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MATHEMATICAL PROBLEMS IN GEAR TRANSMISSIONS

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Mathematical Problems in Gear Transmissions

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Abstract: Two problems are considered: 1. The exact construction of a pair of cog-wheels, where the driving gear is circular and has eccentric bearing with large eccentricity. The shape of the output drive wheel is not known beforehand.
2. In gearboxes all components not being connected to transmit torque may undergo clattering vibrations. A method is presentec so that this type of oscillation can be resolved in a more or less stochastic sequence of impacts taking place at the edges of the gear or bearing tolerances. Some hints are given on why such systems exhibit chaotic behaviour.

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Lots of mathematical problems - very different in nature and in degree of difficulty - arise in connection with gears:

- A method of construction which limits seem to have been reached may have to be analyzed and given a new mathematical description. Relevant quantities may have to be computed in a new manner.
- Dimensions of gears may have to be optimized to obtain a desired behaviour.
- Mathematical considerations may be necessary so that a special gearing may be constructed at all.
- The influence of unavoidable tolerances may have to be examined. Or hints may have to be given in order to construct a gear transmission so that it is as insensitive to tolerances as possible.
- New principles of construction for the tooth surfaces may have to be developed so that the dissipation of energy during the transmission is minimized.

This is only a short selection from the list of problems. In this paper I shall look at two extreme cases of gear transmissions:

1. Very slowly running gear transmissions; in this situation out-of-balance forces, and tolerances within reasonable limits, or noise from clattering vibrations don't matter.

2. Fast running gear transmissions, where the phenomena just mentioned play a more and more important role. The clattering noise for example is not only an additional burden to the environment or place of work but a hearable loss of energy.

I. ECCENTRIC GEARING

The first example comes from the watch-and-clock-making industry. The engineers wanted to construct a pair of gears with a special output torque (fig. 1). This should be realized by a circular driving gear with eccentric bearing and involute tooth system (fig. 2). The output drive wheel should perform a single turn whenever the input wheel has rotated once.

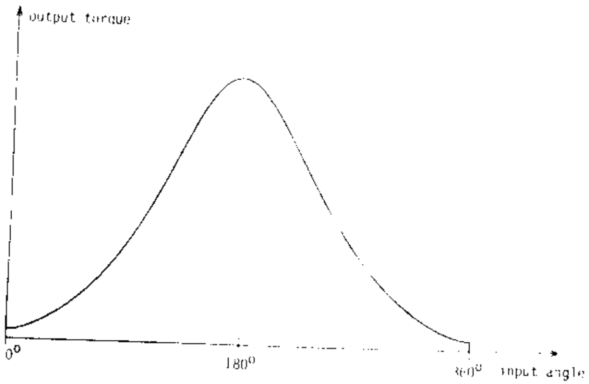

Figure 1.- Graph of the desired output torque.

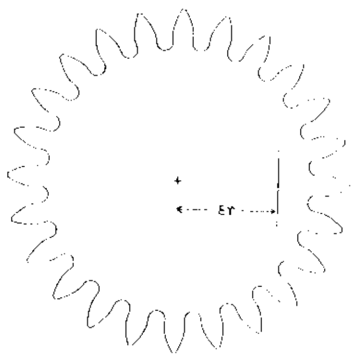

Figure 2.- Circular driving gear with eccentric bearing (radius r, eccentricity ε) and involute tooth system.

Technical and mathematical PROBLEM:

What is the shape of the output drive wheel?

Essentially four problems have to be solved, which can be considered in pairs:

(i) construction of the second pitch-curve (the first pitch-curve is the circle),

(ii) computation of the distance between the pivot points,

(iii) construction of the base curve on which the teeth are "standing",

(iv) construction of the tooth surfaces as involute system.

To (i) and (ii): When considering common arcs of length s which roll one against the other, the condition of contact (fig. 3)

$$\rho(s) + [(c - x(s))^2 + (y(s))^2]^{1/2} = c$$

together with the condition that s shall measure arc length

$$(\dot{x}(s))^2 + (\dot{y}(s))^2 = 1$$

lead to the following first order system of differential equations

$$\dot{x} = \frac{1}{c-\rho}\left(\dot{\rho}(c-x) \pm \sqrt{(1-\dot{\rho}^2)[(c-\rho)^2 - (c-x)^2]}\right)$$

$$\dot{y} = \pm\sqrt{1-\dot{x}^2}\ .$$

(The plus- or the minus-sign has to be chosen suitably, when solving the system.)

