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ANNEX TO LECTURE NOTES ON IDENTIFICATION OF LINEAR DYNAMICAL SYSTEMS

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Parametrization problems for spaces of linear input-output systems

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This note introduces and discusses the general problem of finding good parametrizations of sets of possible models, mainly in the context of finite dimensional dynamic input-output models. The general problem is addressed in particular in the case where it is impossible to find one global parametrization.

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1. Introduction

This note is concerned with a fairly persvasive problem in modeling and identification. Namely the general problem: What is a "good" parametrization for a given model class? Where "good" of course has to be specified and may depend on other factors than just the model class in question. In some of its aspects it is a very old problem and has been with us ever since it was noted that there are several competing cartographic projections which can be used to map the earth and that none of them is perfect (or best) for all purposes.

I shall try to address this question in the context of modeling by means of linear dynamical input-output systems of a priori known state-space dimension (MacMillan degree). That is, we shall assume that our input-output observations are to be modeled by means of a system

$$\dot{x} = Ax + Bu,\ y = Cx,\ x \in \mathbf{R}^n, u \in \mathbf{R}^m, y \in \mathbf{R}^p \quad (\Sigma) \tag{1.1}$$

where A,B,C are constant (unknown) matrices of the appropriate sizes, and where it is assumed that (1.1) is completely reachable (cr) and completely observable (co). (For algebraic criteria for these two conditions c.f. below). A system like (1.1) induces an input-output map V_Σ, which, assuming that the machine (i.e. the system, or the model) starts at $x=0$ at time $t=0$, is given by

$$V_\Sigma : u(\cdot) \mapsto y(\cdot),\ y(t) = \int_0^t Ce^{(t-\tau)A} Bu(\tau)d\tau \tag{1.2}$$

The only data we have available are input-output data. So all that is knowable (identifiable) about (1.1) is the information about $\Sigma = (A,B,C)$ which is encoded in V_Σ. However V_Σ does not determine (A,B,C) uniquely, i.e. the map $(A,B,C) \mapsto V_\Sigma$ is not injective on the space $L^{co,cr}_{m,n,p}$ of all *cr* and *co* matrix triples (A,B,C) of the indicated dimensions. Indeed let $S \in Gl_n(\mathbf{R})$, i.e. S is an invertible real $n \times n$ matrix. Consider

$$\Sigma^S = (A,B,C)^S = (SAS^{-1}, SB, CS^{-1}) \tag{1.3}$$

It is totally elementary to observe that $V_{\Sigma^S} = V_\Sigma$. The transformation (1.3) corresponds to a base change $x'=Sx$ in state space. It is also a fact that this is the only redundancy in the description $(A,B,C) = \Sigma$ with respect to V_Σ. (I.e. if $\Sigma, \Sigma' \in L^{co,cr}_{m,n,p}$, and $V_\Sigma = V_{\Sigma'}$, then $\exists S \in Gl_n(\mathbf{R})$ such that $\Sigma' = \Sigma^S$). The relation

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$$\Sigma \sim \Sigma' \Leftrightarrow \exists S \in Gl_n(\mathbf{R}) \text{ such that } \Sigma' = \Sigma^S \tag{1.4}$$

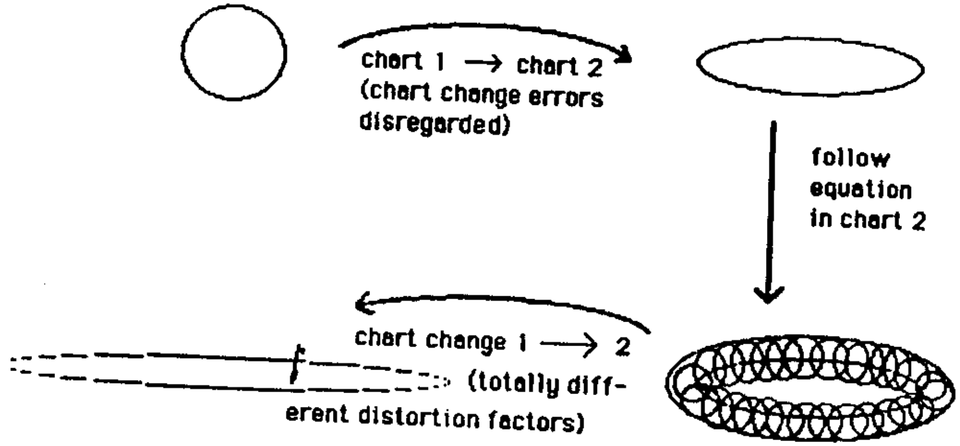
is of course an equivalence relation. The import of the remark above is thus that all we can identify on the basis of input-output data is the equivalence class of a system under this equivalence relation. Or, in other words, what can be identified is a point of the quotient space

$$M^{co,cr}_{m,n,p} = L^{co,cr}_{m,n,p} / \sim\ = L^{co,cr}_{m,n,p} / Gl_n(\mathbf{R}) \tag{1.5}$$

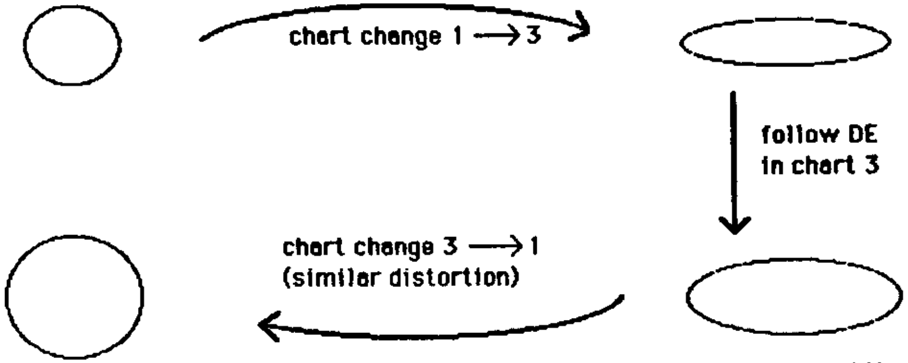
It now turns out that $M^{co,cr}_{m,n,p}$ is in fact quite nice. It is a differentiable manifold (of dimension $mn+np$), c.f. below. That is, it is locally like $\mathbf{R}^{mn+np}$ and can be described by $nm+np$ coordinates ((coordinate) charts) locally, together with correspondence rules, to yield an atlas, very much like an atlas of the world. Cf. below for an explicit atlas of this kind. It also turns out that if $m>1$ and $p>1$ it is not possible to make do with one particular chart. (Similarly it is not possible to have one global coordinate system for the whole earth (the sphere S^2) giving a unique correspondence (continuous both ways) between a part $U \subset \mathbf{R}^2$ and S^2).

Now imagine that we are engaged in a recursive identification procedure. So at time t we have ("best") estimates $\hat{A}_t, \hat{B}_t, \hat{C}_t$) for A,B,C (and $\hat{x}_t$ for the state x at time t). New information comes in and we want to update our estimates. $(\hat{A}_t, \hat{B}_t, \hat{C}_t)$ determines a point in $M^{co,cr}_{m,n,p}$ and we are looking for an optimal nearby point representing our updated estimate. This can be done using a coordinate chart valid at $(\hat{A}_t, \hat{B}_t, \hat{C}_t)$, calculating the relevant numerical coordinates, and calculating the updated versions of these coordinates according to some criterium function as expressed in these same coordinates. Proceeding in this gives a sequence of points $m_t, m_{t+1}, m_{t+2}, \ldots$ in $M^{co,cr}_{m,n,p}$ (represented by, say, $(\hat{A}_t, \hat{B}_t, \hat{C}_t)$, $(\hat{A}_{t+1}, \hat{B}_{t+1}, \hat{C}_{t+1}), \ldots$) and there may come a time when it becomes necessary to switch to another chart, because, say, m_{t+k} is no longer in the domain where the chart we are using is defined, or, in any case, is getting too near the "edge" of this chart to make these chart coordinates very reliable. Think again of using an ordinary street atlas, say, and changing charts when needed. In this framework one can make the general parametrization problem more precise; for instance as follows. Given a differential equation (or class of them), what are good atlases and good switching rules between coordinate charts in order to be able to follow this differential equation well numerically.

To illustrate the point consider the following situation



We start in chart 1 with a point known up to a small uncertainty as indicated. At this point chart 2 is also applicable. Changing coordinates at this point changes the uncertainty circle into an ellipse. (Uncertainty less in y-direction, more in x-direction). Following the equation in chart 2 introduces some additional uncertainty fattening up the ellipse (and even if it did not the difficulties would remain). It now becomes necessary to transfer back to chart 1 again. But now at this point in space the distortion factors may have changed totally. (In the picture a transformation $2 \mapsto 1$ at the first point compresses in the x-direction and magnifies in the y-direction; at the second point it magnifies in the x-direction and compresses in the y-direction). The result is a very elongated ellipse of uncertainty in chart 1 coordinates. Suppose we could also have worked with the coordinates of a chart 3 which as it happened had the following chart change distortion behaviour.



Obviously in this case having chart 3 available was advantageous even though the whole manifold could perhaps have been described in terms of charts 1 and 2 only. (It is by the way very easy to construct examples where this happens).

As described the good-atlases-and-parametrizations-problem seems particularly relevant in the case of recursive identification procedures. The problem however does not go away in the non-recursive case. There remains selecting a best (or good) chart from the several which may be available (and discarding one which turns out to be unsuitable in favour of a new one). And even if one could make do with one chart (on the basis of prior (structural) information concerning the class of models (e.g. in case $p=1$ or $m=1$ this is always possible) this may not be a particularly good one to use for a given problem. (Think of using a map of the earth covering all except the North-pole with in fact the region of interest very near the North-pole but not including it). Algorithms for identification based on overlapping coordinate charts, i.e. atlases, have in fact been developed, c.f. [2,7].

Related to the fact that as a rule it is impossible to use one chart to describe all of $M^{co,cr}_{m,n,p}$ is the fact that it is impossible to select a complete distinguishable class of models in $L^{co,cr}_{m,n,p}$ which is continous with respect to the data. (Nonexistence of continuous canonical forms [3,5]). All this means the following for a class $C \subset L^{co,cr}_{m,n,p}$

(i) Complete: for every input-output operator V (of the type coming from a Σ as in (1.1)) there is in fact a $\Sigma \in C$ such that $V_\Sigma = V$.

(ii) Distinguishable: $\Sigma_1, \Sigma_2 \in C$ and $\Sigma_1 \neq \Sigma_2 \Rightarrow V_{\Sigma_1} \neq V_{\Sigma_2}$.

(iii) Continuous: Let $V \mapsto \Sigma$ be the map determined by (i). Then this map is continuous.

So, roughly speaking it is not possible to select in a nice way one representant of each equivalence class of systems so as to remove the (statistical) indeterminacy of identifying A,B,C on the basis of input-output data alone.

This note which contains material presented at a most stimulating conference in the Pfalz academy in Lambrecht last March, is meant as an introduction to the problem and as an opportunity to introduce to the more applied community the sometimes advantageous possibility (and occasionally the necessity) of using several coordinate charts, and whole atlases. I hope and plan to write a much fuller version in the future. It is a pleasure to thank the organizer of the conference, Prof. H. Neunzert, for bringing this unusual group of scientists together.

2. Description of the spaces of all linear systems of a given degree

As in § 1 above, let $L_{m,n,p}$ be the space of all triples (A,B,C) of matrices of sizes $n\times n, n\times m$ and $p\times n$ respectively. The triple (A,B,C) (in fact the pair (A,B)) is called completely reachable if the $(n+1)m\times n$ reachability matrix

$$R(A,B) = (B \mid AB \mid A^2B \mid \cdots \mid A^nB) = R(A,B,C) = R(\Sigma) \tag{2.1}$$

has rank n. Dually the triple (A,B,C) (in fact the pair (A,C)) is called completely observable if the $n\times(n+1)p$ observability matrix

$$Q(A,C) = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^n \end{bmatrix} = Q(A,B,C) = Q(\Sigma) \tag{2.2}$$

has rank n. The spaces of cr, resp. co, resp. cr and co triples are denoted $L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$, $L^{co,cr}_{m,n,p}$. All three are open dense subspaces of $L_{m,n,p}$ (in the natural topology).

The group of invertible $n\times n$ real matrices $Gl_n(\mathbb{R})$ acts on $L_{m,n,p}$ by the formula given in (1.3) above. The subspaces of cr, co, cr and co systems are stable under this action. Indeed

$$R((A,B,C)^S) = R(SAS^{-1}, SB, CS^{-1}) = SR(A,B,C) \tag{2.3}$$

so that $rk\ R(\Sigma)=n$ iff $rk\ R(\Sigma^S)=n$. And $Q((A,B,C)^S) = S^{-1}Q(A,B,C)$.

The quotient spaces of $L_{m,n,p}$, $L^{co}_{m,n,p}$ and $L^{co,cr}_{m,n,p}$ by this action of $Gl_n(\mathbb{R})$ are denoted $M^{cr}_{m,n,p} = L^{cr}_{m,n,p}/Gl_n(\mathbb{R})$, $M^{co}_{m,n,p} = L^{co}_{m,n,p}/Gl_n(\mathbb{R})$, $M^{co,cr}_{m,n,p} = L^{co,cr}_{m,n,p}/Gl_n(\mathbb{R})$. All these quotient spaces are non-compact, smooth manifolds of dimension $mn+np$.

Below in this section we shall give one detailed description of $M^{cr}_{m,n,p}$ in terms of (coordinate) charts and gluing (= chart correspondence) rules, i.e. in terms of an atlas. To do this we need a few definitions. Consider an array $J_{m,n}$ of $n\times(n+1)m$ dots as indicated below

$$J_{m,n} = \{(i,j) \colon i\in\{0,\ldots,n\},\ j\in\{1,\ldots,m\}\} \tag{2.4}$$

$$J_{5,7} = \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \tag{2.5}$$

The first row of $J_{m,n}$ represents the columns of the matrix B, the second one the columns of AB, etc. Thus $(i,j)\in J_{m,n}$ represents the vector A^ib_j if $B=(b_1,\cdots,b_m)$. A subset α of size n of $J_{m,n}$ is called a *nice selection* if $(i,j)\in\alpha$, $i\geq 1 \Rightarrow (i-1,j)\in\alpha$. Pictorially, if α is depicted as a set of crosses in the array as visualized by (2.5), this means that if a cross appears anywhere then in the column above it there are only crosses. Thus e.g. the left subset of $J_{5,7}$ in (2.6) below is nice, the middle one is not.

$$\begin{array}{ccccc} x & \cdot & x & x & \cdot \\ x & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccc} \cdot & x & \cdot & x & x \\ x & \cdot & \cdot & \cdot & x \\ x & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccc} x & * & x & x & * \\ x & \cdot & * & x & \cdot \\ * & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & * & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{2.6}$$

For a nice selection α and $j\in 1,\dots,m$ let $s(\alpha,j)$ be the element $(k,j)\in J_{n,m}$ determined by $(k,j)\notin\alpha$ and $(i,j)\in\alpha$ for $i\leq k-1$. This one is called the j-th successor index. In (2.6) above the successor indices of the nice selection on the left are indicated by * in the rightmost diagram. Given an $n\times(n+1)m$ matrix R and a subset α of $J_{m,n}$ let R_α denote the matrix obtained from R by removing all columns whose index is not in α.

LEMMA 2.7 *Let $(A,B,C)\in L^{\sigma}_{m,n,p}$, then there is a nice selection α such that the $n\times n$ matrix $R(A,B,C)_\alpha$ is invertible.*

This follows from the special structure of $R(A,B,C)$ given that $R(A,B,C)$ has rank n because (A,B,C) is *cr*.

Let $L_\alpha = \{(A,B,C)\in L^{\sigma}_{m,n,p}\colon R(A,B)_\alpha$ is invertible$\}$. Note that $\Sigma^S\in L_\alpha$ if $\Sigma\in L_\alpha$, for all $S\in Gl_n(\mathbf{R})$. Then by the lemma above

$$\bigcup_{\alpha\ \text{nice}} L_\alpha = L^{\sigma}_{m,n,p} \tag{2.8}$$

LEMMA 2.9 *Let $\Sigma\in L_\alpha$, α a nice selection. Then there is precisely one $S\in Gl_n(\mathbf{R})$ such that $R(\Sigma^S)_\alpha = I_n$, the $n\times n$ identity matrix (and $\Sigma^S\in L_\alpha$ of course).*

This follows immediately from the observation that

$$R(\Sigma^S)_\alpha = S(R(\Sigma)_\alpha) \quad \text{all } \alpha\subset J_{m,n} \tag{2.10}$$

LEMMA 2.11 *Let α be a nice selection. Let $x=(y_1,\dots,y_m,z)$ be an element of $\mathbf{R}^{mn+np}$ written as a sequence of m n-vectors $y_1,\dots,y_m$ and a $p\times n$ matrix z. Then there is precisely one $\Sigma_\alpha(x) = (A_\alpha(x),B_\alpha(x),C_\alpha(x))\in L_\alpha\subset L^{\sigma}_{m,n,p}$ such that*

$$R(\Sigma_\alpha(x))_\alpha = I_n,\ R(\Sigma_\alpha(x))_{s(\alpha,j)} = y_j, C_\alpha(x)=z. \tag{2.12}$$

The matrices $B_\alpha(x),A_\alpha(x)$ are very easy to write down explicitly. They always consist of column vectors which are either equal to one of the standard basis vectors of $\mathbf{R}^n$ or to one of the vectors y_j. Indeed in the case of the example of the nice selection α of (2.6) above we have, writing $e_1,\dots,e_7$ for the standard basis of $\mathbf{R}^7$:

$$B = (e_1,y_2,e_2,e_3,y_5), A = (e_4,y_3,e_5,y_1,e_6,e_7,y_4)$$

$$\begin{array}{ccccc} e_1 & y_2 & e_2 & e_3 & y_5 \\ e_4 & & y_3 & e_5 & \\ y_1 & & & e_6 & \\ & & & e_7 & \\ & & & y_4 & \end{array}$$

I.e. label the crosses $e_1,\dots,e_7$, write in y_j for the successor spots * and read of B and A directly from the resulting pattern remembering that the first row represents the columns of B, the second one the columns of AB, etc.. From these three lemmas there follows immediately the following description of $M^{\sigma}_{m,n,p}$ in terms of local coordinate charts and correspondence rules between these charts.

2.13 Description of the manifold $M^{\sigma}_{m,n,p}$

The manifold $M^{co,\sigma}_{m,n,p}$ is the union of open neighborhoods $V_\alpha,\alpha\subset J_{m,n}$ running through all nice selections. Each V_α is diffeomorphic to $\mathbf{R}^{mn+np}$ via a coordinate chart $\phi_\alpha\colon V_\alpha\xrightarrow{\sim}\mathbf{R}^{mn+np}$. Let $x\in\mathbf{R}^{mn+np} = \psi_\alpha(V_\alpha)$, $x'\in\mathbf{R}^{mn+np} = \psi_\beta(V_\beta)$. Then x and x' correspond to the same element of $M^{co,\sigma}_{m,n,p}$ (i.e. $\psi_\beta(\psi_\alpha^{-1}(x))=x'$) iff

$$R(\Sigma_\beta(x')) = (R(\Sigma_\alpha(x))_\beta)^{-1}R(\Sigma_\alpha(x)),\quad z'=zR(\Sigma_\alpha(x))_\beta \tag{2.14}$$

where as above $x=(y_1,\dots,y_m,z),x'=(y_1',\dots,y_m',z')$. Note that if $x\in\psi_\alpha(V_\alpha)=\mathbf{R}^{mn+np}$ are the α-coordinates of $P\in V_\alpha\subset M^{\sigma}_{m,n,p}$, then the β-coordinates of P are defined iff $R(\Sigma_\alpha(x))_\beta$ is invertible, a condition which is purely in terms of the α-coordinates of P. Note also that because $y_j' = R(\Sigma_\beta(x'))_{s(\beta,j)}$ the β-coordinates of P are then given in terms of explicit rational expressions in the α-coordinates.

Thus (abstractly)

$$M^{\sigma}_{m,n,p} = \coprod_{\alpha\ \text{nice}} V_\alpha'/\sim$$

where $V_\alpha'=\mathbf{R}^{mn+np}$ for each α and $x\sim x'$, $x\in V_\alpha'$, $x'\in V_\beta'$ iff (2.14) holds.

The manifold $M^{co,\sigma}_{m,n,p}$ is an open submanifold of $M^{\sigma}_{m,n,p}$ obtained by gluing together in exactly the same way the open subsets $V^{co}_\alpha\subset V_\alpha$ defined by

$$V^{co}_\alpha = \{x\in V_\alpha=\mathbf{R}^{mn+np}\colon \Sigma_\alpha(x) \text{ is } co\} \tag{2.15}$$

Note that this is an explicit (polynomial) condition in terms of the coordinates of x. For more details and proofs of the above cf. [3,5].

3. $M^{co,\sigma}_{m,n,p}$ AS AN IMBEDDED MANIFOLD

It is perhaps more customary to view a manifold like S^2, the sphere, as imbedded in some euclidean space like $\mathbf{R}^3$ and to view the distortions involved in taking local coordinates as measuring the differences between the geometry of the charts and the (true) geometry of the imbedded manifold (with its notions of distance etc. coming from the ambient euclidean space). As it happens the space $M^{co,\sigma}_{m,n,p}$ does come with a natural imbedding into a euclidean space. This and the relation of this imbedding with various atlases for $M^{co,\sigma}_{m,n,p}$ is the topic of this section.

Let $\mathcal{H}$ be the space of all sequences of $p\times m$ matrices $H_0,H_1,\dots,H_{2n}$ with the normal Euclidean topology. Define a map

$$\nu\colon L_{m,n,p}\to\mathcal{H},\ (A,B,C)\mapsto(CB,CAB,\dots,CA^{2n}B) \tag{3.1}$$

It is elementary to observe that $\nu(\Sigma)=\nu(\Sigma^S)$ for all $S\in Gl_n(\mathbf{R})$ so that ν induces a quotient map also denoted ν which can be restricted to $M^{co,\sigma}_{m,n,p}$

$$\nu\colon M^{co,\sigma}_{m,n,p}\to\mathcal{H} \tag{3.2}$$

THEOREM 3.3 (Kalman). *The map (3.2) is an injection. The image of (3.2) consists precisely of all sequences of matrices $H_0,H_1,\dots,H_{2n}$ such that*

$$rk\begin{pmatrix} H_0 & H_1 & \cdots & H_{n-1} \\ H_1 & H_2 & \cdots & H_n \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ H_{n-1} & H_n & \cdots & H_{2n-2} \end{pmatrix} = n = rk\begin{pmatrix} H_0 & H_1 & \cdots & H_n \\ H_1 & H_2 & \cdots & H_{n+1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ H_n & H_{n+1} & \cdots & H_{2n} \end{pmatrix}. \tag{3.4}$$

In fact the map (3.2) is an imbedding of the differentiable manifold $M^{co,\sigma}_{m,n,p}$ into $\mathcal{X}=\mathbf{R}^{mp(2n+1)}$. It is worth noting that the matrices occuring in $v(\Sigma)$ are directly related to the input-output operator V_Σ associated to Σ. Indeed if $y(t)=V\Sigma u(t)$ and $Y(s),U(s)$ denote the Laplace transforms of $y(t),u(t)$, then

$$Y(s)=T_\Sigma(s)U(s) \tag{3.5}$$

with

$$T_\Sigma(s)=C(sI-A)^{-1}B=CBs^{-1}+CABs^{-2}+CA^2Bs^{-3}+\cdots \tag{3.6}$$

the socalled transferfunction of Σ.

It follows that if $H_0,H_1,\ldots,H_{2n}$ is a sequence of $p\times m$ matrices such that condition (3.4) is fulfilled then there must be a $\Sigma=(A,B,C)\in L^{\sigma,co}_{m,n,p}$ such that $H_i=CA^iB$. An algorithm for finding such an A,B,C is called a realization algorithm. And (clearly) such algorithms are not unrelated to the matter of finding coordinate charts for $M^{co,\sigma}_{m,n,p}$. Here is one ([6]). First observe that if $H_i=CA^iB$ then for the Hankel matrices of $(H_0,\ldots,H_{2n})$ and (A,B,C) we have

$$\begin{pmatrix} H_0 & H_1 & \cdots & H_n \\ H_1 & H_2 & \cdots & H_{n+1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ H_n & H_{n+1} & \cdots & H_{2n} \end{pmatrix} = \begin{pmatrix} CB & CAB & \cdots & CA^nB \\ CAB & CA^2B & \cdots & CA^{n+1}B \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ CA^nB & & \cdots & CA^{2n}B \end{pmatrix}$$

$$=Q(A,B,C)R(A,B,C)=:H(A,B,C)$$

Now because (A,B,C) is cr there is a nice selection α_c of the columns of $R(A,B,C)$ such that $R(A,B,C)_{\alpha_c}$ is invertible. Similary there is a nice selection α_R of the rows of $Q(A,B,C)$ such that $Q(A,B,C)_{\alpha_R}$ is invertible. Now observe that

$$H(A,B,C)_{\alpha_R,\alpha_c}=(Q(A,B,C)_{\alpha_R})(R(A,B,C)\alpha_c) \tag{3.7}$$

where of course H_{α_R,α_c} means the matrix obtained from H by retaining only those columns whose index is in α_R and only those rows whose index is in α_c. The first step of the realization algorithm is hence to find a nice α_R and α_c such that $S=H_{\alpha_R,\alpha_c}$ is invertible. These given (3.4) exist. We now also know that among all the (A,B,C) with this given Hankel matrix there is precisely one with $R_{\alpha_c}=I_n$. This is the one we are going to construct. Then of course $Q_{\alpha_R}=S$ which is now known. Also $H^{\alpha_R}=Q_{\alpha_R}R$ so that we know $R(A,B)=Q^{-1}_{\alpha_R}H_{\alpha_R}$ from which A and B can be recovered (Lemma 2.11). In fact A and B consist of column vectors which are either standard basis vectors or the vectors labelled by the succes indices $s(\alpha_c,j)$ of $Q_{\alpha_R}H_{\alpha_R}$. Finally if p denotes the lables of the first p rows of H we have $C=H_{p,\alpha_c}$.

This particular realization algorithm is clearly much related to the coordinate charts described in § 2 above.

The reader may wonder what the role is of the two rank conditions (3.4) in this algorithm. The first

condition in fact ensures that there are nice α_R and α_c such that H_{α_R,α_c} is invertible. The second one sees to it that the construction in fact yields an A,B,C such that $H_i=CA^iB$ for all i.

4. Can the distortions involved in the coordinate changes be kept under control?

As a start and for the purpose of this note. I shall interpret this question as follows. Consider $M^{co,\sigma}_{m,n,p}$ as imbedded in $\mathcal{X}$. Consider a set of coordinate charts $M=M^{co,\sigma}_{m,n,p}\supset U_\alpha \xrightarrow{\phi_\alpha} \mathbf{R}^{mn+np}$. Give $M^{co,\sigma}_{m,n,p}$ the (Riemanian) metric induced by the imbedding. (This is not the only natural metric on M, *cf.* [4] for another important one). Is it true that one can find an atlas $(U_\alpha,\phi_\alpha)\alpha$ such that for all $P\in M$ there is a good chart in that for a certain predetermined ϵ the Jacobian of ϕ_α at P and its inverse are both at least ϵ away from the subset of singular matrices in the space of all square matrices of size $dimM\times dimM$? This would for example be the case of we could find a finite atlas $(U_\alpha,\phi_\alpha)_\alpha$ (i.e. one with finitely many charts) such that for each α there is a compact set $D_\alpha\subset\phi_\alpha(U_\alpha)$ such that for each $P\in M$ there is an α such that $\phi_\alpha(P)\in D_\alpha$. This, however, would imply that M is compact (as image of $\coprod_\alpha D_\alpha$) which is never the case.

The question is open but is obviously of great relevance for accurate numerical (recursive) identification problems.

The following observation of Bosgra and van der Weiden [1] is probably going to be of importance here. Consider again the realization algorithm described in § 3 above. Because of the Hankel structure of H there are indentical ones among the entries of H which are actually used in constructing (A,B,C). It turns out that in fact precisely $nm+np$ entries of the matrices $H_0,\ldots,H_{2n}$ are used. This means that to each pair of nice selections (α_R,α_c) there is associated a subset of size $nm+np$ of the $mp(2n+1)$ coordinates of $\mathcal{X}$ such that projection onto these $nm+np$ coordinates is in fact a local coordinate chart. And of course the coordinate neighborhoods thus obtained cover all of M. This certainly does not yet give a positive answer to the question asked above but it is a positive indicator in that it is so particularly simple to indicate for a particular $P\in M$ which subsets of the coordinates of $\mathcal{X}$ of the type determined by a pair of nice selections (α_R,α_c) may be used as local coordinate charts around $P\in M$. Of course in itself, abstractly, the fact that for an imbedded manifold dimension r, say $M\subset\mathbf{R}^N$ the r-element set projections $\mathbf{R}^N\to\mathbf{R}^r$ restricted to M may be used as coordinate charts means nothing. Indeed let $P\in M\subset\mathbf{R}^N$. Locally around P the manifold M is then the image of a differentiable map $i:\mathbf{R}^r\to\mathbf{R}^N, 0\mapsto P$, of rank r near 0. That means that the Jacobian matrix $J(i)(0)$ of i at 0 has rank r and so there is a subset α of size r of N such that $J(i)(0)_\alpha$ is invertible. Let $\pi_\alpha:\mathbf{R}^N\to\mathbf{R}^r$ be the projection corresponding to α. Then $\mathbf{R}^r\to\mathbf{R}^N\to\mathbf{R}^r$ is a diffeomorphism near 0 so that π_α is a good coordinate chart for M near P.

