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CONTROL OF ROBOT ARM WITH ELASTIC JOINTS VIA NONLINEAR DYNAMIC FEEDBACK.

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AN APPLICATION OF NONLINEAR MODEL MATCHING TO THE DYNAMIC CONTROL OF ROBOT ARM WITH ELASTIC JOINTS

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ABSTRACT

The paper is concerned with dynamic control of robot arm with non negligible joint elasticity. In this case nonlinear control techniques based on decoupling and nonlinearity compensation via *static* state-feedback cannot be applied for the most common robot structures. This is because the mathematical model associated with these structures is such that the necessary and sufficient conditions for the existence of the decoupling control laws fail to hold. Motivated by these facts, the problem of using a *dynamic* state-feedback is considered. The main purpose of the paper is to show that one can design a dynamic state-feedback compensator, which makes the external behavior of the controlled robot identical to the one of a prescribed decoupled linear model. The particular example considered is a planar robot arm with two elastic joints.

1. INTRODUCTION

The problem of decoupling the dynamics of robot arms via static state-feedback has received large attention in the last years and is an appealing approach for achieving a complete control of the external behavior of an anthropomorphic manipulator.

For robots with rigid joints this control technique has been labeled in a number of different ways in the past [1] and it has been shown that the resulting control is robust [2]; moreover the nonlinear decoupling has the nice property that it completely linearizes the state dynamics [3].

For robots with elastic transmission between actuators and arms, as belts or harmonic drives, the effect of joint elasticities is such that the decoupling technique cannot be applied for the most common robot mechanical structures. This is because the mathematical model associated with these structures is such that the necessary and sufficient conditions for the existence of a decoupling law fail to hold in most cases. There are special mechanical structures for which such conditions hold [4]; however, in these cases, the computation of the relevant control laws is quite cumbersome. In alternative, an approximate decoupling may be obtained using a controller based on singular perturbation theory [5].

In this paper we show that exact decoupling can be achieved for robot with elastic joints via the nonlinear model matching theory [6]. In this case the resulting control law is a dynamic state-feedback which makes the external behavior of the controlled robot identical to the one of a prescribed decoupled linear model. Moreover we show that the system graph representation [7] gives a fruitful insight into the model structure, thus allowing reduction of the dynamic order of the controller and complete linearization of the state dynamics.

2. MATHEMATICAL MODEL OF ELASTIC ROBOTS AND STATIC DECOUPLING TECHNIQUE

The mechanical structure of a robot is constituted by N-1 bodies interconnected through N joints. The body between two joints is called a link. The joints are activated by motors with transmission gears or belts; when the links and the transmissions are assumed to be rigid the dynamical behavior is that of a chain of N rigid bodies. In this case the equation of motion in matrix form is

$$\ddot{q} = B(q)^{-1}\left[m(t) + e(q) + c(q,\dot{q})\right] \qquad (1)$$

where q is the N-vector of joint variables giving the relative displacement between two adjacent links, B(q) is the N × N inertial matrix, m(t) is the N-vector of generalized forces delivered by the motors, e(q) is the N-vector of conservative forces and $c(q,\dot{q})$ is the N-vector collecting centrifugal and Coriolis forces.

When the transmissions are not rigid the N actuating bodies of the motors are elastically coupled to the driven links; therefore the dynamical behavior is that of 2N rigid bodies with 2N constraints, N of which include elasticity; this is the case of interest here. The equation of motion in matrix form is still eq.(1), but with the following peculiarities:

- the number of second order equations is 2N;
- q is a 2N-vector in which q_{2i} denotes the displacement of link i w.r.t. link i-1 and q_{2i-1} denotes the displacement of the driving body of joint i w.r.t. link i-1, for i=1,...,N;
- B(q) is the 2N × 2N inertial matrix of the 2N rigid bodies;
- e(q) and $c(q,\dot{q})$ are 2N-vectors and e(q) includes the effects of elasticity;
- m(t) is a 2N-vector with the even components equal to zero.

Starting from mechanical parameters, the model (1) is given automatically by the DYMIR code [8] both for rigid and elastic robots. Eq.(1) may be rewritten in the standard form

$$\dot{x} = f(x) + g(x)\,u \qquad\qquad (2)$$
$$y = h(x)$$

with state $x = \left[x_p^T\ x_v^T\right]^T = \left[q^T\ \dot{q}^T\right]^T \in X = R^n$, input $u \in R^m$ and output $y \in R^\ell$. In the elastic case $n = 4N$, $m = \ell = N$. The vector f and the m columns $g_1,\dots,g_m$ of the matrix g are smooth vector fields defined on an open subset of R^n; the expressions for f and g are given by:

$$f(x)=\begin{bmatrix} x_v \\ -B(x_p)^{-1}\left[c(x_p,x_v)+e(x_p)\right]\end{bmatrix},\quad g(x)=\begin{bmatrix} 0 \\ B(x_p)^{-1}\mathrm{diag}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\end{bmatrix}$$

Moreover, in (2) h is a smooth vector-valued function; in our case the output y may be defined as the vector of link displacements $x_{2i} = q_{2i}$ (i = 1,...,N). The input u collects only the nonzero components of m(t).

An attractive approach to the control of nonlinear systems is the one based on decoupling state-feedback laws. The *static* state feedback decoupling problem is defined as follows; given a system in the form (2), find a feedback $\alpha(x)$ and a state dependent change of coordinates $\beta(x)$ in the input space such that the closed-loop system formed by the composition of (2) with

$$u = \alpha(x) + \beta(x)\, v \tag{3}$$

has the i-th output dependent only on the i-th component of the new input.

The problem is now well understood [9], [10], [11]; the basic concepts of decoupling theory are summarized in what follows. Given a vector field f on X and smooth function h, recall that the Lie derivative of h in the direction of the field f is the function

$$L_f h = \left(\frac{\partial h}{\partial x_1} \cdots\cdots \frac{\partial h}{\partial x_n}\right) f \ .$$

The k-th order Lie derivative is $L_f^k h = L_f(L_f^{k-1} h)$ while $L_f^0 h = h$.

Definition 1. The characteristic number ρ_j associated with the output y_j is the largest integer such that for all $k < \rho_j$

$$L_{g_i} L_f^k h_j = 0 \quad , \quad \forall i \in \{1,\ldots,m\} \ . \quad \square$$

In the nondegenerate case ($\rho_j \neq \infty$, $\forall_j$) we have:

Definition 2. The decoupling matrix $A(x)$ associated with the system (2), with $m=\ell$, is the $\ell \times \ell$ matrix with smooth entries

$$a_{ji}(x) = L_{g_i} L_f^{\rho_j} h_j \ . \quad \square$$

The main result of static state-feedback decoupling theory is given by:

Theorem 1. [9] A necessary and sufficient condition for the existence of (α,β) which solves the decoupling problem is that the decoupling matrix $A(x)$ is nonsingular. $\square$

When Theorem 1 applies, a possible decoupling control is given by (3) with:

$$\begin{aligned} \alpha(x) &= -A(x)^{-1}\, \mathrm{col}\{L_f^{(\rho_1+1)} h_1, \ldots, L_f^{(\rho_1+1)} h_1\} \\ \beta(x) &= \ A(x)^{-1} \end{aligned} \tag{4}$$

It has been shown [3], [7] that for the rigid robot the corresponding matrix $A(x)$ is nonsingular so that (4) provides a solution to the decoupling problem. On the contrary, in the elastic case considered here the matrix $A(x)$ is in general singular [4], except for particular mechanical structures. We will see in Section 5 that the condition of Theorem 1 fails to hold for the most common robot structures. As a consequence, static state-feedback is no longer sufficient in order to solve the decoupling problem when joint elasticity is taken into account.

3. MODEL MATCHING VIA DYNAMIC STATE FEEDBACK

In this section we summarize some recent results concerned with the use of dynamic feedback compensation in order to match a prescribed input-output behavior and we find, as a byproduct, a sufficient condition for decoupling via dynamic feedback.

A *dynamic* state-feedback is a control mode in which the inputs $u_1,\dots,u_m$ are related to the state x of the process (2) and to other input variables $v_1,\dots,v_\mu$ by means of equations of the form

$$\dot{\xi} = a(x,\xi) + b(x,\xi)\, v$$
$$u = c(x,\xi) + d(x,\xi)\, v. \qquad (5)$$

These equations characterize a dynamical system - the state-feedback compensator - whose state ξ evolves on an open subset of R^ν. The $\nu\times 1$ vector $a(x,\xi)$, the $\nu\times\mu$ matrix $b(x,\xi)$, the $m\times 1$ vector $c(x,\xi)$ and the $m\times\mu$ matrix $d(x,\xi)$ have entries which are smooth functions defined on an open subset of $R^n\times R^\nu$.

The composition of (2) with (5) defines a new dynamical system with inputs $v_1,\dots,v_\mu$, outputs $y_1,\dots,y_\ell$ and state $\hat{x}=(x,\xi)$ given by equations of the form

$$\dot{\hat{x}} = \hat{f}(\hat{x}) + \hat{g}(\hat{x})\, v$$
$$y = \hat{h}(\hat{x}) \qquad (6)$$

with

$$\hat{f}(x,\xi) = \begin{bmatrix} a(x,\xi) \\ f(x)+g(x)c(x,\xi) \end{bmatrix} \quad \hat{g}(x,\xi) = \begin{bmatrix} b(x,\xi) \\ g(x)d(x,\xi) \end{bmatrix}$$

$$\hat{h}(x,\xi) = h(x)$$

We shall see that dynamic compensation enables us to match, if some conditions are verified, a prescribed behavior between inputs and outputs in the composed (closed-loop) system (6). In particular, we will be able to obtain a response of the form

$$y(t) = Q(t,(x^\circ,\xi^\circ)) + \int_0^t W_M(t-\tau)\, v(\tau)\, d\tau \qquad (7)$$

where $W_M(t)$ is the impulse-response of a fixed linear model

$$\dot{z} = Az + Bv$$
$$y = Cz$$

On the first term $Q(t,(x^\circ,\xi^\circ))$ of the right-hand-side of (7), which corresponds to the zero-input response (and clearly depends - possibly in a nonlinear manner - on the initial state), we do not impose at this point any particular constraint.

If the process (2) and the compensator (5) were linear, then a well known necessary and sufficient condition for the existence of solutions to this problem would be the one based on the comparison of the behavior of the transfer functions of the process and of the model as $s\to\infty$ [12]. We recall that the behavior of a strictly proper transfer function W(s) for $s\to\infty$ is fully described by the so-called Smith-McMillan factorization at the infinity [13]

$$W(s) = Q(s)\ \Lambda(s)\ P(s)$$

in which Q(s) and P(s) are biproper rational matrices (i.e. proper rational matrices with an inverse which is also proper) and

$$\Lambda(s) = \mathrm{diag}\{I_{\delta_1}\frac{1}{s}, I_{\delta_2}\frac{1}{s^2}, \dots, I_{\delta_q}\frac{1}{s^q}, 0\} \quad .$$

The sequence $\{\delta_1, \delta_2, \dots\}$ is said to characterize the structure at infinity of the given linear system. Sometimes, instead of the sequence $\{\delta_1, \delta_2, \dots\}$ one considers a sequence $\{r_0, r_1, \dots\}$ related to the former in this way

$$\begin{aligned} r_0 &= \delta_1 \\ r_i &= \delta_{i+1} - \delta_i \qquad i \geq 1 \end{aligned} \tag{8}$$

and whose computation is rather easy on a realization (A,B,C) of the transfer function W(s).

If the process to be compensated is nonlinear, one may still establish a sufficient condition for the existence of solutions to a model matching problem by means of a suitable extension of the notion of structure at infinity [14]. This extension, which is based upon the consideration of the so-called maximal controlled invariant subspace algorithm [9] and is described in full in the Appendix 1, associates with any nonlinear system of the form (2), i.e. with any triplet (f,g,h), a string of integers $\{r_0, r_1, \dots\}$ which shares much of the properties of the sequence defined by means of (8).

A (sufficient) condition for the solvability of the problem of matching the external behavior of a prescribed linear model can be found in the following way. With the given process (2), i.e. with the triplet (f,g,h), we associate its structure at the infinity $\{r_0, r_1, \dots\}$. Moreover, from the triplet (f,g,h)

and from the triplet (A,B,C) which characterizes the model to be matched, we define an *enlarged* system as follows

$$\begin{aligned} \dot{x} &= f(x) + g(x)\,u \\ \dot{z} &= Az + Bv \\ e &= h(x) - Cz. \end{aligned}$$

These equations may be written in more condensed form as

$$\begin{aligned} \dot{x}^E &= f^E(x^E) + g^E(x^E)\,u^E \\ e &= h^E(x^E) \end{aligned} \tag{9}$$

letting $x^E = (x,z)$, $u^E = (u,v)$ and

$$f^E(x,z) = \begin{bmatrix} f(x) \\ Az \end{bmatrix} \qquad g^E(x,z) = \begin{bmatrix} g(x) & 0 \\ 0 & B \end{bmatrix}$$

$$h^E(x,z) = h(x) - Cz.$$

With the triplet (f^E, g^E, h^E) we associate the structure at infinity, denoted $\{r_0^E, r_1^E, \dots\}$.

The coincidence between the structure at infinity of the triplet (f,g,h) and that of the triplet (f^E, g^E, h^E) is exactly the condition we were looking for. As a matter of fact, the following result has been shown to hold.

Theorem 2. Let (f,g,h) and (A,B,C) be given. If

$$r_k^E = r_k \tag{10}$$

for all $k \geq 0$, then there exists a dynamic feedback compensator under which the input-output behavior of (2) becomes

$$y(t) = Q(t,(x^\circ,\xi^\circ)) + \int_0^t C \exp A(t-\tau)\, B\, v(\tau)\, d\tau \ . \quad \square$$

The proof of this Theorem, which is constructive, may be found in [6]. It may be worth observing that the compensator thus determined has the structure

$$\begin{aligned} \dot{\xi} &= A\xi + Bv \\ u &= \alpha(x,\xi) + \gamma(x,\xi)\, v \end{aligned} \tag{11}$$

i.e. incorporates the dynamics of the model to be followed. In particular, its dimension ν is equal to that of the linear model.

The conditions provided by this Theorem may be used, in particular, in order to check the possibility of matching the external behavior of a linear model with transfer function

$$W_M(s) = \begin{bmatrix} \frac{1}{s^\delta} & 0 & \dots & 0 & 0 & .. & 0 \\ 0 & \frac{1}{s^\delta} & \dots & 0 & 0 & .. & 0 \\ & & \dots & & & .. & \\ 0 & 0 & \dots & \frac{1}{s^\delta} & 0 & .. & 0 \end{bmatrix} \tag{12}$$

where δ is a suitable integer.