Here $(x(s),y(s))$ is the unknown parametrization of the second pitch curve and c the unknown distance between the pivot points. The function ρ contains all the information about the driving pitch curve, in this case:

$\rho(s) = r[1+\varepsilon^2 - 2\varepsilon\cos\frac{s}{r}]^{1/2}$ with radius r of the driving wheel and eccentricity ε of the pivot point. The distance c will be determined by the conditions

$$y(0) = y(\pi r) = 0.$$

It turned out that the integral curves are closed curves for exactly one $c > 0$ if r and ε are given. This c has to be computed with high accuracy.

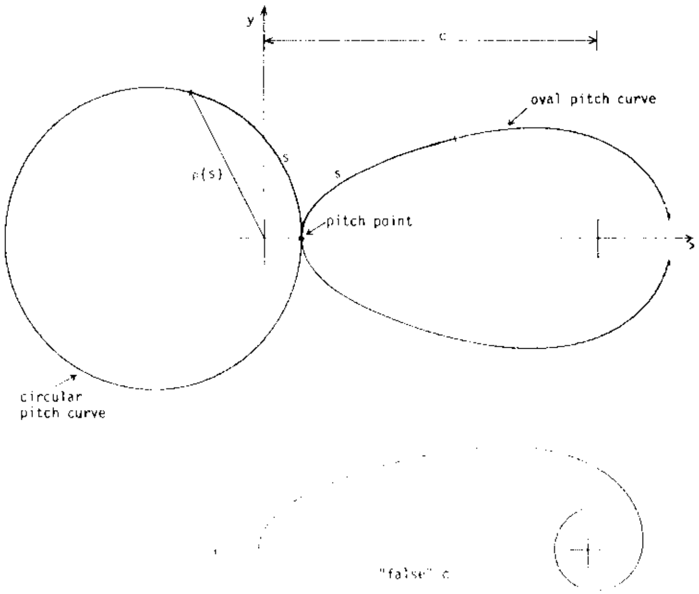

Figure 3.- The oval pitch curve for the "right" c (above) and for a "false" c.

The sensitive dependence of all further constructions of this c was not known to the engineers. They used without comment ("for small eccentricities") an approximation formula:

$$c \approx r \cdot (\nu+1)[1 - \frac{\nu-2}{4\nu} \varepsilon^2] ,$$

where ν is the gear ratio (originally published by L. Burmester, Lehrbuch der Kinematik, 1888).

Figure 3 shows the two pitch-curves for $\varepsilon = 0.75$ and the "right" c, and the second pitch curve when c is chosen arbitrarily.

To (iii) and (iv): The involute tooth system for a cog-wheel is uniquely determined by the following condition:

> At any position of the wheels there is the constant angle α between the common normal in the point of contact of the two tooth surfaces and the tangent in the pitch point. α is called the pressure angle or the angle of mesh.

This means: The tooth surfaces are the involutes of the so-called base curve which is the envelope of all tangents to the pitch curve which have been rotated around the angle α.

The base curve of a circle of radius r is a concentric circle of radius $r \cos \alpha$. In general: If $(x(s),y(s))$ is the parametrization of the pitch curve with respect to the arc length parameter s, then $(X(s),Y(s))$ with

$$X(s) = x(s) - d_2 \frac{d_1\dot{x}(s) + d_2\dot{y}(s)}{\dot{x}(s)\ddot{y}(s) - \ddot{x}(s)\dot{y}(s)} ,$$

$$Y(s) = y(s) + d_2 \frac{d_2\dot{x}(s) - d_1\dot{y}(s)}{\dot{x}(s)\ddot{y}(s) - \ddot{x}(s)\dot{y}(s)}$$

is a parametrization of the base curve.

$D = \begin{pmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{pmatrix}$ is the rotation matrix of α. The denominator of the second summand of X and Y is the curvature in (x,y) and produced numerical and technical difficulties. The base curve of the oval pitch curve, the points of which were given in numerical form, has been computed by this procedure.