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ON THE (INTERNAL) SYMMETRY GROUPS OF LINEAR DYNAMICAL SYSTEMS

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Chapter IX

On the (Internal) Symmetry Groups of Linear Dynamical Systems

M. Hazewinkel

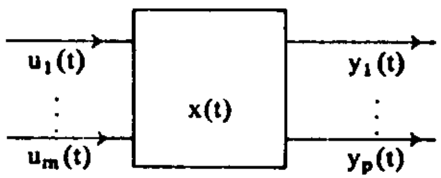
1. Introduction and statement of the main definitions and results

A time invariant linear dynamical system is a set of equations

$$(1.1)\quad \begin{array}{lcl} \dot{x} = Fx + Gu & & x(t+1) = Fx(t) + Gu(t) \\ y = Hx & (\Sigma) & y(t) = Hx(t) \\ \text{(continous time)} & & \text{(discrete time)}, \end{array}$$

where $x \in X = \mathbb{R}^n$, $u \in U = \mathbb{R}^m$, $y \in Y = \mathbb{R}^p$ and where F, G, H are matrices with coefficients in $\mathbb{R}$ of the dimensions $n \times n$, $n \times m$, $p \times n$ respectively. We speak then of a system of dimension n, $\dim(\Sigma) = n$, with m inputs and p outputs. Of cource the discrete time case also makes sense over any field k, (instead of $\mathbb{R}$). The spaces X, U, Y are respectively called state space, input space and output space. The usual picture is a "black box".

(1.2)



That is, the system Σ is viewed as a machine which transforms an m-tuple of input or control functions $u_1(t), \ldots, u_m(t)$ into a p-tuple of output or observation functions $y_1(t), \ldots, y_p(t)$. Many physical systems can be viewed as such a "black box". For instance the box may be a chemical reaction vat. The $u_1(t), \ldots, u_m(t)$ may be concentrations of various chemicals which are inserted and the $y_1(t), \ldots, y_p(t)$ represent certain series of measurements serving as indicators that everything goes as we wish (or not). Especially the output aspect (represented by the matrix H) captures something very often encountered in physics, electronics, chemistry, and also astronomy: only certain functions of the state variables $x_1(t), \ldots, x_n(t)$ are directly observable! Thus in astronomy one has to make do with certain projections (against the sky sphere) of the space variables describing, e.g., the solar system, in atomic physics one may have to rely only on scattering data, and, as a last example, in economics one uses socalled economic indices, which, hopefully, reflect more or less accurately the goings on of the "real" (largely unknown) underlying economic processes.

The formulas expressing $y(t)$ in terms of the $u(t)$ are

$$y(t) = He^{Ft}x(0) + \int_0^t He^{F(t-\tau)}Gu(\tau)d\tau\,,$$

(1.3)

$$y(t) = HF^t x(0) + \sum_{i=0}^{t-1} HF^{t-i-1}Gu(i),$$

where $x(0)$ is the state of the system at time 0 (and where we start putting in input at time $t = 0$). Thus the input-output behaviour of our box depends of course on the initial state $x(0)$. One is particularly interested in the input-output behaviour of Σ when $x(0) = 0$. We shall write $f(\Sigma)$ for the associated input-output operator. Thus

(1.4) $f(\Sigma): u(t) \mapsto \int_0^t He^{F(t-\tau)}Gu(\tau)d\tau,\; f(\Sigma): u(t) \mapsto \sum_{i=0}^{t-1} HF^{t-i-1}Gu(i)$

It is now an important fact that the input-output behaviour description of the machine (1.2) is degenerate, much as, say, energy levels in atomic physics may be degenerate. More precisely the matrices F, G, H (and the initial state $x(0)$) depend on the choice of a basis in state space and from the input-output behaviour of the machine there is (without changing the machine) no way of deciding on a "canonical" basis for the state space $X = \mathbb{R}^n$. More mathematically we have the following. Let $GL_n(\mathbb{R})$ be the group of all invertible real $n \times n$ matrices and let $L_{m,n,p}(\mathbb{R})$ be the space of all triples of matrices (F, G, H) of dimensions $n \times n$, $n \times m$, $p \times n$ respectively. The group $GL_n(\mathbb{R})$ acts on $L_{m,n,p}(\mathbb{R})$ and $\mathbb{R}^n$ = space of initial states, as

(1.5) $(F, G, H)^S = (SFS^{-1}, SG, HS^{-1}),\; x(0)^S = Sx(0)$

and as is easily checked the associated input-output behaviour of the corresponding machine as given by (1.3) and (1.4) is invariant under this action of $GL_n(\mathbb{R})$; i.e., in particular $f(\Sigma^S) = f(\Sigma)$. This action corresponds to base change in state space. Indeed if $x' = Sx$ and $\dot{x} = Fx + Gu$, $y = Hx$ then $S^{-1}\dot{x}' = FS^{-1}x' + Gu$, $y = HS^{-1}x'$ so that $\dot{x}' = SFS^{-1}x' + SGu$, $y = HS^{-1}x'$ and $x'(0) = Sx(0)$.

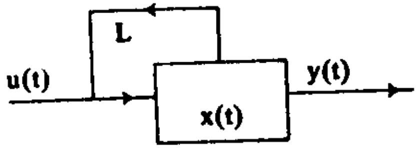
This chapter is concerned with those aspects of the theory of linear dynamical systems which are more or less directly related to the presence of the internal symmetry group $GL_n(\mathbb{R})$ of the internal description of linear dynamical systems by triples of matrices (cf. (1.1)) as compared to the degenerate external description by means of the operator $f(\Sigma)$ (or (1.3)). This is not really a research paper (though it does in fact contain a few new results) but rather a graduate level expository account of some of the material of [3–8] and immediately related matters.

In the remaining part of this introduction we give a slightly informal description of most of the main results of sections 2–8 below.

We shall concentrate on the continuous time case.

1.6 Feedback and how to resolve the external description degeneracy. In the case of atomic physics a degenerate energy level may be split by means of, e.g., a suitable magnetic field. One can ask whether there exists something analogous in our case of degenerate external (= observable) descriptions of linear dynamical systems. There does in fact exist some such thing. It is called state space feedback. Consider the system (1.1). Introduction of state space feedback L changes it to the system $\Sigma(L)$

(1.7)
$$\dot{x} = (F + GL)x + Gu$$
$$y = Hx$$



In thinking about these things the author has found it helpful to visualize a linear dynamical system with (variable) feedback as a set of n-integrators, 1, …, n, interconnected by means of the matrix F, a set of m input points connected to the integrators by means of the matrix G, a set of p output points connected to the integrators by means of the matrix H and a set of connections from the integrators to the input points (feedback) which may be varied in strength by the experimentator (as in atomic physics the splitting magnetic field may be varied). Cf. also the picture below.

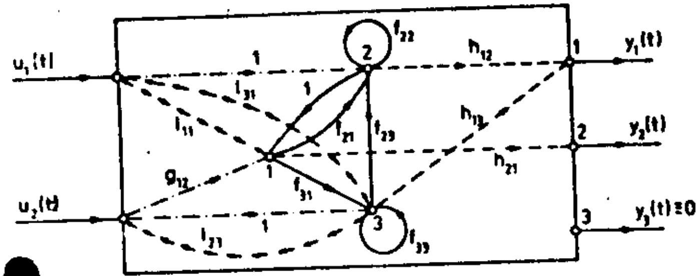


Fig. 1

——▸—— interconnections between the integrators as given by the matrix F

$$F = \begin{pmatrix} 0 & 1 & 0 \\ f_{21} & f_{22} & f_{23} \\ f_{31} & 0 & f_{33} \end{pmatrix}$$

–·–▸–· connections from the input points to integrators as given by the matrix G

$$G = \begin{pmatrix} 0 & g_{12} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

– – → – – connections from the integrators to the output points as given by the matrix H

$$H = \begin{pmatrix} 0 & h_{12} & h_{13} \\ h_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

→ → → connections from the integrators to the input points (can be varied in streng[th] by the experimentator) as given by the matrix L

$$L = \begin{pmatrix} l_{11} & 0 & l_{13} \\ 0 & 0 & l_{23} \end{pmatrix}$$

Now let $\Sigma = (F, G, H)$ and $\Sigma' = (F', G', H')$ be two linear dynamical systems, and suppose that Σ and Σ' are completely reachable and completely observable. (This is an entirely natural restriction in this context, cf. 1.12 below; for a precise definition of the notions, cf. 2.6 below). Suppose that $\Sigma \neq \Sigma'$ but $f(\Sigma) = f(\Sigma')$. Let $\Sigma(L), \Sigma'(L)$ be the systems obtained by introducing the feedback L, i.e. $\Sigma(L) = (F + GL, G, H)$, $\Sigma'(L) = (F' + G'L, G', H')$. Then there is a suitable feedback matrix L, which can be taken arbitrarily small (so that $\Sigma(L)$ and $\Sigma'(L)$ are still completely reachable and observable) such that $f(\Sigma(L)) \neq f(\Sigma'(L))$. I.e. feedback splits the $GL_n(\mathbb{R})$ – degenerate external description of linear dynamical systems.

1.8 **Realization theory.** Let Σ be a linear dynamical system (1.1). Then, if we leave Σ unchanged, from our observations we can deduce the operator $f(\Sigma)$ or, equivalently, we can find the sequence of matrices $A(\Sigma) = (A_0, A_1, A_2, \ldots)$, $A_i = HF^iG$. To obtain these use δ-functions and derivates of δ-functions as inputs. Another way to see this is to apply Laplace transforms to (1.1). This gives

(1.9) $s\hat{x}(s) = F\hat{x}(s) + G\hat{u}(s), \hat{y}(s) = H\hat{x}(s)$,

so that the relation between the Laplace transforms $\hat{y}(s), \hat{u}(s)$ of the outputs $y(t)$ and inputs $u(t)$ is given by multiplication with the socalled transfer matrix $T(s)$

(1.10) $\hat{y}(s) = T(s)\hat{u}(s), T(s) = H(s - F)^{-1}G$.

The power series development of $T(s)$ in powers of s^{-1} (around $s = \infty$) is now

(1.11) $T(s) = A_0 s^{-1} + A_1 s^{-2} + A_2 s^{-3} + \ldots$.

The question now naturally arises: when does a sequence of $p \times m$ matrices $A = (A_0, A_1, \ldots)$ come from a linear dynamical system (1.1), or, as we shall say, when is A *realizable*.

1.12 Theorem (cf. [10]):

(i) If A is realizable by an n-dimensional system Σ then it is also realizable by an $n' \leq n$ dimensional system Σ' which is moreover completely reachable and completely observable.

(ii) The sequence A is realizable by an n dimensional system Σ if and only if rank $(H_s(A)) \leq n$ for all $s \in \mathbb{N} \cup \{0\}$.

Here $H_s(A)$ is the block Hankel matrix

$$H_s(A) = \begin{pmatrix} A_0 & A_1 & \cdots & A_s \\ A_1 & & & \vdots \\ \vdots & & & \\ A_s & \cdots & & A_{2s} \end{pmatrix} .$$

1.13 Invariants and the structure of $M^{cr,co}_{m,n,p}(\mathbb{R}) = L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$.

Let $L_{m,n,p}(\mathbb{R})$ be the space of all triples of matrices (F, G, H) of dimensions $n \times n$, $n \times m$, $p \times n$ respectively. The group $GL_n(\mathbb{R})$ acts on $L_{m,n,p}(\mathbb{R})$ as in (1.5). The input-output matrices $A_i = HF^iG$ are clearly invariants for this action and the question arises whether these are the only invariants. Here an invariant is defined as a function ρ: $L_{m,n,p}(\mathbb{R}) \to \mathbb{R}$ (or possibly a function defined on an invariant open dense subset of $L_{m,n,p}(\mathbb{R})$) such that $\rho((F, G, H)^S) = \rho(F, G, H)$ for all triples (F, G, H) (in the open dense subset).

1.14 Theorem: Every continuous invariant of $GL_n(\mathbb{R})$ acting on $L_{m,n,p}(\mathbb{R})$ is a function of the entries of $A_0, \ldots, A_{2n-1}$.

Let $L^{co,cr}_{m,n,p}(\mathbb{R})$ be the subspace of all triples $(F, G, H) \in L_{m,n,p}(\mathbb{R})$ which are both completely observable and completely reachable. This is an open and dense subspace of $L_{m,n,p}(\mathbb{R})$. On this subspace $GL_n(\mathbb{R})$ acts faithfully and a more precise version of theorem 1.14 describes the quotient space $M^{co,cr}_{m,n,p}(\mathbb{R}) = L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$ explicitly and gives an algorithm for recovering (F, G, H) up-to-$GL_n(\mathbb{R})$-equivalence from $A_0, \ldots, A_{2n-1}$ (cf. 4.25 below). It turns out that $M^{co,cr}_{m,n,p}(\mathbb{R})$ is a smooth differentiable manifold and that the projection $L^{co,cr}_{m,n,p}(\mathbb{R}) \to M^{co,cr}_{m,n,p}(\mathbb{R})$ is a principal $GL_n(\mathbb{R})$-bundle (cf. 6.4 below).

1.15 Canonical forms. For many purposes (prediction, construction of feedbacks, identification and, not least, for proving theorems) an internal description of a black box by means of a triple of matrices (F, G, H) is preferable over knowledge of the input-output calculating some $\Sigma = (F, G, H)$ which realizes $f(\Sigma)$ or $A(\Sigma)$ from the matrices $A_0, \ldots, A_{2n-1}$. One such algorithm is described in 4.25 below. All these algorithms have the drawback that they are discontinuous in general. This is a nontrivial difficulty, because after all one calculates the (F, G, H) because one wants to use them as a basis for further calculations, design, predictions etc., and the $A_0, \ldots, A_{2n-1}$ are after all subject to (small) measurement errors. Thus the question arises whether there exist continuous methods of recovering (F, G, H) up-to-$GL_n(\mathbb{R})$-equivalence from $A_0, \ldots, A_{2n-1}$. Or, in other words, because $M^{co,cr}_{m,n,p}(\mathbb{R})$ is an explicitly describable subspace of the space of all sequences of $p \times m$ matrices and $M^{co,cr}_{m,n,p}(\mathbb{R}) = L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$, the question arises whether there exist continuous canonical forms on $L^{co,cr}_{m,n,p}(\mathbb{R})$, where a continuous canonical form is defined as follows.

1.16 Definition. A continuous canonical form on a $GL_n(\mathbb{R})$-invariant subspace $L' \subset L_{m,n,p}(\mathbb{R})$ is a continuous map $c: L' \to L'$ such that

(i) $c((F, G, H)^S) = c((F, G, H))$ for all $(F, G, H) \in L'$,

(ii) if $c((F, G, H)) = c((F', G', H'))$ then there is a $S \in GL_n(\mathbb{R})$ such that $(F', G', H') = (F, G, H)^S$, and

(iii) for all $(F, G, H) \in L'$ there is an $S \in GL_n(\mathbb{R})$ such that $c(F, G, H) = (F, G, H)^S$.

For some additional remarks on the desirability of *continuous* canonical forms cf. [2] and also [15]. Also our proof of the "feedback suspends degeneracy" theorem mentioned in 1.6 above is based on the use of a suitable canonical form. It turns out that there exist open dense subspaces $U_\alpha \subset L_{m,n,p}(\mathbb{R})$, which together cover $L^{co,cr}_{m,n,p}(\mathbb{R})$, on which continuous canonical forms exist. Cf. 3.10 below. On the other hand.

1.17 Theorem: There exists a continuous canonical form on all of $L^{co,cr}_{m,n,p}(\mathbb{R})$ if and only if $m = 1$ or $p = 1$.

1.18 On the geometry of $M^{co,cr}_{m,n,p}(\mathbb{R})$. Holes. Now suppose we have a black box (1.2) which is to be modelled by a linear dynamical system of dimension n. Then the input-output data give us a point of $M^{co,cr}_{m,n,p}(\mathbb{R})$ and as more and more data come in we find (ideally) a sequence of points in $M^{co,cr}_{m,n,p}(\mathbb{R})$ representing better and better linear dynamical system approximations to the given black box. The same thing happens when one is dealing with a slowly varying black box or linear dynamical system. If this sequence approaches a limit we have "identified" the black box. Unfortunately the space $M^{co,cr}_{m,n,p}(\mathbb{R})$ is never compact so that a sequence of points may fail to converge to anything whatever. There are holes in $M^{co,cr}_{m,n,p}(\mathbb{R})$. Consider for example the following family of 2-dimensional, one input, one output systems

(8)

(1.19) $\quad \ldots \quad F_z = \ldots \quad H_z = (z^2\ 0) \quad z = 1, 2, 3$

Let $u(t), 0 \leq t \leq t_0$ be a smooth input function, then $y(t) = \lim_{z \to \infty} f(\Sigma_z)u(t)$ exists and is equal to $y(t) = \frac{d}{dt} u(t)$. This operator can not be of the form $f(\Sigma)$ for any system Σ of the form (1.1) (because the $f(\Sigma)$ are always bounded operators and $\frac{d}{dt}$ is an unbounded operator). A characteristic feature of this example is that the individual matrices F_z, G_z, H_z do not have limits as $z \to \infty$. (A not unexpected phenomenon, because after all we are taking quotients by the noncompact group $GL_n(\mathbb{R})$). This sort of situation is actually important in practice, e.g. in the study of very high gain state feedback systems $\dot{x} = Fx + Gu$, $u = cLx$, where c is a large scalar gain factor. Cf. [12].

Another type of hole in $M^{co,cr}_{m,n,p}(\mathbb{R})$ corresponds to lower dimensional systems, and in a way these two holes and combinations of them are all the holes there are in the sense of the following definitions and theorems for the case $p = m = 1$. There are similar theorems in the more input/more output cases.

1.20 Definition: We shall say that a family of systems $\Sigma_z = (F_z, G_z, H_z)$ converges in input-output behaviour to an operator B if for every m-vector of smooth input functions $u(t)$ with support in $(0, \infty)$ we have $\lim_{z \to \infty} f(\Sigma_z) u(t) = Bu(t)$ uniformly in t on bounded t intervals.

1.21 Definition: A differential operator of order r is an operator of the form $u(t) \mapsto y(t) = Dy(t) = a_0 u(t) + a_1 \frac{d}{dt} u(t) + \ldots + a_r \frac{d^r}{dt^r} u(t)$, where the $a_0, \ldots, a_r$ are $p \times m$ matrices with coefficients in $\mathbb{R}$, and $a_r \neq 0$. We write $\mathrm{ord}(D)$ for the order of D. By definition $\mathrm{ord}(0) = -1$.

1.22 Theorem: Let $(\Sigma_z)_z$ be a family of systems in $L_{1,n,1}(\mathbb{R})$ which converges in input-output behaviour. Let B be the limit input-output operator. Then there exist a system Σ' and a differential operator D such that

$$Bu(t) = f(\Sigma')u(t) + Du(t)$$

and $\mathrm{ord}(D) + \dim(\Sigma') \leq n - 1$.

1.23 Theorem: Let D be a linear differential operator and $\Sigma' \in L_{1,n,1}(\mathbb{R})$ and suppose that $\mathrm{ord}(D) + \dim(\Sigma') \leq n - 1$. Then there exists a family of systems $(\Sigma_z)_z$, $\Sigma_z \in L^{co,cr}_{1,n,1}(\mathbb{R})$ such that for every smooth input vector $u(t)$

$$\lim_{z \to \infty} f(\Sigma_z)u(t) = f(\Sigma')u(t) + Du(t)$$

uniformely on bounded t-intervals.

1.24 Concluding introductory remarks. Many of the results described above have their analogues in the discrete case and/or the time varying case, cf. [3–8, 9–11, 14]. But not all. For instance the obvious analogues of theorems 1.23 and 1.22 fail utterly in the discrete time case. In this case $\lim_{z \to \infty} f(\Sigma_z)u(t)$ exists for all inputs $u(t)$ if and only if the individual matrices $A_i(z) = H_z F_z^i G_z$ converge for $z \to \infty$. This means that in the case of input-output convergence the limit operator is necessarily of the form $f(\Sigma')$ for some, possibly lower dimensional, system Σ'. The same answer obtains in the continuous time case if besides input-output convergence one also requires that the F_z, G_z, H_z (or more generally the $A_i(z)$) remain bounded.

A number of sections have been marked with a *: these contain additional material and can without endangering one's understanding be omitted the first time through.

2 Complete reachability and complete observability

Let $(F, G, H) \in L_{m,n,p}(\mathbb{R})$ be a real linear dynamical system of state space dimension n, with m inputs and p outputs. We define

(2.1) $R_s(F, G) = (G\ FG \ldots F^sG),\ s = 0, 1, 2, \ldots,\ R(F, G) = R_n(F, G)$

the $n \times (s+1)m$ matrices consisting of the blocks $G, FG, \ldots, F^sG$, and dually

(2.2) $$Q_s(F, H) = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^s \end{pmatrix},\ s = 0, 1, 2, \ldots,\ Q(F, H) = Q_n(F, H).$$

We also define

(2.3) $$\mathcal{H}_s(F, G, H) = \mathcal{H}_s(\Sigma) = \begin{pmatrix} A_0 & A_1 & \ldots & A_s \\ A_1 & & \cdot & \vdots \\ \vdots & \cdot & & \vdots \\ A_s & & \ldots & A_{2s} \end{pmatrix} = Q_s(F, H)R_s(F, G),\ s = 0, 1, 2, \ldots,$$

where $A_i = HF^iG$, $i = 0, 1, 2, \ldots$.

It is useful to notice that

(2.4) $R_k((F, G)^S) = SR_k(F, G),\ Q_k((F, H)^S) = Q_k(F, H)S^{-1}$,

where of course $(F, G)^S = (SFS^{-1}, SG)$, $(F, H)^S = (SFS^{-1}, HS^{-1})$. It follows that

(2.5) $\mathcal{H}_k(\Sigma^S) = \mathcal{H}_k((F, G, H)^S) = \mathcal{H}_k((F, G, H)) = \mathcal{H}_k(\Sigma)$

for all $S \in GL_n(\mathbb{R})$, which is of course also immediately clear from (2.3).

2.6 Definitions of complete reachability of complete observability. The system $(F, G, H) \in L_{m,n,p}(\mathbb{R})$ is said to be completely reachable iff rank $(R(F, G)) = n$. The system (F, G, H) is said to be completely observable iff rank $(Q(F, H)) = n$. These are

(9)

generic conditions; in fact the subspace $L^{co,cr}_{m,n,p}(\mathbb{R})$ of $L_{m,n,p}(\mathbb{R})$ consisting of all systems which are both completely reachable and completely observable is open and dense. We note that (F, G, H) is co (= completely observable) and cr (= completely reachable) iff the matrix $\mathcal{H}_n(F, G, H) = Q(F, H)\, R(F, G)$ is of rank n.

***2.7 Termilogical justification.** Let $(F, G, H) \in L_{m,n,p}(\mathbb{R})$. Then (F, G, H) is completely reachable iff for every $x_1 \in \mathbb{R}^n$ there is an input function $u(t)$ such that the unique solution of

$$\dot{x} = Fx + Gu(t),\ x(0) = 0$$

passes through x_1: i.e. every state is reachable from zero. For a proof cf., e.g., [17, theorem 3.5.3 on page 66] or [10, section 2.3]. Instead of completely reachable one also often finds the terminology (completely state) controllable in the literature.

Dually the system (F, G, H) is completely observable iff the initial state $x(0)$ at time zero is deducible from $y(t)$, $0 \leq t \leq t_1$, $t_1 > 0$ (using zero inputs). Equivalently (F, G, H) is completely observable if the initial state $x(0)$ is deducible from the input-output behaviour of the system on an interval $[0, t_1]$, $t_1 > 0$. Cf., e.g., [14, Ch. V, section 3] or [17, theorem 3.5.26 on page 75].

The following theorem says that as far as input-output behaviour goes every system can be replaced by a system which is co and cr. Thus it is natural to concentrate our investigations on this class of systems.

2.8 Theorem ([10]): Let $\Sigma = (F, G, H) \in L_{m,n,p}(\mathbb{R})$ with input-output operator $f(\Sigma)$. Let $n' = \text{rank}(\mathcal{H}_n(\Sigma))$. Then there exists an

$$\Sigma' = (F', G', H') \in L^{co,cr}_{m,n',p}(\mathbb{R}) \text{ such that } f(\Sigma) = f(\Sigma').$$

Proof: Let $X = \mathbb{R}^n$ be the state space of Σ. Let X^{reach} be the linear subspace of X spanned by the columns of $R(F, G)$. Then, clearly, $G(\mathbb{R}^m) \subset X^{reach}$ and $F(X^{reach}) \subset X^{reach}$ (Because $F^n = a_0I + a_1F + \ldots + a_{n-1}F^{n-1}$ for certain $a_i \in \mathbb{R}$ by the Cayley-Hamilton theorem). Taking a basis for X^{reach} and completing this to a basis for X we see that for suitable $S \in GL_n(\mathbb{R})$, Σ^S is of the form

$$\Sigma^S = \left(\left(\frac{G''}{0}\right), \left(\begin{array}{c|c} F'' & F_{12} \\ \hline 0 & F_{22} \end{array}\right), \left(H'' \mid H''_2\right)\right)$$

where the partition blocks are respectively of the sizes:
$n'' \times m$, $n - n'' \times m$, $n'' \times n''$, $n'' \times n - n''$, $n - n'' \times n''$, $(n - n'') \times (n - n'')$, $p \times n''$, $p \times (n - n'')$ for G'', 0, F'', F_{12}, 0, F_{22}, H'', H''_2 respectively if $n'' = \dim X^{reach}$.

Now clearly

$$He^{F\tau}G = (HS^{-1})e^{SFS^{-1}\tau}SG = H''e^{F''\tau}G''$$

and rank $R(F'', G'')$ = rank $(R(SFS^{-1}, SG))$ = rank $(SR(F, G))$ = rank $R(F, G) = n''$.
It follows, cf. (1.4), that Σ and $\Sigma'' = (F'', G'', H'')$ have the same input-output operator. Thus to prove the theorem it now suffices to prove the theorem under the extra hypothesis that (F, G, H) is cr. Let X_0 be the subspace of all $x \in X$ such that $HF^i x = 0$ for all $i = 0, 1, \ldots, n$; i.e., $X_0 = \mathrm{Ker}(Q(F, H))$. Then $HF^i x = 0$ for all $i = 1, 2, \ldots$, using the Cayley-Hamilton theorem. Hence $FX_0 \subset X_0$ and $HX_0 = 0$. Taking a basis for X_0 and completing it to a basis for X we see that for a suitable $S \in GL_n(\mathbb{R})$, Σ^S is of the form

$$\Sigma^S = \left(\left(\frac{G_1'}{G'}\right), \left(\begin{array}{c|c} F_{11}' & F_{12}' \\ \hline 0 & F' \end{array}\right), (0, H')\right),$$

where G', F', H' are respectively of the sizes $n' \times m$, $n' \times n'$, $p \times n'$, n' = rank $(Q(F, H))$, which is also equal to rank $H_n(F, G, H)$ if (F, G, H) is cr.
Clearly

$$He^{F\tau}G = (HS^{-1})e^{SFS^{-1}\tau}SG = H'e^{F'\tau}G'$$

$$\mathrm{rank}(Q(F, H)) = \mathrm{rank}(Q(SFS^{-1}, SHS^{-1}) = \mathrm{rank}(Q(F', H')),$$

so that $\Sigma' = (F', G', H')$ is completely observable and $f_{\Sigma'} = f_\Sigma$. Also $R(SFS^{-1}, SG)$ is of the form

$$R(SFS^{-1}, SG) = \left(\frac{R'}{R(F', G')}\right).$$

But rank $R(F, G) = n$ so that the n rows of $R(SFS^{-1}, SG) = SR(F, G)$ are independent. It follows that the n' rows of $R(F', G')$ are also independent, proving that Σ' is also completely reachable.

*2.9 **Pole Assignment.** A set Λ of complex numbers with multiplicities is called symmetric if with $\beta \in \Lambda$ also $\bar{\beta} \in \Lambda$ with the same multiplicity. Here $\bar{\beta}$ is the complex conjugate of β. If A is a real $n \times n$ matrix then $\sigma(A)$, the spectrum of A, is a symmetric set.

2.10 **Theorem:** The pair of matrices (F, G), $F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times m}$ is completely reachable iff every symmetric set with multiplicities of size n occurs as the spectrum of $F + GL$ for a suitable (state feedback) matrix L.
I.e. the system (F, G, H) is cr iff we can by means of suitable state feedback arbitrarily reassign the poles of the system. For a proof cf., e.g., [18, section 2.2].