The external behavior of a linear system with transfer function (12) is clearly *decoupled*. Thus, the dynamic state-feedback compensator which enables us to match this model is such as to impose a decoupled behavior between inputs and outputs. As a matter of fact, in the closed-loop system, each component of the output is influenced only by the corresponding component of the input. Note that this is independent of the specific form of the zero-input response $Q(t,(x^\circ,\xi^\circ))$. In the next section we will consider the application of these results to a given class of robot arms.

4. APPLICATION OF THE MODEL MATCHING THEORY TO THE ROBOT ARM

In this section we describe the application of the previous theoretical results to the problem of controlling a robot arm with elastic joints; in particular we will consider as an example the two links planar robot arm.

By means of the Algorithm of the Appendix 1 we will compute the sequence $\{r_0, r_1 \dots\}$ and see that it is possible to match a linear model with transfer function of the form (12), with $\delta = k^* + 1$. This is done by imposing the same structure at infinity of the original system on the system (9) which includes the model to be matched.

The complete mathematical model of the two links robot arm with joint elasticity is described in the Appendix 2. Note that we consider as system outputs the joint position variables; thus, we are looking for decoupling strategies in the joint space. Since the relationship between joint and task space variables, expressed by the robot kinematics, is a continuous mapping almost everywhere invertible, this choice is by no means restrictive.

We begin with the computation of the sequence $\{r_0, r_1, \dots\}$ for our example. In order to do this we have to perform the Algorithm on the triplet (f,g,h) which describes the robot arm system.

In the initial step we make use of the output functions $h_1 = x_2, h_2 = x_4$; we have

$$\Omega_o = \mathrm{sp}\,\{dh_1, dh_2\} = \mathrm{sp}\,\left\{\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{matrix}\right\}$$

so that $s_o = \dim \Omega_o = 2$.

To start the 0-th iteration we need to find r_o i.e. we just have to compute the rank of the matrix

$$A_o = \begin{bmatrix} dh_1 \\ \\ dh_2 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix} = 0 \longrightarrow \quad r_o = 0;$$

Since $r_o=0$, the equations (A1) in Appendix 1 are trivial. To construct Ω_1 we have to pick up functions from the following set

$$\Lambda_o = \{L_f\, h_j,\ L_{g_i}\, h_j \ ; \quad i=1,2 \quad ; \quad j=1,2\}$$

such that their differentials are linear independent with respect to each other and to the set of differentials which span Ω_o. Since we have

$$L_f\, h_1 = x_6 \quad , \quad L_f\, h_2 = x_8 \quad , \quad L_g\, h = 0 \ ,$$

we obtain directly

$$\Omega_1 = \Omega_o \oplus sp\{dL_fh_1, dL_fh_2\} = \quad \Omega_o \oplus sp \left\{ \begin{matrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\}$$

and hence $s_1 = \dim\Omega_1 = 4$. Thus, we set $\lambda_1 = \left[h_1, h_2, L_fh_1, L_fh_2\right]^T$.

In the 1-st iteration, r_1 is given by the rank of the matrix

$$A_1 = \begin{bmatrix} dh_1 \\ dh_2 \\ dL_fh_1 \\ dL_fh_2 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & g_{62} \\ 0 & g_{82} \end{bmatrix} \longrightarrow \quad r_1 = 1.$$

At this point consider the following 4 by 4 permutation matrix

$$P_1 = \begin{bmatrix} P_{11} \\ \hline \\ P_{12} \\ \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is such that $P_{11}A_1$ selects r_1 linearly independent rows from A_1. In this case $r_1=1$ and the third row of A_1 is chosen. Define further

$$B_1 = \begin{bmatrix} dh_1 \\ dh_2 \\ dL_fh_1 \\ dL_fh_2 \end{bmatrix} \cdot f \ = \begin{bmatrix} x_6 \\ x_8 \\ f_6 \\ f_8 \end{bmatrix}$$

and solve for an m-vector α and an $m \times m$ invertible matrix β $(m=2)$ the equations

$$P_{11}\, A_1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = -\, P_{11}\, B_1 \quad , \quad P_{11}\, A_1 \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = K = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

where for K we can choose any matrix of real numbers of rank equal to r_1. We obtain:

$$\alpha_1 = 0,\ \alpha_2 = -f_6/g_{62},\ \beta_{11} = 1,\ \beta_{22} = 1/g_{62},\ \beta_{12} = \beta_{21} = 0$$

from which we can construct a static feedback law

$$u = \alpha(x) + \beta(x)\, w \qquad (13)$$

that gives the new vector fields $\tilde{f}=f+g\alpha$, $\tilde{g}_i=(g\beta)_i$ whose explicit expressions are given in the Appendix 2. The algorithm proceeds by picking up a maximal number of functions from the set

$$\Lambda_1 = \{L_{\tilde f}(L_f h_j)\ ,\ L_{\tilde g_i}(L_f h_j)\ ;\ i=1,2\ ;\ j=1,2\}$$

with linearly independent differentials. Noting that $L_f h = L_{\tilde f} h$, we have

$$L_{\tilde f}^2 h_1 = 0\ ,\ L_{\tilde f}^2 h_2 = \tilde f_8\ ,\ L_{\tilde g_1} L_{\tilde f} h = 0\ ,$$

$$L_{\tilde g_2} L_{\tilde f} h_1 = 1\ ,\ L_{\tilde g_2} L_{\tilde f} h_2 = \tilde g_{82}$$

so that the only candidate is $L_{\tilde f}^2 h_2$ since $\tilde g_{82}$ depends on x_4 only. We obtain in fact

$$\Omega_2 = \Omega_1 \oplus sp\{dL_{\tilde f}^2 h_2\} = \Omega_1 \oplus sp\{0\ \bullet\ \frac{\partial \tilde f_8}{\partial x_3}\ \bullet\ 0\ \bullet\ 0\ 0\}$$

where $(\partial\tilde f_8/\partial x_3)=K_2/N_2A_2\neq 0$ assures the linear independency of the new row w.r.t. the previous ones ($\bullet$ denotes non relevant terms). So $s_2=\dim\Omega_2=5$ and $\lambda_2=\left[h_1,h_2,L_{\tilde f}h_1,L_{\tilde f}h_2,L_{\tilde f}^2h_2\right]^T$.

Next, for the 2-nd iteration compute

$$A_2 = \left[dh_1^T\ dh_2^T\ dL_{\tilde f}h_1^T\ dL_{\tilde f}h_2^T\ dL_{\tilde f}^2h_2^T\right]^T \left[g_1\ \ g_2\right]$$

$$= \begin{bmatrix}0&0&0&0&0\\0&0&g_{62}&g_{82}&\bullet\end{bmatrix}^T \longrightarrow\ r_2 = 1.$$

Again we need to find (α,β) as a solution of a matrix equation similar to the one considered in the previous step. However, as long as $r_k=1$ we obtain the same (α,β) i.e. the same $\tilde f,\tilde g$ and so we will bypass this part of the computations; thus, the new searching set is:

$$\Lambda_2 = \{L_{\tilde f}(L_{\tilde f}^2h_2),\ L_{\tilde g_1}(L_{\tilde f}^2h_2),\ L_{\tilde g_2}(L_{\tilde f}^2h_2)\}.$$

We have

$$L_{\tilde g_1}L_{\tilde f}^2h_2=0,\ L_{\tilde g_2}L_{\tilde f}^2h_2=\partial\tilde f_8/\partial x_6= -2(A_3/A_2)x_6 \sin x_4\ ,$$

$$L_{\tilde f}^3h_2=(\partial\tilde f_8/\partial x_2)x_6+(\partial\tilde f_8/\partial x_3)x_7+(\partial\tilde f_8/\partial x_4)x_8$$

and then

$$\Omega_3=\Omega_2\oplus sp\{dL_{\tilde f}^3h_2\}=\Omega_2\oplus sp\{0\ \bullet\ 0\ \bullet\ 0\ \bullet\ \frac{\partial\tilde f_8}{\partial x_3}\ \bullet\ \}$$

so that $s_3=\dim\Omega_3=6$.

The 3-rd step of the algorithm starts computing

$$A_3=\left[dh_1^T\ dh_2^T\ dL_{\tilde f}h_1^T\ dL_{\tilde f}h_2^T\ dL_{\tilde f}^2h_2^T\ dL_{\tilde f}^3h_2^T\right]^T\left[g_1\ g_2\right]$$

$$=\begin{bmatrix}0&0&0&0&0&0\\0&0&\bullet&\bullet&\bullet&\bullet\end{bmatrix}^T\longrightarrow\ r_3=1.$$

Furthermore

$$\Lambda_3=\{L_{\tilde f}(L_{\tilde f}^3h_2),\ L_{\tilde g_1}(L_{\tilde f}^3h_2),\ L_{\tilde g_2}(L_{\tilde f}^3h_2)\}$$

where $L_{\tilde g_1}L_{\tilde f}^3h_2=0$, $L_{\tilde g_2}L_{\tilde f}^3h_2$ is a function of x_2,x_4,x_6,x_8 only, while

$$L_{\tilde f}^4h_2 = (\partial\tilde f_8/\partial x_3)\ \tilde f_7 + \phi(x_2,x_3,x_4,x_6,x_7,x_8).$$

Notice that $(\partial\tilde f_7/\partial x_1)=K_1/N_1JRZ_2\neq0$; thus, we obtain

$$\Omega_4 = \Omega_3\oplus sp\{dL_{\tilde f}^4h_2\} = \Omega_3\oplus sp\{(\frac{\partial\tilde f_8}{\partial x_3}\ \frac{\partial\tilde f_7}{\partial x_1})\ \bullet\ \bullet\ \bullet\ 0\ \bullet\ \bullet\ \bullet\}$$

and $s_4=\dim\Omega_4=7$.

In the 4-th step,

$$A_4 = \left[dh_1^T\ dh_2^T\ dL_{\tilde f}h_1^T\ dL_{\tilde f}h_2^T\ dL_{\tilde f}^2h_2^T\ dL_{\tilde f}^3h_2^T\ dL_{\tilde f}^4h_2^T\right]^T g$$

$$= \begin{bmatrix}0&0&0&0&0&0&0\\0&0&\bullet&\bullet&\bullet&\bullet&\bullet\end{bmatrix}^T \longrightarrow\ r_4 = 1.$$

We get finally

$$\Lambda_4 = \{L_{\tilde f}(L_{\tilde f}^4 h_2),\ L_{\tilde g_1}(L_{\tilde f}^4 h_2),\ L_{\tilde g_2}(L_{\tilde f}^4 h_2)\}\ .$$

In Λ_4 the only function which depends on x_5 is $L_{\tilde f}^5 h_2$; this has the form

$$L_{\tilde f}^5 h_2 = (\partial \tilde f_8/\partial x_3)\cdot(\partial \tilde f_7/\partial x_1)x_5 + \psi(x_1,x_2,x_3,x_4,x_6,x_7,x_8)\,.$$

At this step we have that

$$\Omega_5 = \Omega_4 \oplus sp\{dL_{\tilde f}^5 h_2\} = \Omega_4 \oplus sp\{*\ \ *\ \ *\ \ *\ (\frac{\partial \tilde f_8}{\partial x_3}\frac{\partial \tilde f_7}{\partial x_1})\ *\ \ *\ \ *\}\ .$$

Thus, $s_5 = \dim \Omega_5 = 8 = \dim x$ and hence the algorithm stops at $k^* = 5$. Last we have to compute the rank of the matrix

$$A_5 = \left[dh_1^T\ dh_2^T\ dL_{\tilde f}h_1^T\ dL_{\tilde f}h_2^T\ dL_{\tilde f}^2 h_2^T\ dL_{\tilde f}^3 h_2^T\ dL_{\tilde f}^4 h_2^T\ dL_{\tilde f}^5 h_2^T\right]^T \cdot g$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & (g_{51}\frac{\partial \tilde f_8}{\partial x_3}\frac{\partial \tilde f_7}{\partial x_1}) \\ 0 & 0 & g_{62} & g_{82} & * & * & * & * \end{bmatrix}^T$$

which gives $r_{k^*} = 2$.

Notice that, as a byproduct of the previous algorithm, we have computed the static feedback law (13) which, in addition, is the same for each step of the algorithm; this feedback is useful for the dynamic decoupling strategy proposed in Section 5. Furthermore, for the particular example considered, we have found that the set of functions whose differentials span Ω_{k^*}, is obtained by taking the Lie derivatives of the system outputs only with respect to the vector field $\tilde f$; thus, these functions define a new set of local coordinates in terms of which the system is described by much simpler expressions.

To conclude this section, we consider the problem of finding what kind of linear and decoupled model it is possible to match. Let the linear model be expressed in a parametrized form

$$\dot z = Az + Bv$$

$$y = Cz$$

with the condition that $c_j A^k b_i = 0$, $\forall k$ if $i \neq j$, i.e. having a decoupled structure of input-output channels. The model matching problem is solvable if the enlarged system robot arm plus model (see (9)) has the same structure at the infinity as the original system (the robot arm alone).

The procedure is then the following: apply the Algorithm to (f^E, g^E, h^E) and compute the matrices A_k^E of the enlarged system; then, imposing the equality between r_k^E (rank of A_k^E) and r_k for $k=1,\ldots,k^*$, derive conditions on the triplet (A,B,C). The computation of Lie derivatives is greatly simplified being the model to be matched a linear one and will be omitted here. After $k^*=5$ steps of the algorithm, we have necessarely

$$C\,A^k\,B = 0\ ,\quad \text{for}\quad k = 0,1,\ldots,4.$$

This means that the input-output behavior of the considered robot arm can be made equal via dynamic feedback to that of a decoupled linear system constituted by two chains of six integrators each. In this case (12) becomes

$$W_M(s) = \begin{bmatrix} \frac{1}{s^6} & 0 \\ 0 & \frac{1}{s^6} \end{bmatrix}\ .$$

Then, a dynamic controller can be constructed just following the arguments of the proof of Theorem 2. The obtained controller will be of order $\ell\cdot(k^*+1)=12$, with a dynamics inherited from the one of the matched model. However, the explicit derivation of this controller will be avoided here; using the fact that such a controller exists, a more direct procedure will be developed in the next sections.