The results were surprising: pressure angle α and eccentricity ε may not be chosen arbitrarily, if tooth surfaces shall be constructed as involutes. If ε increases or α increases, the base curve is no longer simple closed; there are self-intersections; it is no longer smooth and runs into the exterior of the pitch curve (fig. 4).

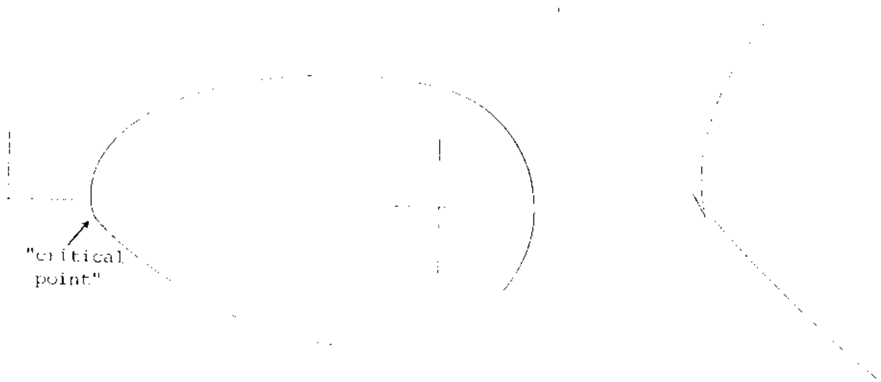

Figure 4.- The base curve for $\varepsilon = 0.627$ and $\alpha = 22^0$ (left) and an enlargement of the critical part for $\alpha = 28^0$ (right).

If one chooses (as usual) [illegible] than the maximum of the output-torque (fig. 1) may not be made as large as one likes. On the other hand, if a greater maximum is needed, the pressure angle has to be decreased. Figure 4 shows for an eccentricity $\epsilon = 0.627$ the base curve for $\alpha = 22^0$, the beginning of the "catastrophe", and for $\alpha = 28^0$ the enlargement of the critical part of the corresponding base curve.

If the base curve has been computed - by decreasing ϵ or α if necessary - the construction of the tooth surfaces as pieces of involutes of the base curve is no longer a problem.

The final result is shown in figure 5.

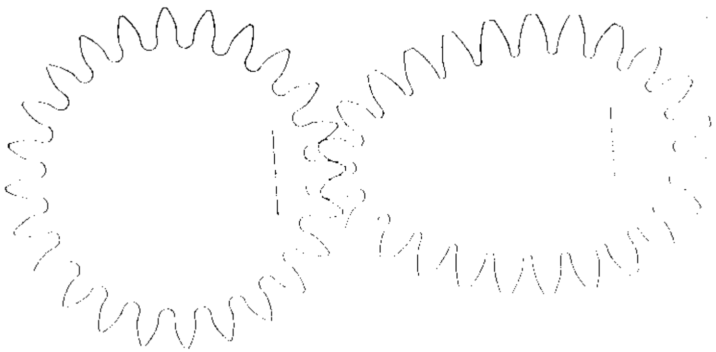

Figure 5.- The final design of the eccentric gearing.

II. CLATTERING VIBRATIONS IN GEARBOXES

In gearboxes of motor vehicles as well as in gear transmissions of all fast running machines (for example sewing machines in industry or household) clattering vibrations arise. It is clear that it is impossible to build such systems without allowing tolerances between the teeth of the cog-wheels and in the various bearings. The vibrations are caused by a sequence of impacts which occur when the components hit the edges of the tolerance intervals.

To be able to optimize the essential parameters, such as tolerances, damping, friction, excitation form, it is first necessary to give a mathematical description of clattering vibrations. F. Küçükay and F. Pfeifer, Munich, developed a "Generalized Impulsive Motion Theory" ([14], [26]) which enables a purely deterministic description while the results look as if they originated in a stochastic motion.

The most simple mechanical model is shown in figure 6 ([14]).

