))_x) = -D'''(u_n)\phi(u_{nx})u_{nx}^3 \qquad \text{in } S_1.$$

By (7.20), $D'(u_n) > 0$ on S_1. Since $D'''D' \geq 0$ and $\phi(s)\cdot s > 0$ it follows that

$$\mathcal{L}((\phi(u_{nx}))_x) \leq 0 \qquad \text{in } S_1.$$

Next we compute $\mathcal{L}(z)$ on S_1. First observe that $|u_{nx}| \leq \nu < \sigma$ and that $u_n \leq 7s_0/8$ on $\bar{S}_1$. Therefore, by the choice of σ, the coefficient of w^2 is strictly positive and the coefficient of w is strictly negative. Let

$$\bar{c} = D'(7s_0/8) \min_{s\in[0,\nu]} \left(2\,\frac{s\phi''(s)}{\phi'(s)} + 3\right).$$

Clearly, $\bar{c}$ is a positive lower bound for the coefficient of w^2. Then in (7.23) we choose $\bar{K} = \max\{K, 1/\bar{c}\}$. An easy computation then shows that $\mathcal{L}(z) > 0$ when $z = -K(n)$ or when $z = -\bar{K}(t^{-1} + t^{-1/2})$. The maximum principle and the boundary estimate (7.24) give

$$(\phi(u_{nx}))_x \geq z \qquad \text{in } \bar{S}_1. \tag{7.25}$$

for all sufficiently large $n \in \mathbb{N}$.

Next we continue for $t \in [\Delta, 2\Delta]$. Let $x_\Delta = \chi_{s_0/2}(\Delta)$ and let

$$S_2 = \{(x,t) \in Q \mid -\infty < x < x_\Delta,\ \Delta < t \leq 2\Delta\}.$$

Again $u(x_\Delta, t) \in (s_0/4, 3s_0/4)$ for $\Delta \leq t \leq 2\Delta$.

Suppose $x_\Delta \leq x_0$. Then by the above result $(\phi(u_{nx}))_x \geq z$ on ∂S_2 and by the maximum principle $(\phi(u_{nx}))_x \geq z$ in $\bar{S}_2$ for all sufficiently large $n \in \mathbb{N}$. Next suppose that $x_\Delta > x_0$. We now use Lemma 7.7 to show that $u(x,\Delta)$ has a positive lower bound on $[x_0, x_\Delta]$ which does not depend on $t = \Delta$. Consider the point $(\chi_{s_0/4}(\Delta), \Delta)$. Since $u(x_0,\Delta) \in (s_0/4, 3s_0/4)$, we have $\chi_{s_0/4}(\Delta) < x_0$. Thus $u(x,\Delta) \geq \underline{f}_{s_0/4}(\Delta) \geq \underline{f}_{s_0/4}(0)$ for $x \in [x_0, x_\Delta]$ (from Lemma 7.7). By definition of x_Δ, also $u(x,\Delta) \leq s_0/2$ for $x \in [x_0, x_\Delta]$. Thus $(\phi(u_{nx}))_x \geq z$ on ∂S_2 for $n \in \mathbb{N}$ large enough, where in z the constant $\bar{K} \geq \max\{K, 1/\bar{c}\}$ is chosen sufficiently large. By the maximum principle

$$(\phi(u_{nx}))_x \geq z \qquad \text{in } \bar{S}_2 \tag{7.26}$$

for all sufficiently large $n \in \mathbb{N}$. We can now repeat this argument with the same constant $\bar{K}$ in the function z on half strips of the form

$$S_i = \{(x,t) \in Q: -\infty < x < x_{(i-1)\Delta},\ (i-1)\Delta < t \leq i\Delta\}$$

where $i = 3,4,\ldots,[T/\Delta]$.

Finally, let $\xi \in C_0^\infty(N_1)$ and $\xi \geq 0$. Then $\operatorname{supp} \xi \subset S_T$ with $T = k\Delta$ and $k \in \mathbb{N}$ sufficiently large. Since $u \in (s_0/4, 3s_0/4)$ on the lateral boundaries of S_i, we have $N_1 \cap S_T \subset \bigcup_{i=1}^{k} S_i$. Thus

$$\int_{N_1} \phi(u_{nx})\xi_x - \int_{N_1} \min\{K(n), \bar{K}(t^{-1} + t^{-1/2})\}\xi \leq 0. \tag{7.27}$$

for all $n \in \mathbb{N}$ large enough. From the existence proof in [9] it follows that $\phi(u_{nx}) \to \phi(u_x)$ in $L^2(\operatorname{supp} \xi)$ when $n \to \infty$. Passing to the limit in (7.27) therefore gives

$$\int_{N_1} \phi(u_x)\xi_x - \int_{N_1} \bar{K}(t^{-1} + t^{-1/2})\xi \leq 0.$$

This concludes the proof of the theorem.

Corollary 7.9. There exists a positive constant $K = K(\nu, u_0)$ such that

$$D^+(u)(\phi(u_x))_x \geq -K(t^{-1} + t^{-1/2}) \qquad \text{on } M(u).$$

Proof. Fix $t \in (0,\infty)$ and let $(k-1)\Delta < t \leq \Delta$ for some $k \in \mathbb{N}$ where $\Delta = \min\{s_0^2/16L^2, (1-s_0)^2/16L^2\}$. From the proof of the proceeding theorem the estimate follows when $\zeta_1(t) < x \leq \chi_{s_0/2}((k-1)\Delta)$ and $\bar{\chi}_{s_0/2+1/2}((k-1)\Delta) \leq x < \zeta_2(t)$. On the intermediate interval we use Lemma 7.7 and the interior estimate. Observe, by the choice of Δ, that

$$\underline{\chi}_{s_0/4}(t) < \underline{\chi}_{s_0/2}((k-1)\Delta)$$

and

$$\bar{\chi}_{s_0/4+3/4}(t) > \bar{\chi}_{s_0/2+1/2}((k-1)\Delta).$$

Therefore Lemma 7.7 gives

$$0 < \underline{f}_{s_0/4}(0) \leq u(x,t) \leq \bar{f}_{s_0/4+3/4}(0) < 1$$

for $x \in (\underline{\chi}_{s_0/2}((k-1)\Delta), \bar{\chi}_{s_0/2+1/2}((k-1)\Delta))$. Corollary 7.5 now gives the desired estimate.

8. THE INTERFACE EQUATIONS

We prove in this section two theorems. In the first we derive differential equations for the interfaces and in the second we prove a monotonicity property.

Theorem 8.1. Let u be the solution of Problem C in which D, ϕ and u_0 satisfy the hypotheses of Theorem 7.8. Then, for every $t > 0$ and for $i = 1,2$

$$\lim_{\substack{x \to \zeta_i(t) \\ (x,t) \in M(u)}} u_x(x,t) = u_x(\zeta_i(t),t)$$

and

$$\lim_{h \downarrow 0} \frac{\zeta_i(t+h) - \zeta_i(t)}{h} = D^+\zeta_i(t)$$

exist and

$$D^+\zeta_i(t) = -D'(i-1)\phi(u_x(\zeta_i(t),t)).$$

Proof. Again we consider here only the case $i = 1$. Let $t_0 > 0$ be fixed. It follows from the first estimate of Theorem 7.8 that

$$\phi(u_x)(x,t_0) + K_1(t_0^{-1} + t_0^{-1/2})x$$

is monotone and bounded for $x \in (\zeta_1(t_0), \zeta_1(t_0) + s_0/4)$. Therefore $\lim_{x \downarrow \zeta_1(t_0)} \phi(u_x)(x,t_0)$ exists and this implies that $\lim_{x \downarrow \zeta_1(t_0)} u_x(x,t_0) = u_x(\zeta_1(t_0),t_0)$ exists. Set $\beta = u_x(\zeta_1(t_0),t_0)$ and $\alpha = D'(0)\phi(\beta)$. We shall prove below that for every $\varepsilon \in (0,\varepsilon_0)$ (with ε_0 chosen sufficiently small), there exists a $\delta > 0$ such that for all $t_0 < t < t_0 + \delta$

$$-\varepsilon < \frac{\zeta_1(t) - \zeta_1(t_0)}{t - t_0} + \alpha < +\varepsilon. \tag{8.1}$$

$\underline{\beta > 0}$. Let $\varepsilon > 0$. Set $K = \varepsilon/2D'(0)b$ and $\beta^{\pm} = \beta \pm K$. If ε_0 is chosen sufficiently small, then $\beta^- > 0$ and $\beta^+ < 1$ for all $\varepsilon \in (0,\varepsilon_0)$. Clearly there exists $\delta' > 0$ such that for $x \in (\zeta_1(t_0), \zeta_1(t_0) + \delta')$

$$\beta^- < u_x(x,t_0) < \beta^+$$

and

$$\beta^-(x - \zeta_1(t_0)) < u(x,t_0) < \beta^+(x - \zeta_1(t_0)).$$

Next let $\mu = \varepsilon/2\phi(\beta)c$ and choose $\delta_0 = \min\{\mu,\delta'\}$. Then consider for $-\infty < x \leq \zeta_1(t_0) + \delta_0$ and $t \geq t_0$ the functions

$$\bar{u}(x,t) = \beta^+(x - \zeta_1(t_0) + D'(0)\phi(\beta^+)(t - t_0))^+,$$

and

$$\underline{u}(x,t) = \beta^-(x - \zeta_1(t_0) + D'(\mu)\phi(\beta^-)(t - t_0))^+.$$

By continuity there exists $\delta > 0$, depending on ε, such that

$$\underline{u}(\zeta_1(t_0) + \delta_0,t) < u(\zeta_1(t_0) + \delta_0,t) < \bar{u}(\zeta_1(t_0) + \delta_0,t)$$

for all $t_0 < t < t_0 + \delta$. Introduce the halfstrip

$$S^- = \{(x,t) \in Q \mid -\infty < x < \zeta_1(t_0) + \delta_0,\ t_0 < t < t_0 + \delta\}.$$

One easily verifies that $\underline{u}$ is a subsolution and $\bar{u}$ is a supersolution on S^-. Hence $\underline{u}(x,t) \leq u(x,t) \leq \bar{u}(x,t)$ on $\overline{S^-}$ and thus

$$-D'(0)\phi(\beta^+) \leq \frac{\zeta_1(t) - \zeta_1(t_0)}{t - t_0} \leq -D'(\mu)\phi(\beta^-),$$

for all $t \in (t_0,t_0 + \delta)$.

The desired estimate (8.1) follows now immediately from the choice of μ and $\beta^{\pm}$.

$\underline{\beta = 0}$. In this case we prove that for every small $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t_0 < t < t_0 + \delta$

$$-\varepsilon < \frac{\zeta_1(t) - \zeta_1(t_0)}{t - t_0} \leq 0.$$

The upper bound follows from the monotonicity ζ_1 (see Proposition 6.2). As in the preceeding case, the lower bound results from a comparison function argument.