5. ANALYSIS OF THE MODEL STRUCTURE VIA SYSTEM GRAPH

As an useful tool in order to explore the decoupling problem and the possibility of reducing the dynamic order of the controller we recall the notion of graph representation of a system, introduced in [15]. Moreover we will consider here, in terms of system graph, the effect of the addition of integrators to the inputs of the system.

The system graph of nonlinear system (2) is a weighted oriented graph $G(N,L)$, where N is the set of nodes representing the input, state and output variables and L is the set of weighted oriented arcs representing the influences among variables.

More precisely, the weight of an arc (u_i,x_k) is given by g_{ki} where g_{ki} is the k-th element of g_i; the weight of an arc (x_k,x_h) is given by $\frac{\partial f_h}{\partial x_k}$, where f_h is the h-th element of f; the weight of an arc (x_h,y_j) is given by $\frac{\partial h_j}{\partial x_h}$, where h_j is the j-th element of h. L is constituted by the nonzero-weighted arcs only. The system graph of the two links planar robot with elastic joints is shown in Fig.1.

Let $d(u_i,y_j)$ be the minimal number of arcs of G forming an oriented path from u_i to y_j and let $d_j=\min_i d(u_i,y_j)$; define length of a path the number of its arcs. The minimal graph G_M is the subgraph of G constituted by all input-output paths of length d_j ending in y_j, for each $j=1,\dots,\ell$.

G_M gives a complete information on the dynamic structure with respect to the decoupling property. Call weight of a path the product of the weights of the arcs forming the path. The entries a_{ji} of the decoupling matrix are given by the sum of the weights of the paths joining u_i to y_j in G_M, when for each output there exists at least an input for which the above sum is not zero. In Fig.1 the inspection of the minimal graph (bold arcs) reveals that the decoupling matrix has only one nonzero column, being thus singular as claimed in Section 2.

Consider now the effects of adding chains of integrators to the inputs of the system; without loss of generality we can limit ourselves to the connection of one integrator to input u_i i.e. $u_i=\xi$, $\dot{\xi}=v$. Modifications occur in the graph only at arcs ending in state nodes directly connected with u_i. More precisely, in the new system graph the weigths of an arc (x_k,x_h), with x_h connected to u_i, becomes $\partial f_h/\partial x_k+\xi\partial g_{hi}/\partial x_k$. New arcs are created if $\partial f_h/\partial x_k=0$ but $\partial g_{hi}/\partial x_k\neq 0$. The remaining parts of the graph are not affected. Notice that no changes occur in the graph if $\partial g_{hi}/\partial x_j=0$ for all h,j. As a matter of fact, the addition of an integrator modifies consistently the structure of the graph. Further additions of integrators to the input u_i leave the graph unchanged.

For the two links arm Figure 1 shows that the singularity of the decoupling matrix is due to the fact that both minimal paths start from input u_2. Since we know that a dynamic controller exists, we may try to extend the system graph with properly connected integrators so as to build a nonsingular decoupling matrix $\hat{A}(\hat{x})$; the obtained *extended* system will then be decoupable via static feedback from the extended state $\hat{x}$.

Therefore to bring into play input u_1 we increase the length of the paths starting from the second input. Adding two cascaded integrators to input u_2, the decoupling matrix becomes full; unfortunately, it turns out to be singular due to the weights on the minimal paths. Moreover, the cascaded addition of further integrators does not change this situation because it does not change the weights of the minimal paths; actually, the addition of integrators to the first input leaves the graph exactly the same (g_1 is constant).

At this point the only possibility in order to modify the graph structure is to use a feedback transformation first and then to work on the obtained graph with the above dynamic extension. However, the iterated application of this procedure generates a trial and error search for a proper extension of the system graph which becomes soon not practical. We can instead make use of the results of the previous sections so as to get a graph which lends itself in a more suitable form for our purposes.

6. COMPUTATION OF THE DECOUPLING FEEDBACK AND STRUCTURE OF THE DYNAMIC CONTROLLER

We saw in Section 4 that during each step of the Algorithm a particular static feedback is computed. Whenever the system is statically decoupable, the feedback obtained at the last step is a decoupling one. In our case we found:

$$u = \alpha(x) + \beta(x)\, w = \begin{bmatrix} 0 \\ -f_6/g_{62} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1/g_{62} \end{bmatrix} w. \quad (14)$$

The application of this feedback law to the system describing the two links planar robot with elastic joints obviously does not achieve a decoupled structure. However, in the system

$$\dot{x} = [f(x) + g(x)\,\alpha(x)] + [g(x)\,\beta(x)]\, w = \tilde{f}(x) + \tilde{g}(x)\, w$$
$$y = h(x) \quad (15)$$

so obtained (see Appendix 2) the first output is decoupled from input w_1, as shown by the system graph representation given in Fig.2. As a matter of fact, the north-east entry of the decoupling matrix will always be zero; therefore, the new system (and its associated graph) is suitable for the dynamic extension procedure.

Since the weights of the arcs starting from input w_2 are constant or depend only on x_4, addition of a cascade of integrators to this input does not add new interactions among the states; following the same reasoning as before (i.e. balancing the minimal paths lengths), it is then straightforward to see that four integrators added to the second input give a nonsingular decoupling matrix.

In fact the following dynamic extension

$$w_1 = \bar{v}_1$$
$$w_2 = \xi_1 \;,\; \dot{\xi}_1 = \xi_2 \;,\; \dot{\xi}_2 = \xi_3 \;,\; \dot{\xi}_3 = \xi_4 \;,\; \dot{\xi}_4 = \bar{v}_2 \quad (16)$$

applied to system (15) leads to the new system

$$\dot{\hat{x}} = \bar{f}(\hat{x}) + \bar{g}(\hat{x})\,\bar{v}$$
$$y = \bar{h}(\hat{x}) = h(x) \quad (17)$$

where the extended state $\hat{x}$ is defined as

$$\hat{x} = [x_1 \; x_2 \; \ldots \; x_8 \; \xi_1 \; \ldots \; \xi_4]^T$$

and the vector fields $\bar{f},\bar{g}_i$ are given respectively by:

$$\bar{f}(\hat{x}) = \begin{bmatrix} x_5 & x_6 & x_7 & x_8 & f_5 & \xi_1 & (\tilde{f}_7+\tilde{g}_{72}\xi_1) & (\tilde{f}_8+\tilde{g}_{82}\xi_1) & \xi_2 & \xi_3 & \xi_4 & 0 \end{bmatrix}^T$$

$$\bar{g}(\hat{x}) = \begin{bmatrix} \bar{g}_1(\hat{x})^T \\ \bar{g}_2(\hat{x})^T \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & g_{51} & 0 & \dots & 0 & 0 \\ 0 & 0 & & & \dots & & & 0 & 1 \end{bmatrix}^T.$$

In Fig.3 the dashed arcs represent the added integrators and the crossbarred arcs are those whose weights are modified by the dynamic extension. The decoupling matrix $\bar{A}(\hat{x})$ for system (17) is:

$$\bar{A}(\hat{x}) = \begin{bmatrix} L_{\bar{g}_1}L_{\bar{f}}^5 h_1 & L_{\bar{g}_2}L_{\bar{f}}^5 h_1 \\ L_{\bar{g}_1}L_{\bar{f}}^5 h_2 & L_{\bar{g}_2}L_{\bar{f}}^6 h_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g_{51}\dfrac{\partial\tilde{f}_7}{\partial x_1}\dfrac{\partial\tilde{f}_8}{\partial x_3} & \tilde{g}_{82} \end{bmatrix}$$

We can compute now the decoupling feedback for the extended system:

$$\bar{v} = \bar{\alpha}(\hat{x}) + \bar{\beta}(\hat{x})\, v. \tag{18}$$

Applying (4) and using the explicit expressions of the terms involved (see Appendix 2) we obtain:

$$\bar{\beta}(\hat{x}) = \bar{A}(\hat{x})^{-1} = \begin{bmatrix} \dfrac{-\tilde{g}_{82}}{g_{51}(\partial\tilde{f}_7/\partial x_1)(\partial\tilde{f}_8/\partial x_3)} & \dfrac{1}{g_{51}(\partial\tilde{f}_7/\partial x_1)(\partial\tilde{f}_8/\partial\tilde{x}_3)} \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -D\,(A_3\cos x_4 + A_2) & D\,A_2 \\ 1 & 0 \end{bmatrix}$$

where $D=N_1N_2JRZ_1JRZ_2/K_1K_2$. Since $L_{\bar{f}}^6 h_1=0$, we have further

$$\bar{\alpha}(\hat{x}) = -\bar{A}(\hat{x})^{-1}\cdot\begin{bmatrix} L_{\bar{f}}^6 h_1 \\ L_{\bar{f}}^6 h_2 \end{bmatrix} = \begin{bmatrix} -D\,A_2\,L_{\bar{f}}^6 h_2 \\ 0 \end{bmatrix}$$

In $\bar{\alpha}(\hat{x})$ the Lie derivative $L_{\bar{f}}^6 h_2$ has a rather long and complex expression composed by trigonometric polynomials. The use of symbolic and algebraic manipulation systems such as MACSYMA or REDUCE is indicated for the easy derivation and simplification of this term.

In any case, combining (14), (16) and (18) we obtain the desired dynamic controller which is of the form (11).

For the reader's convenience we summarize the description of the decoupling feedback. The controller we obtained is a dynamical system with inputs v_1,v_2, outputs u_1,u_2 and 4-dimensional state $\xi=(\xi_1\xi_2\xi_3\xi_4)$, described by equation of the form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= v_1 \\ u_1 &= \bar{\alpha}_1(x,\xi) + \bar{\beta}_{11}(x,\xi)\, v_1 + \bar{\beta}_{12}(x,\xi)\, v_2 \\ u_2 &= -\frac{f_6(x)}{g_{62}(x)} + \frac{1}{g_{62}(x)}\,\xi_1 \end{aligned} \tag{19}$$

with $\bar{\alpha}_1 = -DA_2(L_{\bar{f}}^6 h_2)$, $\bar{\beta}_{11} = -D(A_3\cos x_4 + A_2)$, $\bar{\beta}_{12} = DA_2$.

The order of the controller has been reduced from twelve to four by the joint application of static feedback given by the Algorithm of Appendix 1, dynamic extension and static decoupling feedback from the extended state. Notice however that this may still not be the minimal order. Nevertheless the proposed dynamic controller exhibits a further nice property.

It is known [9] that the static feedback decoupling creates a closed-loop system which has an unobservable part with a possibly nonlinear dynamics of dimension $\hat{n}^*=\hat{n}-(\sum_{j=1}^{2}\hat{\rho}_j+1)$; the remaining part of the system is equivalent to a linear controllable and observable representation. In our case we have $\hat{\rho}_1=\hat{\rho}_2=5$, i.e. two chains of six integrators, $\hat{n}=\dim \hat{x}=12$ and hence $\hat{n}^*=0$, so that the decoupling law is also a linearizing one for the extended system. This is closely related to the fact that the minimal paths on the system graph visit all the state nodes. Thus, the composition of the control law (19) with the robot arm equations (2) yield a dynamical system whose external behavior is decoupled and which is diffeomorphic to a linear and controllable system. As a matter of fact, in the coordinates

$$z_1 = h_1, z_2 = L_{\bar{f}}h_1, z_3 = L_{\bar{f}}^2h_1, z_4 = L_{\bar{f}}^3h_1, z_5 = L_{\bar{f}}^4h_1, z_6 = L_{\bar{f}}^5h_1,$$

$$z_7 = h_2, z_8 = L_{\bar{f}}h_2, z_9 = L_{\bar{f}}^2h_2, z_{10}= L_{\bar{f}}^3h_2, z_{11}= L_{\bar{f}}^4h_2, z_{12}= L_{\bar{f}}^5h_2$$

the closed loop system is described by the equations

$$\dot{z} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} z + \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix} v$$

$$y = \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} z$$

with

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & 0 \\ 0 & \cdot & 0 & 1 & 0 & 0 \\ 0 & \cdot & \cdot & 0 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \qquad B_i = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

$$C_i = [\,1 \; 0 \; \cdot \; \cdot \; \cdot \; 0\,] \qquad i = 1,2.$$

We conclude that in the examined elastic robot we obtain, as a byproduct of the decoupling, the full state linearization. This allows to assign all the dynamic behavior by standard techniques.

7. CONCLUSIONS

In this paper we have shown how nonlinear model matching theory can be applied for the dynamic decoupling control of industrial robots with joint elasticity. The existence of a decoupling controller is guaranteed for the planar two links robot with elastic joints. The model matching approach leads to a twelve-order dynamic controller; the resulting closed-loop system matches the input-output behavior of a prescribed decoupled linear system but includes an unobservable part with a possibly nonlinear dynamics. However, the analysis of the robot model structure allows both to reduce the order of the controller down to four and to fully linearize the closed-loop state dynamics.

The above results can be extended to a three links robot arm (with waist, shoulder and elbow revolute elastic joints), which allows the most general positioning for the robot hand, and to task-oriented decoupling strategies, by considering the robot direct kinematics as system output.

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APPENDIX 1

In this Appendix we describe the Algorithm for the computation of the structure at the infinity of a nonlinear system [16]. We recall that if $\lambda: R^n \to R$ is a smooth function, its differential $d\lambda$ is the $1 \times n$ row vector

$$d\lambda(x) = \left(\frac{\partial\lambda}{\partial x_1} \cdots \frac{\partial\lambda}{\partial x_n}\right)$$

From the components $h_1,\ldots,h_\ell$ of the map h one construct first of all the (x-dependent) subspace (of row vectors)

$$\Omega_0(x) = \text{span}\,\{dh_1(x),\ldots,dh_\ell(x)\}$$

Suppose $\Omega_0(x)$ has dimension $s_0 \le \ell$ in a neighborhood of a point x°. Then there exists an $s_0 \times 1$ column vector λ_0, whose entries $\lambda_{01},\ldots,\lambda_{0s_0}$ are entries of h, with the properties that the differentials $d\lambda_{01},\ldots,d\lambda_{0s_0}$ are linearly independent at all x in a neighborhood of x°.

The algorithm consists of a finite number of iterations, each one defined as follows.