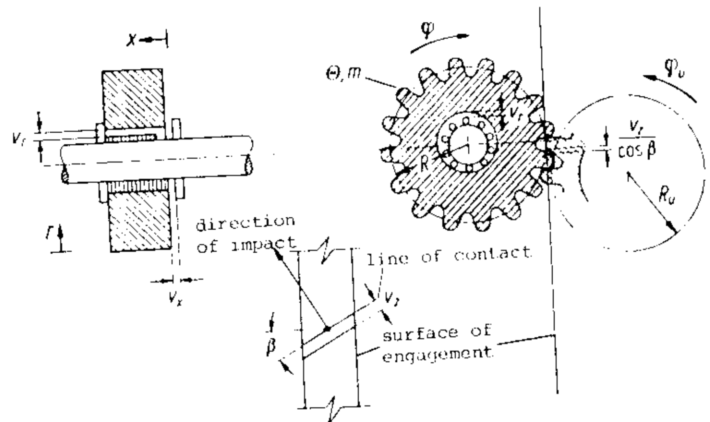

Figure 6.- The most simple mechanical model to examine clattering vibrations: v_r resp. v_x maximal tolerances in radial resp. axial directions; v_z play of teeth; β the angle of helical gearing.

The driven gear allows axial and radial bearing tolerances v_x and v_r and the play v_z between the teeth. Within these tolerances the wheel with mass m and moment of inertia Θ moves as a "flying" body. Thus the equations of motion are

$$\begin{aligned} \Theta\ddot{\varphi} &= T_\varphi - d_\varphi\dot{\varphi} \\ m\ddot{x} &= T_x - d_x\dot{x} \\ m\ddot{r} &= T_r - d_r\dot{r}. \end{aligned} \qquad (1)$$

T_φ, T_x, T_r model the drag due to the oil and d_φ, d_x, d_r are the coefficients of damping. The equations (1) are valid as long as

$$-v_x < x < 0,$$
$$-v_r < r < 0$$

and the relative distance s_z of the teeth surfaces satisfies

$$-v_z(t) < s_z < o_z(t).$$

The edges of the tolerance interval for s_z depend on time t periodically and from purely geometric considerations s_z is given by

$$s_z = e(t) - [(R\cos\beta)\varphi(t) + (\sin\beta)x(t) - (\cos\beta)\cdot r(t)],$$

where e(t) is the excitation function (for example: $e(t) = A\cdot\sin(\omega t)$); e acts in the direction of the normal of the tooth engagement.

When x, r or s_z reach one edge of their tolerance intervals at time t_1 an impact occurs and the initial conditions for solving (1) for $t > t_1$, have to be recalculated. The impact is modelled by

$$\dot{x}^+ = -\varepsilon_x\dot{x}^-,\quad \dot{r}^+ = -\varepsilon_r\dot{r}^-,\quad \dot{s}_z^+ = -\varepsilon_z\dot{s}_z^-; \qquad (2)$$

The minus-sign means the state before the impact and the plus-sign the state after the impact. $\varepsilon_x, \varepsilon_r, \varepsilon_z \in [0,1]$ are the impact coefficients. There are three different situations for impacts

(i) an impact in only one play,
(ii) an impact in two plays,
(iii) an impact of all three plays at the same time.

In the last case the initial conditions for the motion after the impact are known by (2). Interesting are (i) and (ii). For example if there is only an impact between the teeth, the initial conditions also change for x and r. In this situation the kinetic transition equations for the impact give

$$\begin{pmatrix}\dot\varphi^+\\ \dot x^+\\ \dot r^+\end{pmatrix} = \begin{pmatrix}\dot\varphi^-\\ \dot x^-\\ \dot r^-\end{pmatrix} + \eta\begin{pmatrix}mR\cos\beta\\ \Theta\sin\beta\\ \Theta\cos\beta\end{pmatrix}\cdot[\dot e - Q^-], \qquad (3)$$

where

$$\eta = \frac{1+\varepsilon_z}{\Theta + mR^2\cos^2\beta}$$

and

$$Q^- = (R\cos\beta)\cdot\dot\varphi^- + (\sin\beta)\cdot\dot x^- + (\cos\beta)\dot r^-.$$

Thus the description of that simplest case is complete. If one writes $q := \begin{pmatrix}\varphi\\ x\\ r\end{pmatrix}$, $w := \begin{pmatrix}R\cos\beta\\ \sin\beta\\ \cos\beta\end{pmatrix}$, and $M = \mathrm{diag}(\Theta,m,m)$, then (1) may be written as

$$M\cdot\ddot q = F(q,\dot q,t), \qquad (4)$$

and the constraint (3) as

$$Z = Z_0\cdot\dot q^+ + Z_1\cdot\dot q^- + Z_2 = 0 \qquad (5)$$

with $Z_0 = w^T$, $Z_1 = \varepsilon_z w^T$, $Z_2 = -\dot e(1+\varepsilon_z)$.