3. Nice Selections and the Local Structure of $L^{cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$

3.1 **Nice Selections.** Let $(F, G, H) \in L_{m,n,p}(\mathbb{R})$. We use $I(n, m)$ to denote the ordered set of indices of the columns of the matrix $R(F, G)$.
I.e. $I(n, m) = \{(i, j) \mid i = 0, \ldots, n; j = 1, \ldots, m\}$ with the ordening $(0, 1) < (0, 2) < \ldots < (0, m) < (1, 1) < \ldots < (1, m) < \ldots < (n, 1) < \ldots < (n, m)$. A *nice selection* $\alpha \subset I(n, m)$ is a subset of $I(n, m)$ of size $n = \dim \Sigma$ such that $(i, j) \in \alpha \Rightarrow (i-1, j) \in \alpha$ if $i \geq 1$. Pictorially we represent $I(n, m)$ as an $(n + 1) \times m$ rectangular array of which the first row represents the indices of the columns of G, the second row the indices of the columns of $FG, \ldots$ etc .... We indicate the elements of a subset α with crosses. The subset of the picture on the left is then a nice selection ($m = 4$, $n = 5$) and the subset α' of the picture on the right below is not a nice selection

```
. x . x      . . x .
. x . x      . x . x
. x . .      . . x x
. . . .      . . . .
. . . .      . . . .
. . . .      . . . .
```

If β is a subset of $I(n, m)$ we denote with $R(F, G)_\beta$ the matrix obtained from $R(F, G)$ by removing all columns whose index is not in β.
We use $L_{m,n}(\mathbb{R})$ to denote the space of all pairs of real matrices (F, G) of dimensions $n \times n$, $n \times m$ respectively.

3.2 **Lemma:** Let $(F, G) \in L_{m,n}(\mathbb{R})$ be a completely reachable pair of matrices. Then there is a nice selection α such that $R(F, G)_\alpha$ is invertible.

Remark: Complete reachabilitiy means that rank $R(F, G) = n$, so that there is in any case some subset β of size n of $I(n, m)$ such that $R(F, G)_\beta$ is invertible. The lemma says that in that case there is also a *nice* selection for which this holds.

Proof of the lemma: Define a nice subselection of $I(n, m)$ as any subset β (of size $\leq n$) such that $(i, j) \in \beta$, $i \geq 1 \Rightarrow (i-1, j) \in \beta$. Let α be a maximally large nice subselection of $I(n, m)$ such that the columns in $R(F, G)_\alpha$ are linearly independent. We shall show that rank $(R(F, G)_\alpha)$ = rank $(R(F, G))$, which will prove the lemma because by assumption rank $R(F, G) = n$.
Let $\alpha = \{(0, j_1), \ldots, (i_1, j_1); \ldots, (0, j_s), \ldots, (i_s, j_s)\}$. Then by the maximality of α we know the columns of $R(F, G)$ with indices $(0, j)$, $j \in \{1, \ldots, m\} \setminus \{j_1, \ldots, j_s\}$ and the columns of $R(F, G)$ with indices $(i_t + 1, j_t)$, $t = 1, \ldots, s$ are linearly dependent on the columns of $R(F, G)_\alpha$. With induction assume that all columns with indices $(i_t + k, j_t)$, $k < r$,

$t = 1, \ldots, s$ and $(k-1, j)$, $k < r$, $j \in \{1, \ldots, m\} \setminus \{j_1, \ldots, j_s\}$ are linearly dependent on the columns of $R(F,G)_\alpha$. So we have relations

$$F^{r-1} g_j = \sum_{(i,j)\in\alpha} a(i,j) F^i g_j,\ j \in \{1, \ldots, m\} \setminus \{j_1, \ldots, j_s\}$$

$$F^{i_t + r} g_{j_t} = \sum_{(i,j)\in\alpha} b(i,j) F^i g_j,\ t = 1, \ldots, s,$$

where g_j denotes the j-th column of G. Multiplying on the left with F we find

$$F^r g_j = \sum_{(i,j)\in\alpha} a(i,j) F^{i+1} g_j$$

$$F^{i_t + r + 1} g_{j_t} = \sum_{(i,j)\in\alpha} b(i,j) F^{i+1} g_j.$$

We have already seen that the $F^{i+1} g_j$, $(i,j) \in \alpha$ are linear combinations of the columns of $R(F,G)_\alpha$. It follows that also the $F^r g_j$ and $F_t^{i+r+1} g_{j_t}$ are linear combinations of the columns of $R(F,G)_\alpha$. This finishes the induction and hence the proof of the lemma.

3.3 Successor indices. Let $\alpha \subset I(n,m)$ be a nice selection. The successor indices of α are those elements $(i,j) \in I(n,m) \setminus \alpha$ for which $i = 0$ or for which $(i',j) \in \alpha$ for all $i' < i$ if $i \geq 1$. For every $j_0 \in \{1, \ldots, m\}$ there is precisely one successor index of α of the form (i, j_0); this successor index is denoted $s(\alpha, j_0)$. In the picture below the successor indices of α are indiced by $*$'s (and the elements of α with x's).

Columns of G	*	x	*	x		x_1	e_1	x_3	e_2
Columns of FG	.	x	.	x		.	e_3	.	e_4
.	.	x	.	*		.	e_5	.	x_4
.	.	*	.	.		.	x_2	.	
.	.	.	.	.		.	.	.	.
Columns of F^sG	.	.	.	.		.	.	.	.

3.4 Lemma: Let $\alpha \subset I(n,m)$ be a nice selection and $x_1, \ldots, x_m$ an m-tuple of n-vectors. Then there is precisely one pair $(F,G) \in L_{m,n}(\mathbb{R})$ such that

$R(F,G)_\alpha = I_{n\times n}$, the $n \times n$ unit matrix

$R(F,G)_{s(\alpha,j)} = x_j$ for all $j = 1, \ldots, m$.

Proof: Let f_i be the i-th column of the matrix F, $i = 1, 2, \ldots, n$. Then in the example given above the values of the g_j, $j = 1, \ldots, m$ and f_i, $i = 1, \ldots, n$ can simply be read of from the diagram. One has in this case

$$g_1 = x_1, g_2 = e_1, g_3 = x_3, g_4 = e_2$$
$$f_1 = e_3, f_2 = e_4, f_3 = e_5, f_4 = x_4, f_5 = x_2.$$

It is easy to see that this works in general and to write down the general proof though it tends to be notationally cumbersome.

3.5 Local structure of $L^{cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$. Let $\alpha \subset I(n,m)$ be a nice selection. We define

$$(3.6)\quad \begin{aligned} U_\alpha &= \{(F,G,H) \in L_{m,n,p}(\mathbb{R}) \mid \det R(F,G)_\alpha \neq 0\} \\ V_\alpha &= \{(F,G,H) \in L_{m,n,p}(\mathbb{R}) \mid R(F,G)_\alpha = I_{n\times n}\}; \end{aligned}$$

3.7 Lemma:

(i) $U_\alpha \simeq V_\alpha \times GL_n(\mathbb{R})$

(ii) $V_\alpha \simeq \mathbb{R}^{mn+np}$

Proof: (i) Let $(F,G,H) \in U_\alpha$. We assign to (F,G,H) the pair $((F,G,H)^S, S^{-1})$ where $S = R(F,G)_\alpha^{-1}$. Then $(F,G,H)^S \in V_\alpha$ because $R(SFS^{-1}, SG) = SR(F,G)$ and hence $R(SFS^{-1}, SG)_\alpha = SR(F,G)_\alpha$. Inversely given $((F,G,H),S) \in V_\alpha \times GL_n(\mathbb{R})$ we assign to it the element $(F,G,H)^S$. This proves (i). Assertion (ii) follows immediately from lemma 3.4. Indeed, let $z \in \mathbb{R}^{mn+np}$ and view z as an $m+p$ tuple of n-vectors $z = (x_1, \ldots, x_m; y_1, \ldots, y_p)$. Then there are unique F, G, H such that $R(F,G)_\alpha = I_{n\times n}$, $R(F,G)_{s(\alpha,j)} = x_j$, $h_l = y_l$ where h_l is the l-th row of H.

3.8 Local structure of $L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$. Let again α be a nice selection. Then we define in addition.

$$(3.9)\quad U^{co}_\alpha = U_\alpha \cap L^{co,cr}_{m,n,p}(\mathbb{R}),\ V^{co}_\alpha = V_\alpha \cap L^{co,cr}_{m,n,p}(\mathbb{R})$$

Then one has clearly that V^{co}_α is an open dense (algebraic) subset of V_α and that $U^{co}_\alpha \simeq V^{co}_\alpha \times GL_n(\mathbb{R})$.

3.10 The local nice selection canonical forms c_α. Lemma 3.7 defines us a (local) continuous form on U_α for each nice selection α. It is

$$(3.11)\quad c_\alpha((F,G,H)) = (F,G,H)^{S_\alpha} \in V_\alpha,\ S_\alpha = R(F,G)_\alpha^{-1},\ (F,G,H) \in U_\alpha$$

The L_α are open dense subsets of $L^{cr}_{m,n,p}(\mathbb{R})$, and by lemma 3.2 the union of all the U_α, α a nice selection, covers all of $L^{cr}_{m,n,p}(\mathbb{R})$. This is thus a set of local canonical forms which can be useful in identification problems (it leads to statistically and numerically well posed problems, cf. [15, section II].

3.12 **The dual results.** Dually we consider the set $I(n, p)$ of all row indices of $Q(F, H)$, which we also picture as an $(n + 1) \times p$ array of dots. Now the first row represents the rows of H, the second row the rows of HF, ... A nice selection is defined as before and one has the obvious analogues of all the results given above. In particular if $(F, G, H) \in L^{co}_{m,n,p}(\mathbb{R})$ there is a nice selection $\beta \subset I(n, p)$ such that $Q(F, H)_\beta$ is invertible. Here $Q(F, H)_\beta$ is the matrix obtained from $Q(F, H)$ by removing all rows whose index is not in β.

One also has of course local canonical forms $\bar{c}_\beta$ (defined on $\bar{U}_\beta$) for every nice selection $\beta \subset I(n, p)$:

(3.13) $\bar{c}_\beta((F, G, H)) = (F, G, H)^{S_\beta}, S_\beta = Q(F, H)_\beta, (F, G, H) \in \bar{U}_\beta$

(3.14) $\bar{U}_\beta = \{(F, G, H) \in L_{m,n,p}(\mathbb{R}) \mid Q(F, H)_\beta \text{ is invertible}\}$.

4 Realization theory

Let $A = (A_0, A_1, A_2, \ldots)$ be a sequence of $p \times m$ matrices. We shall say that the sequence A is realizable by an n-dimensional linear system if there exist a system $(F, G, H) \in L_{m,n,p}(\mathbb{R})$ such that $A_i = HF^iG$, $i = 0, 1, 2, \ldots$. It follows immediately from (the proof of) theorem 2.8 above that if A is realizable by means of (F, G, H), then there is also a possible lower dimensional system $\Sigma' = (F', G', H') \in L^{co,cr}_{m,n',p}(\mathbb{R})$, $n' < n$, which also realizes A and which is moreover completely reachable and completely observable.

For each sequence of $p \times m$ matrices A we define the block Hankel matrices

$$(4.1)\quad H_s(A) = \begin{pmatrix} A_0 & A_1 & \cdots & A_s \\ A_1 & & \cdot & \\ \vdots & & \cdot & \vdots \\ & \cdot & & \\ A_s & & \cdots & A_{2s} \end{pmatrix}, \quad s = 0, 1, 2, \ldots .$$

4.2 **Theorem:** The sequence of real $p \times m$ matrices $A = (A_0, A_1, \ldots)$ is realizable by means of a completely reachable and completely observable n-dimensional system if and only if rank $H_s(A) = n$ for all large enough s. Moreover if both $\Sigma, \Sigma' \in L^{co,cr}_{m,n,p}(\mathbb{R})$ realize A then $\Sigma' = \Sigma^S$ for some $S \in GL_n(\mathbb{R})$.

This theorem will be proved below. First ...

4.3 **Corollary:** If the sequence of $p \times m$ matrices A is such that rank $H_s(A) = n$ for all sufficiently large s, then rank $H_s(A) = n$ for all $s \geq n - 1$.

Proof. If $\Sigma = (F, G, H)$ realizes A and Σ is co and cr and of dimension n, then rank $R_{n-1}(F, G) = \text{rank } Q_{n-1}(F, H) = n$, so that rank $H_{n-1}(A) = \text{rank}(R_{n-1}(F, G) Q_{n-1}(F, H)) = n$.

A first step in the proof of theorem 4.2 is now the following lemma which says that if rank $H_s(A) = n$ for all $s \geq r - 1$, then the A_i for $i \geq 2r$ are uniquely determined by the 2r matrices $A_0, \ldots, A_{2r-1}$.

4.4 **Lemma:** Let $A = (A_0, A_1, \ldots)$ be a series of $p \times m$ matrices such that rank $H_s(A) = n$ for all $s \geq r - 1$. There are $m \times m$ matrices $S_0, \ldots, S_{r-1}$ and $p \times p$ matrices $T_0, \ldots, T_{r-1}$ such that for all $i = 0, 1, 2, \ldots$.

$$(4.5)\quad A_{i+r} = A_iS_0 + A_{i+1}S_1 + \ldots + A_{i+r-1}S_{r-1} = T_0A_i + T_1A_{i+1} + \ldots + T_{r-1}A_{i+r-1}.$$

Proof: Because rank $H_{r-1}(A) = n$ and rank $H_r(A) = n$ we have

$$n = \text{rank } H_{r-1}(A) = \text{rank}\left(\begin{array}{cccc|c} A_0 & A_1 & \cdots & A_{r-1} & A_r \\ A_1 & & & \vdots & \vdots \\ \vdots & & & & \\ A_{r-1} & & \cdots & A_{2r-2} & A_{2r-1} \end{array}\right)$$

so that there are $m \times m$ matrices $S_0, \ldots, S_{r-1}$ such that

$$A_{i+r} = A_iS_0 + \ldots + A_{i+r-1}S_{r-1}, \; i = 0, \ldots, r - 1 .$$

Similarly, it follows from

$$n = \text{rank } H_{r-1}(A) = \text{rank}\left(\begin{array}{ccc} A_0 & \cdots & A_{r-1} \\ \vdots & & \vdots \\ A_{r-1} & \cdots & A_{2r-2} \\ \hline A_r & \cdots & A_{2r-1} \end{array}\right)$$

that there are matrices $T_0, \ldots, T_{r-1}$ such that

$$(4.6)\quad A_{r+i} = T_0A_i + \ldots + T_{r-1}A_{i+r-1}, \; i = 0, \ldots, r - 1 .$$

Suppose with induction we have already proved (4.5) for $i < k - 1$, $k > r$.

Consider the following submatrix of $H_k(A)$

$$(4.7)\quad \left(\begin{array}{cccc|ccc} A_0 & A_1 & \cdots & A_{r-1} & A_r & \cdots & A_k \\ A_1 & & & & & & \\ \vdots & & & \vdots & \vdots & & \vdots \\ A_{r-1} & & \cdots & A_{2r-2} & A_{2r-1} & \cdots & A_{k+r-1} \\ \hline A_r & & \cdots & A_{2r-1} & A_{2r} & \cdots & A_{k+r} \end{array}\right).$$

Using the relations (4.5) for $i \leq k-1$ we see that the rank of 4.7 is equal to the rank of

$$(4.8)\quad \left(\begin{array}{cccc|ccc} A_0 & A_1 & \cdots & A_{r-1} & 0 & \cdots & 0 \;\; 0 \\ A_1 & & & & & & \\ \vdots & & & \vdots & \vdots & & \vdots \;\; \vdots \\ A_{r-1} & & \cdots & A_{2r-2} & 0 & \cdots & 0 \;\; 0 \\ \hline A_r & & \cdots & A_{2r-1} & 0 & \cdots & 0 \;\; X \end{array}\right),$$

where $X = A_{k+r} - A_k S_0 - \ldots - A_{k+r-1} S_{r-1}$. Using (4.6) we see by means of row operations on (4.8) that the rank of (4.7) is also equal to the rank of

$$\left(\begin{array}{ccc|cccc} A_0 & \cdots & A_{r-1} & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ A_{r-1} & & A_{2r-2} & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & X \end{array}\right).$$

Now the rank of (4.7) is $n = \operatorname{rank} H_{r-1}(A)$. Hence $X = 0$ which proves the induction step. This proves the first half of (4.5); the second half is proved similarly.

More generally one has the following result (which we shall not need in the sequel).

*4.9 **Lemma:** Let $A_0, \ldots, A_s$ be a finite series of matrices and suppose there are $i, j \in \mathbb{N} \cup \{0\}$ such that $i + j = s - 1$ and

$$\operatorname{rank}\begin{pmatrix} A_0 & \cdots & A_i \\ \vdots & & \vdots \\ A_j & \cdots & A_{i+j} \end{pmatrix} = \operatorname{rank}\left(\begin{array}{ccc|c} A_0 & \cdots & A_i & A_{i+1} \\ \vdots & & \vdots & \vdots \\ A_j & \cdots & A_{i+j} & A_{i+j+1} \end{array}\right) = \operatorname{rank}\left(\begin{array}{ccc} A_0 & \cdots & A_i \\ \vdots & & \vdots \\ A_j & \cdots & A_{i+j} \\ \hline A_{j+1} & \cdots & A_{i+j+1} \end{array}\right) = n$$

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for some $n \in \mathbb{N} \cup \{0\}$, then there are unique $A_{s+1}, A_{s+2}, \ldots$ such that

$\operatorname{rank} H_t(A) = n$

for all $t \geq \max(i, j)$.

Proof: By hypothesis we know that there exist matrices $S_0, \ldots, S_i$

$$(4.10)\quad A_{i+r+1} = A_r S_0 + \ldots + A_{r+i} S_i,\; r = 0, \ldots, j.$$

Now define A_t for $t > s$ by the formula

$$(4.11)\quad A_t = A_{t-i-1} S_0 + \ldots + A_{t-1} S_i.$$

Also by hypothesis we know that there exist $T_0, \ldots, T_j$ such that

$$(4.12)\quad A_{j+r+1} = T_0 A_r + \ldots + T_j A_{j+r},\; r = 0, \ldots, i.$$

To prove that $\operatorname{rank} H_t(A) = n$ for all $t \geq \max(i, j)$ it now clearly suffices to show that (4.12) holds in fact for all $r \geq 0$. Suppose this has been proved for $r \leq q-1$, $q \geq i+1$. Consider the matrix

$$(4.13)\quad \left(\begin{array}{ccc|ccc} A_0 & \cdots & A_i & A_{i+1} & \cdots & A_q \\ \vdots & & \vdots & \vdots & & \vdots \\ A_j & \cdots & A_{i+j} & A_{i+j+1} & \cdots & A_{j+q} \\ \hline A_{j+1} & \cdots & A_{i+j+1} & A_{i+j+2} & \cdots & A_{j+q+1} \end{array}\right).$$

By means of column operations, the hypothesis of the lemma, and (4.10)–(4.11) we see that the rank of the matrix (4.13) is n. Using row operations and (4.12) for $r \leq q-1$ (induction hypothesis) we see that the rank of (4.13) is also equal to the rank of

$$(4.14)\quad \left(\begin{array}{ccc|ccc} A_0 & \cdots & A_i & A_{i+1} & \cdots & A_q \\ \vdots & & \vdots & \vdots & & \vdots \\ A_j & \cdots & A_{i+j} & A_{i+j+1} & \cdots & A_{j+q} \\ \hline 0 & \cdots & 0 & 0 & \cdots\; 0 & X \end{array}\right)$$

where X is the matrix $A_{j+q+1} - T_0 A_q - \ldots - T_j A_{j+q}$. Now use column operations and (4.10), (4.11) to see that the rank of (4.14) is also equal to the rank of

$$(4.15)\quad \left(\begin{array}{ccc|cccc} A_0 & \cdots & A_i & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ A_j & \cdots & A_{i+j} & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & X \end{array}\right)$$

4.16 Proof of theorem 4.2 (first step: existence of a co and cr realization; [10]): Let $r \in \mathbb{N}$ be such that $r \geq n$ and rank $H_s(A) = n$ for all $s \geq r-1$. We write

$$H = H_{r-1}(A) = \begin{pmatrix} A_0 & \cdots & A_{r-1} \\ \vdots & & \vdots \\ A_{r-1} & \cdots & A_{2r-2} \end{pmatrix}, H^{(k)} = \begin{pmatrix} A_k & \cdots & A_{r+k-1} \\ \vdots & & \vdots \\ A_{r+k-1} & \cdots & A_{2r+k-1} \end{pmatrix}$$

and for all $s, t \in \mathbb{N}$ we define

$$E_{s\times t} = (I_{s\times s} \mid 0_{s\times(t-s)}) \text{ if } s < t$$

$$E_{s\times s} = I_{s\times s} \qquad \text{if } s = t$$

$$E_{s\times t} = \left(\frac{I_{t\times t}}{0_{(s-t)\times t}}\right) \qquad \text{if } s > t,$$

where $I_{a\times a}$ is the $a \times a$ identity matrix and $0_{a\times b}$ is the $a \times b$ zero matrix. Because H is of rank n, there exist an invertible $pr \times pr$ matrix P and an invertible $mr \times mr$ matrix M such that

$$(4.17) \quad PHM = \left(\begin{array}{c|c} I_{n\times n} & 0_{n\times(mr-n)} \\ \hline 0_{(pr-n)\times n} & 0_{(pr-n)\times(mr-n)} \end{array}\right) = E_{pr\times n}E_{n\times mr}.$$

Now define

$$(4.18) \quad F = E_{n\times pr}PH^{(1)}ME_{mr\times n},\ G = E_{n\times pr}PHE_{mr\times m},$$
$$H = E_{p\times pr}HME_{mr\times n}$$

We claim that then (F, G, H) realizes A, i.e. that

$$(4.19) \quad A_i = HF^iG,\ i = 0, 1, 2, \ldots .$$

To prove this we define

$$D = \begin{pmatrix} 0 & \cdots & 0 & S_0 \\ I & \ddots & \vdots & \vdots \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & \\ 0 & \cdots 0 & I & S_{r-1} \end{pmatrix} \qquad C = \begin{pmatrix} 0' & I' & 0' & \cdots & 0' \\ \vdots & \ddots & & \ddots & \vdots \\ \vdots & & & & 0' \\ 0' & \cdots & 0' & & I' \\ T_0 & \cdots & & & T_{r-1} \end{pmatrix}$$

where $0, I, 0', I'$ are respectively the $m \times m$ zero matrix, the $m \times m$ identity matrix, the $p \times p$ zero matrix and the $p \times p$ identity matrix and where the $S_0, \ldots, S_{r-1}$ and $T_0, \ldots, T_{r-1}$ are such that (4.5) holds for all i. Then

$$(4.20) \quad H^{(k)} = C^kH = HD^k,\ k = 1, 2, \ldots .$$

(14)

Let $H^* = ME_{mr\times n}E_{n\times pr}P$. Then H^* is a pseudoinverse of H in that

$$(4.21) \quad HH^*H = H$$

(Indeed using (4.17) we have $HH^*H = P^{-1}E_{pr\times n}E_{n\times mr}M^{-1}ME_{mr\times n}E_{n\times pr}PP^{-1}E_{pr\times n}E_{n\times mr}M^{-1} = H$ because $M^{-1}M = I$, $PP^{-1} = I$, $E_{n\times mr}E_{mr\times n} = I_{n\times n}$, $E_{n\times pr}E_{pr\times n} = I_{n\times n}$.)

We now first prove that

$$(4.22) \quad E_{n\times pr}PC^kHME_{mr\times n} = F^k,\ k = 1, 2, \ldots .$$

In view of (4.20) this is the definition of F (cf. (4.18)) in the case $k = 1$. So assume (4.22) has been proved for $k \leq t$. We then have

$$\begin{aligned} E_{n\times pr}PC^{t+1}HME_{mr\times n} &= E_{n\times pr}PC^tHDME_{mr\times n} \text{ (by (4.20))} \\ &= E_{n\times pr}PC^tHH^*HDME_{mr\times n} \text{ (by (4.21))} \\ &= E_{n\times pr}PC^tHME_{mr\times n}E_{n\times pr}PHDME_{mr\times n} \\ &\qquad \text{(by the definition of } H^*) \\ &= F^tE_{n\times pr}PCHME_{mr\times n} \text{ (by the induction hypothesis} \\ &\qquad \text{and (4.20))} \\ &= F^tF \text{ (by the definition of F, cf. (4.18) and (4.20))}. \end{aligned}$$

We now have for all $k > 0$

$$\begin{aligned} A_k &= E_{p\times pr}H^{(k)}E_{mr\times m} \text{ (definition of } H^{(k)}) \\ &= E_{p\times pr}C^kHE_{mr\times m} \text{ (by (4.20))} \\ &= E_{p\times pr}C^kHH^*HE_{mr\times m} \text{ (by (4.21))} \\ &= E_{p\times pr}C^kHME_{mr\times n}E_{n\times pr}PHE_{mr\times m} \text{ (by the definition of } H^*) \\ &= E_{p\times pr}HD^kME_{mr\times n}G \text{ (by the definition of G and (4.20))} \\ &= E_{p\times pr}HH^*HD^kME_{mr\times n}G \text{ (by (4.21))} \\ &= E_{p\times pr}HME_{mr\times n}E_{n\times pr}PHD^kME_{mr\times n}G \text{ (by the definition of } H^*) \\ &= HE_{n\times pr}PC^kHME_{mr\times n}G \text{ (by the definition of H and (4.20))} \\ &= HF^kG \text{ (by (4.22))}. \end{aligned}$$

This proves the existence of an n-dimensional system $\Sigma = (F, G, H)$ which realizes A. Now for all $s = 0, 1, 2, \ldots$

$$H_s(A) = Q_s(F, H)R_s(F, G),$$

where

$$Q_s(F, H) = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^s \end{pmatrix},\ R_s(F, G) = (G\ FG\ \ldots\ F^sG).$$

Both $Q_s(F, H)$ and $R_s(F, G)$ have necessarily rank $\leq n$. It follows via the Cayley-Hamilton theorem that (F, G, H) is completely reachable and completely controllable, because rank $H_s(A) = n$ for $s \geq r-1$.

4.23 Proof of the uniqueness statement of theorem 4.2: Let $\Sigma = (F, G, H)$ and $\bar{\Sigma} = (\bar{F}, \bar{G}, \bar{H})$ be two co and cr realizations of A. Then $\dim(\Sigma) = \operatorname{rank} H_{n-1}(A) = \dim(\bar{\Sigma})$. By hypothesis we have

$$(4.24)\qquad A_i = HF^iG = \bar{H}\bar{F}^i\bar{G},\ i = 0, 1, 2, \ldots .$$

According to lemma 3.2 and 3.11 there exists a nice selection α (of size n) of $I(n-1, m)$, the set of column indices of $R_{n-1}(F, G)$ and $H_{n-1}(F, G, H)$, and there exists a nice selection β (of size n) of $I(n-1, p)$, the set of row indices of $Q_{n-1}(F, H)$ and $H_{n-1}(F, G, H)$, such that

$$\operatorname{rank}(R_{n-1}(F, G)_\alpha) = \operatorname{rank}(Q_{n-1}(F, H)_\beta) = n .$$

(Note that a nice selection in $I(n, m)$ (or $I(n, p)$) is always contained in $I(n-1, m)$ (or $I(n-1, p)$).) Let $H_{n-1}(F, G, H)_{\alpha,\beta}$ be the matrix obtained from $H_{n-1}(F, G, H)$ by removing all rows whose index is not in β and all columns whose index is not in α. Then

$$H_{n-1}(F, G, H)_{\alpha,\beta} = Q_{n-1}(F, H)_\beta R_{n-1}(F, G)_\alpha$$

so that $H_{n-1}(F, G, H)_{\alpha,\beta}$ is an invertible $n \times n$ matrix. Also

$$H_{n-1}(F, G, H)_{\alpha,\beta} = H_{n-1}(\bar{F}, \bar{G}, \bar{H})_{\alpha,\beta} = Q_{n-1}(\bar{F}, \bar{H})_\beta R_{n-1}(\bar{F}, \bar{G})_\alpha$$

so that $Q_{n-1}(\bar{F}, \bar{H})_\beta$ and $R_{n-1}(\bar{F}, \bar{G})_\alpha$ are also invertible. Now let

$$\Sigma_1 = (F_1, G_1, H_1) = (F, G, H)^T,\ T = Q_{n-1}(F, H)_\beta$$
$$\bar{\Sigma}_1 = (\bar{F}_1, \bar{G}_1, \bar{H}_1) = (\bar{F}, \bar{G}, \bar{H})^{\bar{T}};\ \bar{T} = Q_{n-1}(\bar{F}, \bar{H})_\beta .$$

Then of course Σ_1 and $\bar{\Sigma}_1$ also realize A. Moreover, using (2.4) we see

$$Q_{n-1}(F_1, H_1)_\beta = I_n = Q_{n-1}(\bar{F}_1, \bar{H}_1)_\beta .$$

It follows that

$$R(F_1, G_1) = H_n(\Sigma_1)_\beta = H_n(\Sigma)_\beta = H_n(\bar{\Sigma})_\beta = H_n(\bar{\Sigma}_1)_\beta = R(\bar{F}_1, \bar{G}_1)$$

and, in turn, this means that $F_1 = \bar{F}_1$ and $G_1 = \bar{G}_1$ by lemma (3.7) (i) combined with lemma (3.4). Further the matrix consisting of the first p rows of $H_n(\Sigma_1) = H_n(\bar{\Sigma}_1)$ is equal to

$$H_1 R(F_1, G_1) = \bar{H}_1 R(\bar{F}_1, \bar{G}_1)$$

so that also $H_1 = \bar{H}_1$ because $R(F_1, G_1) = R(\bar{F}_1, \bar{G}_1)$ is of rank n. This proves that indeed $\bar{\Sigma} = \Sigma^S$ with $S = \bar{T}^{-1}T$.