The following theorem tells us that the interfaces are strictly monotonic after the waiting time.

Theorem 8.2. Suppose ζ_i ($i = 1$ or 2) is constant on some interval $I \subset \mathbb{R}^+$. Then $\inf I = 0$.

Proof. We prove the theorem for $i = 1$. Suppose $t^* = \inf I > 0$. Then there exists $t_0 \in (0,t^*)$ such that ζ_1 is strictly decreasing on (t_0,t^*). Using Proposition 6.3 and Theorem 8.1 this implies

$$\zeta_1'(t) = -D'(0)\phi(u_x(\zeta_1(t),t) < 0 \tag{8.2}$$

for almost every $t \in (t_0,t^*)$. By Theorem 7.8, there exists a constant $K > 0$ such that $u_{xx}(x,t) \geq -K$ for $t \geq t_0$ and $\zeta_1(t) < x < \zeta_1(t) + s_0/4$. Now fix $\tau \in (t_0,t^*)$ such that (8.2) holds and set $\beta = u_x(\zeta_1(\tau),\tau) > 0$. Then for sufficiently large $K^* \geq K$

$$u(x,\tau) \geq \frac{\beta^2}{2K^*}\left(1 - \frac{1}{(\beta/K^*)^2}\left(x - \zeta_1(\tau) - \frac{\beta}{K^*}\right)^2\right)^+, \tag{8.3}$$

for all $x \in \mathbb{R}$.

Next consider the function α as given in Claim 6.5. We easily verify that the rescaled function

$$v(x,t) = p\alpha(qx, pq^2t; qx_0)$$

with $p \in (0,1)$ and $pq \in (0,1)$ also satisfies $Lv \geq 0$ a.e. in Q. Now choose $x_0 = \zeta_1(\tau) + \beta/K^*$, $p = \beta^2/K^*$ and $q = K^*/\beta$. Then $p \in (0,1)$ for K^* sufficiently large and $pq = \beta \in (0,1)$. This and (8.3) imply that $v(x,t-\tau)$ is a subsolution. For the interface ζ_1 this gives

$$\zeta_1(t) \leq \zeta_1(\tau) + \frac{\beta}{K^*} - \frac{\beta}{K^*}\lambda(K^*(t-\tau)).$$

Next choose a fixed $t_1 \in I$, $t_1 > t^*$. Then

$$\zeta_1(t^*) = \zeta_1(t_1) \leq \zeta_1(\tau) + \frac{\beta}{K^*}(1 - \lambda(K^*(t_1 - t^*))). \tag{8.4}$$

Since (8.2) holds at $t = \tau$, we use (5.1) to obtain

$$\zeta_1'(\tau) > -D'(0)a\beta.$$

This inequality, combined with (8.4) and the fact that $\lambda(K^*(t_1 - t^*)) > 1$ gives

$$\zeta_1(t^*) \leq \zeta_1(\tau) + \frac{1}{C}\zeta_1'(\tau)$$

or

$$\left(e^{C\tau}\zeta_1(t^*)\right)' \leq \left(e^{C\tau}\zeta_1(\tau)\right)'$$

for some $C > 0$. Observe that this inequality holds for almost every $\tau \in (t_0,t^*)$. Integration from $t \in (t_0,t^*)$ to t^* gives $\zeta_1(t) \leq \zeta_1(t^*)$. The monotonicity of the interface (Proposition 6.2) then implies $\zeta_1(t) = \zeta_1(t^*)$ for all $t \in (t_0,t^*)$. This is a contradiction. Therefore $t^* = 0$.

REFERENCES

[1] D.G. Aronson, Regularity properties of flows through porous media , SIAM J. Appl. Math., 17 (1969), pp. 461-467.

[2] D.G. Aronson and Ph. Benilan, Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^n$, C.R. Acad. Sci. Paris, 288 (1979), pp. 103-105.

[3] D.G. Aronson, L.A. Caffarelli and S. Kamin, How an initially stationary interface begins to move in porous medium flow, SIAM J. Math. Anal., 14 (1983), pp. 639-658.

[4] D.G. Aronson and J.B. Serrin, Local behaviour of solutions of quasilinear parabolic differential equations, Arch. Rational Mech. Anal., 25 (1967), pp. 81-123.

[5] J. Bear, *Dynamics of Fluids in Porous Media*, American Elsevier Publishing Company, 1972.

[6] L.A. Caffarelli and A. Friedman, Regularity of the free-boundary for one-dimensional flow of gas in a porous medium, Amer. J. Math., 101 (1979), pp. 1193-1218.

[7] J.R. Chan-Hong, C.J. van Duijn, D. Hilhorst and J. van Kester, The interface between fresh and salt groundwater: A numerical study, preprint, Delft (1987).

[8] M.G. Crandall and M. Pierre, Regularizing effects for $u_t + A\Phi(u) = 0$ in L^1, J. Funct. Anal., 45 (1982), pp. 194-212.

[9] C.J. van Duijn and D. Hilhorst, On a doubly nonlinear degenerate equation in hydrology, to appear in Nonlinear Analysis, TMA., 1987.

[10] J.R. Esteban and J.L. Vazquez, On the equation of turbulent filtration in one-dimensional porous media, Nonlinear Analysis, TMA., 10 (1986), pp. 1303-1325.

[11] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[12] B.J. Gilding, Hölder continuity of solutions of parabolic equations, J. London Math. Soc., 13 (1976), pp. 103-106.

[13] M.A. Herrero and J.L. Vazquez, On the propagation properties of a nonlinear degenerate parabolic equation, Comm. P.D.E., 7 (1982), pp. 1381-1402.

[14] D. Hoff, A scheme for computing solutions and interface curves for a doubly-degenerate parabolic equation, SIAM J. Numer. Anal., 22 (1985), pp. 687-712.

[15] De Josselin de Jong, The simultaneous flow of fresh and salt water in aquifers of large horizontal extension determined by shear flow and vortex theory. In: Proceedings of Euromech 143, A. Verruijt & F.B.J. Barends (eds.), Balkema, Rotterdam, 1981.

[16] B.F. Knerr, The porous medium equation in one dimension, Trans. Amer. Math. Soc., 234 (1977), pp. 381-415.

[17] O.A. Ladyženskaja, V.A. Solonnikov and N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, Volume 23, Providence R.I.: American Mathematical Society, 1968.

[18] S. Lang, *Differential Manifolds*, Addison-Wesley Publishing Company, Inc., 1972.

[19] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1967.

[20] J.L. Vazquez, Asymptotic behaviour and propagation properties of the one-dimensional flow of a gas in a porous medium, Trans. Amer. Math. Soc., 277 (1983), pp. 507-527.

CHAPTER 4

Journal of Hydrology, 84 (1986) 55—79
Elsevier Science Publishers B.V., Amsterdam — Printed in The Netherlands

[2]

TRANSVERSE DISPERSION FROM AN ORIGINALLY SHARP FRESH—SALT INTERFACE CAUSED BY SHEAR FLOW

G. DE JOSSELIN DE JONG and C.J. VAN DUIJN

Delft University of Technology, P.O. Box 356, 2600 AJ Delft (The Netherlands)

(Received January 7, 1985; revised and accepted October 5, 1985)

ABSTRACT

De Josselin de Jong, G. and Van Duijn, C.J., 1986. Transverse dispersion from an originally sharp fresh—salt interface caused by shear flow. J. Hydrol., 84: 55—79.

In this paper the influence of transversal dispersion and molecular diffusion on the distribution of salt in a plane flow through a homogeneous porous medium is studied. Since the dispersion depends on the velocity and the velocity on the distribution of salt (through the specific weight) this is a nonlinear phenomenon. In particular for the flow situation considered, this leads to a differential equation which has the character of nonlinear diffusion.

The initial situation (at $t = 0$) is chosen such that the fresh- and salt water are separated by an *interface*, and each fluid has a constant specific weight γ_1 and γ_2, respectively. For this initial situation, the solution of the nonlinear diffusion equation has the form of a similarity solution, depending only on $\zeta/\sqrt{t}$, where ζ denotes the local coordinate normal to the original interface plane and t denotes time.

Properties of this similarity solution are discussed. In particular it is shown how to obtain this solution numerically. The interpretation of these mathematical results in terms of their hydrological significance is given for a number of worked out examples. These examples describe the distribution of salt, as a function of ζ and t, for various flow conditions at the boundaries $\zeta = \pm\infty$. Also examples are given where the molecular diffusion can be disregarded with respect to the transversal dispersion.

INTRODUCTION

When fluids of different densities are present in an aquifer and the density varies in horizontal directions, the fluid motion contains rotation. In the case of a sharp interface this rotation results in a shearflow, which is proportional to the density difference and the interface inclination. Its magnitude was established by Edelman (1940). The rotations and shearflows resulting in the more general case of gradual density variations were treated by De Josselin de Jong (1960).

Specific discharges in an aquifer are accompanied by dispersion. Therefore it can be expected, that at an inclined interface dispersion occurs, which results in changing the abrupt transition from one density to the other, in a

0022-1694/86/$03.50

gradual transition zone. Since dispersion can be described in terms of differential equations (see Bear, 1975) it must be possible to express the spreading from an abrupt interface mathematically in terms of the dispersion parameters.

The governing relations form a coupled system, consisting of equations describing the specific discharge rotations, due to the gradual density variations, and the dispersion, caused by the specific discharge distribution. Solving this system analytically in the general case of arbitrary density distributions and additional superimposed specific discharge distributions seems rather tedious. For practical purposes, therefore, Verruijt (1971) proposed to introduce a new parameter to describe the spreading, without specifying its relation to the dispersion parameters and the density variations.

The purpose of this paper is to show, that it is possible to describe the spreading process analytically, when starting from an abrupt interface in certain simple circumstances, such that the discharge remains parallel and a plane flow situation occurs. For that simplified problem the coupled system can be reduced to one ordinary differential equation of diffusion type. The properties of solutions of this equation are known, because they have been studied extensively from the mathematical standpoint, by Gilding and Peletier (1977), Van Duijn and Peletier (1977) and Van Duijn (1986a).

How these solutions are applied to describe the spreading of density variations from an abrupt interface is shown in this paper.

SIMPLIFIED PROBLEM

In this paper the case is considered of a dispersion zone developing from an originally flat, inclined interface, that extends in all directions to infinity. In order to simplify the analysis, the conditions at the boundaries of the infinite aquifer are assumed to be such, that the flow is constant in planes parallel to the original interface plane.