Iteration (k). consider the $s_k \times m$ matrix $A_k(x)$ whose (i,j)-entry is $d\lambda_{ki}(x)g_j(x)$. Suppose that in a neighborhood of x° the rank of $A_k(x)$ is constant and equal to r_k. Then it is possible to find r_k rows of $A_k(x)$ which, for all x in a neighborhood of x°, are linearly independent. Let

$$P_k = \begin{bmatrix} P_{k1} \\ \hline P_{k2} \end{bmatrix}$$

be an $s_k \times s_k$ permutation matrix, such that the r_k rows of $P_{k1}A_k(x)$ are linearly independent. Let $B_k(x)$ be an s_k-vector whose i-th element is $d\lambda_{ki}(x)f(x)$. As a consequence of the assumptions on P_{k1}, the equations

$$\begin{aligned} P_{k1}A_k(x)\alpha(x) &= -P_{k1}B_k(x) \\ P_{k1}A_k(x)\beta(x) &= K \end{aligned} \tag{A1}$$

(where K is a matrix of real numbers, of rank r_k) may be solved for α and β, an m-vector and an $m \times m$ invertible matrix whose entries are real-valued smooth functions defined in a neighborhood of x°. Set $\tilde{f}=\tilde{g}_0=f+g\alpha$ and $\tilde{g}_i=(g\beta)_i$, $1 \le i \le m$.

Consider the set of functions

$$\Lambda_k = \{\lambda = L_{\tilde{g}_i}\lambda_{kj} \;:\; 1 \le j \le s_k \;,\; 0 \le i \le m\}$$

and the two (x-dependent) subspaces (of row vectors)

$$\Omega_k(x) = \text{span}\,\{d\lambda_{k1}(x),\ldots,d\lambda_{ks_k}(x)\}$$

$$\Omega'_k(x) = \text{span}\,\{d\lambda(x) \;:\; \lambda \in \Lambda_k\}$$

Set $\Omega_{k+1}(x) = \Omega_k(x) + \Omega'_k(x)$.

Suppose $\Omega_{k+1}(x)$ has constant dimension $s_{k+1}(\ge s_k)$ in a neighborhood of x°. Let $\lambda_{k+1,1},\ldots,\lambda_{k+1,s_{k+1}}$ be entries of λ_k and/or elements of Λ_k such that the differentials $d\lambda_{k+1,1},\ldots, d\lambda_{k+1,s_{k+1}}$ are linearly independent at all x in a neighborhood of x°.

Define the s_{k+1}-vector λ_{k+1} whose i-th entry is the function $\lambda_{k+1,i}$. This concludes the k-th iteration.

Note that at each stage of the algorithm two integers are considered

$$s_k = \dim \Omega_k(x)$$

$$r_k = \text{rank}\, A_k(x).$$

Since $s_k \le s_{k+1} \le n$, a dimensionality argument shows that there exists an integer k^* such that

$$s_k = s_{k^*} \;,\; r_k = r_{k^*}$$

for all $k \ge k^*$. The sequence $\{r_0, r_1, \ldots\}$ provides the so-called structure at the infinity associated with the triplet (f,g,h).

APPENDIX 2

We report here the dynamic model of a two links robot arm with joint elasticity, whose possible configurations lie in a vertical plane. Starting from the model in the form (1), obtained using the DYMIR code [8], the state-space representation has been computed via a symbolic manipulation system (REDUCE) and is of the type

$$\dot{x} = f(x) + \sum_{i=1}^{2} g_i(x)\, u_i$$

$$y = h(x)$$

with

$$f(x) = \begin{bmatrix} x_5 & x_6 & x_7 & x_8 & f_5 & f_6 & f_7 & f_8 \end{bmatrix}^T$$

$$g(x) = \begin{bmatrix} g_1(x)^T \\ g_2(x)^T \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & g_{51} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{62} & g_{72} & g_{82} \end{bmatrix}^T$$

$$h(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

where

$$
\begin{aligned}
g_{51} &= G_1 \\
g_{62} &= A_2/DT_1 \\
g_{72} &= (A_3^2\cos^2 x_4 - A_1A_2 + A_2^2)/JRZ_2DT_1 \\
g_{82} &= -DT_2/DT_1 \\
f_5 &= (N_1x_2 - x_1)G_1K_1/N_1^2 \\
f_6 &= \{[K_1(N_1x_2-x_1)+N_1A_5\cos x_2+N_1DT_3]N_2^2A_2 \\
&\quad + [K_2A_2(N_2x_4-x_3)-(K_2(N_2x_4-x_3)+N_2DT_3)N_2DT_2]N_1 \\
&\quad - [x_6^2DT_2+A_2x_8(2x_6+x_8)]N_1N_2^2A_3\sin x_4\}/N_1N_2^2DT_1 \\
f_7 &= \{[x_6^2DT_2+A_2x_8(2x_6+x_8)]N_1N_2^2A_3JRZ_2\sin x_4 \\
&\quad - [K_1(N_1x_2-x_1)+N_1A_5\cos x_2+N_1DT_3]N_2^2A_2JRZ_2 \\
&\quad - [K_2(A_3^2\cos^2x_4+A_2^2-A_1A_2)(x_3-N_2x_4)+(K_2(x_3-N_2x_4)-N_2DT_3)N_2DT_2JRZ_2]N_1\} \\
&\quad /N_1N_2^2DT_1JRZ_2 \\
f_8 &= \{[x_6^2(2DT_2+A_1-JRZ_2-2A_2)+x_8(2x_6+x_8)DT_2]A_3N_1N_2^2\sin x_4 \\
&\quad - [K_1(N_1x_2-x_1)+N_1A_5\cos x_2+N_1DT_3]N_2^2DT_2 \\
&\quad - [(K_2(x_3-N_2x_4)-N_2DT_3)N_2(2DT_2+A_1-JRZ_2-2A_2)-K_2(x_3-N_2x_4)DT_2]N_1\} \\
&\quad /N_1N_2^2DT_1.
\end{aligned}
$$

We defined for compactness and better computing performances the following terms:

$$DT_1 = A_3^2\cos^2 x_4 + A_2(JRZ_2 + A_2 - A_1)$$

$$DT_2 = A_3 \cos x_4 + A_2$$

$$DT_3 = A_4 \cos(x_2 + x_4).$$

At joint i, N_i is the transmission coefficient of the gear box, K_i is the elastic constant, JRZ_i is the inertia of the rotor. The constants $A_1, A_2, \ldots, A_5$ and G_1 include all the kinematic data of the robot (mass and inertia of links and rotors, length and center of gravity of links).

Finally we collect here for convenience the expressions of the vector fields derived in Section 4. With the same notation used therein the vector fields $\tilde{f}, \tilde{g}_i$ computed at the first (and subsequent) step of the algorithm are

$$\tilde{f}(x) = f(x)+g(x)\alpha(x) = \begin{bmatrix} x_5 & x_6 & x_7 & x_8 & f_5 & 0 & \tilde{f}_7 & \tilde{f}_8 \end{bmatrix}^T$$

$$\tilde{g}(x) = g(x)\,\beta(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & g_{51} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \tilde{g}_{72} & \tilde{g}_{82} \end{bmatrix}^T$$

where

$$
\begin{aligned}
\tilde{g}_{72} &= g_{72}/g_{62} = [(A_3^2/A_2)\cos^2 x_4 + A_2 - A_1]/JRZ_2 \\
\tilde{g}_{82} &= g_{82}/g_{62} = -(1+(A_3/A_2)\cos x_4) \\
\tilde{f}_7 &= f_7 - f_6\tilde{g}_{72} = (N_2A_2K_1x_1 - N_1K_2DT_2x_3)/A_2N_1N_2JRZ_2 \\
&\quad +\{A_3\sin x_4[x_6^2DT_2+A_2x_8(x_8+2x_6)]+DT_2[DT_3+K_2x_4] \\
&\quad -A_2[DT_3+A_5\cos x_2+K_1x_2]\}/A_2JRZ_2 \\
\tilde{f}_8 &= f_8 - f_6\tilde{g}_{82} = -[A_3N_2x_6^2\sin x_4+N_2DT_3+K_2(N_2x_4-x_3)]/N_2A_2.
\end{aligned}
$$

We note explicitly that $\tilde{f}_6=0$ and that $\tilde{f}_8$ is a function of x_2, x_3 x_4 and x_6 only; this is reflected in the system graph of Fig.2.

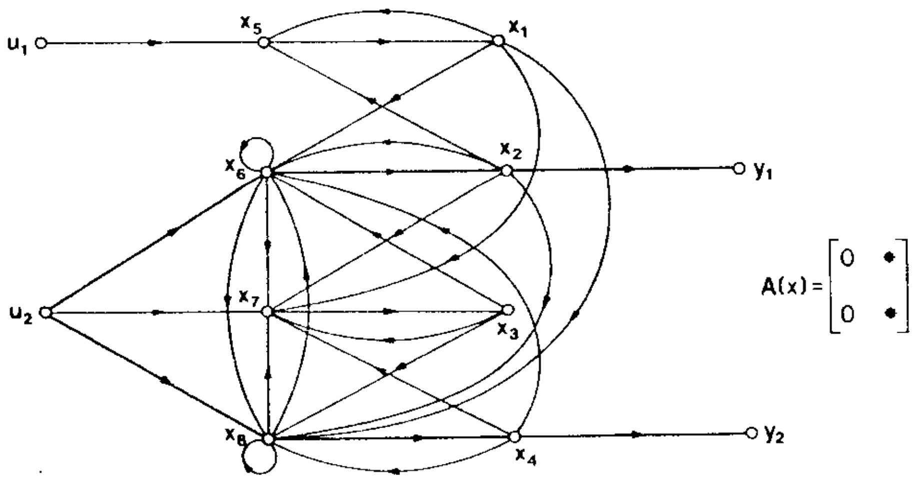

Fig.1 - System graph associated with the two links planar robot arm with joint elasticity (bold arcs = minimal paths)

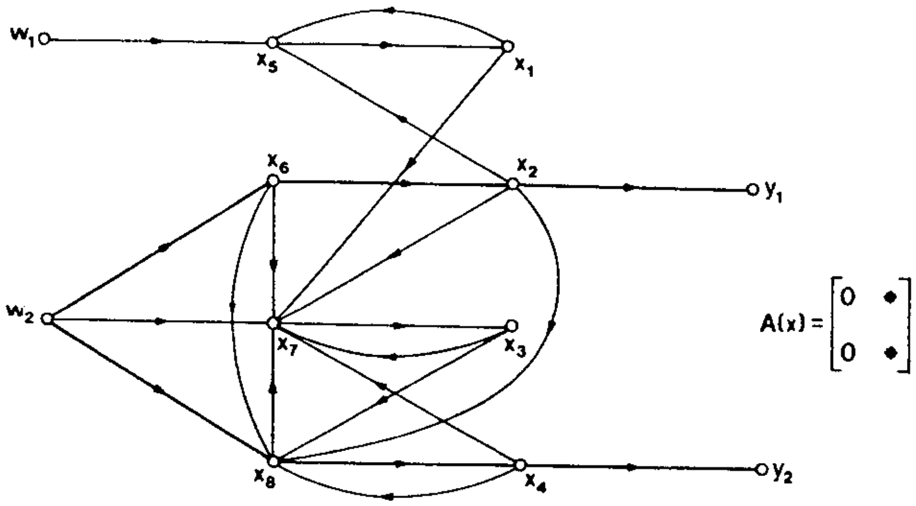

Fig.2 - System graph after the application of the static feedback (14)

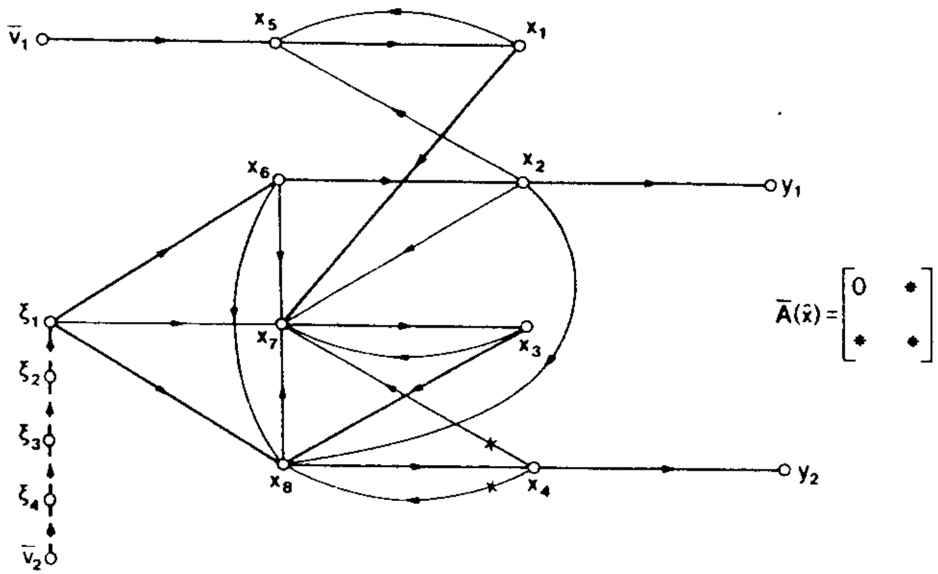

Fig.3 - System graph associated with system (17) (dashed arcs = dynamic extension (16))

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CONTROL OF ROBOT ARM WITH ELASTIC JOINTS VIA NONLINEAR DYNAMIC FEEDBACK

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Abstract

It is known that control of a rigid robot arm can easily be achieved via static state-feedback compensation of the nonlinearities. However, in many practical situations, the elasticity in gear boxes is not negligible. If this is the case, the use of such a control technique is not possible anymore because neither is the system feedback equivalent to a controllable linear one, nor its input-output behavior can be decoupled via static state-feedback.

The purpose of this paper is to show how dynamic state-feedback compensation may be used in order to obtain full state-space linearity, and to present an application to the model of a three link robot arm with elastic joints.

Introduction

The increasing interest for nonlinear control theory in the robotics literature is witnessed by a series of recent papers. Among the others we quote e.g. the works of Freund[1], Tarn and others[2], Singh and Schy[3], Marino and Nicosia[4]. A standard technique proposed for the control of rigid robots is the one based on input-output decoupling and nonlinearity compensation via static state-feedback. For robots with elastic transmission between actuators and arms, as belts or harmonic drives, this control strategy cannot be applied anymore since the associated model is such that the necessary conditions for the existence of the desired feedback fail to hold[4].