4.25 A realization algorithm. Now that we know that A is realizable by a co and cr system of dimension n iff rank $H_s(A) = n$ for all large enough s it is possible to give a rather easier algorithm for calculating a realization than the one used in 4.16 above (which is the algorithm of B.L. Ho). It goes as follows. Because A is realizable by a $\Sigma \in L^{co,cr}_{m,n,p}(\mathbb{R})$ there exist a nice selection $\alpha \subset I(n, m)$, the set of column indices of $R(F, G)$ and $H_n(\Sigma)$, and a nice selection $\beta \subset I(n, p)$, the set of row indices of $Q(F, H)$ and $H_n(\Sigma)$, such that

$$(4.26)\qquad H_n(A)_{\alpha,\beta} = S$$

is an invertible $n \times n$ matrix. Consider

$$S^{-1} H_n(A)_\beta .$$

This $n \times (n+1)\,m$ matrix is necessarily of the form $R(F, G)$ for some $(F, G) \in L^{cr}_{m,n}(\mathbb{R})$ and moreover by (4.26)

$$(S^{-1} H_n(A)_\beta)_\alpha = I_n$$

so that F, G can simply be written down from $S^{-1} H_n(A)_\beta$ as in the proof of lemma 3.4. The matrix H is now obtained as the matrix consisting of the first p rows of $H_n(A)_\alpha$. After choosing α, this algorithm describes the unique triple (F, G, H) which realizes A such that moreover $R(F, G)_\alpha = I_n$.

***4.27 Relation with rational functions.** Suppose that $H_k(A)$ is of rank n for all sufficiently large k. Then by theorem 4.2 the sequence A is realizable. Using Laplace transforms (cf. 1.8 above) we see that this means that the $p \times m$ matrix of power series $\sum_{i=0}^{\infty} A_i s^{-i-1}$ is in fact a matrix of rational functions.

$$(4.28)\qquad \sum_{i=0}^{\infty} A_i s^{-i-1} = (s^n - a_{n-1}s^{n-1} - \ldots - a_1 s - a_0)^{-1} B(s) = d(s)^{-1} B(s),$$

where $B(s)$ is a $p \times m$ matrix of polynomials in s of degree $\leq n-1$. Inversely if

$$(4.29)\qquad \sum_{i=0}^{\infty} A_i s^{-i-1} = d'(s)^{-1} B'(s)$$

for a matrix of polynomials $B'(s)$ and a polynomial $d'(s) = s^r - a'_{r-1}s^{r-1} - \ldots - a'_1 s - a'_0$ with $r = \text{degree}(d'(s)) > \text{degree } B'(s)$, then

$$A_{i+r} = a'_0 A_i + a'_1 A_{i+1} + \ldots + a'_{r-1} A_{i+r-1}$$

for all $i = 0, 1, 2, \ldots$. And this, in turn implies that

$$\operatorname{rank} H_k(A) = \operatorname{rank} H_{r-1}(A)$$

for all $k \geq r-1$, so that A is realizable. It follows that A is realizable iff $\Sigma A_i s^{-i-1}$ represents a rational function which goes to zero as $s \to \infty$.

5 Feedback splits the external description degeneracy

In this section we shall prove the result described in section 1.6. To do this we first discuss still another local canonical form.

5.1 The Kronecker nice selection of a system. Let $(F, G, H) \in L^{cr}_{m,n,p}(\mathbb{R})$. We proceed as follows to obtain a "first" nice selection κ such that $(F, G, H) \in U_\kappa$.
Consider the set of column indices $I(m, n)$ in the order $(0, 1) < (0, 2) < \ldots < (0, m) < (1, 1) < \ldots < (1, m) < \ldots < (n, 1) < \ldots < (n, m)$. For each (i, j) we set $(i, j) \in \kappa \Leftrightarrow F^i g_j$ is linear independent of the $F^{i'} g_{j'}$ with $(i', j') < (i, j)$. We shall call the subset κ of $I(n, m)$ thus obtained, the Kronecker selection of (F, G, H) and denote it with $\kappa(F, G, H)$. It is obvious that κ has n elements if $(F, G, H) \in L^{cr}_{m,n,p}(\mathbb{R})$.

5.2 Lemma: The Kronecker selection κ defined above is a nice selection.

Proof: Let $(i, j) \in \kappa$ and suppose $i \geq 1$. Suppose that $(i', j) \notin \kappa$, $i' < i$. This means that there is a relation

$$F^{i'} g_j = \sum_{(k,l) < (i',j)} b(k, l) F^k g_l .$$

Multiplying with $F^{i-i'}$ on the left one obtains

$$F^i g_j = \sum_{(k,l) < (i',j)} b(k, l) F^{i-i'+k} g_l$$

showing that $F^i g_j$ is linearly dependent on the $F^s g_{j'}$, with $(s, j') < (i, j)$. A contradiction, q.e.d.

5.3 Lemma. Let $(F, G, H) \in L^{cr}_{m,n,p}(\mathbb{R})$ and $S \in GL_n(\mathbb{R})$, then

$$\kappa(F, G, H) = \kappa((F, G, H)^S) .$$

5.4 Lemma. Let $(F, G, H) \in L^{cr}_{m,n,p}(\mathbb{R})$ and let L be an $m \times n$ matrix. Then

$$\kappa(F, G, H) = \kappa(F + GL, G, H) .$$

The proof of lemma 5.3 is immediate, because the dependency relations between the $(SFS^{-1})^i(Sg_j) = S(F^i g_j)$, $(i, j) \in I(n, m)$, are precisely the same as those between the $F^i g_j$, $(i, j) \in I(n, m)$. As to lemma 5.4 we define

$$\begin{aligned} X_0(\Sigma) &= \text{subspace of } X = \mathbb{R}^n \text{ generated by } g_1, \ldots, g_m \\ X_1(\Sigma) &= \text{subspace of } X = \mathbb{R}^n \text{ generated by } g_1, \ldots, g_m, Fg_1, \ldots, Fg_m \\ &\vdots \\ X_n(\Sigma) &= \text{subspace of } X \in \mathbb{R}^n \text{ generated by } g_1, \ldots, g_m, \\ &\quad Fg_1, \ldots, Fg_m, \ldots, F^n g_1, \ldots, F^n g_m . \end{aligned} \tag{5.5}$$

Let $\Sigma(L) = (F + GL, G, H)$ and let $\hat{F} = F + GL$. Then one easily obtains by induction that

(5.6) $X_i(\Sigma(L)) = X_i(\Sigma)$, $i = 0, \ldots, n$

and that

(5.7) $\hat{F}^i g_j \equiv F^i g_j \bmod X^{i-1}(\Sigma)$, $i = 0, 1, \ldots, n$

(where, by definition, $X^{-1}(\Sigma) = \{0\}$). Lemma 5.4 is an immediate consequence of (5.7). (Note that a basis for $X^i(\Sigma)$ is formed by the vectors $F^k g_l$ with $(k, l) \in \kappa(\Sigma)$ and $k \leq i$; the classes of the $F^k g_l$ with $(k, l) \in \kappa(\Sigma)$, $k = i$ are a basis for the quotient space $X^i(\Sigma)/X^{i-1}(\Sigma)$, $i = 0, \ldots, n$).
If $\Sigma = (F, G, H) \in L^{cr,\infty}_{m,n,p}(\mathbb{R})$ then $\kappa(F, G, H)$ can be calculated from $H_n(F, G, H)$. Indeed in that case $Q(F, H)$ is of rank n. Therefore, because $H_n(F, G, H) = Q(F, H)R(F, G)$, the dependency relations between the columns of $H_n(F, G, H)$ and between the columns of $R(F, G)$ are exactly the same.

Remark: If $(F, G, H) \in L^{cr}_{m,n,p}(\mathbb{R})$ then also $(F + GL, G, H) \in L^{cr}_{m,n,p}(\mathbb{R})$ as is easily checked. But if $(F, G, H) \in L^{co}_{m,n,p}(\mathbb{R})$, then $(F + GL, G, H)$ need not also be completely observable. Though of course this will be the case for sufficiently small L (because $L^{co}_{m,n,p}(\mathbb{R})$ is an open subset of $L_{m,n,p}(\mathbb{R})$).

***5.9 The Kronecker control invariants.** The invariant $\kappa(F, G, H)$ depends only on F and G, so that we can also write $\kappa(F, G)$. For each $j = 1, \ldots, m$, let k_j be the number of elements (i, l) in $\kappa(F, G)$ such that $l = j$. Let $\kappa_1(F, G) \geq \ldots \geq \kappa_{m'}(F, G)$, $m' = \text{rank}(G)$, be the sequence of those k_j which are $\neq 0$ ordered with respect to size. It follows from lemma's 5.3 and 5.4 that the $\kappa_i(F, G)$ are invariant for the transformations

(5.10) $(F, G) \mapsto (F, G)^S = (SFS^{-1}, SG)$ (base change in state space)

(5.11) $(F, G) \mapsto (F + GL, G)$ (feedback).

One easily checks that the $\kappa_i(F, G)$ are also invariant under

(5.12) $(F, G) \mapsto (F, GT)$, $T \in GL_m(\mathbb{R})$ (base change in input space).

This can, e.g., be seen as follows. Let $\lambda_i(\Sigma) = \dim X^i(\Sigma) - \dim X^{i-1}(\Sigma)$ for $i = 0, 1, \ldots, n$. Consider an rectangular array of $(n+1) \times m$ boxes with the rows labelled $0, \ldots, n$. Now put a cross in the first $\lambda_i(\Sigma)$ boxes of row i for $i = 0, \ldots, n$. Then $\kappa_j(\Sigma), j = 1, \ldots, m'$ is the number of crosses in column j of the array. Obviously the $\lambda_i(\Sigma)$ do not change under a transformation of type (5.12), proving that also the $\kappa_j(F, G)$ are invariant under 5.12.

The group generated by all these transformations is called the *feedback group*. Thus the $\kappa_i(F, G)$ are invariants of the feedback group acting on $L^{cr}_{m,n}(\mathbb{R})$. It now turns out that these are in fact the only invariants. I.e. if $(F, G), (\bar{F}, \bar{G}) \in L^{cr}_{m,n}(\mathbb{R})$ and $\kappa_i(F, G) = \kappa_i(\bar{F}, \bar{G})$, $i = 1, \ldots, m'$, then $(\bar{F}, G)$ can be obtained from (F, G) by means of a series of transformations from (5.10)–(5.12). Cf. [11] for a proof, or cf. 5.30 below.

The $\kappa_i(F, G)$ are also identifiable with Kronecker's minimal column indices of the singular matrix pencil $(zI_n - F \mid G)$, cf. [11].

Still another way to view the $\kappa_i(F, G)$ is a follows.

Consider the transfer matrix $T(s) = H(sI_n - F)^{-1}G$ of the cr and co linear dynamical system $\Sigma = (F, G, H)$ considered as a $p \times m$ matrix valued function of the complex variable s. One can now prove (cf. [14]):

Theorem: There exist matrices $N(s)$ and $D(s)$ of polynomial functions of s such that (i) $T(s) = N(s)D(s)^{-1}$, (ii) there exist matrices of polynomials such that $X(s)N(s) + Y(s)D(s) = I_m$, (iii) $N(s)$ and $D(s)$ are unique up to multiplication on the right by a unit from the ring of polynomial $m \times m$ matrices. Moreover degree $(\det D(s)) = n = \dim(\Sigma)$.

Now for each $s \in \mathbb{C}$, one defines

$$\phi_\Sigma(s) = \{(N(s)u, D(s)u) \mid u \in \mathbb{C}^m\} \subset \mathbb{C}^{p+m}.$$

If $s \in \mathbb{C}$ is such that $D(s)^{-1}$ exists, then also $\phi_\Sigma(s) = \{(T(s)u, u) \mid u \in \mathbb{C}^m\} \subset \mathbb{C}^{p+m}$. In any case $\phi_\Sigma(s)$ is a p-dimensional subspace of $\mathbb{C}^{p+m}$. In addition one defines $\phi_\Sigma(\infty) = \{(0, u) \mid u \in \mathbb{C}^m\} \subset \mathbb{C}^{p+m}$, which is entirely natural because $\lim_{s \to \infty} T(s) = 0$. This gives a continuous map of the Riemann sphere $\mathbb{C} \cup \{\infty\} = S^2$ to the Grassmann manifold $G_{m,p+m}(\mathbb{C})$ of m-planes in $p + m$ space. Let $\xi_m \to G_{m,p+m}(\mathbb{C})$ be the canonical complex vector bundle whose fibre over $z \in G_{m,p+m}(\mathbb{C})$ is the m-plane represented by z. Pulling back ξ_m along ϕ_Σ gives us a holomorphic complex vector bundle $\xi(\Sigma)$ over S^2.

Now holomorphic vectorbundles over the sphere S^2 have been classified by Grothendieck. The classification result is: every holomorphic vectorbundle over S^2 is isomorphic to a direct sum of line bundles and line bundles are classified by their degrees.

It now turns out that the numbers classifying $\xi(\Sigma)$, the bundle over S^2 defined by the system Σ, are precisely the $-\kappa_i(\Sigma)$, $i = 1, \ldots, m$, where $\kappa_i(\Sigma) = 0$ for $i > m' = \text{rank}(G)$. One also recovers $n = \dim(\Sigma)$, if $\Sigma \in L^{co,cr}_{m,n,p}(\mathbb{R})$, as the intersection number of $\phi_\Sigma(S^2)$ with a hyperplane in $G_{m,m+p}(\mathbb{C})$.

These observations are due to Clyde Martin and Bob Hermann, cf. [13].

(17)

As we have seen the $\kappa_i(\Sigma)$ are invariants for the transformations (5.10), (5.11), (5.12). Being defined in terms of F and G alone they are also obviously invariant under base change in output space: $(F, G, H) \mapsto (F, G, SH)$, $S \in GL_p(\mathbb{R})$. The $\kappa_i(\Sigma)$ are, however, definitely not a full set of invariants for the group $\mathcal{G}$ acting on $L_{m,n,p}(\mathbb{R})$, where $\mathcal{G}$ is the group generated by base changes in state space, input space and output space and the feedback transformations.

5.13 The canonical input base change matrix $T(\Sigma)$. Let $\Sigma = (F, G, H) \in L^{cr}_{m,n,p}(\mathbb{R})$ and let $\kappa = \kappa(\Sigma)$ be the Kronecker nice selection of Σ. Let $(i, j) = s(\kappa, j)$ be a successor index of κ. By the definition of κ we have a unique expression of the form

$$F^i g_j = \sum_{\substack{(i,j') \in \kappa \\ j' < j}} a_j(j') F^i g_{j'} + \sum_{\substack{(k,l) \in \kappa \\ k < i}} a(k, l) F^k g_l \tag{5.14}$$

(where the $a(k, l)$ in the second sum also depend on i and j of course). Now define recursively

$$\hat{g}_j = g_j - \sum_{j' < j} a_j(j') g_{j'}, \quad \hat{G} = (\hat{g}_1, \ldots, \hat{g}_m) \tag{5.15}$$

and

$$T(\Sigma) = (b_{jk}), \tag{5.16}$$

where $b_{jk} = 1$ if $j = k$, $b_{jk} = -a_k(j)$, if $j < k$, and $b_{jk} = 0$ if $j > k$.
Then $\hat{G} = GT(\Sigma)$, and $T(\Sigma)$ is an upper triangular matrix of determinant 1.

5.17 Lemma: Let $\Sigma \in (F, G, H) \in L^{cr}_{m,n,p}(\mathbb{R})$, then

$$T(\Sigma) = T(\Sigma^S), \; T(\Sigma(L)) = T(\Sigma)$$

for all $S \in GL_n(\mathbb{R})$ and all feedback matrices $L \in \mathbb{R}^{m \times n}$.

Proof. Obvious. (Use (5.7)).

5.18 Example: Let $m = 5$, $n = 9$, and let $(F, G, H) \in L^{cr}_{5,9,p}(\mathbb{R})$ have Kronecker selection $\kappa(F, G, H)$ equal to

$$\kappa = \begin{matrix} x & x & . & x & x \\ x & x & . & x & . \\ . & x & . & . & . \\ . & x & . & . & . \\ . & . & . & . & . \end{matrix}$$

where we have omitted ...

Then $T(\Sigma)$ is an upper triangular matrix of the form

$$T(\Sigma) = \begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that $T(\Sigma)^{-1}$ is of precisely the same form.

This is a general phenomon. Indeed by (5.14) and (5.15) (cf. also example (5.18)) $\hat{g}_j$ is of the form

$$(5.19) \quad \hat{g}_j = g_j + \sum_{\substack{k_i > k_j \\ i < j}} b_{ij} g_i, \; T(\Sigma) = (b_{ij}) .$$

So that $b_{ij} = 0$ unless $i = j$ (and then $b_{ij} = 1$) or $i < j$ and $k_i > k_j$

Let $t_1, \dots, t_m$ be the columns of $T(\Sigma)$ and $e_1, \dots, e_m$ the standard basis for $\mathbb{R}^m$. Then

$$(5.20) \quad t_j = e_j + \sum_{\substack{k_i > k_j \\ i < j}} b_{ij} e_i .$$

Using induction with respect to an ordening of the $\{1, \dots, m\}$ satisfying $i < j \Rightarrow k_i \geq k_j$ it readily follows that

$$e_j = t_j + \sum_{\substack{i < j \\ k_i > k_j}} b'_{ij} t_i .$$

which proves that $T(\Sigma)^{-1}$ also has zero entries at all spots (i, j) with $i > j$ or ($i < j$ and $k_i \leq k_j$).

5.21 The block companion canonical form. Let κ be a nice selection. We are going to construct a canonical form on the subspace W_κ of all $\Sigma \in L^{cr,co}_{m,n,p}(\mathbb{R})$ with $\kappa(\Sigma) = \kappa$. We shall do this only in full detail for the case that κ is the nice selection of example 5.18. This special case is, however, general enough to see that this construction works in general. Let $(F, G, H) \in W_\kappa$ and let $\hat{G} = GT(\Sigma)$. Now consider the system $(F, \hat{G}, H)$ which is also in W_κ as is easily checked. This system has the property that for each successor index $s(\kappa, j) = (i, j)$ of κ with $i \neq 0$ we have

$$(5.22) \quad F^i \hat{g}_j = \sum_{\substack{(k,l) \in \kappa \\ k < i}} a'(k, l) F^k \hat{g}_l$$

(i.e. $T(F, \hat{G}, H) = I_m$). Indeed, using (5.14)

$$F^i \hat{g}_j = F^i g_j - \sum_{j' < j} a_j(j') F^i g_{j'} = \sum_{\substack{(k,l) \in \kappa \\ k < i}} a(k, l) F^k g_l = \sum_{\substack{(k,l) \in \kappa \\ k < i}} a'(k, l) F^k \hat{g}_l$$

because, clearly, $X_i(F, G, H) = X_i(F, \hat{G}, H)$ for all $i = 0, 1, 2, \dots, n$, cf. (5.5), and cf. also the remarks just below (5.7).

Now define a new basis for $\mathbb{R}^n$ as follows. Let $\kappa = \{(0, j_1), \dots, (i_1, j_1); \dots; (0, j_r), \dots, (i_r, j_r)\}$. Then $k_t = i_t + 1$, $t = 1, \dots, r$, and $k_1 + \dots + k_r = n$. For the successor indices $s(\kappa, j) = (k_t, j_t)$, $t = 1, \dots, r$, write

$$(5.23) \quad F^{k_t} \hat{g}_{j_t} = - \sum_{\substack{(k,l) \in \kappa \\ k < k_t}} b_t(k, l) F^k \hat{g}_l .$$

Setting $b_t(k, l) = 0$ for all $(k, l) \notin \kappa$ we now define a new basis for $\mathbb{R}^n$ by

$$(5.24) \quad \begin{aligned} e_1 &= F^{k_1 - 1} \hat{g}_{j_1} + \sum_{j=1}^{m} b_1(k_1 - 1, j) F^{k_1 - 2} \hat{g}_j + \dots + \sum_{i=1}^{t} b_1(1, j) \hat{g}_j \\ e_2 &= F^{k_1 - 2} \hat{g}_{j_1} + \sum_{j=1}^{m} b_1(k_1 - 1, j) F^{k_1 - 3} \hat{g}_j + \dots + \sum_{i=1}^{t} b_1(2, j) \hat{g}_j \\ &\;\;\vdots \\ e_{k_1} &= \hat{g}_{j_1} \\ e_{k_1 + 1} &= F^{k_2 - 1} \hat{g}_{j_2} + \sum_{j=1}^{m} b_2(k_2 - 1, j) F^{k_2 - 2} \hat{g}_j + \dots + \sum_{i=1}^{t} b_2(1, j) \hat{g}_j \\ &\;\;\vdots \\ e_{k_1 + k_2} &= \hat{g}_{j_2} \\ &\;\;\vdots \\ e_{k_1 + \dots + k_r} &= \hat{g}_{j_r} . \end{aligned}$$

Let $X_0 \subset \mathbb{R}^n$ be the space spanned by the vectors $\hat{g}_{j_1}, \dots, \hat{g}_{j_r}$ i.e. $X_0 = X_0(F, \hat{G}, H) = X_0(\Sigma)$. Then we see from (5.23) that for the vectors defined by (5.24) above we have

$$Fe_1 \in X_0, \; F(e_i) \equiv e_{i-1} \bmod X_0 \quad \text{for } i = k_1, k_1 - 1, \dots, 2$$

$$Fe_{k_1 + 1} \in X_0, \; F(e_i) \equiv e_{i-1} \bmod X_0 \quad \text{for } i = k_1 + k_2, \dots, k_1 + 2$$

$$\vdots$$

$$Fe_{k_1 + \dots + k_{r-1}} \in X_0, \; F(e_i) \equiv e_{i-1} \bmod X_0 \quad \text{for } i = k_1 + \dots + k_r, \dots, k_1 + \dots + k_{r-1} + 2$$

It follows that which respect to the basis $e_1, \ldots, e_n$, F and $\hat{G}$ are of the form

$$F = \left(\begin{array}{ccc|ccc|c|ccc}
0\;1\;0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\
 & \ddots & 0 & \vdots & & \vdots & \cdots & \vdots & & \vdots \\
0 & \cdots\;0 & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\
* & \cdots & * & * & \cdots & * & & * & \cdots & * \\
\hline
0 & \cdots & 0 & 0\;1\;0 & \cdots & 0 & & 0 & \cdots & 0 \\
\vdots & & \vdots & & \ddots & 0 & \cdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots\;0 & 1 & & 0 & \cdots & 0 \\
* & \cdots & * & * & \cdots & * & & * & \cdots & * \\
\hline
\vdots & & & \vdots & & & \ddots & & \vdots & \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & & 0\;1\;0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & \ddots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots\;0 & 1 \\
* & \cdots & * & * & \cdots & * & & * & \cdots & *
\end{array}\right)
\begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} k_1 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} k_2 \\ \\ \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} k_3 \end{array} \tag{5.25}$$

$\hat{G} = (\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m)$, with

$$\hat{g}_{j_1} = e_{k_1}, \hat{g}_{j_2} = e_{k_1+k_2}, \ldots, \hat{g}_{j_r} = e_{k_1+\ldots+k_r} = e_n, \quad \hat{g}_j = 0 \text{ for } j \in \{1, \ldots, m\} \setminus \{j_1, \ldots, j_r\}. \tag{5.26}$$

In particular in the case that κ is the nice selection of example 5.18 we see that with respect to the basis $e_1, \ldots, e_n$ defined by 5.24 the matrices F and G take the form (cf. 5.18, the inverse of $T(\Sigma)$ is of the same form as $T(\Sigma)$).

$$F' = \left(\begin{array}{cc|cccc|cc|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 \\
\hline
d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9
\end{array}\right) \tag{5.27}$$

(19)

$$G' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & * \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

This does not yet define a canonical form on W_κ. True, for every $\Sigma \in W_\kappa$ there exists an $S \in GL_n(\mathbb{R})$ such that $(F, G)^S$ takes the form (5.27). But for two pairs $(F, G) \neq (\bar{F}, \bar{G})$, both of the form (5.27), there may very well exists an $S \neq I_n$ such that $(F, G)^S = (\bar{F}, \bar{G})$.

In fact, it is now not difficult to check that if S is an $n \times n$ matrix of the form

$$S = \left(\begin{array}{cc|cccc|cc|c}
1 & 0 & s_{13} & s_{14} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & s_{13} & s_{14} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & s_{73} & s_{74} & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & s_{73} & s_{74} & 0 & 0 & 1 & 0 \\
\hline
s_{91} & 0 & s_{93} & s_{94} & s_{95} & 0 & s_{97} & 0 & 1
\end{array}\right)$$

then $SG = G$ and SFS^{-1} is of the same general form as F, if F and G are of the form (5.27). Choosing $s_{13}, s_{14}, s_{73}, s_{74}, s_{91}, s_{93}, s_{94}, s_{95}$ and s_{97} judiciously we see that for every $\Sigma = (F, G, H) \in W_\kappa$, there exists an $S \in GL_n(\mathbb{R})$ such that SFS^{-1} and SG take the forms

$$SFS^{-1} = \left(\begin{array}{cc|cccc|cc|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_1 & a_2 & a_3 & a_4 & 0 & 0 & a_7 & a_8 & a_9 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
c_1 & c_2 & c_3 & c_4 & 0 & 0 & c_7 & c_8 & c_9 \\
\hline
\end{array}\right) \tag{5.28}$$

$$SG = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & c_{13} & 0 & c_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & c_{23} & c_{24} & c_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & c_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$T(\Sigma)^{-1} = \begin{pmatrix} 1 & 0 & c_{13} & 0 & c_{15} \\ 0 & 1 & c_{23} & c_{24} & c_{25} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & c_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The general pattern should be clear: the off-diagonal blocks have zero's in the last row iff there are more columns than rows, in fact in that case the last row ends with (number of columns) – (number of rows) zero's; the structure of the diagonal blocks is clear.
Now suppose that (F', G', H') and (F'', G'', H'') are two systems such that $(F', G')^S = (F'', G'')$ for some S and such that (F', G') and (F'', G'') are both of the forms (5.28). One checks easily that then necessarily $S = I_n$. We have shown

5.29 Proposition: Let κ be the nice selection of example 5.18. Then for every $\Sigma = (F, G, H) \in W_\kappa$ there is precisely one $S \in GL_n(\mathbb{R})$ such that SFS^{-1} and SG have forms (5.28).
This means in particular (in view of the results of section 4 above) that if $\Sigma \in W_\kappa \cap L^{co,cr}_{n,m,p}(\mathbb{R})$, then the real numbers $a_1, \ldots, a_4, a_7, \ldots, a_9, b_1, \ldots, b_9, c_1, \ldots, c_4, c_7, \ldots, c_9, d_1, d_3, d_7, d_9$ can be calculated from $f(\Sigma)$ (or $A_0, \ldots, A_{2n-1}$). Of course these results hold quite generally for all nice selections κ. We note that in general W_κ is not an open subspace of $L^{cr}_{n,m,p}(\mathbb{R})$. In fact $W_\kappa/GL_n(\mathbb{R})$ is a linear subspace of $U_\kappa/GL_n(\mathbb{R}) = \mathbb{R}^{mn+np} \simeq V_\kappa$. In case κ is the nice selection of example 5.18 the codimension of $W_\kappa/GL_n(\mathbb{R})$ in $U_k/GL_n(\mathbb{R})$ is 12. (This number can immediately be read off from κ: g_3 linearly dependent on g_1, g_2 causes $9 - 2 = 7$ linear restrictions; Fg_5 linearly dependent on $g_1, g_2, g_4, g_5, Fg_1, Fg_2, Fg_4$ causes $9 - 7 = 2$ extra linear restrictions; F^2g_1 linearly dependent on $g_1, g_2, g_4, g_5, Fg_1, Fg_2, Fg_4$ causes $9 - 7 = 2$ more linear restrictions; and finally F^2g_4 dependent on $g_1, g_2, g_4, g_5, Fg_1, Fg_2, Fg_4, F^2g_2$ causes $9 - 8 = 1$ more linear restriction; $7 + 2 + 2 + 1 = 12$).