Let coordinates ξ, η be in the original interface with η horizontal and ξ pointing upwards at an angle α with the horizontal, see Fig. 1. The coordinate ζ is normal to the original interface plane and points upwards. Flow conditions then are such, that the specific discharge components q_ξ, q_η, q_ζ satisfy:

$$q_\eta = q_\zeta = 0 \qquad \partial q_\xi/\partial \xi = \partial q_\xi/\partial \eta = 0 \tag{1}$$

indicating that only q_ξ is nonzero and a function of only ζ and the time t, i.e. $q_\xi = q_\xi(\zeta, t)$.

At time $t = 0$ the interface is assumed to be sharp, such that the plane $\zeta = 0$ separates the aquifer into two regions in which the density of the fluids is a constant. Above it is freshwater with density ρ_1 and below it salt water with density ρ_2. In the description below the specific weight $\gamma = \rho g$ is used instead of density ρ. So in formula the situation is initially:

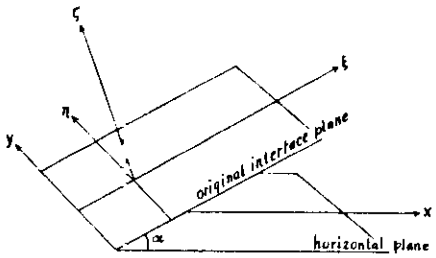


Fig. 1. Interface plane at angle α with horizontal, separating freshwater above it from salt water below it, at time $t = 0$.

$$\left.\begin{aligned} \gamma &= \gamma_1 \quad \text{for} \quad \zeta > 0,\ t = 0 \\ \gamma &= \gamma_2 \quad \text{for} \quad \zeta < 0,\ t = 0 \end{aligned}\right| \tag{2}$$

Under influence of this specific weight difference, initially an Edelman (1940) shear flow $\dot q$ exists at the interface of magnitude:

$$\dot q = (q_{\xi_1} - q_{\xi_2}) = (\kappa/\mu)(\gamma_2 - \gamma_1)\sin\alpha \tag{3}$$

(see e.g. De Josselin de Jong, 1981). In this expression q_{ξ_1}, q_{ξ_2} are the specific discharge components parallel to the interface in the fresh- and salt water regions, respectively. κ is the intrinsic permeability of the aquifer, considered to be a constant, and μ is the dynamic viscosity of the fluids. For fresh- and salt water the viscosity differs by an amount small enough to disregard its influence on the results of the analysis below: for a justification see e.g. Verruijt (1980).

Superimposed on the shear flow an average specific discharge in ξ direction of magnitude $\beta \dot q$ is considered to occur in this paper, with β a number, that remains constant in time. The initial flow conditions are then in accordance with eqn. (3) given by:

$$\left.\begin{aligned} q_{\xi_1} &= (\beta - \tfrac{1}{2})\dot q \quad \text{for} \quad \zeta > 0,\ t = 0 \\ q_{\xi_2} &= (\beta - \tfrac{1}{2})\dot q \quad \text{for} \quad \zeta < 0,\ t = 0 \end{aligned}\right| \tag{4}$$

The following situations are to be distinguished, see Fig. 2:

$\beta < -\tfrac{1}{2}$	both fresh- and salt water flow downward
$\beta = -\tfrac{1}{2}$	the fresh water is stationary
$-\tfrac{1}{2} < \beta < \tfrac{1}{2}$	fresh flows up, salt flows down
$\beta = \tfrac{1}{2}$	the salt water is stationary
$\beta > \tfrac{1}{2}$	both fresh- and salt water flow upwards.

As time proceeds the transition from γ_1 to γ_2, which originally is sharp at $\zeta = 0$, will spread by hydraulic dispersion and molecular diffusion. Salt water

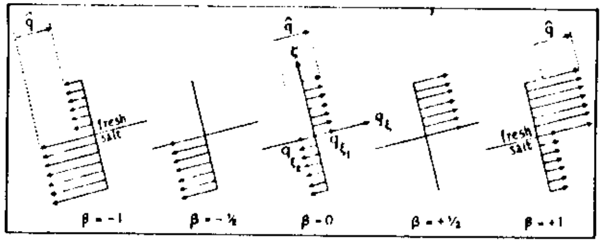


Fig. 2. Distribution of q_ξ at time $t = 0$ for different values of β.

will mix with freshwater, and the specific weight will become a variable function of position and time. Because of the plane character of the case considered, γ will be a function of ζ and t only, i.e. $\gamma = \gamma(\zeta, t)$, which implies that:

$$\partial\gamma/\partial\xi = \partial\gamma/\partial\eta = 0 \tag{6}$$

At infinity the specific weights will tend towards the original values γ_1 at $\zeta \to +\infty$ and γ_2 at $\zeta \to -\infty$.

The mathematical description of the spreading process is developed below from basic equations, describing the changes in specific weight and specific discharge q_ξ in course of time. The boundary conditions at infinity are assumed to be such, that the specific discharges remain constant there and equal to the initial values eqn. (4). Thus:

$$\left.\begin{aligned} \gamma &= \gamma_1, \quad q_\xi = (\beta + \tfrac{1}{2})\hat{q} \quad \text{for} \quad \zeta \to +\infty, t > 0 \\ \gamma &= \gamma_2, \quad q_\xi = (\beta - \tfrac{1}{2})\hat{q} \quad \text{for} \quad \zeta \to -\infty, t > 0 \end{aligned}\right| \tag{7}$$

The result of the analysis will be a distribution of specific discharge and specific weight as functions of ζ and t. An example is shown in Fig. 3 for the

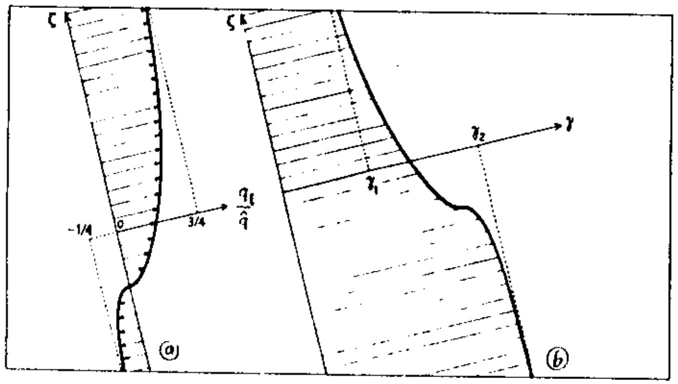


Fig. 3. Distribution of specific discharge (a) and specific weight (b) for the case $m = 0$, $\beta = 1/4$ at some time, $t > 0$.

case $\beta = \frac{1}{4}$ at some later time, $t > 0$. A more detailed description of this figure is given in the sections below.

BASIS EQUATIONS

Flow rule

When an aquifer contains fluids of different specific weight, the flow rule is obtained by considering equilibrium of forces. Let K_q be the force required to act on a unit volume of pore fluid in order to maintain the specific discharge q through a porous medium. According to Darcy's experiments this force equals $K_q = (\mu/\kappa)q$. Let K_p be the force exerted on a unit volume of pore fluid by the gradient in the pressure p and the influence of gravity on the specific weight, γ. This force equals $K_p = -\operatorname{grad} p + \gamma e$, where e is a unit vector in the direction of gravity, i.e. downward. Equating K_q and K_p results in:

$$(\mu/\kappa)q = -\operatorname{grad} p + \gamma e \tag{8}$$

By taking the curl of this relation, the pressure p is eliminated. This leads to:

$$(\mu/\kappa) \operatorname{curl} q = \operatorname{curl} (\gamma e) \tag{9}$$

which, accounting for eqns. (1) and (6), reduces to:

$$(\mu/\kappa)(\partial q_\xi/\partial\zeta) = -(\partial\gamma/\partial\zeta) \sin\alpha \tag{10}$$

Integration of eqn. (10) is possible, because ζ is the only independent variable. This gives:

$$q_\xi = -(\kappa/\mu)\gamma \sin\alpha + \text{constant}$$

where the integration constant can be determined by use of eqn. (7). This gives, taking account of eqn. (3):

$$\left.\begin{aligned} q_\xi &= (\kappa/\mu)(\gamma_1 - \gamma) \sin\alpha + (\beta + \tfrac{1}{2})\hat{q} \\ &\text{or:} \\ q_\xi/\hat{q} &= \beta + (\gamma_1 + \gamma_2 - 2\gamma)/2(\gamma_2 - \gamma_1) \end{aligned}\right| \tag{11}$$

In this equation q_ξ and γ are the variables, which are both functions of ζ and t. It may be remarked here, that eqn. (11) is a linear relationship between these two variables. This is reflected in Fig. 3, where the curves for q_ξ and γ, respectively, are shown to be each others mirror image, when drawn on appropriate scales.

Continuity of fluids

When both fluid and porous medium can be considered to be incompressible, continuity of fluid is satisfied, when div q is zero. Using eqn. (1) it can be verified, that all terms of div q vanish and so continuity of fluid is guaranteed, identically.

Continuity of salt

In stationary groundwater salt is spread by molecular diffusion. In addition, this spreading is enhanced when the fluid moves, because inhomogeneities in the pore space scatter and recombine fluid elements. In the periods of being adjacent, salt is transmitted by molecular diffusion to neighbouring streamlines and carried off in directions deviating from the average flow paths. This process is called mechanical dispersion (see e.g. Bear, 1975).

Averaged over the pore space a salt flux F occurs, which expressed as weight transport per unit time and unit area of the aquifer is:

$$F = -\mathrm{D}\ \mathrm{grad}\ \gamma \tag{12}$$

where D is a second rank tensor. It is the dispersion tensor consisting of terms due to molecular diffusion and mechanical dispersion.

Continuity of salt is satisfied, when the divergence of the exchange flux is balanced by the local rate of change of the specific weight $\partial\gamma/\partial t$ and its convective rate of change $\mathrm{div}(q\gamma)$. When n denotes the porosity of the porous medium, this balance is expressed by:

$$n(\partial\gamma/\partial t) + \mathrm{div}(q\gamma) = -\mathrm{div}\, F = \mathrm{div}(\mathrm{D}\ \mathrm{grad}\ \gamma)$$

Taking account of eqns. (1) and (6) this expression reduces to:

$$n(\partial\gamma/\partial t) = \partial[(nD_{\mathrm{mol}} + \alpha_{\mathrm{T}}|q_\xi|)\ \partial\gamma/\partial\zeta]/\partial\zeta \tag{13}$$

where D_{mol} is the molecular diffusion coefficient and α_{T} is the transverse dispersion length in the direction of ζ (i.e. in the direction perpendicular to q_ξ). A special feature of the dispersion is, that not the specific discharge itself, but its absolute value has to be taken into account.