The results can be applied immediately to gear transmissions with several gears. Then q has n coordinates $\varphi_1, x_1, r_1, \varphi_2, x_2, r_2, \ldots$, where n is the number of degrees of freedom which may be involved in impacts. M is a symmetric (nxn)-matrix; Z_0, Z_1 are (mxn)-matrices, Z_2 the "excitation vector" with m coordinates; m is the number of constraints, i.e. the number of degrees of freedom involved in the actual impact. With (5) and the general form of the transition equations

$$M(\dot q^+ - \dot q^-) + \left(\frac{\partial Z}{\partial \dot q^+}\right)^T\Lambda = 0$$

follows the general form of (3):

$$\dot q^+ = [E - M^{-1}Z_0^T(Z_0M^{-1}Z_0^T)^{-1}(Z_0 + Z_1)]\dot q^- - M^{-1}Z_0^T(Z_0M^{-1}Z_0^T)^{-1}Z_2. \qquad (6)$$

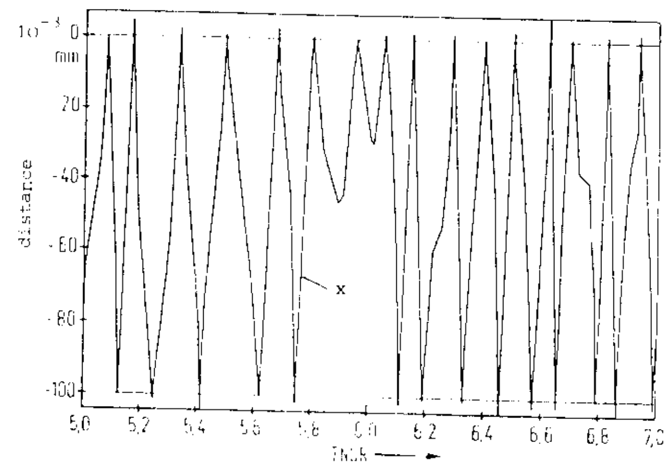

Figure 7.- The axial oszillations of the bearing.

Figures 7, 8 and 9 from [14] show the run of the relative distance s_z of the second gear of a gearbox with five-gearshifting and the axial and radial oscillations of the same wheel ($v_x = 0.1$ mm, $v_r = 0.05$ mm, $e(t) = 0.15 \sin(\omega t)$, average play of teeth = 0.1 mm, $\varepsilon_z = 0.87$).