*5.30. Using the results above, it is now easy to prove that the $\kappa_1(F, G), \ldots, \kappa_{m'}(F, G)$ are the only invariants of the feedback group acting on $L^{cr}_{m,n}(\mathbb{R})$. Indeed, we have already shown that the $\kappa_i(F, G)$, $i = 1, \ldots, m'$ are invariants.
Inversely, using first of all a transformation of type (5.12) we can see to it that (F, GT) has $k_1 \geq k_2 \geq \ldots \geq k_m$, and then $\kappa_1(F, G) = k_1, \ldots, \kappa_{m'}(F, G) = k_{m'}$, $k_i = 0$ for $i > m'$. Then, using transformations of type (5.10) and (5.12), we can change (F, GT) into a pair (F', G') with F' and G' of the type (5.25), (5.26). A final transformation of type (5.11) then changes F' into a matrix of type (5.25) with all stars equal to zero. The final pair (F'', G'') thus obtained depends only on the numbers $\kappa_1(F, G), \ldots, \kappa_{m'}(F, G)$.

5.31 Feedback breaks all symmetry. We are now in a position to prove the result mentioned in 1.6 that feedback splits the degenerate external description of systems. We shall certainly have proved this if we have proved.

5.32 Theorem: Let $\Sigma \in L^{co,cr}_{m,n,p}(\mathbb{R})$. Then Σ is completely determined by the input-output maps $f(\Sigma(L))$ for small L. More precisely let $\Sigma = (F, G, H)$ and $A_i(L) = H(F + GL)^iG$ for $i = 0, 1, \ldots, 2n - 1$. Then the entries of $A_i(L)$ are differentiable functions of L, and F, G and H can be calculated from $A_0, \ldots, A_{2n-1}$ and the numbers

$$\frac{\partial A_i(L)}{\partial l_{jk}}\Big|_{L=0}, \quad i = 0, \ldots, 2n-1,\ j = 1, \ldots, m,\ k = 1, \ldots, n.$$

Proof: Let $\kappa = \kappa(\Sigma)$. Recall that κ can be calculated from $A_0, \ldots, A_{2n-1}$ (because Σ is co and cr). Now assume that κ is the nice selection of example 5.18. (This is sufficiently general, I hope, to make it clear that the theorem holds in general). Let $\Sigma' = (F', G', H')$ be the block companion canonical form of (F, G, H) (Σ' is obtained as follows: first calculate any realization $\Sigma'' = (F'', G'', H'')$ of $A_0, \ldots, A_{2n-1}$, e.g. by means of the algorithm of 4.25 above and then put Σ'' in block companion canonical form as in 5.21 above).
Then

$$\Sigma' = \Sigma^{S^{-1}}$$

for a certain $S \in GL_n(\mathbb{R})$, and it remains to calculate S. With this aim in mind we examine $\Sigma(L) = (F + GL, G, H)$ and its block companion canonical form. Consider

$$\begin{aligned} \Sigma(L)^{S^{-1}} &= (S^{-1}FS + S^{-1}GLS, S^{-1}G, HS) \\ &= (F' + G'LS, G', H'). \end{aligned}$$

Now assume that L is of the form

$$(5.33)\quad L = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ l_{21} & \cdot & \cdot & \cdot & l_{29} \\ 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \\ \ldots \end{pmatrix}.$$

Then if F' is of the form (5.28) we see that if $S = (s_{ij})$

$$F' + G'LS = \left(\begin{array}{cc|cccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & 0 & 0 & a_7 & a_8 & a_9 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ b'_1 & b'_2 & b'_3 & b'_4 & b'_5 & b'_6 & b'_7 & b'_8 & b'_9 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & c_4 & 0 & 0 & c_7 & c_8 & c_9 \\ \hline d_1 & 0 & d_3 & 0 & 0 & 0 & d_7 & 0 & d_9 \end{array}\right)$$

with $b'_i = b_i(L) = b_i + \sum_{j=1}^{9} l_{2j}s_{ji}$, $i = 1, \ldots, 9$. Thus the block companion canonical form of $\Sigma(L)$ is always $\Sigma(L)^{S^{-1}}$ if L is of the form (5.33). Note that the number of the row which has nonzero entries is determined by $\kappa(\Sigma)$; it is the smallest i for which k_i is maximal; note also that if j is such that k_j is maximal then the j-th vector of G' is always the $(k_1 + \ldots + k_j)$-th standard basis vector (cf. just below (5.19).

So to find S we proceed as follows. Calculate the block companion canonical forms of $\Sigma(L)$ from $A_0(L), \ldots, A_{2n-1}(L)$ for small L. (This can be done because for small enough L, $\Sigma(L)$ is still co). This gives us in particular the functions $b_i(L)$. Then

$$s_{ji} = \frac{\partial b_i(L)}{\partial l_{2j}}\Big|_{L=0} .$$

This determines S and gives us Σ as $\Sigma = (\Sigma')^S$. q.e.d.

6 Description of $L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$. Invariants

6.1 Local structure of $L^{co,cr}_{m,n,p}(\mathbb{R})$. Let $\alpha \subset I(n, m)$ be a nice selection. We recall that $U_\alpha = \{(F, G, H) \in L_{m,n,p}(\mathbb{R}) \mid \det R(F, G)_\alpha \neq 0\}$, that $V_\alpha = \{(F, G, H) \in L_{m,n,p}(\mathbb{R}) \mid R(F, G)_\alpha = I_n\}$ and that $U_\alpha/GL_n(\mathbb{R}) \simeq V_\alpha \cong \mathbb{R}^{nm+np}$, cf. section 3.

For each $x \in \mathbb{R}^{nm+np}$ let $(F_\alpha(x), G_\alpha(x), H_\alpha(x)) \in V_\alpha$ be the unique system corresponding to x according to the isomorphism of 3.7 above.

6.2 The quotient manifold $M^{cr}_{m,n,p}(\mathbb{R}) = L^{cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$. Now that we know what $U_\alpha/GL_n(\mathbb{R})$ looks like it is not difficult to describe $L^{cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$. Recall that the union of the U_α for α nice covers $L^{cr}_{m,n,p}(\mathbb{R})$). We only need to figure out how the $V_\alpha \simeq \mathbb{R}^{mn+np}$ should be glued together. This is not particularly difficult because if $(F, G, H)^S = (F', G', H')$ for some S and $(F, G, H) \in U_\alpha$ then $S = R(F', G')_\alpha R(F, G)_\alpha^{-1}$. It

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follows that the quotient space $M^{cr}_{m,n,p}(\mathbb{R}) = L^{cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$ can be constructed as follows.

For each nice selection α let $\bar{V}_\alpha = \mathbb{R}^{mn+np}$ and for each second nice selection β let

$$\bar{V}_{\alpha\beta} = \{x \in \bar{V}_\alpha \mid \det R(F_\alpha(x), G_\alpha(x))_\beta \neq 0\} .$$

We define

$$\phi_{\alpha\beta} : \bar{V}_{\alpha\beta} \to \bar{V}_{\beta\alpha}$$

by the formula

(6.3) $\phi_{\alpha\beta}(x) = y \Leftrightarrow R(F_\alpha(x), G_\alpha(x))_\beta^{-1} R(F_\alpha(x), G_\alpha(x)) = R(F_\beta(y), G_\beta(y))$.

Let $M^{cr}_{m,n,p}(\mathbb{R})$ be the topological space obtained by glueing together the $\bar{V}_\alpha$ by means of the isomorphisms $\phi_{\alpha\beta}$.

Then $M^{cr}_{m,n,p}(\mathbb{R}) = L^{cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$. If we denote also with $\bar{V}_\alpha$ the isomorphic image of $\bar{V}_\alpha$ in $M^{cr}_{m,n,p}(\mathbb{R})$ then the quotient map $\pi : L^{cr}_{m,n,p}(\mathbb{R}) \to M^{cr}_{m,n,p}(\mathbb{R})$ can be described as follows. For each $\Sigma = (F, G, H) \in L^{cr}_{m,n,p}(\mathbb{R})$, choose a nice selection α such that $\Sigma \in U_\alpha$. Then $\pi(\Sigma) = x \in \bar{V}_\alpha \subset M^{cr}_{m,n,p}(\mathbb{R})$ where x is such that $\Sigma^S = (F_\alpha(x), G_\alpha(x), H_\alpha(x))$ with $S = R(F, G)_\alpha^{-1}$.

6.4 Theorem: $M^{cr}_{m,n,p}(\mathbb{R})$ is a differentiable manifold and $\pi : L^{cr}_{m,n,p}(\mathbb{R}) \to M^{cr}_{m,n,p}(\mathbb{R})$ is a principal $GL_n(\mathbb{R})$ fibre bundle.

For a proof, cf. [5].

6.5 The quotient manifold $M^{co,cr}_{m,n,p}(\mathbb{R}) = L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$. Let $M^{co,cr}_{m,n,p}(\mathbb{R}) = \pi(L^{co,cr}_{m,n,p}(\mathbb{R}))$. Then $M^{co,cr}_{m,n,p}(\mathbb{R})$ is an open submanifold of $M^{cr}_{m,n,p}(\mathbb{R})$. It can be described as follows. For each nice selection α let $\bar{V}^{co}_\alpha = \{x \in \bar{V}_\alpha \mid (F_\alpha(x), G_\alpha(x), H_\alpha(x))$ is completely observable$\}$, and for each second nice selection β let $\bar{V}^{co}_{\alpha\beta} = \bar{V}^{co}_\alpha \cap \bar{V}_{\alpha\beta}$. Then $\phi_{\alpha\beta}(\bar{V}^{co}_{\alpha\beta}) = \bar{V}^{co}_{\beta\alpha}$ and $M^{co,cr}_{m,n,p}(\mathbb{R})$ is the differentiable manifold obtained by glueing together the $\bar{V}^{co}_\alpha$ by means of the isomorphisms $\phi_{\alpha\beta} : \bar{V}^{co}_{\alpha\beta} \to \bar{V}^{co}_{\beta\alpha}$.

6.6 $M^{co,cr}_{m,n,p}(\mathbb{R})$ as a submanifold of $\mathbb{R}^{2nmp}$. Let $(F, G, H) \in L^{co,cr}_{n,m,p}(\mathbb{R})$. We associate to (F, G, H) the sequence of 2n $p \times m$ matrices $(A_0, \ldots, A_{2n-1}) \in \mathbb{R}^{2nmp}$, where $A_i = HF^iG$, $i = 0, \ldots, 2n-1$. The results of section 4 above (realization theory) prove that this map is injective and prove that its image consists of those elements $(A_0, \ldots, A_{2n-1}) \in \mathbb{R}^{2nmp}$ such that rank $H_{n-1}(A)$ = rank $H_n(A) = n$. We thus obtain $M^{co,cr}_{m,n,p}(\mathbb{R})$ as a (nonsingular algebraic) smooth submanifold of $\mathbb{R}^{2nmp}$.

6.7 Invariants. By definition a smooth invariant for $GL_n(\mathbb{R})$ acting on $L_{m,n,p}(\mathbb{R})$ is a smooth function $f: U \to \mathbb{R}$, defined on an open dense subset $U \subset L$ [illegible]

Now $L^{co,cr}_{m,n,p}(\mathbb{R})$ is open and dense in $L_{m,n,p}(\mathbb{R})$. It now follows from 6.6 that every invariant can be written as a smooth function of the entries of the invariant matrix valued functions $A_0, \ldots, A_{2n-1}$ on $L_{m,n,p}(\mathbb{R})$.

7 On the (non) existence of canonical forms

7.1 Canonical forms: Let L' be a $GL_n(\mathbb{R})$-invariant subspace of $L_{m,n,p}(\mathbb{R})$. A canonical form for $GL_n(\mathbb{R})$ acting on L' is a mapping $c: L' \to L'$ such that the following three properties hold

(7.2) $c(\Sigma^S) = c(\Sigma)$ for all $\Sigma \in L', S \in GL_n(\mathbb{R})$

(7.3) for all $\Sigma \in L'$ there is an $S \in GL_n(\mathbb{R})$ such that $c(\Sigma) = \Sigma^S$.

(7.4) $c(\Sigma) = c(\Sigma') \Rightarrow \exists S \in GL_n(\mathbb{R})$ such that $\Sigma' = \Sigma^S$

(Note that (7.4) is implied by (7.3)).

Thus a canonical form selects precisely one element out of each orbit of $GL_n(\mathbb{R})$ acting on L'. We speak of a continuous canonical form if c is continuous.

Of course, there exist canonical forms on, say $L^{co,cr}_{m,n,p}(\mathbb{R})$, e.g. the following one, $\bar{c}_\kappa : L^{co,cr}_{m,n,p}(\mathbb{R}) \to L^{co,cr}_{m,n,p}(\mathbb{R})$ which is defined as follows: let $\Sigma \in L^{co,cr}_{m,n,p}(\mathbb{R})$, calculate $\kappa(\Sigma)$ and let $\bar{c}_\kappa(\Sigma)$ be the block companion canonical form of Σ as described in section 5.21 above.

This canonical form is not continuous, however (, though still quite useful, as we saw in section 5.31). As we argued in 1.15 above, for some purposes it would be desirable to have a continuous canonical form (cf. also [2]). In this connection let us also remark that the Jordan canonical form for square matrices under similarity transformations ($M \to SMS^{-1}$) is also not continuous, and this causes a number of unpleasant numerical difficulties, cf. [16].

*7.5 **Continuous canonical forms and sections.** Let L' be a $GL_n(\mathbb{R})$-invariant subspace of $L^{cr}_{m,n,p}(\mathbb{R})$. Let $M' = \pi(L') \subset M^{cr}_{m,n,p}(\mathbb{R})$ be the image of L' under the projection π (cf. 6.2 above). Now let $c: L' \to L'$ be a continuous canonical form on L'. Then $c(\Sigma^S) = c(\Sigma)$ for all $\Sigma \in L'$ so that c factorizes through M' to define a continuous map $s: M' \to L'$ such that $c = s \circ \pi$. Because of (7.3) we have $\pi \circ c = \pi$ so that $\pi = \pi \circ s \circ \pi$. Because π is surjective it follows that $\pi \circ s = \mathrm{id}$, so that s is a continuous section of the (principal $GL_n(\mathbb{R})$) fibre bundle $\pi: L' \to M'$. Inversely let $s: M' \to L'$ be a continuous section of π. Then $s \circ \pi : L' \to L'$ is a continuous canonical form on L'.

7.6 **(Non) existence of global canonical forms.** In this section we shall prove theorem 1.17 which says that there exists a continuous canonical form on all of $L^{cr,co}_{m,n,p}(\mathbb{R})$ if and

First suppose that $m = 1$. Then there is only one nice selection in $I(n, m)$, viz. ((0, 1), (1, 1), ..., (n − 1, 1)). We have already seen that there exists a continuous canonical form $c_\alpha : U_\alpha \to U_\alpha$ for all nice selections α (cf. 3.10). This proves the theorem for $m = 1$. The case $p = 1$ is treated similarly (cf. 3.11). It remains to prove that there is no continuous canonical form on $L^{co,cr}_{m,n,p}(\mathbb{R})$ if $m \geq 2$ and $p \geq 2$. To do this we construct two families of linear dynamical systems as follows for all $a \in \mathbb{R}, b \in \mathbb{R}$ (We assume $n \geq 2$; if $n = 1$ the examples must be modified somewhat).

$$G_1(a) = \left(\begin{array}{cc|ccc} a & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \hline 2 & 1 & & & \\ \vdots & \vdots & & B & \\ 2 & 1 & & & \end{array}\right) \quad G_2(b) = \left(\begin{array}{cc|ccc} 1 & b & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \hline 2 & 1 & & & \\ \vdots & \vdots & & B & \\ 2 & 1 & & & \end{array}\right),$$

where B is some (constant) $(n-2) \times (m-2)$ matrix with coefficients in $\mathbb{R}$

$$F_1(a) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n \end{pmatrix} = F_2(b)$$

$$H_1(a) = \left(\begin{array}{cc|ccc} y_1(a) & 1 & 2 & \cdots & 2 \\ y_2(a) & 1 & 1 & \cdots & 1 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & C & \\ 0 & 0 & & & \end{array}\right) \quad H_2(b) = \left(\begin{array}{cc|ccc} x_1(b) & 1 & 2 & \cdots & 2 \\ x_2(b) & 1 & 1 & \cdots & 1 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & C & \\ 0 & 0 & & & \end{array}\right)$$

where C is some (constant real $(p-2) \times (n-2)$ matrix. Here the continuous functions $y_1(a), y_2(a), x_1(b), x_2(b)$ are e.g. $y_1(a) = a$ for $|a| \leq 1$, $y_1(a) = a^{-1}$ for $|a| \geq 1$, $y_2(a) = \exp(-a^2)$, $x_1(b) = 1$ for $|b| \leq 1$, $x_1(b) = b^{-2}$ for $|b| \geq 1$, $x_2(b) = b^{-1}\exp(-b^{-2})$ for $b \neq 0$, $x_2(0) = 0$. The precise form of these functions is not important. What is important is that they are continuous, that $x_1(b) = b^{-1}y_1(b^{-1})$, $x_2(b) = b^{-1}y_2(b^{-1})$ for all $b \neq 0$ and that $y_2(a) \neq 0$ for all a and $x_1(b) \neq 0$ for all b.

For all $b \neq 0$ let $T(b)$ be the matrix

(7.7)
$$T(b) = \begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let $\Sigma_1(a) = (F_1(a), G_1(a), H_1(a))$, $\Sigma_2(b) = (F_2(b), G_2(b), H_2(b))$. Then one easily checks that

(7.8) $ab = 1 \Rightarrow \Sigma_1(a)^{T(b)} = \Sigma_2(b)$.

Note also that $\Sigma_1(a), \Sigma_2(b) \in L^{co,cr}_{m,n,p}(\mathbb{R})$ for all $a, b \in \mathbb{R}$; in fact

(7.9) $\Sigma_1(a) \in U_\alpha, \alpha = ((0,2),(1,2),\ldots,(n-1,2))$ for all $a \in \mathbb{R}$

(7.10) $\Sigma_2(b) \in U_\beta, \beta = ((0,1),(1,1),\ldots,(n-1,1))$ for all $b \in \mathbb{R}$

which proves the complete reachability. The complete observability is seen similarly.

Now suppose that c is a continuous canonical form on $L^{co,cr}_{m,n,p}(\mathbb{R})$. Let $c(\Sigma_1(a)) = (\bar F_1(a), \bar G_1(a), \bar H_1(a))$, $c(\Sigma_2(b)) = (\bar F_2(b), \bar G_2(b), \bar H_2(b))$. Let $S(a)$ be such that $c(\Sigma_1(a)) = \Sigma_1(a)^{S(a)}$ and let $\bar S(b)$ be such that $c(\Sigma_2(b)) = \Sigma_2(b)^{\bar S(b)}$. It follows from (7.9) and (7.10) that

$$\begin{aligned} S(a) &= R(\bar F_1(a), \bar G_1(a))_\alpha R(F_1(a), G_1(a))_\alpha^{-1} \\ \bar S(b) &= R(\bar F_2(b), \bar G_2(b))_\beta R(F_2(b), G_2(b))_\beta^{-1} . \end{aligned} \tag{7.11}$$

Consequently $S(a)$ and $\bar S(b)$ are (unique and are) continuous functions of a and b. Now take $a = b = 1$. Then $ab = 1$ and $T(b) = I_n$ so that (cf (7.7), (7.8) and (7.11)) $S(1) = \bar S(1)$. It follows from this and the continuity of $S(a)$ and $\bar S(b)$ that we must have

(7.12) $\operatorname{sign}(\det S(a)) = \operatorname{sign}(\det \bar S(b))$ for all $a, b \in \mathbb{R}$.

Now take $a = b = -1$. Then $ab = 1$ and we have, using (7.8),

$$\begin{aligned} \Sigma_1(-1)^{(\bar S(-1)T(-1))} &= (\Sigma_1(-1)^{T(-1)})^{\bar S(-1)} \\ &= \Sigma_2(-1)^{\bar S(-1)} = c(\Sigma_2(-1)) \\ &= c(\Sigma_1(-1)) = \Sigma_1(-1)^{S(-1)} . \end{aligned}$$

It follows that $S(-1) = \bar S(-1)T(-1)$, and hence by (7.7), that

$$\det(S(-1)) = -\det(\bar S(-1))$$

which contradicts (7.12). This proves that there does not exists a continuous canonical form on $L^{co,cr}_{m,n,p}(\mathbb{R})$ if $m \geq 2$ and $p \geq 2$.

*7.13 **Acknowledgement and remarks.** By choosing the matrices B and C in $G_1(a)$, $G_2(b)$, $H_1(a)$, $H_2(b)$ judiciously we can also ensure that $\operatorname{rank}(G_1(a) = m = \operatorname{rank} G_2(b)$ if $m < n$ and $\operatorname{rank} H_1(a) = p = \operatorname{rank} H_2(b)$ if $p < n$.

As we have seen in 7.5 above there exists a continuous canonical form on $L^{co,cr}_{m,n,p}(\mathbb{R})$ if and only if the prinicpal $GL_n(\mathbb{R})$ fibre bundle $\pi: L^{co,cr}_{m,n,p}(\mathbb{R}) \to M^{co,cr}_{m,n,p}(\mathbb{R})$ admits a section. This, in turn is the case if and only if this bundle is trivial. The example on which the proof in 7.6 above is based is precisely the same example we used in [5] to prove that

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the fibre bundle π is in fact nontrivial if $p \geq 2$ and $m \geq 2$, and from this point of view the example appears somewhat less "ad hoc" than in the present setting. The idea of using the example to prove nonexistence as done above is due to R. E. Kalman.

8 On the geometry of $M^{co,cr}_{m,n,p}(\mathbb{R})$. Holes and (partial) compactifications

As we have seen in the introduction (cf. 1.19) the differentiable manifold $M^{co,cr}_{m,n,p}(\mathbb{R})$ is full of holes, a situation which is undesirable in certain situations. In this section we prove theorems 1.22 and 1.23 but, for the sake of simplicity, only in the case $m = 1$ and $p = 1$.¹)

8.1 **An addendum to realization theory.** Let $T(s) = d(s)^{-1}b(s)$ be a rational function, with degree $d(s) = n >$ degree $b(s)$. Then we know by 4.27 that there is a one input, one output system Σ with transfer function $T_\Sigma(s)$. We claim that we can see to it that $\dim(\Sigma) \leq n$. Indeed if

$$T_\Sigma(s) = a_0 s^{-1} + a_1 s^{-2} + a_2 s^{-3} + \ldots$$

then, if $d(s) = s^n - d_{n-1}s^{n-1} - d_1 s - d_0$, we have

$$a_{i+n} = d_0 a_i + d_1 a_{i+1} + \ldots + d_{n-1} a_{i+n-1}$$

for all $i \geq 0$. It follows that if $A = (a_0, a_1, a_2, \ldots)$, then $\operatorname{rank} H_r(A) = \operatorname{rank} H_{n-1}(A)$ for all $r \geq n-1$. But $H_{n-1}(A)$ is an $n \times n$ matrix and hence $\operatorname{rank} H_r(A) \leq n$ for all s, which by section 4 means that there is a realization of A (or $T(s)$) of dimension $\leq n$.

It follows that a cr and co system Σ of dimension n has a transfer function $T_\Sigma(s) = d(s)^{-1}b(s)$ with degree $(d(s)) = n$ and no common factors in $d(s)$ and $b(s)$, and inversely if $T(s) = d(s)^{-1}b(s)$, degree $b(s) < n =$ degree $(d(s))$, and $b(s)$ and $d(s)$ have no common factors, then all n-dimensional realizations of $T(s)$ are co and cr.

Indeed if $d(s)$ and $b(s)$ have a common factor, then $T_\Sigma(s) = d'(s)^{-1}b'(s)$ with degree $(d'(s)) \leq n-1$ and it follows as above that $\operatorname{rank} H_r(A) \leq n-1$ so that Σ is not cr and co. Inversely if Σ is not cr and co there is a Σ' of dimension $\leq n-1$ which also realizes A so that $T(s) = T_{\Sigma'}(s) = h'(sI - F')^{-1}g' = \det(sI - F')^{-1}B(s) = d'(s)^{-1}B(s)$ with degree $(d'(s)) \leq n-1$.

*8.2. There is a more input, more output version of 8.1. But it is not perhaps the most obvious possibility. E.g. the lowest dimensional realization of $s^{-1}\binom{1\ 2}{1\ 1}$ has dimension 2. The right generalization is: Let $T(s) = D(s)^{-1}N(s)$, where $D(s)$ and $N(s)$ are as in the theorem mentioned in section 5.9. Then there is a co and cr realization of $T(s)$ of dimension degree $(\det(D(s))$.

¹) Added in proof. For the analogous results in the multivariable case and a more careful, easier and more detailed treatment cf M. Hazewinkel, "Families of systems: degeneration phenomena", Report 7918. Econometric Inst., Erasmus Univ. Rotterdam.

8.3 Theorem: Let $D = a_0 + a_1 \frac{d}{dt} + \ldots + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}$, $a_i \in \mathbb{R}$ be a differential operator of order $\leq n-1$. Then there exists a family of systems $(\Sigma_z)_z \subset L^{co,cr}_{1,n,1}(\mathbb{R})$ such that the $f(\Sigma_z)$ converge to D in the sense of definition 1.20.

To prove this theorem we need to do some exercises concerning differentiation, determinants and partial integration. They are

(8.4) Let $k \in \mathbb{Z}$, $k \geq -1$ and let $B_{n,k}$ be the $n \times n$ matrix with (i, j)-th entry equal to the binomial coefficient $\binom{i+j+k}{i+k+1}$. Then $\det(B_{n,k}) = 1$.

(8.5) Let $u^{(i)}(t) = \frac{d^i u(t)}{dt^i}$. Then $\int_0^t z^n e^{-z(t-\tau)} u(\tau)d\tau =$

$$= z^{n-1}u(t) + \ldots + (-1)^{n-1}u^{(n-1)}(t) + 0(z^{-1})$$

if $\mathrm{supp}(u) \subset (0, \infty)$, where 0 is the Landau symbol.

(8.6) Let $\phi(\tau) = (t-\tau)^m u(\tau)$, $\phi^{(i)}(\tau) = \frac{d^i\phi(\tau)}{d\tau^i}$. Then $\phi^{(i)}(t) = 0$ for $i < m$ and

$$\phi^{(i)}(t) = (-1)^m i(i-1)\ldots(i-m+1)u^{(i-m)}(t) \text{ if } i \geq m.$$

And finally, combining (8.5) and (8.6),

$$(8.7) \int_0^t e^{-z(t-\tau)} z^n (t-\tau)^m u(\tau)d\tau = (-1)^m m! \sum_{i=m+1}^{n} (-1)^{i+1} z^{n-i}\binom{i-1}{m} u^{(i-1-m)}(t) + 0(z^{-1})$$

8.8 Proof of theorem 8.3: We consider the following family of n dimensional systems (with one output and one input),

$$g_z = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ z^m \end{pmatrix}, \quad F_z = \begin{pmatrix} -z & z & 0 & \ldots & 0 \\ 0 & -z & & \ddots & \vdots \\ \vdots & \ddots & \ddots & & 0 \\ \vdots & & \ddots & & z \\ 0 & \ldots & & 0 & -z \end{pmatrix}, h_z = (0, \ldots, 0, x_m, \ldots, x_1)$$

where the $x_1, \ldots, x_m$, $m \leq n$, are some still to be determined real numbers. One calculates

$$e^{sF_z} = \begin{pmatrix} 1 & sz & \frac{s^2z^2}{2!} & \ldots & \frac{(sz)^{n-1}}{(n-1)!} \\ 0 & 1 & & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{s^2z^2}{2!} \\ \vdots & & & 1 & sz \\ 0 & \ldots & & 0 & 1 \end{pmatrix}$$

Hence

$$h_z e^{(t-\tau)F_z} g_z = \sum_{i=1}^{m} x_i z^{m+i}(i!)^{-1}(t-\tau)^i e^{-z(t-\tau)}$$

and, using (8.7),

$$\int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)\,d\tau = \sum_{i=1}^{m} (i!)^{-1} x_i \sum_{j=i+1}^{m+i} (-1)^i (i!) (-1)^{j+1} \binom{j-1}{i} z^{m+i-j} u^{(j-i-1)}(t) + O(z^{-1})$$

$$= \sum_{l=0}^{m-1} (-1)^{m-l+1} z^l \left(\sum_{i=1}^{m} x_i \binom{m+i-l-1}{i} u^{(m-l-1)}(t) + O(z^{-1}) \right.$$

Now, by (8.4) we know that $\det\left(\left(\binom{m+i-l-1}{i}\right)_{i,l}\right) = 1$, so that we can choose $x_1, \ldots, x_m$ in such a way that

$$\int_0^t h_z e^{(t-\tau)F_z} g_z u(\tau)\,d\tau = a_{m-1}u^{(m-1)}(t) + O(z^{-1})$$

where a_{m-1} is any pregiven real number.