Equation (13) can be simplified by taking advantage of the linear relationship (11) between γ and q_ξ. This permits to write it in terms of q_ξ only as:

$$n(\partial q_\xi/\partial t) = \alpha_{\mathrm{T}}\ \partial[(\hat{q}m + |q_\xi|)\ \partial q_\xi/\partial\zeta]/\partial\zeta \tag{14}$$

where m is a dimensionless parameter representative for the ratio between the influence of molecular diffusion and mechanical dispersion. It is given by:

$$m = nD_{\mathrm{mol}}/\alpha_{\mathrm{T}}\hat{q} \tag{15}$$

Expressed in these terms the salt flux vector F given by eqn. (12) has only one component F_ζ of magnitude:

$$F_\zeta = +\alpha_{\mathrm{T}}(\gamma_2 - \gamma_1)[(m + |q_\xi|/\hat{q})\ \partial q_\xi/\partial\zeta] \tag{16}$$

Equation (14) has to be solved subject to the initial and boundary conditions, eqns. (4) and (7). The mathematical implications of this system of equations is treated in the next sections.

SIMILARITY TRANSFORMATION

The partial differential equation (14) can be converted into an ordinary differential equation with simple boundary conditions, by subjecting it to the Boltzmann (1894) similarity transformation. This means, that the two variables ζ and t are replaced by the independent similarity variable r according to:

$$r = \zeta/(\alpha_{\mathrm{T}}\hat{q}t/n)^{1/2} \tag{17}$$

Introduction of this variable implies that $\partial/\partial\zeta = (\partial r/\partial\zeta)\mathrm{d}/\mathrm{d}r$ ... etc., so that eqn. (14) is transformed into the following ordinary differential equation:

$$-\tfrac{1}{2}\, r(\mathrm{d}q_\xi/\mathrm{d}r) = \mathrm{d}[(m + |q_\xi|/\hat{q})\ \mathrm{d}q_\xi/\mathrm{d}r]/\mathrm{d}r \tag{18}$$

A solution of eqn. (18), which only depends on r is called a similarity solution of the original equation (14). Let the new variable $w = w(r)$ be defined by:

$$w = q_\xi/\hat{q} \tag{19}$$

Then eqn. (18) reduces to:

$$\tfrac{1}{2}\, r(\mathrm{d}w/\mathrm{d}r) + \mathrm{d}[(m + |w|)\ \mathrm{d}w/\mathrm{d}r]/\mathrm{d}r = 0 \tag{20}$$

This is a nonlinear, ordinary differential equation in which the relevant coefficient has the form $(m + |w|)$. By its dependence on the absolute value of w, this coefficient creates a special nonlinear character of the problem.

The boundary conditions (4) and (7) reduce by introduction of eqns. (17) and (19) to:

$$\left.\begin{array}{lll} w = \beta + \tfrac{1}{2} & \text{for} & r \to +\infty \\ w = \beta - \tfrac{1}{2} & \text{for} & r \to -\infty \end{array}\right| \tag{21}$$

Problem P(m, β)

The problem, defined by the differential equation (20) with the boundary conditions (21), will be referred to as $P(m, \beta)$, indicating that m and β are the essential parameters. Since these parameters are related to the molecular diffusion (see eqn. 15) and to the specific discharge at infinity (see eqn. 7), it is justified physically to assume that they are real numbers, satisfying:

$$0 \leqslant m < \infty \qquad \text{and} \qquad -\infty < \beta < \infty \tag{22}$$

Using the substitutions (17) and (19) it is possible to write the salt flux from eqn. (16) in the form:

$$F_\zeta = (\gamma_2 - \gamma_1)(\alpha_T \dot{q} n/t)^{1/2}[(m + |w|)dw/dr] \tag{23}$$

Application of similarity solution

A solution of $P(m, \beta)$ is called a similarity solution of eqns. (14), (4) and (7). In Fig. 4 it is shown, how such a solution w as a function of r is related to the curves for q_ζ as a function of ζ and t. The heavy curve in Fig. 4a is the solution $w(r)$ of $P(0, \frac{1}{4})$, determined in a manner that is explained in example 3-ii in the section on practical application.

This curve intersects the axis $r = 0$ in $w_0 = 0.372$. This means that according to eqns. (17) and (19) $q_\zeta = 0.372\, \dot{q}$ at $\zeta = 0$ for every time, $t > 0$. Thus, all the curves in Fig. 4b, which represent the distribution of q_ζ over the height ζ at different time $t > 0$, have the intersection with the axis $\zeta = 0$ in common.

The heavy curve in Fig. 4a intersects the axis $w = 0$ in $r_0 = -0.503$. This means according to eqns. (17) and (19) that the plane where the discharge q_ζ is zero, is located in $\zeta_0 = -0.503\ (\alpha_T \dot{q} t/n)^{1/2}$. As a consequence this plane descends with time proportional to $t^{1/2}$ as shown in Fig. 4c. The curves in Fig. 4b all have the same shape as the heavy curve in Fig. 4a but are stretched with a factor, that is proportional to $t^{1/2}$.

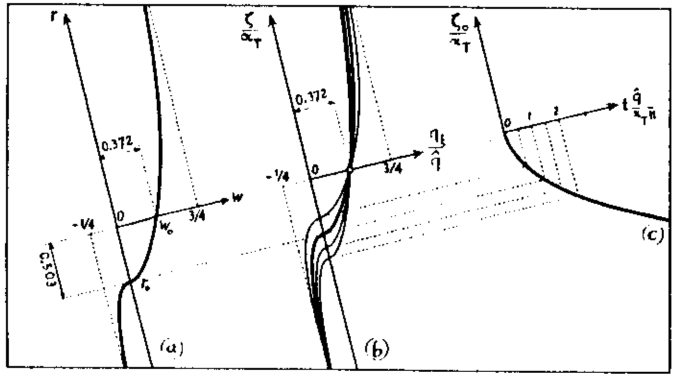


Fig. 4. Relation between a solution $w(r)$ in (a) and the specific discharge q_ζ (ζ, t) in (b). The plane, where $q_\zeta = 0$ descends in course of time according to $\sqrt{t}$, see (c). The curves are for the case $m = 0, \beta = 1/4$.

AUXILIARY PROBLEM Q

The problem $P(m, \beta)$ is not explicitly solvable, because of its nonlinear character, although much is known about its solutions. A special difficulty mentioned already is the occurrence of the absolute value $|w|$, which causes solutions to consist of combinations of parts, where $w > 0$ and $w < 0$.

This difficulty is solved by considering first the auxiliary problem Q, which is defined by the differential equation:

$$\tfrac{1}{2} s(du/ds) + d[u(du/ds)]/ds = 0 \tag{24}$$

with boundary condition:

$$u = 1 \qquad \text{for} \qquad s \to +\infty \tag{25}$$

and the additional condition:

$$0 \leqslant u(s) \leqslant 1 \qquad \text{for} \qquad -\infty < s < +\infty \tag{26}$$

In this section the solution set of this problem Q is considered. In subsequent sections it is shown, that with these solutions it is possible to produce the solutions of $P(m, \beta)$ for all m and β by application of an appropriate rescaling procedure.

Solutions of Q

The solutions required in this paper are given by the family of curves represented in Fig. 5.

Solutions of equations similar to eqn. (24) were studied extensively by Gilding and Peletier (1977), Van Duijn and Peletier (1977), Gilding (1980) and Van Duijn (1986a). They gave rigorous mathematical proofs about existence and uniqueness of solutions and they studied their behaviour. Using ideas developed in these papers, the following basic facts about solutions of Q can be established, see Appendix A.

All the curves from Fig. 5 are strictly increasing, with $du/ds > 0$, at points where $u > 0$. Different curves cannot intersect. Further, all curves approach the upper boundary $u = 1$ as a complementary error function such that:

$$1 - u = O[\mathrm{erfc}\,(+\tfrac{1}{2} s)] \qquad \text{for} \qquad s \to +\infty \tag{27}$$

Expressions for the curves in Fig. 5 cannot be given in closed form. They were constructed with a shooting method, which was executed by using a finite difference approximation of eqn. (24), described in Appendix B.

The curves are indicated by type numbers I and II, such that type I curves are located above the heavy line in Fig. 5, and type II curves below that line. The separation line is called *separatrix*. In the description below a few remarks are inserted on the construction of the curves. These are included for those readers who wish to dispose of more accurate values than can be inferred from Fig. 5.

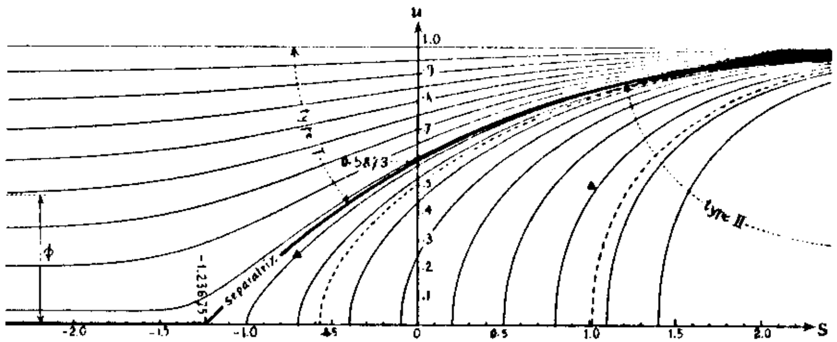


Fig. 5. Family of curves representing solutions of problem Q. Triangles are for example 3-i, where $m = 1/4$, $\beta = 1/4$. Dashed lines are for example 3-ii, where $m = 0$, $\beta = 1/4$.

Type I curves

These remain strictly positive in the entire interval for s, such that u tends to a positive value ϕ as a lower boundary, i.e.:

$$u = \phi \quad \text{with} \quad 0 < \phi < 1 \qquad \text{for} \qquad s \to -\infty \tag{28}$$

This lower boundary is also approached as a complementary error function, in this case such that:

$$u - \phi = O[\text{erfc}\,(-\tfrac{1}{2}s/\phi^{1/2})] \qquad \text{for} \qquad s \to -\infty \tag{29}$$

The numerical procedure for establishing the type I curves is to start from different values u_0 on the vertical axis, $s = 0$. Next u_0', the value of $(du/ds)_0$ in each starting point, is chosen in such a manner that constructing the curve to the right, it reaches the value $u = +1$ asymptotically as a complementary error function according to eqn. (27). Subsequently starting with these values u_0 and u_0' the curves are constructed to the left. The asymptotic value ϕ mentioned in eqn. (28) is established using again the complementary error function approximation, now with eqn. (29).

For every value of ϕ, the corresponding starting values are assembled in Fig. 6, u_0 on the left-hand scale and u_0' on the right-hand scale. For example, the value $\phi = 0.580$ corresponds to $u_0 = 0.8$, $u_0' = 0.132$.