In a recent paper[5], the authors suggested the use of dynamic state-feedback and, applying the nonlinear model matching theory[6], solved the noninteracting control problem for the case of a two-link planar robot with elastic joints. The dynamic compensator thus found was such as to induce full linearity in suitable local coordinates for the resulting closed-loop system. This suggested further investigations addressed to the problem of getting full linearization via *dynamic* state-feedback. This control problem is apparently a new one, a natural generalization of that originally posed by Brockett and fully solved by Jacubczyk and Respondek[7] and independently by Hunt and others[8] by means of *static* state-feedback. As a matter of fact, a set of sufficient conditions for the solvability of this problem has been found, described in the first half of this paper. These conditions turn out to be satisfied for a three-link robot arm with elastic joints, which is considered as an example in the second part of the paper.

Exact Linearization via Dynamic State-Feedback

Consider a control system described by differential equations of the form:

$$\dot{x} = f(x) + g(x)u \qquad y = h(x) \tag{1}$$

with state x evolving on an open subset M of $\mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^\ell$. The vector f, the m columns of the matrix g, and the vector h are assumed throughout the paper to be analytic on M.

In what follows we will let the control u depend on the state x and on a reference variable v through equations of the form:

$$\dot{z} = a(x,z) + b(x,z)v \qquad u = c(x,z) + d(x,z)v. \tag{2}$$

These equations characterize a dynamical system - a state-feedback compensator - whose state z evolves on an open subset N of $\mathbb{R}^\nu$. The vector a, the m columns of the matrix b, the vector c and the m columns of the matrix d are assumed to be analytic on an open subset of M × N.

The purpose of this section is to show how to design the compensator in such a way that the closed-loop system resulting from the composition of (1) and (2) becomes locally diffeomorphic to a linear controllable system.

In doing so we will make use to a large extent of some basic results from nonlinear differential geometric feedback control theory; some background material in this field is assumed to be known[9]. In particular, most of our results will rely upon certain properties of the so-called maximal controlled invariant distribution Algorithm[10].

We recall that with any system of the form (1), one may associate a sequence of codistribution defined in the following way:

$$\Omega_o(x) = \text{span}\{dh_1(x),\ldots,dh_\ell(x)\}$$

$$\Omega_k(x) = \Omega_{k-1}(x) + (L_f(\Omega_{k-1} \cap G^\perp))(x) + \sum_{i=1}^{m}(L_{g_i}(\Omega_{k-1} \cap G^\perp))(x) \tag{3}$$

where $G(x) = \text{span}\{g_1(x),\ldots,g_m(x)\}$. This sequence is clearly increasing and, if $\Omega_{k^*} = \Omega_{k^*+1}$ for some k^*, then $\Omega_k = \Omega_{k^*}$ for all $k > k^*$.

For practical purpose, we shall henceforth assume that the codistributions involved in this Algorithm have constant dimension around the point of interest x^o. This is precised in the following terms.

Definition. The point x^o is a regular point for the Algorithm (3) if for all x in a neighborhood of x^o.

(i) the dimension of G(x) is constant
(ii) the dimension of $\Omega_k(x)$ is constant, for all $k \geq 0$
(iii) the dimension of $(\Omega_k \cap G^\perp)(x)$ is constant, for all $k \geq 0$. □

Note that if x^o is a regular point for the Algorithm (3), then there exists an integer $k^* < n$ such that $\Omega_{k^*} = \Omega_{k^*+1}$ and this implies the convergence of the Algorithm, in a neighborhood of x^o, in a finite number of stages. The codistribution Ω_{k^*} will be sometimes denoted by the simpler symbol Ω_* and its annihilator by

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$$\Delta^* = \Omega_*^{\perp}$$

The Algorithm in question will be used in the sequel in order to compute the distribution Δ^*, to check some suitable structural conditions-stated in terms of properties of the codistributions Ω_k - and also in order to compute the so-called structure at infinity[11] of the system (1). We recall that the latter is defined in the following terms. Set

$$r_k = \dim \frac{\Omega_k}{\Omega_k \cap G^{\perp}} \quad , \qquad k \geq 0$$

and

$$\delta_1 = r_o \ , \ \delta_{i+1} = r_i - r_{i-1} \ , \ i \geq 1.$$

Then the system (1) is said to have δ_i (formal) zeros at infinity of multiplicity i.

The ingredients summarized so for enable us to give an answer to the problem of exact linearization via dynamic state feedback. The key tool in the procedure that follows is a nice canonical form under feedback-equivalence[7] which exists under the specific conditions stated hereafter. For the sake of notational simplicity we will restrict our considerations to the particular case of systems with three inputs and three outputs.

Theorem 1. Suppose $\ell = m = 3$ in (1). Moreover let the following assumptions be satisfied:

(A1) $\Delta^* = 0$

(A2) $\sum_{i=1}^{m} (L_{g_i}(\Omega_{k-1} \cap G^{\perp}))(x) \subset \Omega_{k-1}(x), \ \ k \geq 1.$

Then system (1) has exactly $\ell = 3$ (formal) zeros at infinity, of multiplicity $\mu_1 \leq \mu_2 \leq \mu_3$, and

$$\mu_1 + \mu_2 + \mu_3 = n.$$

Moreover, there exists a feedback $u = \alpha(x) + \beta(x)w$, with α and β defined in a neighborhood of x^o, such that

$$\begin{aligned} \dot{x} &= (f+g\alpha)(x) + (g\beta)(x)w \\ y &= h(x) \end{aligned} \qquad (4)$$

via the local diffeomorphism

$$\phi(x) = (\xi_1, \xi_2, \ldots, \xi_{\mu_1}, \eta_1, \eta_2, \ldots, \eta_{\mu_2}, \zeta_1, \zeta_2, \ldots, \zeta_{\mu_3})$$

where

$$\xi_i = L^{i-1}_{(f+g\alpha)} h_{j_1}$$

$$\eta_i = L^{i-1}_{(f+g\alpha)} h_{j_2}$$

$$\zeta_i = L^{i-1}_{(f+g\alpha)} h_{j_3}$$

and (j_1, j_2, j_3) is a permutation of (1,2,3), becomes

$$\dot{\xi}_1 = \xi_2$$

$$\ldots$$

$$\dot{\xi}_{\mu_1-1} = \xi_{\mu_1}$$

$$\dot{\xi}_{\mu_1} = w_1$$

$$\dot{\eta}_1 = \eta_2$$

$$\ldots$$

$$\dot{\eta}_{\mu_1-1} = \eta_{\mu_1}$$

$$\dot{\eta}_{\mu_1} = \eta_{\mu_1+1} + \gamma_{\mu_1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_1}, \zeta_1, \ldots, \zeta_{\mu_1}) w_1$$

$$\dot{\eta}_{\mu_1+1} = \eta_{\mu_1+2} + \gamma_{\mu_1+1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_1+1}, \zeta_1, \ldots, \zeta_{\mu_1+1}) w_1$$

$$\ldots$$

$$\dot{\eta}_{\mu_2-1} = \eta_{\mu_2} + \gamma_{\mu_2-1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2-1}, \zeta_1, \ldots, \zeta_{\mu_2-1}) w_1$$

$$\dot{\eta}_{\mu_2} = w_2$$

$$\dot{\zeta}_1 = \zeta_2$$

$$\ldots$$

$$\dot{\zeta}_{\mu_1-1} = \zeta_{\mu_1}$$

$$\dot{\zeta}_{\mu_1} = \zeta_{\mu_1+1} + \delta_{\mu_1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_1}, \zeta_1, \ldots, \zeta_{\mu_1}) w_1$$

$$\dot{\zeta}_{\mu_1+1} = \zeta_{\mu_1+2} + \delta_{\mu_1+1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_1+1}, \zeta_1, \ldots, \zeta_{\mu_1+1}) w_1$$

$$\ldots$$

$$\dot{\zeta}_{\mu_2-1} = \zeta_{\mu_2} + \delta_{\mu_2-1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2-1}, \zeta_1, \ldots, \zeta_{\mu_2-1}) w_1$$

$$\dot{\zeta}_{\mu_2} = \zeta_{\mu_2+1} + \delta_{\mu_2}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_2}) w_1 + \varepsilon_{\mu_2}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_2}) w_2$$

$$\dot{\zeta}_{\mu_2+1} = \zeta_{\mu_2+2} + \delta_{\mu_2+1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_2+1}) w_1 + \varepsilon_{\mu_2+1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_2+1}) w_2$$

$$\ldots$$

$$\dot{\zeta}_{\mu_3-1} = \zeta_{\mu_3} + \delta_{\mu_3-1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_3-1}) w_1 + \varepsilon_{\mu_3-1}(\xi_1, \ldots, \xi_{\mu_1}, \eta_1, \ldots, \eta_{\mu_2}, \zeta_1, \ldots, \zeta_{\mu_3-1}) w_2$$

$$\dot{\zeta}_{\mu_3} = w_3$$

$$y_{j_1} = \xi_1$$

$$y_{j_2} = \eta_1$$

$$y_{j_3} = \zeta_1. \qquad (5)$$

□

The proof of this Theorem may be found elsewhere[12]. Anyway, the interested reader may recover the fundamental steps of this proof from the application to the robot equations discussed in the second half of the paper.

The possibility of having exact linearization via dynamic feedback is shown in the following Corollary, whose proof is an easy consequence of the existence of the canonical form (5).

Corollary. Suppose $\ell = m = 3$. Moreover, let the assumptions (A1),(A2) be satisfied. Consider the following dynamic extension of system (4)

$$\begin{array}{ll} \dot{z}_{11} = z_{12} & \dot{z}_{21} = z_{22} \\ \dots & \dots \\ \dot{z}_{1,\mu_3-\mu_1-1} = z_{1,\mu_3-\mu_1} & \dot{z}_{2,\mu_3-\mu_2-1} = z_{2,\mu_3-\mu_2} \\ \dot{z}_{1,\mu_3-\mu_1} = \bar{w}_1 & \dot{z}_{2,\mu_3-\mu_2} = \bar{w}_2 \end{array} \tag{6}$$

$$w_1 = z_{11}\ , \quad w_2 = z_{21}\ , \quad w_3 = \bar{w}_3 .$$

Then the composition of (4) and (6) yields a dynamical system which is feedback-equivalent to a system of the form

$$\begin{aligned} \dot{\bar{\xi}}_{i1} &= \bar{\xi}_{i2} \\ &\dots \\ \dot{\bar{\xi}}_{i,\mu_3-1} &= \bar{\xi}_{i,\mu_3} \qquad , \quad i = 1,2,3, \\ \dot{\bar{\xi}}_{i,\mu_3} &= v_i \\ y_{j_i} &= \bar{\xi}_{i1} \end{aligned} \tag{7}$$

Proof. Let $\bar{x} = (x,z)$ and

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}) + \bar{g}(\bar{x})\bar{w} \\ y &= \bar{h}(\bar{x}) \end{aligned} \tag{8}$$

denote the composition of (4) and (6). A direct computation based on the canonical form (5) shows that

$L_{\bar{g}}L_{\bar{f}}^{k}\bar{h}_i = 0$, $i = 1,2,3$; $k = 0,\dots,\mu_3-2$.

and that the 3×3 matrix

$$\bar{A}(\bar{x}) = L_{\bar{g}}L_{\bar{f}}^{\mu_3-1}\bar{h}$$

is nonsingular. Then, there exist a feedback $\bar{w}=\bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})v$ which makes (8) input-output-wise linear and decoupled[10]. Moreover, since the dimension of $\bar{x}$ is $3\mu_3$, the mapping

$$\bar{\phi}(\bar{x}) = \{\bar{\xi}_{ij},\ j = 1,\dots,\mu_3;\ i = 1,2,3\}$$

with $\bar{\xi}_{ij} = L_{\bar{f}}^{j-1}\bar{h}_i(x)$, is a local diffeomorphism, which brings the system

$$\begin{aligned} \dot{\bar{x}} &= (\bar{f}+\bar{g}\bar{\alpha})(\bar{x}) + (\bar{g}\bar{\beta})(\bar{x})v \\ y &= \bar{h}(\bar{x}) \end{aligned}$$

to the form (7). □

A series of remarks are now in order.

Remark 1. The composition of the feedback $u=\alpha(x)+\beta(x)w$, the dynamic extension (6) and the feedback $\bar{w}=\bar{\alpha}(\bar{x})+\bar{\beta}(\bar{x})v$ characterizes a dynamic compensator of the form (2) which solves the exact linearization problem. The structure of this compensator is shown in Fig. 1. Note that system (1) has dimension $n = \mu_1+\mu_2+\mu_3$, the dynamic compensator has dimension $\nu = 2\mu_3-\mu_2-\mu_1$. The closed loop system has dimension $n+\nu = 3\mu_3$ and in suitable local coordinates appears as three decoupled chains of μ_3 integrators each.

Remark 2. It is well known[10] that a system in which the noninteracting control problem (via static state-feedback) is solvable, if $\Delta^* = 0$, is feedback-equivalent to a linear controllable system. As a matter of fact, the same feedback which yields noninteraction makes the system diffeomorphic to a linear controllable system.

If $\Delta^* \neq 0$, the above feedback yields input-output linearity but a possibly nonlinear unobservable part is left. In the present case we keep the assumption $\Delta^* = 0$ (see (A1)) but we replace the condition needed for solvability of the noninteracting control problem by the *weaker* assumption (A2). We still get full linearity at the state-space level and noninteraction but using now a dynamic, rather than static, state-feedback.

Remark 3. Note that in the canonical form (5) the drift vector field is linear and all the nonlinearity is concentrated in the vector fields which multiply the inputs. The triangular structure of the latter and the specific dependencies of their entries from the local coordinates is a direct consequence of the structural assumption (A2). The most important feature of the canonical form (5) is that the addition of integrators to any input channel does not destroy the condition $\Delta^* = 0$ (this is not always the case for nonlinear systems[13]). This explains why the composition of (4) with the dynamic extension (6), having still $\Delta^* = 0$, and being such that the noninteracting control problem is solvable, is feedback-equivalent to a linear (and decoupled) system.

Remark 4. In the applications one might be interested in a further, now linear, feedback from the state variables $\bar{\xi}_{ij}$, in order to place all the $n+\nu$ eigenvalues of the resulting closed loop system.

Remark 5. If two or three of the indexes μ_i are equal, the canonical form (5) particularizes in an obvious way. If, for instance, $\mu_1 = \mu_2$, not only the dynamics of the ξ_i's but also that of the η_i's is fully linear. It may be worth noting the relation between the μ_i's and the so-called characteristic numbers ρ_i's (the least integer such that $L_g L_f^{\rho_i} h_i \neq 0$). Assuming $\rho_1 \leq \rho_2 \leq \rho_3$ one has $\rho_1 = \mu_1-1$, $\rho_2 \leq \mu_2-1$, $\rho_3 \leq \mu_3-1$, equalities being true if and only if the noninteracting control problem is solvable via static state-feedback.