It follows that $\lim_{z\to\infty} f(\Sigma_z) = a_{m-1}\frac{d^{m-1}}{dt^{m-1}}$

Let $\Sigma_z(i) = (F_z(i), g_z(i), h_z(i))$, $i = 0, \ldots, n-1$ be systems constructed as above with limiting input/output operator equal to $a_i \frac{d^i}{dt^i}$. Now consider the n^2-dimensional systems $\hat{\Sigma}_z$ defined by

$$\hat{F}_z = \begin{pmatrix} F_z(0) & 0 & \ldots & 0 \\ 0 & & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & F_z(n-1) \end{pmatrix}, \hat{g}_z = \begin{pmatrix} g_z(0) \\ \vdots \\ g_z(n-1) \end{pmatrix}, \hat{h}_z = (h_z(0), \ldots, h_z(n-1)).$$

Then clearly $\lim_{z\to\infty} f(\hat{\Sigma}_z) = D$. Let $T_z^{(i)}(s)$ be the transfer function of $\Sigma_z(i)$. Then for certain polynomials $B_z^{(i)}(s)$ we have

(8.9) $T_z^{(i)}(s) = d_z(s)^{-1} B_z^{(i)}(s)$, with $d_z(s)$ *independent of i*

The transfer function of $\hat{\Sigma}_z$ is clearly equal to

$$(8.10) \quad T_z(s) = \sum^{n-1} T_z^{(i)}(s) = d_z(s)^{-1} B_z(s), B_z(s) = \sum^{n-1} B_z^{(i)}(s)$$

By 8.1 it follows from (8.10) that $T_z(s)$ can also be realized by an n-dimensional system, Σ_z'. Then also $\lim_{z\to\infty} f(\Sigma_z') = D$. Finally we can change Σ_z' slightly to Σ_z for all z to find a family $(\Sigma_z)_z \subset L^{co,cr}_{1,n,1}(IR)$ such that $\lim_{z\to\infty} f(\Sigma_z) = D$. This proves the theorem.

8.11 **Corollary:** Let Σ' be a system of dimension i and let D be a differential operator of order $n - i - 1$ (where order $(0) = -1$). Then there exists a family $(\Sigma_z)_z \subset L^{co,cr}_{1,n,1}(IR)$ such that $\lim_{z\to\infty} f(\Sigma_z) = D + f(\Sigma')$

Proof: Let $\Sigma_z'' = (F_z'', g_z'', h_z'')$ be a family in $L_{1,n-i,1}(IR)$ such that $\lim_{z\to\infty} f(\Sigma_z'') = D$. Let $\Sigma' = (F', g', h')$. Let $\hat{\Sigma}_z$ be the n-dimensional system defined by the triple of matrices

$$\hat{F}_z = \begin{pmatrix} F_z'' & 0 \\ 0 & F' \end{pmatrix},\ g_z = \begin{pmatrix} g_z'' \\ g' \end{pmatrix},\ h_z = (h_z'', h').$$

Then $\lim_{z\to\infty} f(\hat{\Sigma}_z) = D + f(\Sigma')$. Now perturb $\hat{\Sigma}_z$ slightly for each z to Σ_z, to find a completely reachable and completely observable family $(\Sigma_z)_z$ such that $\lim_{z\to\infty} f(\Sigma_z) = D + f(\Sigma')$.

8.12 **Theorem:** Let $(\Sigma_z)_z \subset L_{1,n,1}(IR)$ be a family of systems which converges in input-output behaviour in the sense of definition 1.20. Then there exist a system Σ' and a differential operator D such that $\dim(\Sigma') + \mathrm{ord}(D) \leq n - 1$ and $\lim_{z\to\infty} f(\Sigma_z) = f(\Sigma') + D$

Proof: Consider the relation

$$y_z(t) = f(\Sigma_z)u(t)$$

for smooth input functions u(t). Let $\hat{u}(s)$ and $\hat{y}_z(s)$ be the Laplace transforms of u(t) and $y_z(t)$. Then we have

$$\hat{y}_z(s) = T_z(s)\hat{u}(s),$$

where $T_z(s)$ is the transferfunction of Σ_z. Because the $f(\Sigma_z)$ converge as $z \to \infty$ (in the sense of definition 1.20), and because the Laplace transform is continuous, it follows that there is a rational function $T(s) = d(s)^{-1}b(s)$ with degree $d(s) \leq n$, degree $b(s) \leq n - 1$ such that

$$\lim_{z\to\infty} T_z(s) = T(s)$$

pointwise in s for all but finitely many s. Write

$$T(s) = e_0 + e_1 s + \ldots + e_{n-i-1}s^{n-i-1} + \frac{b'(s)}{d'(s)}$$

with degree $d'(s) = i$, degree $(b'(s)) < i$. Let Σ' be a system of dimension $\leq i$ with transfer function equal to $d'(s)^{-1}b'(s)$ and let D be the differential operator $e_0 + e_1 \frac{d}{dt} + \ldots + e_{n-i-1}\frac{d^{n-i-1}}{dt^{n-i-1}}$. The Laplace transform of the relation

$$y(t) = f(\Sigma')u(t) + Du(t)$$

for smooth input functions u(t), is

$$\hat{y}(s) = T(s)\hat{u}(s).$$

Because the Laplace transform is injective (on smooth functions) it follows that

$$\lim_{z\to\infty} f(\Sigma_z) = f(\Sigma') + D.$$

*8.13 **Remarks on compactification, desingularization, symmetry breaking, etc.** There are more input, more output versions of theorems 8.3 and 8.12. To prove them it is more convenient to use another technique which is based on a continuity property of the inverse Laplace transform for certain sequences of functions. (The inverse Laplace transform is certainly not continuous in general; also it is perfectly possible to have a sequence of systems Σ_z such that their transfer functions $T_z(s)$ converge for $z \to \infty$, but such that the $f(\Sigma_z)$ do not converge, e.g. $T_z(s) = z(z-s)^{-1}$).

Let Σ be a co and cr system of dimension n with one input and one output. Let T(s)

$$T(s) = \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0} = \frac{b(s)}{d(s)}$$

be the transfer function of Σ. Assign to T(s) the point

$$(b_0 : \ldots : b_{n-1} : d_0 : \ldots : d_{n-1} : 1) \in \mathbb{P}^{2n}(IR),$$

real projective space of dimension 2n. This defines an embedding of $M^{co,cr}_{1,n,1}(IR)$ into $\mathbb{P}^{2n}(IR)$. The image is obviously dense so that $\mathbb{P}^{2n}(IR)$ is a smooth compactification of $M^{co,cr}_{1,n,1}(IR)$.

Let $\bar{M}_{1,n,1}(IR)$ be the subspace of $\mathbb{P}^{2n}(IR)$ consisting of those points $(x_0 : \ldots : x_{n-1} : y_0 : y_1 : \ldots : y_n) \in \mathbb{P}^{2n}(IR)$ for which at least one y_i, $i = 0, \ldots, n$ is different from zero. For these points

$$\frac{x_0 + x_1 s + \ldots + x_{n-1}s^{n-1}}{y_0 + y_1 s + \ldots + y_n s^n}$$

has meaning and this rational function is then the transfer function of a generalized linear dynamical system:

$$(8.14)\quad \begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx + Du \end{aligned}$$

where D is a differential operator. (The points in $\mathbb{P}^{2n}(\mathbb{R}) \setminus \overline{M}_{1,n,1}$ correspond to "systems" which tend to give infinite outputs for finite inputs; they are interpretable, however, in terms of correspondences $y(t) \mapsto u(t)$).

Further let $\hat{M}_{1,n,1}$, consist of those $(x_0 : \ldots : x_{n-1} : y_0 : \ldots : y_n)$ for which if $y_i = 0$ for $i > r$, then also $x_{i-1} = 0$ for $i \geq r$. For these points the D in (8.14) is zero and these points thus yield transfer functions of systems of dimension $\leq n$. (But many points in $\hat{M}_{1,n,1}$ have the same transfer functions). Assigning to a point in $\hat{M}_{1,n,1}$ the first $2n+1$ coefficients of

$$\frac{x_0 + x_1 s + \ldots + x_{n-1} s^{n-1}}{y_0 + y_1 s + \ldots + y_n s^n} = a_0 s^{-1} + a_1 s^{-2} + a_2 s^{-3} + \ldots$$

we find the following situation

$$\begin{array}{ccc} M^{co,cr}_{1,n,1} & \subset & \hat{M}_{1,n,1} \\ \downarrow H & & \downarrow \hat{H} \\ \mathbb{R}^{2n+1} & = & \mathbb{R}^{2n+1} \end{array} .$$

Here H is an embedding and its image is the subspace of all sequences $A = (a_0, \ldots, a_{2n})$ such that $\operatorname{rank} H_{n-1}(A) = \operatorname{rank} H_n(A) = n$. The image of $\hat{H}$ is the space of all sequences A such that $\operatorname{rank} H_n(A) = \operatorname{rank} H_{i-1}(A) = i$ for some $i \leq n$. This is a singular submanifold of $\mathbb{R}^{2n+1}$ and $\hat{H}$ is a resolution of singularities.

The points of $(\hat{M}_{1,n,1} \setminus M^{co,cr}_{1,n,1})$ correspond to transfer functions of lower dimensional co and cr systems. If a sequence $x_z \in M^{co,cr}_{1,n,1}$ converges to such a point, the internal symmetry group $GL_n(\mathbb{R})$ of x_z suddenly contracts to some $GL_m(\mathbb{R}) \subset GL_n(\mathbb{R})$ with $m < n$.

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(FINE) MODULI (SPACES) FOR LINEAR SYSTEMS: WHAT ARE THEY AND WHAT ARE THEY GOOD FOR

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(FINE) MODULI (SPACES) FOR LINEAR SYSTEMS: WHAT ARE THEY AND WHAT ARE THEY GOOD FOR.

Michiel Hazewinkel

Lectures given at the NATO-AMS Adv. Study Inst. and Summer Sem. in Appl. Math. on Algebraic and Geometric Methods in Linear System Theory (Harvard Univ., June, 1979).

ABSTRACT.

This tutorial and expository paper considers linear dynamical systems $\dot{x} = Fx + Gu$, $y = Hx$, or, $x(t+1) = Fx(t) + Gu(t)$, $y(t) = Hx(t)$; more precisely it is really concerned with families of such, i.e., roughly speaking, with systems like the above where now the matrices F,G,H depend on some extra parameters σ. After discussing some motivation for studying families (delay systems, systems over rings, n-d systems, perturbed systems, identification, parameter uncertainty) we discuss the classifying of families (fine moduli spaces). This is followed by two straightforward applications: realization with parameters and the nonexistence of global continuous canonical forms. More applications, especially to feedback will be discussed in Chris Byrnes' talks at this conference and similar problems as in these talks for networks will be discussed by Tyrone Duncan. The classifying fine moduli space cannot readily be extended and the concluding sections are devoted to this observation and a few more related results.

CONTENTS.

1. INTRODUCTION.

The basic object of study in these lectures (as in many others at this conference) is a constant linear dynamical system, that is a system of equations

$$(1.1) \qquad \begin{matrix} \dot{x} = Fx + Gu \\ y = Hx \\ \text{(a): continuous time} \end{matrix} \qquad (\Sigma) \qquad \begin{matrix} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \\ \text{(b): discrete time} \end{matrix}$$

with $x \in k^n$ = space space, $u \in k^m$ = input or control space, $y \in k^p$ = output-space, and F,G,H matrices with coefficients in k of the appropriate sizes; that is, there are m inputs and p outputs and the dimension of the state space, also called the dimension of the system Σ and denoted $\dim(\Sigma)$, is n. Here k is an appropriate field (or possibly ring). In the continuous time case of course k should be such that differentiation makes sense for (enough) functions $\mathbb{R} \to k$, e.g. $k = \mathbb{R}$ or $\mathbb{C}$. Often one adds a direct feedthrough term Ju, giving $y = Hx + Ju$ in case (a) and $y(t) = Hx(t) + Ju(t)$ in case (b) instead of $y = Hx$ and $y(t) = Hx(t)$ respectively; for the mathematical problems to be discussed below the presence or absence of J is essentially irrelevant.

More precisely what we are really interested in are families of objects (1.1), that is sets of equations (1.1) where now the matrices F,G,H depend on some extra parameters σ. As people have found out by now in virtually all parts of mathematics and its applications, even if one is basically interested only in single objects, it pays and is important to study families of such objects depending on a small parameter ε (deformation and perturbation considerations). This could be already enough motivation to study families, but, as it turns out, in the case of (linear) systems theory there are many more circumstances where families turn up naturally. Some of these can be briefly summed up as delay-differential systems, systems over rings, continuous canonical forms, 2-d and n-d systems, parameter uncertainty, (singularly) perturbed systems. We discuss these in some detail below in section 2.

To return to single systems for the moment. The equations (1.1) define input/output maps $f_\Sigma : u(t) \mapsto y(t)$ given respectively by

$$(1.2a) \qquad y(t) = \int_0^t He^{F(t-\tau)}Gu(\tau)d\tau, \qquad t \geq 0$$

$$(1.2b) \qquad y(t) = \sum_{i=1}^{t} A_i u(t-1-i), \; A_i = HF^{i-1}G, \; i = 1,2,\ldots, \; t = 1,2,3 \ldots$$

where we have assumed that the system starts in $x(0) = 0$ at time 0. In both cases the input/output operator is uniquely determined by the sequence of matrices $A_1, A_2, \ldots$. Inversely, realization theory studies when a given sequence $A_1, A_2, \ldots$ is such that there exist F,G,H such that $A_i = HF^{i-1}G$ for all i. Realization with parameters is now the question: given a sequence of matrices $A_1(\sigma), A_2(\sigma), A_3(\sigma), \ldots$ depending polynomially (resp. continuously, resp. analytically, resp. ...) on parameters σ, when do there exist matrices F,G,H depending polynomially (resp. continuously, resp. analytically, resp....) on the parameters σ such that $A_i(\sigma) = H(\sigma)F^{i-1}(\sigma)G(\sigma)$ for all i. And to what extend are such realizations unique? Which brings us to the next group of questions one likes to answer for families.

A single system Σ given by the triple of matrices F,G,H is completely reachable if the matrix R(F,G) consisting of the blocks G, FG, ..., F^nG

$$(1.3) \qquad R(\Sigma) = R(F,G) = (G \,|\, FG \,|\, \ldots \,|\, F^nG)$$

has full rank n. (This means that any state x can be steered to any other state x' by means of a suitable input). Dually the system Σ is said to be completely observable if the matrix Q(F,G) consisting of the blocks H, HF, ..., HF^n

$$(1.4) \qquad Q(\Sigma) = Q(F,H) = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^n \end{pmatrix}$$

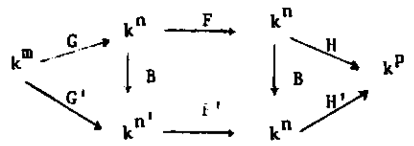
has full rank n. (This means that two different states x(t) and x'(t) of the system can be distinguished on the basis of the output $y(\tau)$ for all $\tau \geq t$). As is very well known if $A_1, A_2, \ldots$ can be realized then it can be realized by a co and cr system and any two such realizations are the same up to base change in state space. That is, if $\Sigma = (F,G,H)$ and $\Sigma' = (F',G',H')$ both realize $A_1, A_2, \ldots$ and both are cr and co then $\dim(\Sigma) = \dim(\Sigma') = n$ and there is an invertible n x n matrix S such that $F' = SFS^{-1}$, $G' = SG$, $H' = HS^{-1}$. (It is obvious that if Σ and Σ' are related in this way then they give the same input/output map). This transformation

$$(1.5) \qquad \Sigma = (F,G,H) \mapsto \Sigma^S = (F,G,H)^S = (SFS^{-1}, SG, HS^{-1})$$

corresponds of course to the base change in state space $x' = Sx$. This argues that at least one good notion of isomorphism of systems is: two systems Σ, Σ' over k are isomorphic iff $\dim(\Sigma) = \dim(\Sigma')$ and there is an $S \in GL_n(k)$, the group of invertible matrices with coefficients in k, such that $\Sigma' = \Sigma^S$

A corresponding notion of homomorphism is: a homomorphism from $\Sigma = (F,G,H)$, $\dim\Sigma = n$, to $\Sigma' = (F',G',H')$, $\dim\Sigma = n'$, is an $n \times n'$ matrix B (with coefficients in k) such that $BG = G'$, $BF = F'B$, $H'B = H$.

Or, in other words, it is a linear map from the state space of Σ to the state space of Σ' such that the diagram below commutes.

(1.6) 

The obvious corresponding notion of isomorphism for families $\Sigma(\sigma)$, $\Sigma'(\sigma)$ is a family of matrices $S(\sigma)$ such that $\Sigma(\sigma)^{S(\sigma)} = \Sigma'(\sigma)$, where, of course, $S(\sigma)$ should depend polynomially, resp. continuously, resp. analytically, resp.... on σ if Σ and Σ' are polynomial, resp. continuous, resp. analytical, resp....families. One way to look at the results of section 3 below is as a classification result for families, or, even, as the construction of canonical forms for families, under the notion of isomorphism just described. As it happens the classification goes in terms of a universal family, that is, a family from which, roughly speaking, all other families (up to isomorphism) can be uniquely obtained via a transformation in the parameters.

Let $L_{m,n,p}(k)$ be the space of all triples of matrices (F,G,H) of dimensions nxn, nxm, pxn, and let $L^{co,cr}_{m,n,p}$ be the subspace of cr and co triples. Then the parameter space for the universal family is the quotient space $L^{co,cr}_{m,n,p}(k)/GL_n(k)$, which turns out to be a very nice space.

The next question we shall take up is the existence or nonexistence of continuous canonical forms. A continuous canonical form on $L^{co,cr}_{m,n,p}$ is a continuous map $(F,G,H) \mapsto c(F,G,H)$ such that $c(F,G,H)$ is isomorphic to (F,G,H) for all $(F,G,H) \in L^{co,cr}_{m,n,p}$ and such that (F,G,H) and (F',G',H') are isomorphic if and only if $c(F,G,H) = c(F',G',H')$ for all (F,G,H), $(F',G',H') \in L^{co,cr}_{m,n,p}$. Obviously if one wants to use canonical forms to get rid of superfluous parameters in an identification problem the canonical form had better be continuous. This does not mean that (discontinuous) canonical forms are not useful. On the contrary, witness e.g. the Jordan canonical form for square matrices under similarity. On the other hand, being discontinuous, it also has very serious drawbacks; cf. e.g. [GWi] for a discussion of some of these. In our case it turns out that

there exists a continuous canonical form on all of $L^{co,cr}_{m,n,p}$ if and only if $m = 1$ or $p = 1$.

Now let, again, Σ be a single system. Then there is a canonical subsystem $\Sigma^{(r)}$ which is completely reachable and a canonical quotient system Σ^{co} which is completely observable. Combining these two constructions one finds a canonical subquotient (or quotient sub) which is both cr and co. The question arises naturally whether (under some obvious necessary conditions) these constructions can be carried out for families as well and also for single time varying systems. This is very much related to the question of whether these constructions are continuous. In the last sections we discuss these questions and related topics like: given two families Σ and Σ' such that $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ are isomorphic for all (resp. almost all) values of the parameters σ; what can be said about the relation between Σ and Σ' as families (resp. about $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ for the remaining values of σ).

2. WHY SHOULD ONE STUDY FAMILIES OF SYSTEMS.

For the moment we shall keep to the intuitive first approximation of a family of systems as a family of triples of matrices of fixed size depending in some continuous manner on a parameter σ. This is the definition which we also used in the introduction.

2.1. <u>(Singular)</u> <u>perturbation</u>, <u>deformation</u>, <u>approximation</u>.

This bit of motivation for studying families of objects, rather than just the objects themselves, is almost as old as mathematics itself. Certainly (singular) perturbations are a familiar topic in the theory of boundary value problems for ordinary and partial differential equations and more recently also in optimal control, cf. e.g. [OMa].

For instance in [OMa], chapter VI, O'Malley discusses the singularly perturbed regulator problem which consists of the following set of equations, initial conditions and quadratic cost functional which is to be minimized for a control which drives the state $x = \binom{y}{z}$ to zero at time $t = 1$.

(2.1.1)
$$\begin{aligned}
\dot y &= A_1(\varepsilon)y + A_2(\varepsilon)z + B_1(\varepsilon)u \qquad & y(0,\varepsilon) &= y^o(\varepsilon)\\
\varepsilon\dot z &= A_3(\varepsilon)y + A_4(\varepsilon)z + B_2(\varepsilon)u \qquad & z(0,\varepsilon) &= z^o(\varepsilon)
\end{aligned}$$
$$J(\varepsilon) = x^T(1,\varepsilon)\pi(\varepsilon)x(1,\varepsilon) + \int_0^1 (x^T(t,\varepsilon)Q(\varepsilon)x(t,\varepsilon) + u^T(t,\varepsilon)R(\varepsilon)u(t,\varepsilon))dt$$

with positive definite $R(\varepsilon)$, and $Q(\varepsilon),\pi(\varepsilon)$ positive semidefinite. Here the upper T denotes transposes. The matrices $A_i(\varepsilon)$, $i = 1,2,3,4$, $B_i(\varepsilon)$, $i = 1,2$,

$H(t)$, $Q(t)$, $R(t)$ may also depend on t. For fixed small $\varepsilon > 0$ there is a unique optimal solution. Here one is interested, however, in the asymptotic solution of the problem as ε tends to zero, which is, still quoting from [OMa] a problem of considerable practical importance, in particular in view of an example of Hadlock et al. [HJK] where the asymptotic results are far superior to the physically unacceptable results obtained by setting $\varepsilon = 0$ directly.

Another interesting problem arises maybe when we have a system

$$\dot{x} = Fx + G_1u + G_2v, \quad y = Hx \tag{2.1.2}$$

where v is noise, and where F, G_1, G_2, H depend on a parameter ε. Suppose we can solve the disturbance decoupling problem for $\varepsilon = 0$. I.e. we can find a feedback matrix L such that in the system with state feedback loop L

$$\dot{x} = (F+GL)x + G_1u + G_2v, \quad y = Hx$$

the disturbances v do not show up any more in the output y, (for $\varepsilon = 0$). Is it possible to find a disturbance discoupler $L(\varepsilon)$ by "perturbation" methods, i.e. as a power series in ε which converges (uniformly) for ε small enough, and such that $L(0) = L$.

In this paper we shall not really pay much more attention to singular perturbation phenomena. For some more systems oriented material on singular perturbations cf. [KKU] and also [Haz 4].

2.2. Systems over rings.

Let R be an arbitrary commutative ring with unit element. A linear system over R is simply a triple of matrices (F,G,H) of sizes n x n, n x m, p x n respectively with coefficients in R. Such a triple defines a linear machine

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t), \; t = 0,1,2,\ldots, \; x \in R^n, \; u \in R^m \\ y(t) &= Hx(t), \qquad\qquad y \in R^p \end{aligned} \tag{2.2.1}$$

which transformes input sequences $(u(0),u(1),u(2),\ldots)$ into output sequences $(y(1),y(2),y(3),\ldots)$ according to the convolution formula (1.2.b).

It is now absolutely standard algebraic geometry to consider these data as a family over Spec(R), the space of all prime ideals of R with the Zariski topology. This goes as follows. For each prime ideal $\mathfrak{p}$ let $i_{\mathfrak{p}} : R \to Q(R/\mathfrak{p})$ be the canonical map of R into the quotient field $Q(R/\mathfrak{p})$ of the integral domain $R/\mathfrak{p}$. Let $(F(\mathfrak{p}), G(\mathfrak{p}), H(\mathfrak{p}))$ be the triple of matrices over $Q(R/\mathfrak{p})$ obtained by applying $i_{\mathfrak{p}}$ to the entries of F,G,H. Then $\Sigma(\mathfrak{p}) = (F(\mathfrak{p}),G(\mathfrak{p}),H(\mathfrak{p}))$ is a family of systems parametrized by Spec(R).

Let me stress that, mathematically, there is no difference between a system over R as in (2.2.1) and the family $\Sigma(\mathfrak{p})$. As far as intuition goes there is quite a bit of difference, and the present author e.g. has found it helpful to think about families of systems over Spec(R) rather then single systems over R. Of course such families over Spec(R) do not quite correspond to families as one intuitively thinks about them. For instance if $R = \mathbb{Z}$ = the integers, then Spec($\mathbb{Z}$) consists of (0) and the prime ideals (p), p a prime number, so that a system over $\mathbb{Z}$ gives rise to a certain collection of systems: one over $\mathbb{Q}$ = rational numbers, and one each over every finite field $\mathbb{F}_p = \mathbb{Z}/(p)$. Still the intuition one gleans from thinking about families as families parametrized continuously by real numbers seems to work well also in these cases.

2.3. Delay-differential systems.

Consider for example the following delay-differential system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t-2) + x_2(t-\alpha) + u(t-1) + u(t) \\ \dot{x}_2(t) &= x_1(t) + x_2(t-1) + u(t-\alpha) \\ y(t) &= x_1(t) + x_2(t-2\alpha) \end{aligned} \tag{2.3.1}$$

where α is some real number incommensurable with 1. Introduce the delay operators σ_1, σ_2 by $\sigma_1\beta(t) = \beta(t-1)$, $\sigma_2\beta(t) = \beta(t-\alpha)$. Then we can rewrite (2.3.1) formally as

$$\dot{x}(t) = Fx(t) + Gu(t), \; y(t) = Hx(t) \tag{2.3.2}$$

with

$$F = \begin{pmatrix} \sigma_1^2 & \sigma_2 \\ 1 & \sigma_1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 + \sigma_1 \\ \sigma_2 \end{pmatrix}, \quad H = (1 \quad \sigma_2^2) \tag{2.3.3}$$

and, forgetting so to speak where (2.3.2), (2.3.3) came from, we can view this set of equations as a linear dynamical system over the ring $\mathbb{R}[\sigma_1,\sigma_2]$, and then using 2.2 above also as a family of systems parametrized by the (complex) parameters σ_1, σ_2 , a point of view which has proved fruitfull e.g. in [By 4]. This idea has been around for some time now, [ZW, An, Yo , RMY], though originally the tendency was to consider these systems as systems over the fields $\mathbb{R}(\sigma_1,\ldots,\sigma_2)$; the idea to consider them over the rings $\mathbb{R}[\sigma_1,\ldots,\sigma_2]$ instead is of more recent vintage([Mo,Kam]).

There are, as far as I know no relations between the solutions of (2.3.1) and the solutions of the family of systems (2.3.2), (2.3.3). Still many of the interesting properties and constructions for (2.3.1) have their counterpart for (2.3.2), (2.3.3) and vice versa. For example to construct a stabilizing state feedback loop for the family (2.3.2) - (2.3.3) depending polynomially on the parameters σ_1, σ_2 that is finding a stabilizing state feedback loop for the system over $\mathbb{R}[\sigma_1,\sigma_2]$, means finding an m x n matrix $L(\sigma_1,\sigma_2)$ with entries in $\mathbb{R}[\sigma_1,\sigma_2]$ such that for all complex σ_1,σ_2 $\det(s-(F+GL))$ has its roots in the left half plane. Reinterpreting σ_1 and σ_2 as delays so that $L(\sigma_1,\sigma_2)$ becomes a feedback matrix with delays one finds a stabilizing feedback loop for (the infinite dimensional) system (2.3.1). (cf. [BC], cf. also [Kam], which works out in some detail some of the relations between (2.3.1) and (2.3.2) - (2.3.3) viewed as a system over the ring $\mathbb{R}[\sigma_1,\sigma_2]$)

As another example a natural notion of isomorphism for systems $\Sigma = (F,G,H)$, $\Sigma' = (F',G',H')$ over a ring R is: Σ and Σ' are isomorphic if there exists an n x n matrix S over R, which is invertible over F, i.e. such that det(S) is a unit of R, such that $\Sigma' = \Sigma^S$. Taking $R = \mathbb{R}[\sigma_1,\sigma_2]$ and reinterpreting the σ_i as delays we see that the corresponding notion for the delay-differential systems is coordinate transformations with time delays which is precisely the right notion of isomorphism for studying for instance degeneracy phenomena, cf[Kap].

Finally applying the Laplace transform to (2.3.1) we find a transfer function $T(s,e^{-s},e^{-\alpha s})$, which is rational in s,e^{-s} and $e^{-\alpha s}$. It can also be obtained by taking the family of transfer functions $T_{\sigma_1,\sigma_2}(s) = H(\sigma_1,\sigma_2)(s-F(\sigma_1,\sigma_2))^{-1}G(\sigma_1,\sigma_2)$ and then substituting e^{-s} for σ_1 and $e^{-\alpha s}$ for σ_2. Inversely given a transfer function T(s) which is rational in $s,e^{-s},e^{-\alpha s}$ one way ask whether it can be realized as a system with delays which are multiples of 1 and α. Because the functions s, e^{-s}, $e^{-\alpha s}$ are algebraically independant (if α is incommensurable with 1), there is a unique

rational function $\tilde{T}(s,\sigma_1,\sigma_2)$ such that $T(s) = \tilde{T}(s,e^{-s},e^{-\alpha s})$ and the realizability of T(s) by means of a delay system, say a system with transmission lines, is now mathematically equivalent with realizing the two parameter family of transfer functions $T(s,\sigma_1,\sigma_2)$ by a family of systems which depends polynomally on σ_1, σ_2.

2.4. 2-d and n - d systems.

Consider a linear discrete time system with direct feed-through term

(2.4.1) $$x(t+1) = Fx(t) + Gu(t),\quad y(t) = Hx(t) + Ju(t)$$

The associated input/output operator is a convolution operator, viz. (cf.(1.2.b))

(2.4.2) $$y(t) = \sum_{i=o}^{t} A_i u(t-i),\quad A_o = J,\ A_i = HF^{i-1}G \text{ for } i = 1,2, \ldots$$

Now there is an obvious (north-east causal) more dimensional, generalization of the convolution operator (2.4.2), viz.