Type II curves

These remain zero for all values of s below a value s_0, specific for each curve, i.e.:

$$u = 0 \qquad \text{for} \qquad -\infty < s \leqslant s_0 \tag{30}$$

At s_0 the curves start on the base line, $u = 0$, with a vertical tangent $u' \to +\infty$, such that:

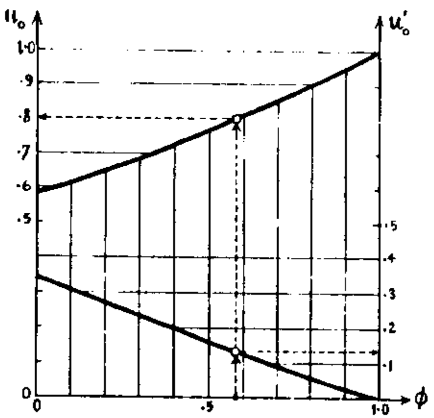


Fig. 6. Startvalues u_0 and $u_0' = (du/ds)$ in $s = 0$ corresponding to different endvalues ϕ for type-I curves.

$$2u(du/ds) = \lambda \qquad u = 0 \qquad \text{in} \qquad s = s_0 \tag{31}$$

with λ a value specific for each curve.

The numerical procedure for establishing the type II curves is to start from different s_0 values on the base line, to choose a λ and to use the series expansion in terms of λ mentioned in Appendix C for small values of $(s - s_0)$. Subsequently, the curves are constructed towards the right, using the finite difference scheme of Appendix B. The asymptotic end value of u is established by use of the complementary error function approximation (eqn. 27). The value of λ is finally chosen in such a manner that the end value equals $u = 1$. The λ values corresponding to different start values s_0 are assembled in Fig. 7.

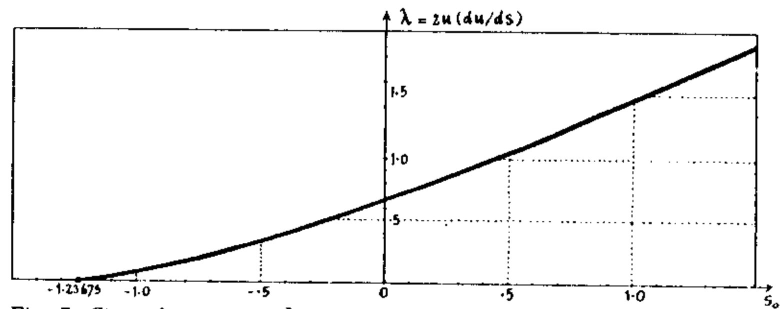


Fig. 7. Startvalues s_0 and $\lambda = 2u(du/ds)$ on $u = 0$ for type-II curves ending in $u = 1$, $s = +\infty$.

Separatrix

The separatrix is the limiting case between the two types of curves. It is a type I curve with ϕ reduced to zero, and a type II curve with λ reduced to zero. The series expansion for small values of $(s - s_0)$ of the separatrix differs from the series applicable to the type II curves (see Appendix C).

PRACTICAL APPLICATION

In this section it is shown how curves of Fig. 5, representing the solutions of problem Q, can be used for solving problem $P(m, \beta)$ in practical situations. It is assumed here, that the relevant hydraulic parameters m and β, defined by eqns. (15), (4) and (7) are known. The situations where $|\beta| \geq \frac{1}{2}$ and $|\beta| < \frac{1}{2}$ are distinguished.

When $|\beta| \geq \frac{1}{2}$ the boundary conditions (eqn. 21) show that either $w(\infty) > 0$ and $w(-\infty) \geq 0$ (case 1) or $w(\infty) \leq 0$ and $w(-\infty) < 0$ (case 2). In both cases the corresponding solution of $P(m, \beta)$ does not change sign on the entire interval $(-\infty, \infty)$. It will be shown that in both cases a solution of $P(m, \beta)$ can be obtained from a solution of problem Q (i.e. a curve from Fig. 5 for that matter) by applying an elementary transformation involving rescaling and displacing the curve in u-direction.

When $|\beta| < \frac{1}{2}$ the boundary conditions (eqn. 21) have opposite sign such that $w(-\infty) < 0$ and $w(-\infty) > 0$ (case 3). Consequently, the solution w changes sign on the interval $(-\infty, \infty)$. This introduces an additional difficulty caused by the absolute value of w in the differential equation (20). Let r_0 be the value of r where the solution vanishes: i.e. $w(r_0) = 0$, then $w(r) < 0$ for $r < r_0$ and $w(r) > 0$ for $r > r_0$. The solution thus consists of two parts ($w > 0$ and $w < 0$). Each part is an appropriate transformed solution of problem Q. They are joined together at r_0 using the continuity of the concentration (or velocity q_t) and the continuity of the salt flux.

The treatment here is aimed to describe the required procedure of rescaling and combining the curves appropriately, without too much mathematical details. For more detailed information the reader is referred to Van Duijn (1986b).

Case 1: $\beta \geq \frac{1}{2}$

When β is larger than $\frac{1}{2}$, both fresh- and salt water flow upwards. When β equals $\frac{1}{2}$, only the freshwater flows upwards while the salt water is stagnant. Both situations are shown in Fig. 2. Now let $u(s)$ be a solution of problem Q and consider for $\sigma > 0$ the transformation:

$$r = \sigma s \tag{32}$$

$$w(r) = \sigma^2 u(r/\sigma) - m \tag{33}$$

This transformation consists of a rescaling (caused by σ) and a displacement (over m) of a relevant curve from Fig. 5. By the transformation the differential equation (24) becomes:

$$\tfrac{1}{2} r\, dw/dr + d[(m + w)\, dw/dr]/dr = 0 \tag{34}$$

because the σ cancels.

Next let $u(s)$ be a solution of problem Q of type I or the separatrix. Then $u(\infty) = 1$ and $u(-\infty) = \phi \geq 0$. Using this in eqn. (33) gives:

$$w(\infty) = \sigma^2 - m \tag{35}$$

and:

$$w(-\infty) = \sigma^2\phi - m \tag{36}$$

Thus when choosing σ such that:

$$\beta + \tfrac{1}{2} = \sigma^2 - m \quad \text{or} \quad \sigma = (\beta + \tfrac{1}{2} + m)^{1/2} \tag{37}$$

and ϕ such that:

$$\beta - \tfrac{1}{2} = \sigma^2\phi - m \quad \text{or} \quad \phi = (\beta - \tfrac{1}{2} + m)/(\beta + \tfrac{1}{2} + m) \tag{38}$$

it follows that the function $w(r)$ satisfies the boundary conditions (eqn. 21). Moreover $w(r) \geq 0$ for all $-\infty < r < \infty$, implies $w(r) = |w(r)|$. Therefore eqn. (34) is identical to eqn. (20). Thus the function $w(r)$, defined according to eqns. (32), (33) and (37), (38) is a solution of $P(m, \beta)$.

Example 1-i: $m = 1$ and $\beta = 0.881$

Then from eqns. (37) and (38) there results $\sigma = 1.543$ and $\phi = 0.580$. Using Fig. 6, the value of ϕ indicates that the relevant curve is a type I curve of Fig. 5 passing through the point $u_0 = 0.8$, $s = 0$ with inclination $u_0' = 0.132$. Transformed with eqns. (32) and (33), the u, s values of this type I curve produce the following w, r values:

$$w = \sigma^2 u - m = 2.381\, u - 1$$

and:

$$r = 1.54\, s$$

These values form the curve of Fig. 8, which is readily verified to satisfy the boundary conditions (eqn. 21). The curve in Fig. 8 resembles a complementary error function. Indeed this kind of function is to be expected as a solution of problem $P(m, \beta)$ in the limit-case where m is large with respect to one.

Example 1-ii: $m = 0$ and $\beta = \frac{1}{2}$

Then the molecular diffusion can be disregarded with respect to the effect of the lateral dispersion and the salt water is stagnant. This situation can be considered as the limit of the case where $\beta > \frac{1}{2}$ and $m > 0$. From eqns. (37)

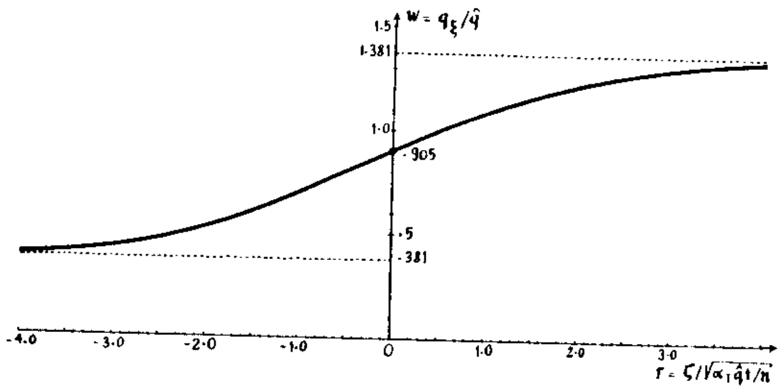


Fig. 8. Similarity solution for $m = 1, \beta = 0.881$.

and (38) it follows that:

$$\sigma = 1 \quad \text{and} \quad \phi = 0 \tag{39}$$

As mentioned above, this value of ϕ indicates that the relevant curve is the separatrix of Fig. 5 and the value of σ shows, that the similarity solution is the undeformed separatrix. Since this simple result produces a clarifying example, the corresponding specific weight and specific discharge distributions are shown in Fig. 9. This figure is to be interpreted in the manner as explained for Figs. 3 and 4.

In Fig. 5 the separatrix is specified by the intersections with the coordinate axes. These are:

$$\left.\begin{array}{ll} s = 0 & u_0 = 0.5873 \\ s_0 = -1.23675 & u = 0 \end{array}\right| \tag{40}$$

These values are reencountered in Fig. 9 in the following manner. Since the scale factor in this case is $\sigma = 1$, see eqn. (39), the s, u values are directly the r, w values, which are related to physical quantities by eqns. (17) and (19).

Using these relations it follows that $s = 0$ corresponds to $\zeta = 0$ for all time t which is the original height of the fresh—salt interface. The first line of eqn. (40) therefore indicates that the specific discharge at the original interface height is constant for all t and equal to $q_\xi = 0.5873\,\hat{q}$, see Fig. 9b.

The second line of eqn. (40) indicates that the depth ζ_0, below which the groundwater is still stationary and the specific weight is not yet reduced, equals $-1.23675\ (\alpha_T \hat{q} t/n)^{1/2}$. This means that this depth increases proportional to root time, see Fig. 9c.