Exact Linearization of the Robot Arm with Elastic Joints

In this section we will apply the results described before to the control of a robot arm with elastic joints. The mathematical model of this kind of robot arm is briefly summarized hereafter[14].

Consider the mechanical structure of a robot as being constituted by an open chain of N+1 bodies (links) interconnected through N rotational/translational joints. The joints are activated by motors with transmission gears or belts; when the links and the transmissions are assumed to be rigid the dynamical behavior is that of a chain of N rigid bodies. In this case the Lagrangian formulation[15] leads to equations of motion in the form:

$$B(q)\ddot{q} + c(q,\dot{q}) + e(q) = m(t) \tag{9}$$

where q is the N-vector of joint variables giving the relative displacement between two adjacent links, $B(q)$ is the N×N nonsingular inertial matrix, $m(t)$ is the N-vector of generalized forces delivered by the motors, $e(q)$ is the N-vector of conservative forces and $c(q,\dot{q})$ is the N-vector collecting centrifugal and Coriolis forces.

When the transmissions are not rigid the N actuating bodies of the motors are elastically coupled to the driven links. Therefore, the dynamical behavior is that of 2N rigid bodies, N of which are directly actuated while the other N include elasticity; this is the case of interest here. The equations of motion are still given by (9), but with the following peculiarities:

- the number of second order equations is 2N;
- q is a 2N-vector in which q_{2i} denotes the displacement of link i w.r.t. link i-1 and q_{2i-1} denotes the displacement of the driving body of joint i w.r.t. link i-1, for i = 1,...,N;
- B(q) is the 2N×2N inertial nonsingular matrix of the 2N rigid bodies;
- e(q) and $c(q,\dot{q})$ are 2N-vectors and e(q) includes the effects of elasticity;
- m(t) is a 2N-vector with the even components equal to zero.

Starting from mechanical parameters, the model (9) is given automatically by the DYMIR code both for rigid and elastic robots[16]; (9) may be rewritten in the standard form

$$\dot{x} = f(x) + g(x)u \qquad y = h(x) \tag{10}$$

with state $x \triangleq \begin{bmatrix} x_p^T & x_v^T \end{bmatrix}^T \triangleq \begin{bmatrix} q^T & \dot{q}^T \end{bmatrix}^T \in M \subset R^n$, input $u \in R^m$ and output $y \in R^\ell$. In the elastic case n = 4N; moreover, the input u collects only the nonzero components of m(t) while the output y may be defined as the vector of link displacements $x_{2i} = q_{2i}$ (i=1,...,N). Thus, $m = \ell = N$. The expressions for f and g are given by:

$$f(x) = \begin{bmatrix} x_v \\ -B(x_p)^{-1}[c(x_p,x_v) + e(x_p)] \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 \\ B(x_p)^{-1}\mathrm{diag}\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right] \end{bmatrix}. \tag{11}$$

The equations of a PUMA-like three-link robot arm with elastic joints (see Fig. 2) are reported in Appendix 1.

It is well known[1,2,15] that the rigid robot can be decoupled and linearized via static state-feedback, whereas this is no longer the case whenever joint elasticity is not negligible[4]. In view of this we consider now the problem of achieving linearity via dynamic state-feedback. To this end the first thing to do is to perform the maximal invariant distribution Algorithm on the equations of the robot under consideration. All computations may be found with full details in Appendix 2.

As a result of these computations we find that assumptions (A1) and (A2) of Theorem 1 are satisfied. Moreover, since

$$r_o = 0,\ r_1 = 1,\ r_2 = 1,\ r_3 = 2,\ r_4 = 2,\ r_5 = r_{k*} = 3$$

we have

$$\delta_1 = 0,\ \delta_2 = 1,\ \delta_3 = 0,\ \delta_4 = 1,\ \delta_5 = 0,\ \delta_6 = 1$$

and thus $\mu_1 = 2$, $\mu_2 = 4$, $\mu_3 = 6$. In addition we see that the set of functions

$$\xi_i = L^{i-1}_{(f+g\alpha)}h_2 \qquad i = 1,2;$$

$$\eta_i = L^{i-1}_{(f+g\alpha)}h_1 \qquad i = 1,\dots,4;$$

$$\zeta_i = L^{i-1}_{(f+g\alpha)}h_3 \qquad i = 1,\dots,6$$

qualifies a new set of local coordinates in the state space. The function α(x) is given by:

$$\alpha(x) = \begin{bmatrix} \left(\frac{\phi_1(x)f_{10}(x)}{g_{10,3}(x)} + \phi_2(x)\right)/\left(g_{71}\cdot\frac{\partial f_8(x)}{\partial x_1}\right) \\ 0 \\ -f_{10}(x)/g_{10,3}(x) \end{bmatrix} \tag{12}$$

where all terms involved may be found in either Appendices. The choice of this α(x) together with a β(x) given by:

$$\beta(x) = \begin{bmatrix} L_g L_{(f+g\alpha)}h_2 \\ L_g L^3_{(f+g\alpha)}h_1 \\ L_g L^5_{(f+g\alpha)}h_3 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & g_{10,3}(x) \\ g_{71}\frac{\partial f_8(x)}{\partial x_1} & 0 & \phi_1(x) \\ \phi_3(x) & \phi_4 & \phi_5(x) \end{bmatrix}^{-1} \tag{13}$$

in system (4) yields, in the local coordinates ξ_i, η_i, ζ_i, the canonical form (5). The dynamic extension (6) considered in the Corollary of Theorem 1 consists here of the addition of $\mu_3 - \mu_1 = 4$ integrators on the input w_1 and of $\mu_3 - \mu_2 = 2$ integrators on the input w_2 i.e.

$$\begin{aligned} &\dot{z}_{11} = z_{12},\ \dot{z}_{12} = z_{13},\ \dot{z}_{13} = z_{14},\ \dot{z}_{14} = \bar{w}_1 \\ &\dot{z}_{21} = z_{22},\ \dot{z}_{22} = \bar{w}_2 \\ &w_1 = z_{11},\ w_2 = z_{21},\ w_3 = \bar{w}_3 \end{aligned} \tag{14}$$

The robot model (10) subject to a feedback $u = \alpha(x) + \beta(x)w$, with α and β specified by (12) and (13), together with the dynamic extension (14) is now a system which can be decoupled and fully linearized by a static state-feedback of the form $\bar{w} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})v$. In the notation of the previous section (recall that (8) indicates the composition of (4) and (6)) the functions $\bar{\alpha}$ and $\bar{\beta}$ are now given by

$$\bar{\beta}(\bar{x}) = \left[L_{\bar{g}} L^5_{\bar{f}} \bar{h}(\bar{x}) \right]^{-1}$$

$$\bar{\alpha}(\bar{x}) = -\bar{\beta}(\bar{x}) \cdot L^6_{\bar{f}} \bar{h}(\bar{x}) .$$

The resulting closed-loop system is locally diffeomorphic to three chains of $\mu_3 = 6$ integrators each.

Conclusions

In this paper we have shown how, under suitable assumptions, dynamic state-feedback can be used in order to make a given nonlinear system diffeomorphic to a linear controllable (and decoupled) one. The assumptions in question are indeed weaker than the ones which guarantee the achievement of the same result via static state-feedback. In particular, the assumption of nonsingularity of the so-called decoupling matrix has been replaced by the structural assumption (A2) which characterizes a specific property of the sequence of codistribution generated by means of the maximal controlled invariant distribution Algorithm. Intuitively speaking, the structural assumption (A2) simply means that, from the point of view of its formal structure at infinity, the system under consideration essentially behaves like a linear one.

The technique of dynamic extension used here in order to achieve decoupling is similar to the one proposed by Descusse and Moog[17]. The replacement of their conditions with the stronger assumptions (A1),(A2) provides the required state-space full linearization. Related results based on Hirschorn's inversion algorithm are due to Singh[18].

In the second part of the paper we applied our synthesis procedure to the case of a three-link robot arm with elastic joints. On the DYMIR-generated model[16] we checked the fulfillment of assumptions (A1),(A2) and showed how to compute all the relevant functions associated with the dynamic compensator. The complexity of the actual computations requires symbolic manipulation systems like MACSYMA or REDUCE. We considered as outputs the joint coordinates but the proposed approach is likewise successfull for task-oriented synthesis problems. Moreover, we conjecture that any robot model with joint elasticity satisfies the assumptions (A1) and (A2). The idea of using nonlinear feedback in order to compensate nonlinearities and to achieve noninteraction dates back to early works of Porter[19] and Singh and Rugh[20] ; similar techniques have been simultaneously and independently developed in the robotic field dealing with the case of rigid robots[21] . The solution of the same kind of problems for robots with joint elasticity can still be accomplished but now requires, as shown here, full exploitment of nonlinear differential geometric control techniques.

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Appendix 1

We report here the dynamic model of a three-link robot arm with joint elasticity (see Fig. 2). The state space representation has been obtained by means of a symbolic manipulation system (REDUCE) starting from the DYMIR code[16] which outputs the matrix and vector entries in (9). We have:

$$\dot{x} = f(x) + \sum_{i=1}^{3} g_i(x)u_i = f(x) + g(x)u,$$

$$y = h(x)$$

with

$$f(x) = [x_7 \; x_8 \; x_9 \; x_{10} \; x_{11} \; x_{12} \,\vdots\, f_7 \; f_8 \; f_9 \; f_{10} \; f_{11} \; f_{12}]^T,$$

$$g(x) = \begin{bmatrix} & \vdots & g_{71} & 0 & 0 & 0 & 0 & 0 \\ 0_{3\times 6} & \vdots & 0 & 0 & g_{92} & 0 & 0 & 0 \\ & \vdots & 0 & 0 & 0 & g_{10,3} & g_{11,3} & g_{12,3} \end{bmatrix}^T,$$

$$h(x) = [x_2 \; x_4 \; x_6]^T$$

where

$$g_{71} = G_1$$

$$g_{92} = G_3$$

$$g_{10,3} = -G_5H_8/\omega_1$$

$$g_{11,3}=[4H_8(H_3\cos x_6+H_7)-(H_3\cos x_6+2H_8)^2]/4\omega_1$$

$$g_{12,3} = G_5(H_3\cos x_6+2H_8)/2\omega_1$$

$$f_7 = (N_1x_2-x_1)K_1G_1/N_1^2$$

$$f_8=\{N_1x_8[x_{10}(H_1\sin(2x_4)+H_2\sin(2x_4+2x_6)+H_3\sin(2x_4+x_6))$$
$$+x_{12}(H_2\sin(2x_4+2x_6)+H_3\cos x_4\sin(x_4+x_6))]$$
$$-K_1(N_1x_2-x_1)\}/N_1\omega_2$$

$$f_9 = (N_2x_4-x_3)K_2G_3/N_2^2$$

$$f_{10}=G_5\{N_2N_3^2[(H_3\cos x_6+2H_8)(x_8^2(H_2\sin(2x_4+2x_6)$$
$$+H_3\cos x_4\sin(x_4+x_6))+x_{10}^2H_3\sin x_6)$$
$$+2H_8(x_{12}(2x_{10}+x_{12})H_3\sin x_6$$
$$-x_8^2(H_1\sin(2x_4)+H_2\sin(2x_4+2x_6)+H_3\sin(2x_4+x_6))]$$
$$+2N_2[N_3(H_3\cos x_6+2H_8)(N_3\omega_3+K_3(N_3x_6-x_5))$$
$$-2H_8K_3(N_3x_6-x_5)]$$
$$-4N_3^2H_8[N_2H_5\cos x_4+N_2\omega_3+K_2(N_2x_4-x_3)]\}/4N_2N_3^2\omega_1$$

$$f_{11}=-\{G_5N_2N_3^2[(H_3\cos x_6+2H_8)(x_8^2(H_2\sin(2x_4+2x_6)$$
$$+H_3\cos x_4\sin(x_4+x_6))+x_{10}^2H_3\sin x_6)$$
$$+2H_8(x_{12}(2x_{10}+x_{12})H_3\sin x_6$$
$$-x_8^2(H_1\sin(2x_4)+H_2\sin(2x_4+2x_6)$$
$$+H_3\sin(2x_4+x_6)))]$$
$$-4G_5N_3^2H_8[N_2H_5\cos x_4+N_2\omega_3+K_2(N_2x_4-x_3)]$$
$$+N_2[2G_5N_3(H_3\cos x_6+2H_8)(N_3\omega_3+K_3(N_3x_6-x_5))$$
$$-K_3(N_3x_6-x_5)(4H_8(H_3\cos x_6+H_7)$$
$$-(H_3\cos x_6+2H_8)^2)]\}/4N_2N_3^2\omega_1$$

$$f_{12}=G_5\{-N_2N_3^2[2(H_3\cos x_6+H_7-G_5)(x_8^2(H_2\sin(2x_4+2x_6)$$
$$+H_3\cos x_4\sin(x_4+x_6))+x_{10}^2H_3\sin x_6)$$
$$+(H_3\cos x_6+2H_8)(x_{12}(2x_{10}+x_{12})H_3\sin x_6$$
$$-x_8^2(H_1\sin(2x_4)+H_2\sin(2x_4+2x_6)$$
$$+H_3\sin(2x_4+x_6)))]$$
$$+2N_3^2(H_3\cos x_6+2H_8)[N_2H_5\cos x_4+N_2\omega_3+K_2(N_2x_4-x_3)]$$
$$-2N_2[2N_3(H_3\cos x_6+H_7-G_5)(N_3\omega_3+K_3(N_3x_6-x_5))$$
$$-K_3(N_3x_6-x_5)(H_3\cos x_6+2H_8)]\}/4N_2N_3^2\omega_1.$$

In the expressions above we defined for compactness the terms:

$$\omega_1(x_4) = H_9 + H_{10}\cos^2 x_6$$

$$\omega_2(x_4,x_6)=H_1\cos^2 x_4+H_2\cos^2(x_4+x_6)+H_3\cos x_4\cos(x_4+x_6)+H_4$$

$$\omega_3(x_6) = H_3\cos x_6+H_7.$$

The constants $H_1 \ldots H_{10}$ and G_1, G_3, G_5 depend on the robot data which include length, mass, inertia tensor and center of mass for each link, mass and inertia tensor for each rotor; furthermore at joint i, N_i is the reduction ratio of the gear box and K_i is its elastic constant.