(2.4.3) $$y(h,k) = \sum_{i=o}^{h}\sum_{j=o}^{k} A_{i,j}u(h-i,k-j),\quad h,k = 0,1,2,\ldots$$

A (Givone-Roesser) realization of such an operator is a "2-d system"

(2.4.4) $$\begin{aligned} x_1(h+1,k) &= F_{11}x_1(h,k) + F_{12}x_2(h,k) + G_1u(h,k) \\ x_2(h,k+1) &= F_{21}x_1(h,k) + F_{22}x_2(h,k) + G_2u(h,k) \\ y(h,k) &= H_1x_1(h,k) + H_2x_2(h,k) + Ju(h,k) \end{aligned}$$

which yields an input/output operator of the form (2.4.3) with the $A_{i,j}$ determined by the power series development of the 2-d transfer function $T(s_1,s_2)$

(2.4.5) $$\sum_{i,j} A_{i,j}s_1^{-i}s_2^{-j} = T(s_1,s_2) = (H_1\ H_2)\left(\begin{pmatrix} s_1I_{n_1} & 0 \\ 0 & s_2I_{n_2}\end{pmatrix} - \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22}\end{pmatrix}\right)^{-1}\begin{pmatrix} G_1 \\ G_2\end{pmatrix} +.$$

where I_r is the r x r unit matrix and n_1 and n_2 are the dimensions of the state vectors x_1 and x_2. There are obvious generalisations to n-d systems, $n \geq 3$. The question now arises whether every proper 2-d matrix transfer function can indeed be so realized. (cf.[Eis] or [So2] for a definition of proper. A way to approach this is to treat one of the s_i as a parameter, giving us a realization with parameters problem.

More precisely let R_g be the ring of all proper rational functions in s_1. In the 2-d case this is a principal ideal domain which simplifies things considerably. Now consider $T(s_1,s_2)$ as a proper rational function in s_2 with coefficients in R_g. This transfer function can be realized giving us a discrete time system over R_g defined by the quadruple of matrices $(F(s_1), G(s_1), H(s_1), J(s_1))$. Each of these matrices is proper as a function of s_1 and hence can be realized by a quadruple of constant matrices. Suppose that

$$(F_F,G_F,H_F,J_F) \text{ realizes } F(s_1)$$

$$(F_g,G_g,H_g,J_g) \text{ realizes } G(s_1)$$

$$(F_H,G_H,H_H,J_H) \text{ realizes } H(s_1)$$

$$(F_J,G_J,H_J,J_J) \text{ realizes } J(s_1)$$

Then, as is easily checked, a realization in the sense of (2.4.4) is defined by

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \left(\begin{array}{c|cccc} J_F & H_F & H_G & 0 & 0 \\ \hline G_F & F_F & 0 & 0 & 0 \\ 0 & 0 & F_G & 0 & 0 \\ G_H & 0 & F_H & 0 & 0 \\ 0 & 0 & 0 & 0 & F_J \end{array}\right), \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} J_G \\ \hline 0 \\ G_G \\ 0 \\ G_J \end{pmatrix}$$

$$H = (H_1 \;\; H_2) = (J_H \mid 0 \;\; 0 \;\; H_H \;\; H_J), \quad J = J_J$$

This is the procedure followed in [Eis] ; a somewhat different approach, with essentially the same initial step (i.e. realization with parameters, or realization over a ring) is followed in [So2].

2.5. Parameter Uncertainty.

Suppose that we have a system $\Sigma = (F,G,H)$ but that we are uncertain about some of its parameters, i.e. we are uncertain about the precise value of some of the entries of F,G or H. That is, what we really have is a family of systems $\Sigma(\beta)$, where β runs through some set B of parameter values, which we assume compact. For simplicity assume that we have a one input-one output system. Let the transfer function of $\Sigma(\beta)$ be $T_\beta(s) = f_\beta(s)/g_\beta(s)$. Now suppose we want to stabilize Σ by a dynamic output feedback loop with

transfer function $P(s) = \phi(s)/\psi(s)$, still being uncertain about the value of β. The transferfunction of the resulting total system is $T(s)/(1-T(s)P(s))$. So we shall have succeeded if we can find polynomials $\phi(s)$ and $\psi(s)$ such that for all $\beta \in B$ all roots of

$$g_\beta(s)\psi(s) + f_\beta(s)\phi(s)$$

are in the left halfplane, possibly with the extra requirement that P(s) be also stable. The same mathematical question arises from what has been named the blending problem, cf [Ta1]. It cannot always be solved. In the special but important case where the uncertainty is just a gain factor, i.e. in the case that B is an interval $[b_1,b_2]$, $b_2 > b_1 > 0$ and $T_\beta(s) = \beta T(s)$, where T(s) is a fixed transforfunction, the problem is solved completely in [Ta1].

3. THE CLASSIFICATION OF FAMILIES. FINE MODULI SPACES.

3.1. Introductory and Motivational Remarks.

(Why classifying families is essentially more difficult than classifying systems and why the set of isomorphism classes of (single) systems should be topologized).

Obviously the first thing to do when trying to classify families up to isomorphism is to obtain a good description of the set of isomorphism classes of (single) systems over a field k, that is to obtain a good description of the sets $L_{m,n,p}(k)/GL_n(k) = M_{m,n,p}(k)$ and of the quotient map $L_{m,n,p}(k) \to M_{m,n,p}(k)$. This will be done below in section 3.2 for the subset of isomorphism classes (or sets of orbits) of completely reachable systems. This is not particularly difficult (and also well known) nor is it overly complicated to extend this to a description of all of $M_{m,n,p}(k) = L_{m,n,p}(k)/GL_n(k)$, cf. [Haz6]. Though, as we shall see, there are, for the moment, good mathematical reasons, to limit ourselves to cr systems and families of cr systems, or, dually, to limit ourselves to co systems.

Now let us consider the classification problem for families of systems. For definiteness sake suppose we are interested (cf. 2.1 and 2.3 above e.g.) in real families of systems $\Sigma(\sigma) = (F(\sigma), G(\sigma), H(\sigma))$ which depend continuously on a real parameter $\sigma \in \mathbb{R}$. The obvious, straightforward and in fact right thing to do is to proceed as follows. For each $\sigma \in \mathbb{R}$ we have a system $\Sigma(\sigma)$, and hence a point $\phi(\sigma) \in M_{m,n,p}(\mathbb{R}) = L_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$, the set of isomorphism classes or, equivalently, the set of orbits in $L_{m,n,p}(\mathbb{R})$ under the action $(\Sigma,S) \mapsto \Sigma^S$ of $GL_n(\mathbb{R})$ on $L_{m,n,p}(\mathbb{R})$. This defines a map $\phi(\Sigma): \mathbb{R} \to M_{m,n,p}(\mathbb{R})$, and one's first guess would be that two families Σ,Σ' are isomorphic iff their associated maps $\phi(\Sigma)$, $\phi(\Sigma')$ are equal. However, things are not that simple as the following example in $L_{1,2,1}(\mathbb{R})$ shows.

(3.1.1) Example. $\Sigma(\sigma) = (\begin{pmatrix} 1 & 0 \\ \sigma^2 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1,2))$,

$\Sigma'(\sigma) = (\begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1,2\sigma))$

For each $\sigma \in \mathbb{R}$, $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ are isomorphic via $T(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & \sigma^{-1} \end{pmatrix}$ if $\sigma \neq 0$ and via $T(\sigma) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ if $\sigma = 0$. Yet they are not isomorphic as continuous families, meaning that there exists no continuous map $\mathbb{R} \to GL_2(\mathbb{R})$, $\sigma \mapsto T(\sigma)$, such that $\Sigma'(\sigma) = \Sigma(\sigma)^{T(\sigma)}$ for all $\sigma \in \mathbb{R}$. One might guess that part of the problem is topological. Indeed, it is in any case sort of obvious that one should give $M_{m,n,p}(\mathbb{R})$ as much structure as possible. Otherwise the map $\phi(\Sigma): \mathbb{R} \to M_{m,n,p}(\mathbb{R})$ does not tell us whether it could have come from a continuous family. (Of course if $\Sigma(\sigma)$ is a continuous family over $\mathbb{R}$ giving rise to $\phi(\Sigma)$ and $S \in GL_n(\mathbb{R})$ is such that $\Sigma(0)^S \neq \Sigma(0)$ then the discontinuous family $\Sigma'(\sigma)$, $\Sigma'(\sigma) = \Sigma(\sigma)$ for $\sigma \neq 0$, $\Sigma'(0) = \Sigma(0)^S$ gives rise to the same map). Similarly we would like to have $\phi(\Sigma)$ analytic if Σ is an analytic family, polynomial if Σ is polynomial, differentiable if Σ is differentiable, ...

One reason to limit oneselve to cr systems is now that the natural topology (which is the quotient topology for $\pi: L_{m,n,p}(\mathbb{R}) \to M_{m,n,p}(\mathbb{R})$) will not be Hausdorff unless we limit ourselves to cr systems. (It is clear that one wants to put in at least all co,cr systems).

There are more reasons to topologize $M_{m,n,p}(\mathbb{R})$ and more generally $M_{m,n,p}(k)$, where k is any field. For one thing it would be nice if $M_{m,n,p}(\mathbb{R})$ had a topology such that the isomorphism classes of two systems Σ and Σ' were close together if and only if their associated input/output maps were close together (in some suitable operator topology; say the weak topology); a requirement which is also relevant to the consistency requirement of maximum likelyhood identification of systems, cf. [De,DDH,DH,DS,Han].

Jet topologizing $M_{m,n,p}(\mathbb{R})$ does not remove the problem posed by example (3.1.1). Indeed, giving $M_{m,n,p}(\mathbb{R})$ the quotient topology inherited from $L_{m,n,p}(\mathbb{R})$ the maps defined by the families Σ and Σ' of example (3.1.1) are both continuous.

Restricting ourselves to families consisting of cr systems (or dually to families of co systems), however, will solve the problem posed by example (3.1.1). This same restriction will also see to it that the quotient topology is Hausdorff and it will turn out that $M^{cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$ is naturally a smooth differentiable manifold. From the algebraic geometric point of view we shall see that the quotient $L^{cr}_{m,n,p}/GL_n$ exists as a smooth scheme defined over $\mathbb{Z}$. It is also pleasant to notice that for pairs of matrices (F,G) the prestable ones (in the sense of [Mu]) are precisely the completely reachable ones ([Ta2]) and they are also the semi-stable points of weight one, [BH].

Ideally it would also be true that every continuous, differentiable, polynomial,... map $\phi : \mathbb{R} \to M^{cr}_{m,n,p}(\mathbb{R})$ comes from a continuous, differentiable, polynomial,... family. This requires assigning to each point of $M^{cr}_{m,n,p}(\mathbb{R})$ a system represented by that point and to do this in an analytic manner. This now really requires a slightly more sophisticated definition of family then we have used up to now, cf. 3.4. below. And indeed to obtain e.g. all continuous maps of say the circle into $M_{m,n,p}(\mathbb{R})$ as maps associated to a family one also needs the same more general conceptof families of system over the circle.

3.2. Description of the quotient set (or set of orbits) $L^{cr}_{m,n,p}(k)/GL_n(k)$.

Let k be any field, and fix $n,m,p \in \mathbb{N}$. Let

(3.2.1) $$J_{n,m} = \{(0,1),(0,2),\ldots,(0,m);(1,1),\ldots,(1,m);\ldots;(n,1),\ldots,(n,m)\},$$

lexicographically ordered (which is the order in which we have written down the (n+1)m elements of $J_{n,m}$). We use $J_{n,m}$ to label the columns of the matrix $R(F,G)$, $F \in k^{n\times n}$, $G \in k^{n\times m}$, cf. 1.3 above, by assigning the label (i,j) to the j-th column of the block F^iG.

A subset $\alpha \subset J_{n,m}$ is called nice if $(i,j) \in \alpha \Rightarrow (i-1,j) \in \alpha$ or $i = 0$ for all i,j. A nice subset with precisely n elements is called a nice selection.

Given a nice selection α, a successor index of α is an element $(i,j) \in J_{n,m}$ such that $\alpha \cup \{(i,j)\}$ is nice. For every $j_0 \in \{1,\ldots,m\}$ there is precisely one successor index (i,j) of α with $j = j_0$. This successor index will be denoted $s(\alpha,j_0)$.

Pictorially these definitions look as follows. We write down the elements of $J_{n,m}$ in a square as follows ($m=4, n=5$)

(0,1)	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(0,2)	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)
(0,3)	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)
(0,4)	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)

Using dots to represent elements of $J_{n,m}$ and x's to represent elements of α the following pictures represent respectively a nice subset, a not nice subset and a nice selection.

```
. . . . . .     . x x . . .     . . . . . .
x x x . . .     x . . x . .     x x . . . .
x . . . . .     . x . . . .     . . . . . .
. . . . . .     x x . . . .     x x x . . .
```

The successor indices of the nice selection α of the third picture above are indicated by *'s in the picture below

```
          * . . . . .
          x x * . . .
(3.2.2)
          * . . . . .
          x x x * . .
```

We shall use $L_{m,n}(k)$ to denote the set of all pairs of matrices (F,G) over k of sizes $n \times n$ and $n \times m$ respectively; $L^{cr}_{m,n}(k)$ denotes the subset of completely reachable pairs (cf. 1.3 above). For each subset $\beta \in J_{n,m}$ and each $(F,G) \in L_{m,n}(k)$ we shall use $R(F,G)_\beta$ to denote the matrix obtained from $R(F,G)$ by removing all columns whose index is not in β.

With this terminology and notation we have the following lemma.

3.2.3. Nice Selection Lemma.

Let $(F,G) \in L^{cr}_{m,n}(k)$. Then there is a nice selection α such that $\det(R(F,G)_\alpha) \neq 0$.

Proof. Let α be a nice subset of $J_{n,m}$ such that the columns of $R(F,G)_\alpha$ are linearly independant and such that α is maximal with respect to this property.

Let $\alpha = \{(0,j_1), \ldots, (i_1,j_1); (0,j_2), \ldots, (i_2,j_2); \ldots ; (0,j_s), \ldots, (i_s,j_s)\}$.

By the maximality of α we know that the successor indices $s(\alpha,j)$, $j = 1, \ldots, m$ are linearly dependant on the columns of $R(F,G)_\alpha$. I.e. the columns with indices $(i_1+1,j_1), \ldots, (i_s+1,j_s)$ and $(0,t)$, $t \in \{1,\ldots,m\} \setminus \{j_1,\ldots,j_s\}$ are linearly dependant on the columns of $R(F,G)_\alpha$. Suppose now that with induction we have proved that all columns with indices $(i_r+\ell,j_r)$, $r = 1, \ldots, s$ and $(\ell-1,t)$, $t \in \{1,\ldots,m\} \setminus \{j_1,\ldots,j_s\}$ are linearly dependant on the columns of $R(F,G)_\alpha$, $\ell \geq 1$. This gives us certain relations

$$F^{\ell-1}G_t = \sum_{(i,j)\in\alpha} a(i,j)F^iG_j, \quad F^{i_r+\ell}G_{j_r} = \sum_{(i,j)\in\alpha} b(i,j)F^iG_j$$

(where G_t denotes the t-th column of G). Multiplying on the left with F we find expressions

$$F^{\ell}G_t = \sum_{(i,j)\in\alpha} a(i,j)F^{i+1}G_j, \quad F^{i_r+\ell+1}G_{j_r} = \sum_{(i,j)\in\alpha} b(i,j)F^{i+1}G_j$$

expressing $F^\ell G_t$ and $F^{i_r+\ell+1}G_{j_r}$ as linear combination of those columns of $R(F,G)$ whose indices are either in α or a successor index of α. The latter are in turn linear combinations of the columns of $R(F,G)_\alpha$, so that we have proved that all columns of $R(F,G)$ are linear combinations of the columns of $R(F,G)_\alpha$. Now (F,G) is cr so that $\operatorname{rank}(R(F,G)) = n$, so that α must have had n elements, proving the lemma.

For each nice selection α we define

(3.2.4) $\qquad U_\alpha(k) = \{(F,G,H) \in L_{m,n,p}(k) \mid \det(R(F,G)_\alpha) \neq 0\}$

Recall that $GL_n(k)$ acts on $L_{m,n,p}(k)$ by $(F,G,H)^S = (SFS^{-1}, SG, HS^{-1})$.

3.2.5. Lemma. U_α is stable under the action of $GL_n(k)$ under $L_{m,n,p}(k)$. For each $\Sigma \in (F,G,H) \in U_\alpha$ there is precisely one $S \in GL_n(k)$ such that $R(\Sigma^S)_\alpha = R(SFS^{-1},SG)_\alpha = I_n$, the $n \times n$ identity matrix.

Proof. We have

(3.2.6) $\qquad R(\Sigma^S) = R(SFS^{-1},SG) = SR(F,G) = S\,R(\Sigma)$

It follows that $R(\Sigma^S)_\alpha = SR(\Sigma)_\alpha$, which proves the first statement. It also follows that if we take $S = R(F,G)_\alpha^{-1}$ then $R(\Sigma^S)_\alpha = I_n$ and this is also the only S which does this because in the equation $S\,R(\Sigma)_\alpha = R(\Sigma^S)_\alpha$, $R(\Sigma)_\alpha$ has rank n.

3.2.7. Lemma. Let $x_1, \ldots, x_m$ be an arbitrary m-tuple of n-vectors over k, and let α be a nice selection. Then there is precisely one pair $(F,G) \in L^{cr}_{m,n}(k)$ such that $R(F,G)_\alpha = I_n$, $R(F,G)_{s(\alpha,j)} = x_j$, $j = 1, \ldots, m$.

Proof (by sufficiently complicated example). Suppose $m = 4$, $n = 5$ and that α is the nice selection of (3.2.2) above. Then we can simply read off the desired F,G. In fact we find $G_1 = x_1$, $G_2 = e_1$, $G_3 = x_3$, $G_4 = e_2$, $F_1 = e_3$, $F_2 = e_4$, $F_3 = x_2$, $F_4 = e_5$, $F_5 = x_4$. Writing down a fully general proof is a bit tedious and notationally a bit cumbersome and it should now be trivial exercise.

3.2.8. Corollary. The set of orbits $U_\alpha(k)/GL_n(k)$ is in bijective correspondence with $k^{nm} \times k^{pn}$, and $U_\alpha(k) \tilde{\to} GL_n(k) \times (k^{nm} \times k^{pn})$ (as sets with $GL_n(k)$-action, where $GL_n(k)$ acts on $GL_n(k) \times (k^{nm} \times k^{pn})$ by multiplication on the left on the first factor).

Proof. This follows immediately from lemma 3.2.5 together with lemma 3.2.6. Indeed given $\Sigma = (F,G,H) \in U_\alpha$. Take $S = R(F,G)_\alpha^{-1}$ and let $(F',G',H') = \Sigma^S$. Now define $\phi : U_\alpha(k) \to GL_n(k) \times (k^{nm} \times k^{pn})$ by assigning to (F,G,H) the matrix S^{-1}, the m n-vectors $R(\Sigma^S)_{s(\alpha,j)}$, $j = 1, \ldots, m$ and the $p \times n$ matrix H'. Inversely given a $T \in GL_n(k)$, m n-vectors x_j, $j = 1, \ldots, m$ and a $p \times n$ matrix y. Let $(F',G') \in L^{cr}_{m,n}(k)$ be the unique pair such that $R(F',G')_\alpha = I_n$, $R(F',G')_{s(\alpha,j)} = x_j$, $j = 1, \ldots, m$. Take $H' = y$ and define $\psi : GL_n(k) \times (k^{nm} \times k^{pn}) \to U_\alpha(k)$ by $\psi(T,(x,y)) = (F',G',H')^T$. It is trivial to check that $\psi\phi = \mathrm{id}$, $\phi\psi = \mathrm{id}$. It is also easy to check that ϕ commutes with the $GL_n(k)$-actions.

3.2.9. The $c_{\#\alpha}$ (local) canonical forms. For each $\Sigma \in U_\alpha(k)$ we denote with $c_{\#\alpha}(\Sigma)$ the triple:

(3.2.10) $c_{\#\alpha}(\Sigma) = \Sigma^S$ with $S = R(\Sigma)_\alpha^{-1}$

i.e. $c_{\#\alpha}(\Sigma)$ is the unique triple Σ' in the orbit of Σ such that

$R(\Sigma') = I_n$. Further if $z \in k^{mn} \times k^{np}$, then we let $(F_\alpha(z), G_\alpha(z), H_\alpha(z))$ be the triple $\psi(I_n,z)$; that is if $z = ((x_1, \ldots, x_m), y)$ $(F_\alpha(z), G_\alpha(z), H_\alpha(z))$ is the unique triple such that:

(3.2.11) $R(F_\alpha(z), G_\alpha(z))_\alpha = I_n$, $R(F_\alpha(z), G_\alpha(z))_{s(\alpha,j)} = x_j$, $H_\alpha(z) = y$

$z \in ((x_1, \ldots, x_m), y) \in k^{nm} \times k^{pn}$

3.2.12. Remark. Let $\pi : U_\alpha(k) \to k^{nm} \times k^{pn}$ be equal to $\psi: U_\alpha(k) \to GL_n(k) \times (k^{nm} \times k^{pn})$ followed by the projection on the second factor. Then $\tau_\alpha: z \to (F_\alpha(Z), G_\alpha(z), H_\alpha(z))$ is a section of π_α (meaning that $\pi_\alpha\tau_\alpha = \mathrm{id}$), and $c_{\#\alpha}(\tau_\alpha) = \tau_\alpha$. Of course, π_α induces a bijection $U_\alpha(k)/GL_n(k) \to k^{nm} \times k^{pn}$.

3.2.13. Description of the set of orbits $L^{cr}_{m,n,p}(k)/GL_n(k)$. Order the set of all nice selections from $J_{n,m}$ in some way. For each $\Sigma \in L^{cr}_{m,n,p}$ let $\alpha(\Sigma)$ be the first nice selection in this ordening. Now assign to Σ the triple $c_{\#\alpha(\Sigma)}(\Sigma)$. This assigns to each $\Sigma \in L^{cr}_{m,n,p}(k)/GL_n(k)$ one particular well defined element in its orbit and this hence gives complete description of the set of orbits $L^{cr}_{m,n,p}(k)/GL_n(k)$.

3.3. Topologizing $L^{cr}_{m,n,p}(k)/GL_n(k) = M^{cr}_{m,n,p}(k)$

3.3.1. A more "homogeneous" description of $M^{cr}_{m,n,p}(k)$. The description of the set of orbits of $GL_n(k)$ acting on $L^{cr}_{m,n,p}(k)$ given in 2.3.13 is highly lopsized in the various possible nice selections α. A more symmetric description of $M^{cr}_{m,n,p}(k)$ is obtained as follows. For each nice selection α, let $V_\alpha(k) = k^{nm} \times k^{pn}$ and let for each second nice selection β:

(3.3.2) $V_{\alpha\beta}(k) = \{z \in V_\alpha \mid \det(R(F_\alpha(z), G_\alpha(z))_\beta) \neq 0\}$

That is, under the section $\tau_\alpha: V_\alpha(k) \to U_\alpha(k)$ of 3.2 above which picks out precisely one element of each orbit in $U_\alpha(k)$ $V_{\alpha\beta}(k)$ corresponds to those orbits which are also in $U_\beta(k)$; or, equivalently $V_{\alpha\beta}(k) = \pi_\alpha(U_\alpha(k) \cap U_\beta(k))$. We now glue the $V_\alpha(k)$, α nice, together along the $V_{\alpha\beta}(k)$ by means of the identifications:

$$\phi_{\alpha\beta}: V_{\alpha\beta}(k) \to V_{\beta\alpha}(k),\ \phi_{\alpha\beta}(z) = z' \Leftrightarrow$$

(3.3.3)

$$(F_\alpha(z),\ G_\alpha(z),\ H_\alpha(z))^S = (F_\beta(z'),\ G_\beta(z'),\ H_\beta(z')),\ S = R(F_\alpha(z), G_\alpha(z))_\beta^{-1}$$

Then, as should be clear from the remarks made just above, $M^{cr}_{m,n,p}(k)$ is the union of the $V_\alpha(k)$ with for each pair of nice selections α, β, $V_{\alpha\beta}(k)$ identified with $V_{\beta\alpha}(k)$ according to (3.3.3).

3.3.4. The analytic varieties $M^{cr}_{m,n,p}(\mathbb{R})$ and $M^{cr}_{m,n,p}(\mathbb{C})$. Now let $k = \mathbb{R}$ or $\mathbb{C}$ and give $V_\alpha(k) = $ for $k^{nm} \times k^{pn}$ its usual (real) analytic structure. The subsets $V_{\alpha\beta}(k) \subset V_\alpha(k)$ are then open subsets and the $\phi_{\alpha\beta}(k)$ are analytic diffeomorphisms. It follows that $M^{cr}_{m,n,p}(\mathbb{R})$ and $M^{cr}_{m,n,p}(\mathbb{C})$ will be respectively a real analytic (hence certainly C^∞) manifold and a complex analytic manifold, provided we can show that they are Hausdorff.

First notice that if we give $L_{m,n,p}(\mathbb{R})$ and $L_{m,n,p}(\mathbb{C})$ the topology of $\mathbb{R}^{mn+n^2+np}$ and $\mathbb{C}^{n^2+nm+np}$ respectively and the open subsets $U_\alpha(k)$ and $L^{cr}_{m,n,p}(k)$, $k = \mathbb{R}$, $\mathbb{C}$ the induced topology, then the quotient topology for $\pi_\alpha: U_\alpha(k) \to V_\alpha(k)$ is precisely the topology resulting from the identification $V_\alpha(k) \simeq k^{nm} \times k^{pn}$. It follows that the topology of $M^{cr}_{m,n,p}(k)$ is the quotient topology of $L^{cr}_{m,n,p}(k) \to L^{cr}_{m,n,p}(k)/GL_n(k) = M^{cr}_{m,n,p}(k)$.

Now let $G_{n,m(n+1)}(k)$ be the Grassmann variety of n-planes in m(n+1)-space. For each (F, G), R(F,G) is an $n \times m(n+1)$ matrix of rank n which hence defines a unique point of $G_{n,m(n+1)}(k)$. Because $R(SFS^{-1}, SG) = SR(F, G)$ we have that (F, G) and $(F, G)^S$ define the same point in Grassmann space. It follows that by forgetting H we have defined a map:

(3.3.5) $\tilde{R}: M^{cr}_{m,n,p}(k) \to G_{n,m(n+1)}(k)$, $(F, G) \mapsto$ subspace spanned by the rows of R(F,G).

In addition we let $h: M^{cr}_{m,n,p}(k) \to k^{(n+1)^2mp}$ be the map induced by:

$$(3.3.6)\quad \tilde{h}(F,G,H) = \begin{bmatrix} A_1 & A_2 & \cdots & A_{n+1} \\ A_2 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \\ A_{n+1} & & \cdots & A_{2n+1} \end{bmatrix},\quad A_i = HF^{i-1}G,\ i = 1, \ldots, 2n+1$$

It is not particularly difficult to show ([Haz 1-3], cf. also the realization algorithm in 5.2 below) that the combined map $(\tilde{R},h): M^{cr}_{m,n,p}(k) \to G_{n,m(n+1)}(k) \times k^{(n+1)^2mp}$ is injective. By the quotient topology remarks above it is then a topological embedding, proving that $M^{cr}_{m,n,p}(k)$ is a Hausdorff topological space. So we have:

3.3.7. Theorem. $M^{cr}_{m,n,p}(\mathbb{R})$ and $M^{cr}_{m,n,p}(\mathbb{C})$ are smooth analytic manifolds. The sets $M^{cr,co}_{m,n,p}(\mathbb{R})$ and $M^{cr,co}_{m,n,p}(\mathbb{C})$ are analytic open sub-manifolds. (These are the sets of orbits of the cr and co systems, or equivalently, the images of $L^{cr,co}_{m,n,p}(k)$ under $\pi: L^{cr}_{m,n,p}(k) \to M^{cr}_{m,n,p}(k)$, $k = \mathbb{R}, \mathbb{C}$).

3.3.8. Remark. A completely different way of showing that the quotient space $M^{cr}_{m,n,p}(\mathbb{R})$ is a differentiable manifold is due to Martin and Krishnaprasad, [MK]. They show that with respect to a suitable invariant metric on $L^{cr,co}_{m,n,p}(k)$, $GL_n(k)$ acts properly discontinuously.

3.3.9. The algebraic varieties $M^{cr}_{m,n,p}(k)$. Now let k be any algebraically closed field. Giving $L_{m,n,p}(k) = k^{n^2+nm+np}$ the Zariski topology and $U_\alpha(k)$ the induced topology for each nice selection α. Then $U_\alpha(k) \simeq GL_n(k) \times V_\alpha(k)$, $V_\alpha(k) = k^{nm+np}$ also as algebraic varieties. The $V_{\alpha\beta}(k)$ are open subvarieties and the $\phi_{\alpha\beta}(k): V_{\alpha\beta}(k) \to V_{\beta\alpha}(k)$ are isomorphisms of algebraic varieties. The map $(\tilde{R},h)$ is still injective and it follows that $M^{cr}_{m,n,p}(k)$ has a natural structure of a smooth algebraic variety, with $M^{cr,co}_{m,n,p}(k)$ an open subvariety.