Case 2: $\beta \leqslant -\frac{1}{2}$

When β is smaller than $-\frac{1}{2}$, both fresh- and salt water move downwards. When β equals $-\frac{1}{2}$, only the salt water flows downward while the freshwater

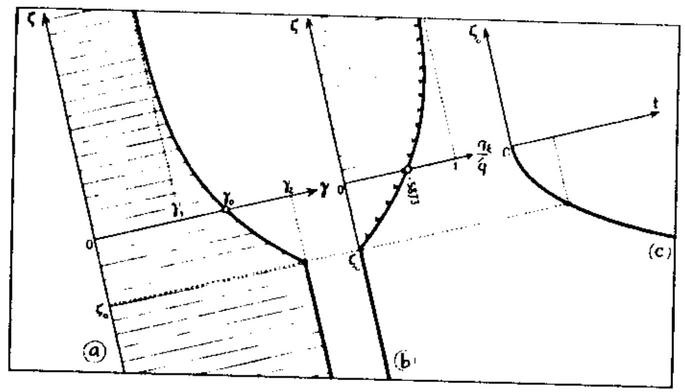


Fig. 9. Example 1-ii, case $m = 0, \beta = 1/2$. Development of brackish zone, when molecular diffusion can be disregarded and the salt water is stationary.

is stagnant. Both situations are shown in Fig. 2. Now the appropriate transformation is:

$$r = -\sigma s \tag{41}$$

$$-w(r) = \sigma^2 u(-r/\sigma) - m \tag{42}$$

Applying this transformation, eqn. (24) becomes:

$$\tfrac{1}{2} r\, dw/dr + d[(m - w)\, dw/dr]/dr = 0 \tag{43}$$

Let again $u(s)$ be of type I or the separatrix. Then eqns. (41) and (42) imply that:

$$-w(-\infty) = \sigma^2 - m \tag{44}$$

and:

$$-w(+\infty) = \sigma^2\phi - m \tag{45}$$

Now σ must be chosen such that:

$$-\beta + \tfrac{1}{2} = \sigma^2 - m \quad \text{or} \quad \sigma = (-\beta + \tfrac{1}{2} + m)^{1/2} \tag{46}$$

and ϕ such that:

$$-\beta - \tfrac{1}{2} = \sigma^2\phi - m \quad \text{or} \quad \phi = (-\beta - \tfrac{1}{2} + m)/(-\beta + \tfrac{1}{2} + m) \tag{47}$$

in order that $w(r)$ satisfies eqn. (21). Moreover, $w(r) \leqslant 0$, implies that $-w(r) = |w(r)|$. Thus eqn. (43) is identical to eqn. (20), showing that $w(r)$ in this case is in fact a solution of $P(m, \beta)$.

Summarizing, for this case the solution of $P(m, \beta)$ again consists of a type I or separatrix curve from Fig. 5. Since both eqns. (41) and (42) contain a minus sign the relevant curve from Fig. 5 is rotated over 180° to produce the similarity solution in the r, w plane.

Case 3: $-\frac{1}{2} < \beta < +\frac{1}{2}$

In this case fresh- and salt water flow in opposite directions (see Fig. 2) and so q_ξ and therefore also w from eqn. (19) change sign in the region of integration. The positive part of the solution is denoted by w^+ and its negative part by w^-. Then w^+ resembles case 1 and w^- resembles case 2.

Before showing the practical elaboration of two examples in the sub-section "use of Fig. 10" below, a few concepts required in the procedure are mentioned here first. The positive part of $w(r)$ is defined according to:

$$w^+(r) = \sigma^{+2}u^+(r/\sigma^+) - m \qquad \text{with} \qquad \sigma^+ = (m + \beta + \tfrac{1}{2})^{1/2} \tag{48}$$

and the negative part by:

$$-w^-(r) = \sigma^{-2}u^-(-r/\sigma^-) - m \qquad \text{with} \qquad \sigma^- = -(m - \beta + \tfrac{1}{2})^{1/2} \tag{49}$$

In eqns. (48) and (49), u^+ and u^- are parts of two different curves from Fig. 5. Because of the minus signs in eqn. (49), the part u^- is rotated over 180° in the rescaling process. By choosing σ according to eqns. (48) and (49), it follows that $w^+(\infty) = \beta + \frac{1}{2}$ and $w^-(-\infty) = \beta - \frac{1}{2}$.

It remains to organize the solution in such a manner that the two parts w^+ and w^- match together at the point where $w = 0$. More precisely, it remains to select r_0 and curves $u^+(s)$ and $u^-(s)$ from Fig. 5 so that the composite function:

$$w(r) = \begin{cases} w^+(r) & \text{for} \quad r > r_0 \\ w^-(r) & \text{for} \quad r < r_0 \end{cases} \tag{50}$$

satisfies certain continuity properties.

In a study of the more general nonlinear partial differential equation $\partial u/\partial t = \partial^2(|u|^{k-1}u)/\partial x^2$ with $k > 1$, it is shown by Van Duijn (1986a) how to join the solutions on the base line, $u = 0$ in his case. He points out, that the fitting conditions are to be deduced from additional physical considerations. In the present problem the condition to be satisfied is that the salt flux should be continuous. Using expression (23) this condition can be written as:

$$2(m + w^+)\mathrm{d}w^+/\mathrm{d}r = 2(m - w^-)\mathrm{d}w^-/\mathrm{d}r = \Lambda \qquad \text{for}$$
$$w^+ = w^- = 0 \quad \text{at} \quad r = r_0 \tag{51}$$

Elaboration of this condition is rather involved, since the relation between Λ and r_0 cannot be determined directly. In order, however, to provide a first approximation of the solution, Fig. 10 is added here, in which Λ and r_0 values are assembled for a relevant region of m, β values.

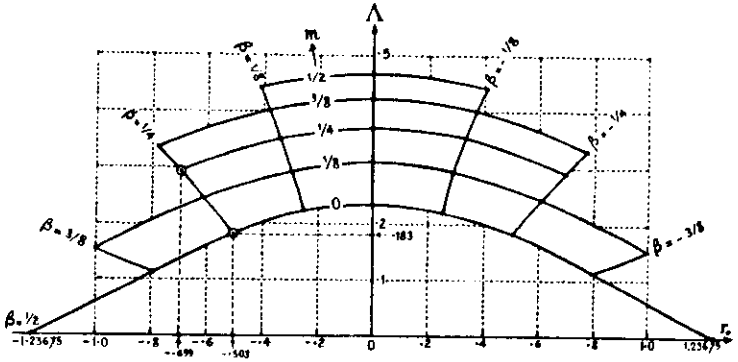


Fig. 10. Values for r_0 and Λ for different m, β combinations.

Use of Figure 10

Example 3-i: $m = \frac{1}{4}$ *and* $\beta = \frac{1}{4}$

The similarity solution in r, w coordinates is the dashed line in Fig. 11. This curve is obtained as follows. From Fig. 10 the appropriate values for this example are found to be $\Lambda = 0.292$, $r_0 = -0.699$. The value of r_0 means that the dashed curve in Fig. 11 intersects the axis $w = 0$ in the point $r = r_0 = -0.699$.

The value of Λ indicates with eqn. (51) that the inclination of the curve is:

$$\mathrm{d}w/\mathrm{d}r = \Lambda/2m = 0.585 \qquad \text{in} \qquad w = 0, \qquad r_0 = -0.699$$

This information is sufficient to construct the two parts of the curve with the finite difference scheme of Appendix B, by extending them from the starting point $r = r_0$, $w = 0$ in both directions up to infinity.

It is also possible to obtain the curve by transforming two curves from Fig. 5. From eqns. (48) and (49) it follows that the scale factors are $\sigma^+ = 1$ for the positive part and $\sigma^- = -(\frac{1}{2})^{1/2} = -0.707$ for the negative part, respectively. Again from eqns. (48) and (49) it can be deduced that the starting points of the corresponding curves in Fig. 5 are:

$$u^+ = \tfrac{1}{4} \qquad \text{and} \qquad s_0^+ = r_0/\sigma^+ = -0.699$$

and:

$$u^- = \tfrac{1}{2} \qquad \text{and} \qquad s_0^- = r_0/\sigma^- = +0.989$$

The points are indicated by two triangles in Fig. 5. It is readily verified, that the upper part of the dashed curve in Fig. 11 is identical to the undeformed curve starting in the left triangle in Fig. 5, undeformed because $\sigma^+ = 1$. The other one is rescaled by $\sigma^- = -0.707$.

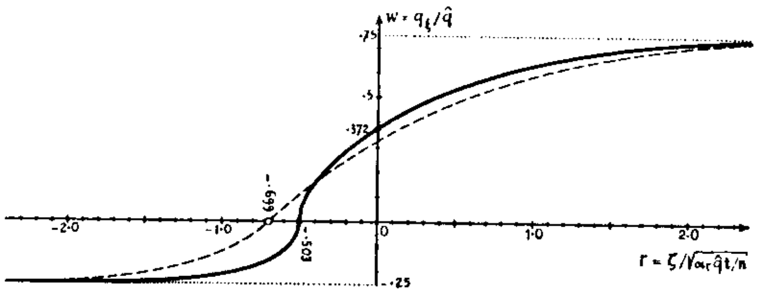


Fig. 11. Similarity solutions for the example 3-i, case $m = 1/4$, $\beta = 1/4$ (dashed line) and example 3-ii, case $m = 0$, $\beta = 1/4$ (full line).

Example 3-ii: $m = 0$ and $\beta = \frac{1}{4}$

The solution in r, w coordinates is the full line in Fig. 11. This curve is obtained as follows. From Fig. 10 the appropriate values for this example are found to be $\Lambda = 0.183$ and $r_0 = -0.503$. The value of r_0 indicates that the full line in Fig. 11 intersects the $w = 0$ axis in the point $r = r_0 = -0.503$.

The value of Λ being nonzero, it follows from eqn. (51) in this case where $m = 0$, that dw^+/dr and dw^-/dr are infinite. The two parts of the curve have a vertical tangent in the intersection point with the axis $w = 0$. They can be constructed, however, by starting on either side of the intersection point with the series expansion of Appendix C because $2w^+dw^+/dr = -2w^-(r)\, dw^-/dr = \Lambda$ is finite and known in this case. Using the finite difference scheme of Appendix B the curve can be extended towards $r \to \pm\infty$.

It is also possible to obtain the curve by transforming two appropriate curves of Fig. 5. From eqns. (48) and (49) it follows that the starting points of the curves are:

$$u^+ = 0 \quad \text{and} \quad s_0^+ = r_0/\sigma^+ = -0.503/(3/4)^{1/2} = -0.581$$

and:

$$u^- = 0 \quad \text{and} \quad s_0^- = r_0/\sigma^- = -0.503/-(1/4)^{1/2} = +1.006$$

The corresponding curves are indicated by dashed lines in Fig. 5.

It may be remarked, that u^+ intersects the axis $s = 0$ in $u^+ = 0.496$. Rescaled with eqn. (48) this point becomes $r = 0, w_0 = 0.372$. This value and $r_0 = -0.503$ mentioned above are reencountered in Fig. 4, which was dicussed in the subsection "Application of Similarity Solution".