We collect in this Appendix also the relevant terms which are computed during the application of the maximal controlled invariant distribution Algorithm to the robot arm under consideration (see Appendix 2 and formulas (12) and (13) in the text):

$$\tilde{g}_{11,3}=g_{11,3}/g_{10,3}=-[1+\omega_1/H_8G_5^2]$$

$$\tilde{g}_{12,3}=g_{12,3}/g_{10,3}=-[1+(H_3/2H_8)\cos x_6]$$

$$\tilde{f}_{11}=f_{11}-f_{10}\tilde{g}_{11,3}$$
$$=-\{[x_8^2(H_2\sin(2x_4+2x_6)+H_3\cos x_4\sin(x_4+x_6))$$
$$+x_{10}^2H_3\sin x_6]/2H_8+[N_3\omega_3+K_3(N_3x_6-x_5)]/N_3H_8\}$$

$$\tilde{f}_{12}=f_{12}-f_{10}\tilde{g}_{12,3}=\{[(x_{10}+x_{12})^2H_3\sin x_6)$$
$$-x_8^2(H_1\sin(2x_4)+H_3\sin x_4\cos(x_4+x_6))]/2$$
$$+H_3\cos x_6[x_8^2(H_2\sin(2x_4+2x_6)+H_3\cos x_4\sin(x_4+x_6))$$
$$+x_{10}^2H_3\sin x_6+2\omega_3]/4H_8+(1+H_3\cos x_6/2H_8)(N_3x_6-x_5)K_3/N_3$$
$$-(N_2x_4-x_3)K_2/N_2-H_5\cos x_4\}/G_5$$

$$\phi_1(x)=g_{10,3}\Big(\frac{\partial f_8}{\partial x_4}+x_{10}\frac{\partial^2 f_8}{\partial x_4\partial x_{10}}+x_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{10}}+f_8\frac{\partial^2 f_8}{\partial x_8\partial_{10}}$$
$$+\frac{\partial f_8}{\partial x_8}\frac{\partial f_8}{\partial x_{10}}+\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde{f}_{12}}{\partial x_{10}}\Big)+g_{12,3}\Big(\frac{\partial f_8}{\partial x_6}+x_{10}\frac{\partial^2 f_8}{\partial x_4 x_{12}}$$
$$+x_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{12}}+f_8\frac{\partial^2 f_8}{\partial x_8\partial x_{12}}+\frac{\partial f_8}{\partial x_8}\frac{\partial f_8}{\partial x_{12}}\Big)$$

$$\phi_2(x)=-\{x_7\Big(2x_{10}\frac{\partial^2 f_8}{\partial x_1\partial x_4}+2x_{12}\frac{\partial^2 f_8}{\partial x_1\partial x_6}+\frac{\partial f_8}{\partial x_1}\frac{\partial f_8}{\partial x_8}\Big)$$
$$+x_8\Big(2x_{10}\frac{\partial^2 f_8}{\partial x_2\partial x_4}+2x_{12}\frac{\partial^2 f_8}{\partial x_2\partial x_6}+\frac{\partial f_8}{\partial x_2}\frac{\partial f_8}{\partial x_8}\Big)$$
$$+x_{10}\Big(x_{10}\frac{\partial^2 f_8}{\partial x_4^2}+2x_{12}\frac{\partial^2 f_8}{\partial x_4\partial x_6}+\frac{\partial f_8}{\partial x_4}\frac{\partial f_8}{\partial x_8}+2f_8\frac{\partial^2 f_8}{\partial x_4\partial x_8}$$
$$+\tilde{f}_{12}\frac{\partial^2 f_8}{\partial x_4\partial x_{12}}+\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde{f}_{12}}{\partial x_4}\Big)+x_{11}\Big(\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde{f}_{12}}{\partial x_5}\Big)$$
$$+x_{12}\Big(x_{12}\frac{\partial^2 f_8}{\partial x_6^2}+\frac{\partial f_8}{\partial x_6}\frac{\partial f_8}{\partial x_8}+2f_8\frac{\partial^2 f_8}{\partial x_6\partial x_8}$$
$$+\tilde{f}_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{12}}+\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde{f}_{12}}{\partial x_6}\Big)+f_7\Big(\frac{\partial f_8}{\partial x_1}\Big)$$
$$+f_8\Big(\frac{\partial f_8}{\partial x_2}+\Big(\frac{\partial f_8}{\partial x_8}\Big)^2+f_{12}\frac{\partial^2 f_8}{\partial x_8\partial x_{12}}+\frac{\partial f_8}{\partial x_{12}}+\frac{\partial \tilde{f}_{12}}{\partial x_8}\Big)$$

$$+f_{10}\left(\frac{\partial f_8}{\partial x_4}+x_{10}\frac{\partial^2 f_8}{\partial x_4\partial x_{10}}+x_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{10}}+f_8\frac{\partial^2 f_8}{\partial x_8\partial x_{10}}+\frac{\partial f_8}{\partial x_8}\frac{\partial f_8}{\partial x_{10}}+\frac{\partial f_8}{\partial x_{12}}\frac{\partial \tilde f_{12}}{\partial x_{10}}\right)+f_{12}\left(\frac{\partial f_8}{\partial x_6}+x_{10}\frac{\partial^2 f_8}{\partial x_4\partial x_{12}}+x_{12}\frac{\partial^2 f_8}{\partial x_6\partial x_{12}}+f_8\frac{\partial^2 f_8}{\partial x_8\partial x_{12}}+\frac{\partial f_8}{\partial x_8}\frac{\partial f_8}{\partial x_{12}}\right)$$

$$\phi_3(x)=G_1\Big\{\frac{K_1}{N_1}\Big[x_{10}\frac{\partial}{\partial x_4}\Big(\frac{1}{\omega_2}\frac{\partial\tilde f_{12}}{\partial x_8}\Big)+x_{12}\frac{\partial}{\partial x_6}\Big(\frac{1}{\omega_2}\frac{\partial\tilde f_{12}}{\partial x_8}\Big)+\frac{f_8}{\omega_2}\frac{\partial^2\tilde f_{12}}{\partial x_8^2}\Big]-\frac{K_1}{\omega_2}\Big[2x_{10}\frac{\partial^2\tilde f_{12}}{\partial x_4\partial x_8}+2x_{12}\frac{\partial^2\tilde f_{12}}{\partial x_6\partial x_8}+2f_8\frac{\partial^2\tilde f_{12}}{\partial x_8^2}+\frac{\partial\tilde f_{12}}{\partial x_8}\frac{\partial f_8}{\partial x_8}\Big]+\frac{K_1}{\omega_2^2}\frac{\partial\tilde f_{12}}{\partial x_8}\Big[x_{10}\frac{\partial\omega_2}{\partial x_4}-\frac{x_{12}}{N_1}\frac{\partial\omega_2}{\partial x_6}\Big]\Big\}$$

$$\phi_4 = G_3K_2K_3/N_2N_3G_5H_8$$

$$\phi_5(x)=g_{10,3}\frac{\partial L^5_{\tilde f}h_3}{\partial x_{10}}+g_{11,3}\frac{\partial L^5_{\tilde f}h_3}{\partial x_{11}}+g_{12,3}\frac{\partial L^5_{\tilde f}h_3}{\partial x_{12}}$$

$$\tilde g_{71} = g_{71}\tilde\beta_{11} = N_1\omega_2/K_1$$

$$\tilde g_{73} = g_{71}\tilde\beta_{13} = N_1\omega_1\omega_2\phi_1/K_1G_5H_8$$

$$\tilde f_7 = f_7+g_{71}\tilde\alpha_1 = f_7+\frac{N_1\omega_2}{K_1}\Big(\phi_2-\frac{\omega_1\phi_1}{G_5H_8}f_{10}\Big).$$

Appendix 2

In this Appendix we apply the maximal controlled invariant distribution Algorithm, in the form suggested by Krener[22], to the three-link robot arm with non negligible joint elasticity whose model is reported in Appendix 1; we will show that this model satisfies the assumptions (A1) and (A2) of Theorem 1. For the sake of completeness we report here the above Algorithm.

From the components $h_1,\dots,h_\ell$ of the map h one constructs first of all the (x-dependent) subspace (of row vectors)

$$\Omega_o(x) = \mathrm{span}\{dh_1(x),\dots,dh_\ell(x)\}$$

Suppose $\Omega_o(x)$ has dimension $s_o \le \ell$ in a neighborhood of a point x^o. Then there exists an $s_o\times 1$ column vector λ_o, whose entries $\lambda_{01},\dots,\lambda_{0s_o}$ are entries of h, with the property that the differentials $d\lambda_{01},\dots,d\lambda_{0s_o}$ are linearly independent at all x in a neighborhood of x^o. The Algorithm consists of a finite number of iterations, each one defined as follows.

Iteration (k). Consider the $s_k\times m$ matrix $A_k(x)$ whose (i,j)-entry is $d\lambda_{ki}(x)g_j(x)$. Suppose that in a neighborhood of x^o the rank of $A_k(x)$ is constant and equal to r_k. Then it is possible to find r_k rows of $A_k(x)$ which, for all x in a neighborhood of x^o, are linearly independent. Let $P_k^T=[P_{k1}^T \mid P_{k2}^T]$ be an $s_k\times s_k$ permutation matrix, such that the r_k rows of $P_{k1}A_k(x)$ are linearly independent. Let $B_k(x)$ be an s_k-vector whose i-th element is $d\lambda_{ki}(x)f(x)$. As a consequence of the assumptions on P_{k1}, the equations

$$P_{k1}A_k(x)\alpha(x) = -P_{k1}B_k(x)$$
$$P_{k1}A_k(x)\beta(x) = K \qquad (15)$$

(where K is a matrix of real numbers, of rank r_k) may be solved for α and β, an m-vector and an m×m invertible matrix whose entries are real-valued smooth functions defined in a neighborhood of x^o. Set $\tilde f=\tilde g_o = f+g\alpha$ and $\tilde g_i=(g\beta)_i$, $1\le i\le m$.

Consider the set of functions

$$\Lambda_k=\{\lambda=L_{\tilde g_i}\lambda_{kj}:\ 1\le j\le s_k,\ 0\le i\le m\}$$

and the two (x-dependent) subspaces (of row vectors)

$$\Omega_k(x) = \mathrm{span}\{d\lambda_{k1}(x),\dots,d\lambda_{ks_k}(x)\}$$
$$\Omega_k'(x) = \mathrm{span}\{d\lambda(x):\ \lambda\in\Lambda_k\}$$

Set $\Omega_{k+1}(x)=\Omega_k(x)+\Omega_k'(x)$.

Suppose $\Omega_{k+1}(x)$ has constant dimension $s_{k+1}(\ge s_k)$ in a neighborhood of x^o. Let $\lambda_{k+1,1},\dots,\lambda_{k+1,s_{k+1}}$ be entries of λ_k and/or elements of Λ_k such that the differentials $d\lambda_{k+1,1},\dots,d\lambda_{k+1,s_{k+1}}$ are linearly independent at all x in a neighborhood of x^o. Define the s_{k+1}-vector λ_{k+1} whose i-th entry is the function $\lambda_{k+1,i}$. This concludes the k-th iteration. At each stage of the Algorithm two integers are considered

$$s_k=\dim\Omega_k(x),\quad r_k=\mathrm{rank}\,A_k(x).$$

Since $s_k\le s_{k+1}\le n$, a dimensionality argument shows that there exists an integer k^* such that $s_k=s_{k^*}$, $r_k=r_{k^*}$ for all $k\ge k^*$. The sequence $\{r_0,r_1,\dots\}$ provides the structure at the infinity associated with the triplet (f,g,h).

We perform next this Algorithm on the triplet (f,g,h) which describes the robot arm system dynamics. In the *initial step* we use the output functions h_1,h_2 and h_3 and we get

$$\Omega_o=\mathrm{sp}\{dh_1dh_2dh_3\}=\mathrm{sp}\left\{\begin{array}{cccccc|c}0&1&0&0&0&0&\\0&0&0&1&0&0&O_{3\times 6}\\0&0&0&0&0&1&\end{array}\right\}$$

and thus $s_o=3$ and $\lambda_o=h$.

In the 0-*th iteration* we have:

$$A_o=d\lambda_o\cdot g=L_gh=0\ ,\quad r_o=0$$

and hence $\Omega_o\cap G^\perp=\Omega_o$ so that it is easy to see that assumption (A2) holds for k = 1. Furthermore,

$$\Omega_1=\Omega_o\oplus\mathrm{sp}\{dL_fh_1dL_fh_2dL_fh_3\}=\Omega_o\oplus\mathrm{sp}\left\{\begin{array}{c|cccccc}&0&1&0&0&0&0\\O_{3\times 6}&0&0&0&1&0&0\\&0&0&0&0&0&1\end{array}\right\}$$

giving $s_1 = 6$ and $\lambda_1 = [h^T\ L_f h^T]^T = [x_2\ x_4\ x_6\ x_8\ x_{10}\ x_{12}]$. This way of "translating" dependencies from the first group of states (x_p) to the second one (x_v) reflects the Newtonian structure of the considered system.

In the 1-*st iteration*, the matrix

$$A_1 = d\lambda_1 \cdot g = \left[\begin{array}{c|ccc} 0_{3\times 3} & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & g_{10,3} & g_{12,3} \end{array}\right]^T$$

has rank $r_1 = 1$; thus, we have to compute a feedback pair (α,β) from equation (15). Choosing P_1 such that $P_{11} = [0\ 0\ 0\ 0\ 1\ 0]$, since $B_1 = d\lambda_1 \cdot f =$
$= [x_8\ x_{10}\ x_{12}\ f_8\ f_{10}\ f_{12}]$ we have as a solution:

$\alpha_1 = \alpha_2 = 0$, $\alpha_3 = -f_{10}/g_{10,3}$,

$\beta_{11} = \beta_{22} = 1$, $\beta_{33} = 1/g_{10,3}$, $\beta_{ij} = 0\ (i \neq j)$.