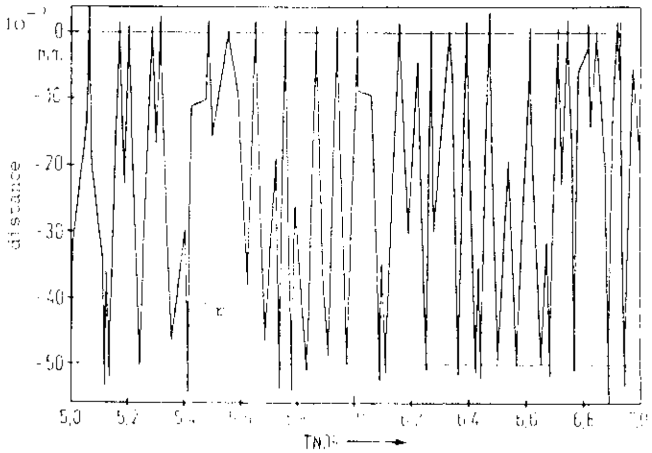

Figure 8.- The radial oscillations of the bearing

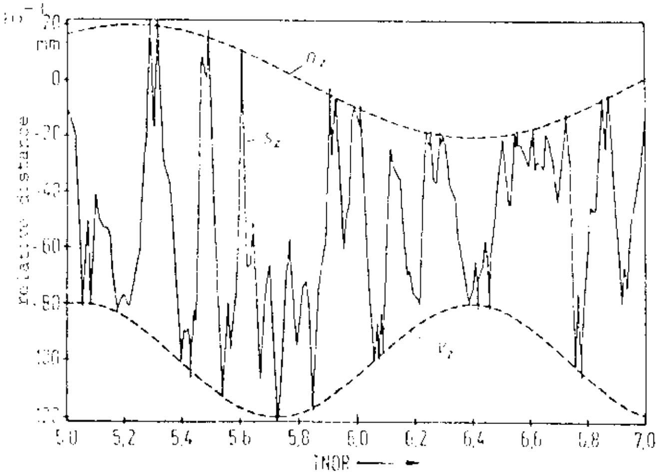

Figure 9.- The relative distance s_z and the edges of tolerance v_z and ν_z.

Though the description of the motion is purely deterministic, the results show an almost chaotic behaviour. Such phenomena are well known. How can they be explained in this context?

Let us again consider the most simple equations (1), (2), (3), and additionally (as in [8]):

(i) $\beta = 0$, which means: no impact in the axial direction;
(ii) only impacts between teeth.

Then we have $q = \begin{pmatrix}\varphi \\ r\end{pmatrix}$. Let $t_1 < t_2 < \ldots$ be the ordered time-series where t_k stands for the time at which the k-th impact occurs, and let $\dot\varphi_k = \dot\varphi(t_k)$ and $\dot r_k = \dot r(t_k)$ just before the impact, then (3) takes the simpler form

$$\begin{pmatrix}\dot\varphi_k^+ \\ \dot r_k^+\end{pmatrix} = M^* \begin{pmatrix}\dot\varphi_k \\ \dot r_k\end{pmatrix} + \begin{pmatrix}\eta m R \dot e(t_k) \\ \eta \Theta \dot e(t_k)\end{pmatrix}, \qquad (7)$$

where

$$M^* = \begin{pmatrix} \frac{\Theta - \varepsilon_z m R^2}{\Theta + m R^2} & -\eta m R \\ -\eta \Theta R & \frac{m R^2 - \varepsilon_z \Theta}{m R^2 + \Theta} \end{pmatrix} .$$

Solving the system (1) and using (7) successively, the sequence (t_k) can be calculated. In the special case $R = m = \Theta = 1$, hence $\eta = \frac{1+\varepsilon_z}{2}$, $d_\varphi = d_r = d$ and $o_z = v_z$ one gets the recursive equation

$$\pi_{k+1} = \pi_k + \frac{T}{d}(t_{k+1} - t_k) + \left(\frac{T}{d^2} - \frac{J_k}{d}\right)\left[e^{-d(t_{k+1} - t_k)} - 1\right], \qquad (8)$$

where $T = T_\varphi + T_r$, $J_k = \dot\varphi_k + \dot r_k$, and π_k has to be chosen "suitably" from $(e(t_k) - v_z(t_k), e(t_k) + v_z(t_k))$. Moreover

$$J_{k+1} = -\varepsilon_z J_k + A\omega \cos(\omega t_k), \qquad (9)$$

if e is chosen as mentioned at the beginning.

The equations (8) and (9) define a discrete mapping

$$(t_k, J_k) \longmapsto (t_{k+1}, J_{k+1}) \qquad (10)$$

which is usually taken as the starting point for the study of nonlinear dynamical systems with chaotic behaviour ([15]).
Under additional assumptions the physical meaning of which is immediately clear:

$$d \ll 1, \; T \ll 1, \; \omega(t_{k+1}-t_k) \ll 1, \; \varepsilon_z < 1, \; \varepsilon_z \approx 1,$$

(8) can be solved approximately and the mapping (10) reads

$$\begin{aligned} J_{k+1} &= -\varepsilon_z J_k + A\omega \cos(\omega t_k) \\ t_{k+1} &= t_k + \left|\frac{v_z}{J_{k+1}}\right| . \end{aligned} \qquad (11)$$

The equations (11) belong to the class of Fermi accelerator systems ([15]). Such dynamical systems exhibit strange attractors for particular sets of external parameters. On a strange attractor the motion is chaotic.

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