ANISOTROPY

From the practical standpoint the case of anisotropy is of importance The results of the analysis presented here are still valid in that case. Only the

coefficients κ for the intrinsic permeability and α_T for the transverse dispersion, have to be adjusted. It was shown (De Josselin de Jong, 1981) that for the case of a soil with a horizontal permeability κ_h deviating from κ_v the vertical permeability, the relation (3) for the shear flow, $\hat{q}$, remains valid. Only κ has to be replaced by κ_s, the intrinsic permeability in the direction of the interface, such that $(1/\kappa_s) = (\cos^2\alpha/\kappa_h) + (\sin^2\alpha/\kappa_v)$.

With respect to the transverse dispersion it may be remarked here, that the custom to attribute α_T a value of one tenth of the longitudinal dispersion length α_L, which is common in these days, is not justified in general. It is certainly an overestimation in the case of anisotropy created by lenses of more permeable material, that are elongated in horizontal direction.

The longitudinal dispersion length in the direction of the lenses may be of the order of the lens lengths and/or their mutual distance. The longitudinal dispersion length perpendicular to the lenses, may be of the size of the lens thickness and/or spacing. But the transverse dispersion may be much smaller because that effect is due to the possibility for the groundwater to exchange salt with fluid elements in neighbouring streamlines. Since the elaboration of streamline patterns and the ensuing exchange possibilities is rather involved, this point is not pursued in detail here. For practical use it may suffice to mention, though, that it is more realistic to envisage a value much smaller than one tenth for the ratio between α_T and α_L.

RECAPITULATION OF THE RESULTS

In the preceding sections, the mixing process of fresh- and salt groundwater due to molecular diffusion and transverse dispersion was discussed. This was done for plane flow under several different conditions. In all cases, the initial distribution (at time $t = 0$) of the specific weight is that the fresh and salt fluids have constant specific weights, γ_1 and γ_2 respectively, and are separated by a sharp interface. The difference is in the specific discharges q_ξ of the two unmixed fluids. These are considered to have a constant value in each of the two regions above and below the interface at $t = 0$, and to keep that same magnitude at infinity both above and below, $\zeta = \pm\infty$, for all later times $t \geq 0$. For $t > 0$ the fluids become more or less mixed and the specific weight γ becomes a function of the local height ζ and the time t. The specific discharge q_ξ changes accordingly because it depends linearly on γ, see eqn. (11).

Because of the plane flow and other simplifying assumptions, the system of partial differential equations, that describes the spreading process can be reduced to the single differential equation (20). This is achieved by introducing the similarity variables w and r. From these, w is related to the specific discharge q_ξ by eqn. (19) and to the specific weight γ by using eqn. (11). The variable r is related to the height ζ and the time t by eqn. (17).

The governing equation (20) has as relevant solutions the family of curves shown in Fig. 5. By rescaling, displacing and combining in various manners the appropriately chosen curves of this family, it is possible to construct the solution for various values of molecular diffusion and flow conditions at infinity. Molecular diffusion in comparison to dispersion is described by the parameter m, see eqn. (15), the flow conditions originally and at infinity by the parameter β, see eqns. (4) and (7).

The differential equation (20) is a nonlinear diffusion equation with $(m + |w|)$ as diffusion coefficient. The absolute value of w in this coefficient is unusual and requires a special treatment when w changes sign in the integration interval. By eqn. (19), this occurs when the fluids flow in opposite directions.

In this paper three cases are considered that differ in the way the two unmixed fluids flow. In the cases 1 and 2 both fluids flow initially and at infinity in the same direction. In case 1, the choice $\beta \geqslant \frac{1}{2}$ guarantees that both fluids flow upwards or only the freshwater flows upwards and the salt water is stagnant ($q_\xi \geqslant 0$). In case 2 ($\beta \leqslant -\frac{1}{2}$), both flow downwards or only the salt water flows downwards and the freshwater is stagnant ($q_\xi \leqslant 0$). In case 3 ($-\frac{1}{2} < \beta < \frac{1}{2}$), the two fluids flow in opposite directions.

When $\beta \geqslant \frac{1}{2}$ or $\beta \leqslant -\frac{1}{2}$, it follows from eqn. (21) that the function w is nonnegative ($w \geqslant 0$) or nonpositive ($w \leqslant 0$), respectively. In these two cases the solution consists of one rescaled and displaced curve of Fig. 5 from the subfamily called type I curves. In the section "Practical Application" two numerical examples (1-i and 1-ii) are elaborated to demonstrate the procedure.

Mathematically of more interest is case 3 where w changes sign according to eqn. (21) and $-\frac{1}{2} < \beta < \frac{1}{2}$. The solution then consists of two rescaled and displaced curves of Fig. 5. The procedure is now more involved because the curves have to be selected in such a manner that they fit together correctly in the point, where $w = 0$. The transition condition is derived from the requirement of continuity of salt flux. In case 3 both curves of type I and type II from Fig. 5 can be required to produce the end result. Examples 3-i and 3-ii show these results explicitly.

In practical situations the flow conditions are in general not as simple as assumed in this study. Plane flow is an exception, the interface is generally not flat and flow is not necessarily parallel to the original interface. However, being an exact solution of a simplified situation, the results of this analysis may be useful for verifying numerical procedures that describe variable density flow with dispersion of a more general purpose character.

ACKNOWLEDGEMENT

The authors are indebted to J. van Kan for a number of clarifying discussions about the numerical approach.

APPENDIX A

In this appendix, some elementary properties of solutions of problem Q are discussed. For convenience, the notation $u' = du/ds$ is being used here.

Uniqueness

Consider eqn. (24). Suppose that at some point s_0 the values:

$$u(s_0) = C_1 \quad \text{and} \quad u'(s_0) = C_2 \tag{A1}$$

are prescribed, where C_1 and C_2 are given constants. Taking $C_1 > 0$, it was shown by Atkinson and Peletier (1971) that for any $-\infty < C_2 < \infty$, there exists a unique solution of eqn. (24), which satisfies the conditions (A1) and which exists on the largest possible interval, where it is positive. A consequence of the uniqueness is monotonicity of solutions of Q.

Monotonicity

Let u be a solution of problem Q and suppose that there exists a point s_0 where:

$$u(s_0) = C_1 \quad \text{with} \quad 0 < C_1 < 1 \quad \text{and} \quad u'(s_0) = 0 \tag{A2}$$

Now observe, that the constant function $\hat{u}(s) = C_1$ for $-\infty < s < \infty$ also satisfies eqns. (24) and (A2). Then the uniqueness requires, that the solutions u and $\hat{u}$ must be identical, which implies that $u(s) = C_1$ for all $-\infty < s < \infty$. This contradicts the boundary condition (25). Therefore the only solution of Q, which is not strictly increasing is the constant $u = 1$. All other solutions with $0 < u < 1$ must satisfy $u' > 0$ in order to satisfy the boundary condition (25).

Intersection

Next it is shown, that two solutions of Q cannot intersect. Let u_1 and u_2 be two solutions of Q and suppose, that there exists an intersection point s_0, where $u_1(s_0) = u_2(s_0)$. By the uniqueness, $u_1'(s_0) \neq u_2'(s_0)$, because otherwise u_1 and u_2 would be identical. Without loss of generality it is assumed here, that $u_1'(s_0) > u_2'(s_0)$. Then two situations can arise:

(1) There exists an other intersection point s_1, such that:

$$u_1(s_1) = u_2(s_1) \quad \text{and} \quad u_1(s) > u_2(s) \quad \text{for} \quad s_0 < s < s_1$$

(2) $u_1(s) > u_2(s)$ for all $s > s_0$ and both solutions satisfy eqn. (25).

Ad (1)

Integration of eqn. (24) with respect to s from s_0 to s_1 gives for u_1 and u_2, respectively:

$$u_1(s_1)u_1'(s_1) - u_1(s_0)u_1'(s_0) + \tfrac{1}{2}s_1u_1(s_1) - \tfrac{1}{2}s_0u_1(s_0) - \tfrac{1}{2}\int_{s_0}^{s_1} u_1(s)\mathrm{d}s = 0 \tag{A3}$$

and:

$$u_2(s_1)u_2'(s_1) - u_2(s_0)u_2'(s_0) + \tfrac{1}{2}s_1u_2(s_1) - \tfrac{1}{2}s_0u_2(s_0) - \tfrac{1}{2}\int_{s_0}^{s_1} u_2(s)\mathrm{d}s = 0 \tag{A4}$$

Subtracting these equations gives:

$$u_1(s_1)[u_1' - u_2'](s_1) - u_1(s_0)[u_1' - u_2'](s_0) - \tfrac{1}{2}\int_{s_0}^{s_1} [u_1(s) - u_2(s)]\mathrm{d}s = 0 \tag{A5}$$

However, by the above assumptions $[u_1' - u_2'](s_1) < 0$, $[u_1' - u_2'](s_0) > 0$ and $\int_{s_0}^{s_1} [u_1(s) - u_2(s)]\mathrm{d}s > 0$. This contradicts the equal sign in eqn. (A5). Therefore case (1) cannot arise.

Ad (2)

A similar argument gives for this case also a contradiction. Thus both (1) and (2) cannot arise and thus no intersection point exists.

Asymptotic behaviour

Next an argument due to Peletier (1970) is used to obtain eqns. (27) and (29). The starting point is the following observation. At points where $u > 0$, eqn. (24) can be written as:

$$\tfrac{1}{2}(s/u)(u^2)' + (u^2)'' = 0 \tag{A6}$$

which can be integrated to give:

$$(u^2)'(s) = (u^2)'(s_0)\exp\left\{-\tfrac{1}{2}\int_{s_0}^{s}[z/u(z)]\mathrm{d}z\right\} \tag{A7}$$

Here s_0 is an arbitrary chosen point, where $u(s_0) > 0$.

Since $u(s) < 1$, it follows from eqn. (A7), that:

$$(u^2)'(s) \leqslant (u^2)'(s_0)\exp\left[-\tfrac{1}{4}(s^2 - s_0^2)\right] \qquad \text{for} \qquad s \geqslant s_0 \tag{A8}$$

Using $u'(s) > 0$ and thus $u(s) > u(s_0)$ for $s > s_0$ in eqn. (A8) gives:

$$0 < u'(s) \leqslant u'(s_0)\exp\left[-\tfrac{1}{4}(s^2 - s_0^2)\right] \qquad \text{for} \qquad s \geqslant s_0 \tag{A9}$$

Integration of eqn. (A9) with respect to s from a point $\bar{s} \geqslant s_0$ to ∞ gives:

$$1 - u(\bar{s}) \leqslant u'(s_0)\exp\left(\tfrac{1}{4}s_0^2\right)\int_{\bar{s}}^{\infty}\exp\left(-\tfrac{1}{4}s^2\right)\mathrm{d}s = u'(s_0)\exp\left(\tfrac{1}{4}s_0^2\right)\sqrt{\pi}\,\mathrm{erfc}\left(\tfrac{1}{2}\bar{s}\right) \tag{A10}$$

where $\mathrm{erfc}\,(x) = (2/\sqrt{\pi})\int_x^{\infty}\exp(-z^2)\mathrm{d}z$.