This gives:

$$\tilde{f} = f + g\alpha = [x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12} \mid f_7\ f_8\ f_9\ 0\ \tilde{f}_{11}\ \tilde{f}_{12}]^T$$

$$\tilde{g} = g\beta = \left[\begin{array}{c|cccccc} 0_{3\times 6} & g_{71} & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & g_{82} & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & \tilde{g}_{11,3} & \tilde{g}_{12,3} \end{array}\right]^T.$$

The complete expression of the new terms involved is reported in Appendix 1; notice only that the new vector fields $\tilde{f}, \tilde{g}_i$ have much simpler forms than the original ones. Furthermore since $L_f h = L_{\tilde{f}} h$, the set Λ_1 where we have to look for functions with linear independent differentials is the following:

$$\Lambda_1 = \{L^2_{\tilde{f}} h_j,\ L_{\tilde{g}_i} L_{\tilde{f}} h_j;\ i,j = 1,2,3\}.$$

We get:

$$L_{\tilde{g}_1} L_{\tilde{f}} h = L_{\tilde{g}_2} L_{\tilde{f}} h = 0,$$

$$L_{\tilde{g}_3} L_{\tilde{f}} h = [0\ \ 1\ \ \tilde{g}_{12,3}]^T,$$

$$L^2_{\tilde{f}} h = [f_8\ \ 0\ \ \tilde{f}_{12}].$$

From $\tilde{g}_{12,3} = \tilde{g}_{12,3}(x_4 x_6)$ we have at this step that $\sum_{i=1}^{3} L_{\tilde{g}_i}(\Omega_1 \cap G^\perp) \subset \Omega_1$ (assumption (A2) for k = 2) holds. Thus,

$$\Omega_2 = \Omega_1 \oplus \mathrm{sp}\{dL^2_{\tilde{f}} h_1\ dL^2_{\tilde{f}} h_3\}$$

$$= \Omega_1 \oplus \mathrm{sp}\left\{\begin{array}{cccccc|cccccc} \partial f_8/\partial x_1 & 0 & * & 0 & 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & * & \partial\tilde{f}_{12}/\partial x_5 & * & 0 & * & 0 & * & 0 & 0 \end{array}\right\}$$

where $*$ denotes non relevant terms and $\partial f_8/\partial x_1 = K_1/N_1\omega_2 \neq 0$, $\partial\tilde{f}_{12}/\partial x_5 = K_3/N_3H_8 \neq 0$ (a constant). Note that ω_2 is always nonzero being the second diagonal element of the inertia matrix $B(x_p)$ of the robot, which is positive definite for all x_p. So $s_2 = 8$ everywhere and $\lambda_2 = [h^T\ L_{\tilde{f}} h^T\ L^2_{\tilde{f}} h_1\ L^2_{\tilde{f}} h_3]^T$. Moreover, the characteristic numbers for the second and third outputs are $\rho_2 = \rho_3 = 1$ while $\rho_1 > 1$.

In the 2-*nd iteration*,

$$A_2 = d\lambda_2 \cdot g = \left[\begin{array}{c|ccccc} 0_{3\times 3} & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & g_{10,3} & g_{12,3} & * & * \end{array}\right]^T,\ r_2 = 1.$$

As long as r_k remains constant we do not need to recompute a feedback pair (α,β). The functions in the set Λ_2 are the following:

$$L_{\tilde{g}} L^2_{\tilde{f}} h_1 = [0\ \ 0\ \ (\partial f_8/\partial x_{10} + \tilde{g}_{12,3}\partial f_8/\partial x_{12})]$$

$$L_{\tilde{g}} L^2_{\tilde{f}} h_3 = [0\ \ 0\ \ (\partial\tilde{f}_{12}/\partial x_{10})]$$

$$L^3_{\tilde{f}} h_1 = x_7(\partial f_8/\partial x_1) + \psi_1(x)$$

$$L^3_{\tilde{f}} h_3 = x_{11}(\partial\tilde{f}_{12}/\partial x_5) + \psi_2(x)$$

where $L_{\tilde{g}_3} L^2_{\tilde{f}} h_1$, $L_{\tilde{g}_3} L^2_{\tilde{f}} h_3$, ψ_1 and ψ_2 are all independent from $x_3 x_7 x_9 x_{11}$.

Again we have

$$\sum_{i=1}^{3} L_{\tilde{g}_i}(\Omega_2 \cap G^\perp) \subset \Omega_2$$

and

$$\Omega_3 = \Omega_2 \oplus \mathrm{sp}\{dL^3_{\tilde{f}} h_1\ \ dL^3_{\tilde{f}} h_3\}$$

$$= \Omega_2 \oplus \mathrm{sp}\left\{\begin{array}{cccccc|cccccc} * & * & 0 & * & * & * & \partial f_8/\partial x_1 & * & 0 & * & 0 & * \\ * & * & 0 & * & 0 & * & 0 & * & 0 & * & \partial\tilde{f}_{12}/\partial x_5 & * \end{array}\right\}$$

giving $s_3 = 10$ everywhere and $\lambda_3 =$
$= [h^T\ L_{\tilde{f}} h^T\ L^2_{\tilde{f}} h_1\ L^2_{\tilde{f}} h_3\ L^3_{\tilde{f}} h_1\ L^3_{\tilde{f}} h_3]^T$.

We can see that $\rho_1 = 2$; the rank of the decoupling matrix[10] $A(x)$ - which is a feedback invariant - is thus:

$$\mathrm{rank}\ A(x) = \mathrm{rank}\begin{bmatrix} L_{\tilde{g}} L^2_{\tilde{f}} h_1 \\ L_{\tilde{g}} L_{\tilde{f}} h_2 \\ L_{\tilde{g}} L_{\tilde{f}} h_3 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} = 1$$

and we conclude that, as expected, the system is not decoupable by static state-feedback. It is also worth mentioning that this system does not satisfy the necessary and sufficient conditions[23] for the existence of a static state-feedback law which makes the input-dependent part of the response of the closed loop system linear in the input and independent from the initial state, as shown by Marino and Nicosia[4].

Coming back to the 3-*rd iteration* of the Algorithm we have:

$$A_3 = d\lambda_3 \cdot g = \left[\begin{array}{c|ccc|cccc} 0_{3\times 3} & 0 & 0 & 0 & 0 & 0 & (g_{71}\partial f_8/\partial x_1) & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & g_{10,3} & g_{12,3} & * & * & \phi_1(x) & * \end{array}\right]^T$$

and $r_3 = 2$. Choose the permutation matrix P_3 so that P_{31} picks up rows 5 and 9 from A_3. Then $-P_{31}B_3 =$
$= -P_{31} \cdot d\lambda_3 \cdot f = [-f_{10}\ \ \phi_2(x)]^T$ and a feedback pair $(\tilde{\alpha},\tilde{\beta})$ is obtained solving the matrix equation (15) which gives:

$\tilde{\alpha}_1 = [(\phi_1(x) f_{10}/g_{10,3}) + \phi_2(x)]/(g_{71} \cdot \partial f_8/\partial x_1)$

$\tilde{\alpha}_2 = 0 \ , \quad \tilde{\alpha}_3 = -f_{10}/g_{10,3} \ ,$

$\tilde{\beta}_{11} = 1/(g_{71} \cdot \partial f_8/\partial x_1), \ \tilde{\beta}_{22} = 1, \ \tilde{\beta}_{33} = 1/g_{10,3} \ ,$

$\tilde{\beta}_{13} = -\phi_1(x)/(g_{71} g_{10,3} \cdot \partial f_8/\partial x_1), \ \tilde{\beta}_{ij} = 0$ (else).

The new vector fields $\tilde{f}, \tilde{g}$ are then:

$$\tilde{f} = f + g\tilde{\alpha} = [\, x_7 \ x_8 \ x_9 \ x_{10} \ x_{11} \ x_{12} \mid \tilde{f}_7 \ f_8 \ f_9 \ 0 \ \tilde{f}_{11} \ \tilde{f}_{12} \,]^T ,$$

$$\tilde{g} = g\tilde{\beta} = \left[\begin{array}{c|cccccc} & \tilde{g}_{71} & 0 & 0 & 0 & 0 & 0 \\ 0_{3\times 6} & \tilde{0} & 0 & g_{92} & 0 & 0 & 0 \\ & \tilde{g}_{73} & 0 & 0 & 1 & \tilde{g}_{11,3} & \tilde{g}_{12,3} \end{array} \right]^T .$$

Again, the expression of the new terms involved are given in full in Appendix 1.

Compute next the functions in the set

$$\Lambda_3 = \{ L_{\tilde f} L_{\tilde f}^3 h_j , \ L_{\tilde g_i} L_{\tilde f}^3 h_j \ ; \ j = 1,3; \ i = 1,2,3 \}$$

which have the following expressions:

$$L_{\tilde g_1} L_{\tilde f}^3 h_1 = 1, \ L_{\tilde g_1} L_{\tilde f}^3 h_1 = L_{\tilde g_2} L_{\tilde f}^3 h_1 = L_{\tilde g_2} L_{\tilde f}^3 h_3 = 0,$$

$$L_{\tilde g_3} L_{\tilde f}^3 h_1 = \psi_3(x), \ L_{\tilde g_3} L_{\tilde f}^3 h_3 = \psi_4(x)$$

$$L_{\tilde f} L_{\tilde f}^3 h_1 = \psi_5(x), \ L_{\tilde f} L_{\tilde f}^3 h_3 = x_3 (\partial \tilde f_{11}/\partial x_3)(\partial \tilde f_{12}/\partial x_5) + \psi_6(x)$$

with $\partial \tilde f_{11}/\partial x_3 = K_2/N_2 G_5$ (constant) $\neq 0$ and

$\psi_3(x), \ldots, \psi_6(x)$ all independent from x_3, x_9. Thus we still find $\sum_{i=1}^{3} L_{\tilde g_i}(\Omega_3 \cap G^{\perp}) \subset \Omega_3$ and

$$\Omega_4 = \Omega_3 \oplus sp\{d(L_{\tilde f} L_{\tilde f}^3 h_1)\} - \Omega_3 \oplus sp\{* * (\frac{\partial \tilde f_{11}}{\partial x_3} \frac{\partial \tilde f_{12}}{\partial x_5}) * * * \mid * * 0 * 0 *\}$$

with $s_4 = 11$, globally; finally

$$\lambda_4 = [\, h^T \ L_{\tilde f} h^T \ L_{\tilde f}^2 h_1 \ L_{\tilde f}^2 h_3 \ L_{\tilde f}^3 h_1 \ L_{\tilde f}^3 h_3 \ L_{\tilde f} L_{\tilde f}^3 h_3 \,]^T .$$

In the 4-*th iteration* the rank r_4 of the matrix $A_4 = d\lambda_4 \cdot g$ is again 2; after similar computation as in the previous steps we can see that the only "new" function from the set Λ_4 is

$$L_{\tilde f}^2 L_{\tilde f}^3 h_3 = x_9 (\partial \tilde f_{11}/\partial x_3)(\partial \tilde f_{12}/\partial x_5) + \psi_7(x)$$

where $\psi_7(x)$ does not depend on x_9. The structural assumption (A2) is again satisfied at this step (and hence for all $k \geq 1$) and

$$\Omega_5 = \Omega_4 \oplus sp\{dL_{\tilde f}^2 L_{\tilde f}^3 h_3\} = \Omega_4 \oplus sp\{* * * * * * \mid * * (\frac{\partial \tilde f_{11}}{\partial x_3} \frac{\partial \tilde f_{12}}{\partial x_5}) * * *\}$$

giving $s_5 = 12 = \dim M$. Thus $k^* = 5$, $\Omega_* = T^* M$ for any x and $\Delta^* = \Omega_*^{\perp} = 0$, assumption (A1) of Theorem 1.

Last,

$$\lambda_5 = [\, h^T \ L_{\tilde f} h^T \ L_{\tilde f}^2 h_1 \ L_{\tilde f}^2 h_3 \ L_{\tilde f}^3 h_1 \ L_{\tilde f}^3 h_3 \ L_{\tilde f} L_{\tilde f}^3 h_3 \ L_{\tilde f}^2 L_{\tilde f}^3 h_3 \,]^T$$

and we need to compute the matrix

$$A_5 = d\lambda_5 g = \left[\begin{array}{ccc|ccc|cc|cc|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{71}\partial f_8/\partial x_1 & 0 & * & \phi_3(x) \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_4 \\ 0 & 0 & 0 & 0 & g_{10,3} & * & * & * & \phi_1(x) & * & * & \phi_5(x) \end{array} \right]^T$$

which has rank $r_5 = 3$, being $\phi_4 = g_{92}(\partial \tilde f_{11}/\partial x_3)(\partial \tilde f_{12}/\partial x_5)$ a nonzero constant.

To conclude this Appendix, note that the following identities hold:

$$L_{\tilde f} h_i = L_f h_i = x_{6+2i} \ , \ i = 1,2,3;$$

$$L_{\tilde f}^2 h_1 = L_f^2 h_1 = f_8 \ , \ L_{\tilde f}^2 h_3 = L_f^2 h_3 = \tilde f_{12} \ ;$$

Furthermore $L_{\tilde f}^3 h_1 = L_f^3 h_1$ which is due to the fact that the vector fields $\tilde f$ and f differ only in the seventh component while f_8 is independent from x_7; for the same reason $L_{\tilde f}^3 h_3 = L_f^3 h_3$.

Thus Ω_* can also be spanned by the differentials of the following set of functions:

$$\xi_1 = h_2, \ \xi_2 = L_{\tilde f} h_2;$$

$$\eta_1 = h_1, \ \eta_2 = L_{\tilde f} h_1 , \ \eta_3 = L_{\tilde f}^2 h_1 , \ \eta_4 = L_{\tilde f}^3 h_1 ;$$

$$\zeta_1 = h_3, \ \zeta_2 = L_{\tilde f} h_3 , \ \zeta_3 = L_{\tilde f}^2 h_3 , \ \zeta_4 - L_{\tilde f}^3 h_3 , \ \zeta_5 - L_{\tilde f}^4 h_3 ,$$

$$\zeta_6 - L_{\tilde f}^5 h_3 .$$

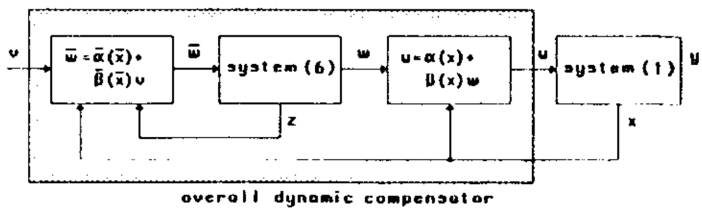

Fig. 1.

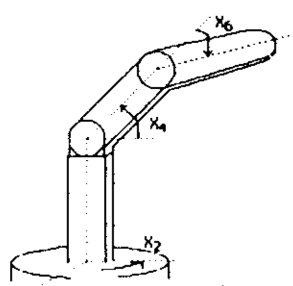

Fig. 2.