From eqn. (A10), condition (27) follows when $\bar{s} \to \infty$. A similar argument gives eqn. (29), whenever $\phi > 0$.

APPENDIX B

A finite difference method to obtain the solutions of problem Q as shown in Fig. 5, is discussed here.

The solutions of problem Q were distinguished in type-I curves, type-II curves and a separatrix. In all three cases, first a value of u and $\mathrm{d}u/\mathrm{d}s$ was chosen at a point where u is positive. For curves of type I this was done at $s = 0$ and the appropriate values for $u(0)$ and $\mathrm{d}u/\mathrm{d}s(0)$ were found by a try and error method. For curves of type II and for the separatrix a series expansion was used to approximate the value of u and $\mathrm{d}u/\mathrm{d}s$ at a point s_1 which was chosen sufficiently close to the point s_0 where the solution vanishes, see Appendix C.

Thus for the finite-difference approximation one has to solve an initial value problem of the form:

$$(u^2)'' + su' = 0 \qquad s > \hat{s} \tag{B1}$$

$$u(\hat{s}) = \hat{u} \tag{B2}$$

$$u'(\hat{s}) = \hat{u}' \tag{B3}$$

where eqn. (B1) is a rewritten version of eqn. (24) and where $\hat{s}$, $\hat{u}$ and $\hat{u}'$ are three given numbers such that $\hat{u} > 0$ and $\hat{u}' > 0$.

Let $s_i = \hat{s} + i\Delta s$, where Δs denotes the discretization interval. Integration of eqn. (B1) from s_i to s_{i+1} gives:

$$(u^2)'(s_{i+1}) - (u^2)'(s_i) + \int_{s_i}^{s_{i+1}} su'(s)\mathrm{d}s = 0 \tag{B4}$$

Integrating the third term in eqn. (B4) by parts and setting $u' = p$ gives:

$$2u_{i+1}p_{i+1} = 2u_ip_i + s_iu_i - s_{i+1}u_{i+1} + \int_{s_i}^{s_{i+1}} u(s)\mathrm{d}s \tag{B5}$$

where $u_i = u(s_i)$ and $p_i = u'(s_i)$.

A second equation is needed to solve eqn. (B5) and this can be:

$$u_{i+1} = u_i + \int_{s_i}^{s_{i+1}} p(s)\mathrm{d}s \tag{B6}$$

The integrals in eqns. (B5) and (B6) are approximated by a third-degree Hermite polynomial, see Ralston (1965, p. 60). Using the notation $u'' = q$ this leads to the finite-difference scheme:

$$2u_{i+1}p_{i+1} = 2u_ip_i + s_iu_i - s_{i+1}u_{i+1} + \frac{\Delta s}{2}(u_i + u_{i+1}) + \frac{\Delta s^2}{12}(p_i - p_{i+1}) \tag{B7}$$

and:

$$u_{i+1} = u_i + \frac{\Delta s}{2}(p_i + p_{i+1}) + \frac{\Delta s^2}{12}(q_i - q_{i+1}) \tag{B8}$$

This scheme has a local truncation error of $O(\Delta s^5)$, see Ralston (1965, p. 212), and it is therefore expected that the global accuracy is of $O(\Delta s^4)$.

The values of q at s_i and s_{i+1} are obtained by using the differential equation (B1) at $s = s_i$ and $s = s_{i+1}$. For q_i this gives:

$$q_i = -\frac{p_i^2}{u_i} - s_i\frac{p_i}{2u_i} \tag{B9}$$

Substituting this expression and the corresponding expression for q_{i+1} into eqn. (B8) yields:

$$u_{i+1} = u_i + \frac{\Delta s}{2}(p_i + p_{i+1}) - \frac{\Delta s^2}{12}\left(\frac{p_i^2}{u_i} + s_i\frac{p_i}{2u_i} - \frac{p_{i+1}^2}{u_{i+1}} - s_{i+1}\frac{p_{i+1}}{2u_{i+1}}\right) \tag{B10}$$

Thus for given values of s_i, s_{i+1}, u_i and p_i, the two nonlinear equations (B7) and (B10) have to be solved to obtain u_{i+1} and p_{i+1}. However, because of the nonlinearity this cannot be done directly and the following iteration process was used to obtain approximate values for u_{i+1} and p_{i+1}. Let $u_{i+1}^{(0)} = u_i$. Substituting this value into eqn. (B7) and solving the resulting linear equation for p_{i+1}, gives the approximate value $p_{i+1}^{(0)}$.

This value in turn is substituted into eqn. (B10). The resulting quadratic equation in u_{i+1} can be solved directly to obtain the value $u_{i+1}^{(1)}$ as a next approximation for u_{i+1}. Since this process converges rapidly only a few of these iteration steps were needed at each value of $i = 0, 1, 2, \ldots$

APPENDIX C

In this appendix an approximation is considered of curves of type II and the separatrix from Fig. 5 for small values of u. This approximation has the form of a series expansion which solves the initial value problem defined by the differential equation (24) for $s > s_0$ and the initial conditions (eqn. 31).

Type-II curves

For type-II curves, $\lambda > 0$ and a series expansion of the form:

$$\bar{u}(s) = [\lambda(s-s_0)]^{1/2} + a(s-s_0) + b(s-s_0)^{3/2} + c(s-s_0)^2 \quad \text{(C1)}$$

is chosen as an approximation in a sufficiently small neighbourhood of s_0. Clearly, any approximation of the form (C1) satisfies eqn. (31). The values of the constants a, b and c are obtained by substituting eqn. (C1) into the differential equation (24). This gives the following set of equations:

from the term with $(s-s_0)^{-1/2}$: $2a\sqrt{\lambda} = -\frac{2}{3}s_0\sqrt{\lambda}$

from the term with $(s-s_0)^0$: $a^2 + 2b\sqrt{\lambda} = -\frac{1}{2}as_0$

from the term with $(s-s_0)^{1/2}$: $2ab + 2c\sqrt{\lambda} = \frac{2}{3}\lambda - \frac{2}{5}bs_0 - \frac{4}{5}\sqrt{\lambda}$

from the term with $(s-s_0)$: $b^2 + 2ac = \frac{1}{3}c - \frac{2}{3}a$

from the term with $(s-s_0)^{3/2}$: $2bc = -\frac{4}{7}b$

from the term with $(s-s_0)^2$: $c^2 = -\frac{1}{2}c$

Thus for $\lambda > 0$, the choice:

$$a = -\tfrac{1}{3}s_0 \qquad b = s_0^2/36\sqrt{\lambda} \qquad c = (s_0^3/270\lambda) - (1/15) \quad \text{(C2)}$$

guarantees, that the approximation (C1) satisfies equation (24) up to terms with $(s-s_0)^{1/2}$.

The first point on the curve (u_1, s_1) is chosen in such a manner, that $(s_1 - s_0)$ is a sufficiently small number for approaching the value of u_1 by the series expansion. The value of u'_1, which represents (du/ds) in the first point, is taken from the derivative of the series expansion. The values u_1, u'_1, s_1 are the starting values for establishing the rest of the curve upwards and to the right by applying the finite-difference procedure of Appendix B. The parameter to adjust in this case is λ from eqn. (31).

Separatrix

For the separatrix, an approximation of the form (C1) is chosen with $\lambda = 0$ and $s_0 < 0$. Then the following two sets of values for a, b and c occur:

$$a = 0 \qquad b = 0 \qquad c = 0 \quad \text{(C3)}$$

and:

$$a = -\tfrac{1}{2}s_0 \qquad b = 0 \qquad c = -s_0/(3s_0 + 1) \quad \text{(C4)}$$

The set of values (C3) leads to an approximation, which is identically zero. This solution corresponds to the part of the separatrix in the region $s \leqslant s_0$. In the region $s > s_0$, the set (C4) gives:

$$\bar{u}(s) = -\tfrac{1}{2}s_0(s-s_0) - s_0(s-s_0)^2/(3s_0+1) \quad \text{(C5)}$$

and this approximation satisfies eqn. (24) up to terms with $(s-s_0)^{3/2}$. To obtain numerically the solution, that starts at this value of s_0 an identical procedure as for the case of type-II curves was used. The parameter to adjust in this case is s_0. It is found, that for $s_0 = -1.23675$ the corresponding solution reaches the value $u = +1$ asymptotically, when $s \to +\infty$.

REFERENCES

Atkinson, F.V. and Peletier, L.A., 1971. Similarity profiles of flow through porous media. Arch. Ration. Mech. Anal., 42: 363–379.

Bear, J., 1975. Dynamics of Fluids in Porous Media. Elsevier, New York, N.Y., 2nd ed.

Boltzmann, L., 1894. Zur Integration der Diffusionsgleichung bei variabeln Diffusionscoefficienten. Ann. Phys. Chem., 53: 959–964.

De Josselin de Jong, G., 1960. Singularity distributions for the analysis of multiple fluid flow through porous media. J. Geophys. Res., 65 (11): 3739–3758.

De Josselin de Jong, G., 1981. The simultaneous flow of fresh and salt water in aquifers of large horizontal extension determined by shear flow and vortex theory. In: A. Verruijt and F.B.J. Barends (Editors), Proc. Euromech. 143, Balkema, Rotterdam, pp. 75–82.

Edelman, J.H., 1940. Strooming van zoet en zout grondwater. Rapport 1940 inzake de watervoorziening van Amsterdam, Bijl. 2, pp. 8–14.

Gilding, B.H., 1980. On a class of similarity solutions of the porous media equation III. J. Math. Anal. Appl., 77: 381–402.

Gilding, B.H. and Peletier, L.A., 1977. On a class of similarity solutions of the porous media equation II. J. Math. Anal. Appl., 57: 522–538.

Peletier, L.A., 1970. Asymptotic behaviour of temperature profiles of a class of nonlinear heat conduction problems. Q. J. Mech. Appl. Math., 23: 441–447.

Ralston, A., 1965. A First Course in Numerical Analysis. McGraw-Hill, New York, N.Y.

Van Duijn, C.J., 1986a. On a class of similarity solutions of the porous media equation. (In prep.)

Van Duijn, C.J., 1986b. A mathematical analysis of density dependent dispersion in fresh–salt groundwater flow. Proc. 9th Salt Water Intrusion Meeting, Delft. (In prep.)

Van Duijn, C.J. and Peletier, L.A., 1977. A class of similarity solutions of the nonlinear diffusion equation. Nonlinear Anal., Theory, Methods, Appl., 1: 223–233.

Verruijt, A., 1971. Steady dispersion across an interface in a porous medium. J. Hydrol., 14: 337–347.

Verruijt, A., 1980. The rotation of a vertical interface in a porous medium. Water Resour. Res., 16: 239–240.