3.3.10. The scheme $M^{cr}_{m,n,p}$. As a matter of fact, the defining pieces of the algebraic varieties $M^{cr}_{m,n,p}(k)$, that is the $V_\alpha(k)$, and the glueing isomorphisms $\phi_{\alpha\beta}(k)$ are all defined over $\mathbb{Z}$. So there exists a scheme $M^{cr}_{m,n,p}$ over $\mathbb{Z}$ such that for all fields k the rational points over k, $M^{cr}_{m,n,p}(k)$ are precisely the orbits of $GL_n(k)$ acting on $L^{cr}_{m,n,p}(k)$. For details cf. section 4 below.

3.4. A universal family of linear dynamical systems

3.4.1. As has been remarked above it would be nice if we could attach in a continuous way to each point of $M_{m,n,p}(k)$ a system over k representing that point. Also it would be pleasant if every appropriate map from a parameter space V to $M^{cr}_{m,n,p}$ came from a family over V. Recalling from 2.2 above that systems over a ring R can be reinterpreted as families over Spec(R), this would mean that the isomorphism classes of systems over R would correspond bijectively with the R-rational points $M^{cr}_{m,n,p}(R)$ of the scheme $M^{cr}_{m,n,p}$ over $\mathbb{Z}$, cf. 3.3.10.

Both wishes, if they are to be fulfilled require a slightly more general definition of system then we have used up to now. In the case of systems over a ring R the extra generality means that instead of considering three matrices F, G, H over R, that is three homomorphisms $G: R^m \to R^n$, $F: R^n \to R^n$, $H: R^n \to R^p$ we now generalize to the definition: a projective system over R consists of a projective module X as state module together with three homomorphisms $G: R^m \to X$, $F: X \to X$, $H: X \to R^p$. Thus the extra generality sits in the fact that the state R-module X is not required to be free, but only projective. The geometric counterpart of this is a vectorbundle, cf. below in 3.4.2 for the precise definition of a family and the role the vectorbundle plays.

In some circumstances it appears to be natural, in any case as an intermediate step to consider even more general families. Thus over a ring R it makes perfect sense to consider arbitrary modules as state modules, and indeed these turn up naturally when doing "canonical" realization theory, cf. [Eil, Ch. XVI], which in terms of families means that one may need to consider more general fibrations by vector spaces than locally trivial ones.

3.4.2. Families of linear dynamical systems (over a topological space). Let V be a topological space. A continuous family Σ of real linear dynamical stystems over V (or parametrized by V) consists of:

(a) a vectorbundle E over V

(b) a vectorbundle endomorphism $F: E \to E$

(c) a vectorbundle morphism $G: V \times \mathbb{R}^m \to E$

(d) a vectorbundle morphism $H: E \to V \times \mathbb{R}^p$

For each $v \in V$ let E(v) be the fibre of E over v. Then we have homomorphisms of vector spaces $G(v): \{v\} \times \mathbb{R}^m \to E(v)$, $F(v): E(v) \to E(v)$, $H(v): E(v) \to \{n\} \times \mathbb{R}^p$. Thus choosing a basis in E(v), and taking the obvious bases in $\{v\} \times \mathbb{R}^m$ and $\{v\} \times \mathbb{R}^p$ we find a triple of matrices $\bar{F}(v)$, $\bar{G}(v)$, $\bar{H}(v)$. Thus the data listed above do define a family over V in the sense that they assign to each $v \in V$ a linear system. Note however that there is no natural basis for E(v) so that the system is really only defined up to base change, i.e. up to the $GL_n(\mathbb{R})$ action, so that what the data (a)-(d) really do is assign a point of $M_{m,n,p}(\mathbb{R})$ to each point $v \in V$.

As E is a vectorbundle we can find for each $v \in V$ an open neighborhood W and n-sections $s_1, \ldots, s_n: W \to E|_W$ such that $s_1(w), \ldots, s_n(w) \in E(w)$ are linearly independent for all $w \in W$. Writing out matrices for F(w), G(w), H(w) with respect to the basis $s_1(w), \ldots, s_n(w)$ (and the obvious bases in $\{w\} \times \mathbb{R}^m$ and $\{w\} \times \mathbb{R}^p$), we see that over W the family Σ can indeed be described as a triple of matrices depending continuously on parameters. Inversely if $(\bar{F}, \bar{G}, \bar{H})$ is a triple of matrices depending continuously on a parameter $v \in V$, then $E = V \times \mathbb{R}^n$, $F(v,x) = (v, \bar{F}(v)x)$, $F(v,u) = (v,\bar{G}(v)u)$, $H(v,x) = (v,\bar{H}(v)x)$ define a family as described above. Thus locally the new definition agrees (up to isomorphism) with the old intuitive one we have been using up to now; globally it does not.

Here the appropriate notion of isomorphism is of course: two families $\Sigma = (E; F, G, H)$ and $\Sigma' = (E'; F', G', H')$ over V are isomorphic if there exists a vectorbundle isomorphism $\phi: E \to E'$ such that $F'\phi = \phi F$, $\phi G = G'$, $H = H'\phi$.

3.4.3. Other kinds of families of systems. The appropriate definitions of other kinds of families are obtained from the one above by means of minor and obvious adjustments. For instance, if V is a differentiable (resp. real analytic) manifold then a differentiable (resp. real analytic) family of systems consists of a differentiable vector bundle E with differentiable morphisms F, G, H (resp. an analytic vectorbundle with analytic morphisms F, G, H). And of course isomorphisms are supposed to be differentiable (resp. analytic).

Similarly if V is a scheme (over k) then an algebraic family consists of an algebraic vectorbundle E over V together with morphisms of algebraic vectorbundles $F: E \to E$, $G: V \times \mathbb{A}^m \to E$, $H: E \to V \times \mathbb{A}^p$, where $\mathbb{A}^r$ is the (vectorspace) scheme $\mathbb{A}^r(R) = R^r$ (with the obvious R-module structure).

Still more variations are possible. E.g. a complex analytic family (or holomorphic family) over a complex analytic space V would consist of a complex analytic vectorbundle E with complex analytic vecto-bundle homomorphisms $F: E \to E$, $G: V \times \mathbb{C}^m \to E$, $H: E \to V \times \mathbb{C}^p$.

3.4.4. Convention. From now one whenever we speak about a family of systems it will be a family in the sense of (3.4.2) and (3.4.3) above.

3.4.5. The canonical bundle over $G_{n,r}(k)$. Let $G_{n,r}(k)$ be the Grassmann manifold of n-planes in r-space ($r > n$). Let $E(k) \to G_{n,r}(k)$ be the fibre bundle whose fibre over $x \in G_{n,r}(k)$ is the n-plane in k^r represented by the point x. If $k = \mathbb{R}$ or $\mathbb{C}$ this is an analytic vector bundle over $G_{n,r}(k)$. More generally this defines an algebraic vectorbundle E over the scheme $G_{n,r}$.

In terms of trivial pieces and glueing data this bundle can be described as follows. Let $M^{n\times r}_{reg}(k)$ be the space of all $n \times r$ matrices of rank n and let $\pi: M^{n\ r}_{reg}(k) \to G_{n,r}(k)$ be the map which associates to each $n \times r$ matrix of rank n, the n-space in k^{nr} spanned by its row vectors. Then the fibre over E(x) of E over $x \in G_{n,r}(k)$ is precisely the vector space of all linear combinations of any element in $\pi^{-1}(x)$. From this there results the following local pieces and glueing data description of $G_{n,r}(k)$ and E(k). For each subset α of size n of $\{1, 2, \ldots, r\}$ let $U'_\alpha(k)$ be the set of all $n \times r$ matrices A such that A_α is invertible, let $V'_\alpha(k) = k^{n(r-n)}$ and for each $z \in V'_\alpha(k)$, $z = (z_1, \ldots, z_{r-n})$, $z_i \in k^n$, let $A_\alpha(z)$ be the unique $n \times r$ matrix such that $(A_\alpha(z))_\alpha = I_n$ and $A_\alpha(z)_{t(j)} = z_j$ where t(j) runs through the elements of $\{1, 2, \ldots, r\} - \alpha$ in the natural order, $j = 1, \ldots, r-n$. Then $G_{n,r}(k)$ consists of the $V'_\alpha(k)$ glued together along the $V'_{\alpha\beta}(k) = \{z \in V'_\alpha(k) \mid A_\alpha(z)_\beta \text{ is invertible}\}$ by means of the isomorphisms:

$$(3.4.7)\quad \phi'_{\alpha\beta}(k): V'_{\alpha\beta}(k) \to V'_{\beta\alpha}(k),\ z \leftrightarrow z' \Leftrightarrow (A_\alpha(z)_\beta)^{-1}A_\alpha(z) = A_\beta(z')$$

(Note how very similar this is to the pieces and patching data description of $M^{cr}_{m,n,p}(k)$ given in 3.3.1 above; the reason is understandable if one observes that the map $R: L^{cr}_{m,n,p}(k) \to M^{n\times(n+1)m}_{reg}(k)$, induces a map $\tilde{R}: M^{cr}_{m,n,p}(k) \to G_{n,(n+1)m}(k)$, which is compatible with the local pieces and patching data for the two spaces).

The bundle E(k) over $G_{n,r}(k)$ can now be described as follows. Over each $V'_\alpha(k) \subset G_{n,r}(k)$ we can trivialize E(k) as follows:

$$(3.4.8)\quad V'_\alpha(k) \times k^n \xrightarrow{\sim} E(k)|_{V'_\alpha(k)},\ (z,x) \mapsto x^T A_\alpha(z).$$

It follows that the bundle E(k) over $G_{n,r}(k)$ admits the following local pieces and patching data description which is compatible with the local pieces and patching data description given above for $G_{n,r}(k)$. The bundle E(k) consists of the local pieces $E_\alpha(k) = V'_\alpha(k) \times k^n$ glued together along the $E_{\alpha\beta}(k) = V'_{\alpha\beta}(k) \times k^n$ by means of the isomorphisms:

$$(3.4.9)\quad \tilde{\phi}'_{\alpha\beta}: V'_{\alpha\beta}(k) \times k^n \xrightarrow{\sim} V'_{\beta\alpha}(k) \times k^n$$
$$(z,x) \mapsto (\phi'_{\alpha\beta}(z),\ (A_\alpha(z)_\beta)^T x)$$

The bundle which is really of interest to us is the dual bundle E^d to E described by the local pieces $E^d_\alpha(k) = V'_\alpha(k) \times k^n$ glued together by the patching data:

$$(3.4.10)\quad \tilde{\phi}^d_{\alpha\beta}: V'_{\alpha\beta}(k) \times k^n \xrightarrow{\sim} V'_{\beta\alpha}(k) \times k^n$$
$$(z,x) \mapsto (\phi'_{\alpha\beta}(z),\ (A_\alpha(z)_\beta)^{-1} x)$$

(Note that the glueing isomorphisms $\tilde{\phi}^d_{\alpha\beta}$ are compatible with the projections $E^d_\alpha(k) \to V'_\alpha(k)$ and the glueing isomorphisms $\phi'_{\alpha\beta}$ for $G_{n,r}(k)$; note also that all three sets of glueing data $\phi'_{\alpha\beta}$, $\tilde{\phi}'_{\alpha\beta}$, $\tilde{\phi}^d_{\alpha\beta}$ are transitive in the sense that $\tilde{\phi}^d_{\beta\gamma} \circ \tilde{\phi}^d_{\alpha\beta} = \tilde{\phi}^d_{\alpha\gamma}$ are similarly for the $\tilde{\phi}'$ and ϕ').

3.4.11. <u>The underlying vector bundle of the universal family over</u> $M^{cr}_{m,n,p}(k)$. The map $R: L^{cr}_{m,n,p}(k) \to M^{n\times(n+1)m}_{reg}(k)$, $(F,G,H) \mapsto R(F,G)$ induces a map.

$$(3.4.12)\quad \tilde{R}: M^{cr}_{m,n,p}(k) \to G_{n,(n+1)m}(k)$$

(because $R(\Sigma^S) = SR(\Sigma)$, $S \in GL_n(k)$).

If $k = \mathbb{R}$ or $\mathbb{C}$, (3.4.12) is a morphism between analytic manifolds. In general (3.4.12) defines a morphism between the schemes $M^{cr}_{m,n,p}$ and $G_{n,(n+1)m}$. Now let $E^u = \tilde{R}^! E^d$, the pullback by means of $\tilde{R}$ of the "canonical" bundle E^d described above in (3.4.5).

Now recall that $M^{cr}_{m,n,p}(k)$ was obtained by glueing the various pieces $V_\alpha(k) = k^{nm} \times k^{pn}$ together, where α runs through all nice selections from $J_{n,m}$. In terms of this description $E^u(k)$ can be described as follows: $E^u(k)$ consists of pieces $E^u_\alpha(k) = V_\alpha(k) \times k^n = k^{nm} \times k^{pn} \times k^n$, one for each nice selection α. For each pair of nice selections $E^u_{\alpha\beta}(k) = V_{\alpha\beta}(k) \times k^n \subset V_\alpha(k) \times k^n$. Now for each pair of nice selections α, β let $\tilde{\phi}_{\alpha\beta}(k): E^u_{\alpha\beta}(k) \to E^u_{\beta\alpha}(k)$ be the isomorphism:

$$(3.4.13)\quad \tilde{\phi}_{\alpha\beta}(k)\ (z,x) = (\phi_{\alpha\beta}(z),\ (R(F_\alpha(z),\ G_\alpha(z))_\beta)^{-1}\ x)$$

where $\phi_{\alpha\beta}: V_{\alpha\beta}(k) \to V_{\beta\alpha}(k)$ is the isomorphism of 3.3 above (which describes how the $V_\alpha(k)$ should be glued together to give $M_{m,n,p}(k)$, and $V_\alpha(k) \to U_\alpha(k)$, $z \mapsto (F_\alpha(z), G_\alpha(z), H_\alpha(z))$ is the section τ_α described above in (3.2.12). Then $E^u(k)$ is obtained by glueing together the $E^u_\alpha(k)$ along the $E^u_{\alpha\beta}(k)$ by means of the isomorphisms (3.4.13).

3.4.14. <u>Construction of a universal family of cr systems</u>. Let $E^u(k)$ over $M^{cr}_{m,n,p}(k)$ be the bundle described above and view it as obtained via the patching data (3.4.13). Recall also that, cf. (3.3.3) above:

$$(3.4.15)\quad \phi_{\alpha\beta}(z) = z' \Leftrightarrow (F_\alpha(z),\ G_\alpha(z),\ H_\alpha(z))^S = (F_\beta(z'),\ G_\beta(z'),\ H_\beta(z'))$$
$$\text{with } S = R(F_\alpha(z),\ G_\alpha(z))^{-1}_\beta$$

For each nice selection α we now define a bundle endomorphism $F^u_\alpha(k)$ of $E^u_\alpha(k) = V_\alpha(k) \times k^n$ and bundle morphisms $G^u_\alpha(k)$: $V_\alpha(k) \times k^m \to E^u_\alpha(k)$, $H^u_\alpha(k)$: $E^u_\alpha(k) \to V_\alpha(k) \times k^p$. These are defined as follows:

$$F^u_\alpha(k)\ (z,x) = (z,\ F_\alpha(z)x)$$
$$(3.4.16)\quad G^u_\alpha(k)\ (z,u) = (z,\ G_\alpha(z)x)$$
$$H^u_\alpha(k)\ (z,x) = (z,\ H_\alpha(z)x)$$

We now claim that these bundle morphisms are compatible with the glueing isomorphisms (3.4.13), which means that we must prove the commutativity of the diagram below for each pair of nice selections α, β.

$$(3.4.17)\quad \begin{array}{ccccccc}
V_{\alpha\beta}\times k^m & \xrightarrow{G^u_\alpha} & E^u_{\alpha\beta} & \xrightarrow{F^u_\alpha} & E^u_{\alpha\beta} & \xrightarrow{H^u_\alpha} & V_{\alpha\beta}\times k^p \\
\downarrow \phi_{\alpha\beta}\times id & & \downarrow \tilde{\phi}_{\alpha\beta} & & \downarrow \tilde{\phi}_{\alpha\beta} & & \downarrow \phi_{\alpha\beta}\times id \\
V_{\beta\alpha}\times k^m & \xrightarrow{G^u_\beta} & E^u_{\beta\alpha} & \xrightarrow{F^u_\beta} & E^u_{\beta\alpha} & \xrightarrow{H^u_\beta} & V_{\beta\alpha}\times k^p
\end{array}$$

where we have abbreviated various notations in obvious ways. Now

$$\begin{aligned}
\tilde{\phi}_{\alpha\beta}G^u_\alpha(z,u) &= \tilde{\phi}_{\alpha\beta}(z,\ G_\alpha(z)\ u) && \text{by (3.4.16)}\\
&= (\phi_{\alpha\beta}(z),\ R(F_\alpha(z),\ G_\alpha(z))^{-1}_\beta\ G_\alpha(z)\ u) && \text{by (3.4.13)}\\
&= (\phi_{\alpha\beta}(z),\ G_\beta(z')\ u) && \text{by (3.4.15)}\\
&= G^u(\phi_{\alpha\beta}\times id(z,\ u))
\end{aligned}$$

proving the commutativity of the left most square of (3.4.17). Similarly:

$$\begin{aligned}
\tilde{\phi}_{\alpha\beta}F^u_\alpha(z,x) &= \tilde{\phi}_{\alpha\beta}(z,\ F_\alpha(z)\ x) && \text{by (3.4.16)}\\
&= (\phi_{\alpha\beta}(z),\ R(F_\alpha(z),\ G_\alpha(z))^{-1}_\beta\ F_\alpha(z)\ x) && \text{by (3.4.13)}\\
&= (\phi_{\alpha\beta}(z),\ F_\beta(z')\ R(F_\alpha(z),\ G_\alpha(z))^{-1}_\beta x) && \text{by (3.4.15)}\\
&= F^u_\beta\ \tilde{\phi}_{\alpha\beta}(z,x)
\end{aligned}$$

proving the commutativity of the middle square of (3.4.17). And finally, and completely analogously:

$$H^u_\beta \phi_{\alpha\beta}(z,x) = H^u_\beta(\phi_{\alpha\beta}(z),\ R(F_\alpha(z),\ G_\alpha(z))^{-1}\ x) \qquad \text{by (3.4.13)}$$
$$= (\phi_{\alpha\beta}(z),\ H_\beta(z)\ R(F_\alpha(z),\ G_\alpha(z))^{-1}\ x) \qquad \text{by (3.4.16)}$$
$$= (\phi_{\alpha\beta}(z),\ H_\alpha(z)\ x) \qquad \text{by (3.4.15)}$$
$$= (\phi_{\alpha\beta} \times \text{id})\ (H^u_\alpha(z,\ x))$$

proving the commutativity of the last square of (3.4.17).

Thus the F^u_α, G^u_α, H^u_α combine to define bundle morphisms $F^u(k)$: $E^u(k) \to E^u(k)$, G^u: $M^{cr}_{m,n,p}(k) \times k^m \to E^u(k)$, $H^u(k)$: $E^u(k) \to M^{cr}_{m,n,p}(k) \times k^p$.

If $k = \mathbb{R}$ or $\mathbb{C}$, $F^u(k)$, $G^u(k)$, $H^u(k)$ are morphisms of analytic vector bundles. Algebraically speaking the $F^u(k)$, $G^u(k)$, $H^u(k)$ for varying k are part of a morphism of algebraic vector bundles over the scheme $M^{cr}_{m,n,p}$, which are defined over $\mathbb{Z}$.

3.4.18. The pullback construction. Let V be a topological space and ϕ: $V \to M^{cr}_{m,n,p}(\mathbb{R})$ a continuous map. Let $\Sigma^u = (E^u;\ F^u,\ G^u,\ H^u)$ be the universal family of systems constructed above. Then associated to ϕ we have an induced family $\phi^!\Sigma^u$ over V (obtained by pullback). The precise formulas are as follows:

- $\phi^! E^u = \{(v,x) \in V \times E^u | \phi(v) = \pi(x)\}$, where π: $E^u \to M^{cr}_{m,n,p}(\mathbb{R})$ is the bundle projection; the bundle projection of $\phi^! E^u$ is defined by $(v,x) \mapsto v$;
- $\phi^! F^u$: $(v,x) \to (v,\ F^u x) \in \phi^! E^u$
- $\phi^! G^u$: $(v,u) \to (v,\ G^u u) \in \phi^! E^u$
- $\phi^! H^u$: $(v,x) \to (v,\ H^u x) \in \phi^!(M^{cr}_{m,n,p}(\mathbb{R}) \times (\mathbb{R}^p) = V \times \mathbb{R}^p$

Obivously $\phi^! E^u$ is (up to isomorphism) the family of systems over V such that the system over $v \in V$ is (up to isomorphism) the system over $\phi(v)$ in the family Σ^u.

If V and ϕ are differentiable (resp. real analytic) there results a differentiable (resp. real analytic) family over V. If ϕ: $V \to M^{cr}_{m,n,p}(\mathbb{C})$ is a morphism of complex analytic manifolds there results a complex analytic family and on the algebraic-geometric side of things if ϕ: $V \to M^{cr}_{m,n,p}$ is a morphism of schemes one finds thus an algebraic family over the scheme V.

3.4.19. The topological fine moduli theorem. Let V be a topological space and Σ a continuous family of completely reachable systems over V. Then there exists a unique continuous map ϕ: $V \to M^{cr}_{m,n,p}(\mathbb{R})$ such that Σ is isomorphic to $\phi^!\Sigma^u$ (as continuous families; i.e. there is a bijective correspondence between continuous maps $V \to M^{cr}_{m,n,p}(\mathbb{R})$ and isomorphism classes of continuous families over V).

3.4.20. The algebraic-geometric fine moduli theorem. Let V be a scheme and Σ an algebraic family of cr systems over V. Then there exists a unique morphism of schemes ϕ: $V \to M^{cr}_{m,n,p}$ such that Σ is isomorphic to $\phi^!\Sigma^u$ over V.

3.4.21. On the proof of theorem 3.4.19. First consider the topological case. The map ϕ associated to Σ is defined as follows. For each $v \in V$ we have a system $\Sigma(v)$, which uniquely determines an isomorphism class of linear dynamical systems (cf. (3.4.2)); that is it uniquely defines a point $\phi(v)$ of $M^{cr}_{m,n,p}(\mathbb{R})$ which is the space of all isomorphism classes of cr systems (of the dimensions under consideration). This ϕ is obviously continuous. Now $\Sigma^u(z)$ for all $z \in M^{cr}_{m,n,p}(\mathbb{R})$ respresents z. So, by 3.4.18, Σ and $\phi^!\Sigma^u$ are two continuous families of cr systems over V such that for all $v \in V$, $\Sigma(v)$ and $\phi^!\Sigma^u(v)$ are isomorphic. It follows that the families Σ and $\Sigma' = \phi^!\Sigma^u$ are isomorphic as continuous families. The reason is the following rigidity property: if $(F, G, H), (F', G', H') \in L^{cr}_{m,n,p}(\mathbb{R})$ are isomorphic then the isomorphism is unique. Indeed, if S is an isomorphism then we must have $SR(F, G) = R(F', G')$ so that if α is a nice selection such that $R(F,G)$ is invertible, then $S = R(F', G')_\alpha\ (R(F, G)_\alpha)^{-1}$. The statement that Σ and Σ' over V are isomorphic if they are pointwise isomorphic results as follows. For every $v \in V$ there is a $V' \ni v$ such that the bundles E and E' of Σ and Σ' are trivial over V' so that over V' the families Σ and Σ' are simply (up to isomorphism) continuously varying triples of matrices $(F(v'),\ G(v'),\ H(v'))$, $(F'(v'),\ G'(v'),\ H'(v'))$, $v' \in V'$. Let α be a nice selection such that $R(F(v),G(v))_\alpha$ is invertible. Restricting V' a bit more if necessary we can assume that $R(F(v'),G(v'))_\alpha$ is invertible for all $v' \in V'$. Then $S(v') = R(F'(v'),G'(v'))_\alpha\ (R(F(v'),\ G(v'))_\alpha)^{-1}$ is a continuous family of invertible matrices taking $\Sigma(v')$ into $\Sigma'(v')$ for all $v' \in V'$. Thus Σ and Σ' are isomorphic over some small neighborhood of every point of V. The isomorphisms in question must agree on the intersections of these neighborhoods, again by the rigidity property. It follows that these local isomorphisms combine to define a global isomorphism over all of V from Σ to Σ'.

A more formal and also more formula based version of this argument can be found in [Haz1]. The scheme theoretic version (theorem 3.4.20) is based on the same rigidity property, cf section 4 below for some details.

3.4.22. Remark. In [HK] I claimed that the underlying bundle E^u of the universal family Σ^u was the pullback by means of $\tilde{R}$ (cf. (3.3.5)) of the bundle E over $G_{n,(n+1)m}$ whose fibre over z was the n-plane represented by z. As we have seen it is not; instead E^u is the pullback of the dual bundle E^d of E. Now the determinant bundle of E^d is a very ample line bundle (rather then the determinant bundle of E) so that the argument in [HK] to prove that $M_{m,n}$ is not quasi affine is correct modulo two errors which cancel each other.

4. THE CLASSIFYING "SPACE" $M^{cr}_{m,n,p}$ IS DEFINED OVER $\mathbb{Z}$ AND CLASSIFIES OVER $\mathbb{Z}$.

Mainly for completeness and tutorial reasons I give in this section the details algebraic-geometric details of the remarks 3.3.10 and 3.4.20 that there exists a scheme $M^{cr}_{m,n,p}$ over $\mathbb{Z}$ of which the varieties $M^{cr}_{m,n,p}(k)$, cf. 3.3.9, k an algebraically closed field, are obtained by base change and that this scheme is classifying for algebraic families of cr systems, and thus in particular classifying for cr systems over rings (with possibly a projective module as state module).

Those who are not particularly interested in the algebraic-geometric details can skip this section without consequences for their understanding of the remainder of this paper. There is in any case nothing difficult about what follows below and anyone who has once seen, say, the construction of the Grassmann schemes or projective spaces over $\mathbb{Z}$, will have no difficulty in supplying all details for himself from what has been said in section 3 above. All we are really doing below is rewriting a number of formulas of section 3 above using capital letters instead of small ones. This does take a certain number of pages, though. It seemed desirable to include these, as, judging from the audience's remarks during the oral presentation of these lectures, there is, perhaps rightly so, a distinct unwillingness in accepting without further proof a statement on the part of the lecturer like "the algebraic-geometric version of this theorem is proved similarly".

4.1. Definition of the scheme $M^{cr}_{m,n,p}$. For each nice selection $\alpha \subset J_{n,m}$ let

$$(4.1.1)\quad V_\alpha = \mathrm{Spec}(\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs};\ i = 1, \ldots, n,\ j = 1, \ldots, m,\ r = 1, \ldots, p,\ s = 1, \ldots, n])$$

Let $H_\alpha(Y)$ be the $p \times n$ matrix (Y^α_{rs}), and let $(F_\alpha(X), G_\alpha(X))$ be the unique pair of matrices over $\mathbb{Z}[X^\alpha_{ij}]$ such that

$$(4.1.2)\quad R(F_\alpha(X),G_\alpha(X))_\alpha = I_n,\quad R(F_\alpha(X),G_\alpha(X))_{s(\alpha,j)} = \begin{pmatrix} X^\alpha_{1j} \\ \vdots \\ X^\alpha_{nj} \end{pmatrix},\quad j = 1, \ldots, m$$

(where the $s(\alpha,j)$ are the m successor indices of α, cf. 3.2). Finally for each pair of nice selections α,β let $d_{\alpha\beta}(X) \in \mathbb{Z}[X^\alpha_{ij}]$ be the element

$$(4.1.3)\quad d_{\alpha\beta}(X) = \det(R(F_\alpha(X),G_\alpha(X))_\beta)$$

and let $V_{\alpha\beta}$ be the open subscheme of V_α obtained by localizing with respect to $d_{\alpha\beta}(X)$, i.e.

$$(4.1.4)\quad V_{\alpha\beta} = \mathrm{Spec}(\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, d_{\alpha\beta}(X)^{-1}])$$

Now for each pair of nice selections α,β write down the formulas

$$(4.1.5)\quad \begin{aligned} &S_{\alpha\beta}(X)^{-1}F_\alpha(X)S_{\alpha\beta}(X) = F_\beta(X) \\ &S_{\alpha\beta}(X)^{-1}G_\alpha(X) = G_\beta(X),\quad H_\alpha(Y)S_{\alpha\beta}(X) = H_\beta(Y) \end{aligned}$$

where

$$(4.1.6)\quad S_{\alpha\beta}(X) = R(F_\alpha(X),G_\alpha(X))_\beta$$

Because the entries of $F_\beta(X)$ and $G_\beta(X)$ are equal to zero, 1 or X^β_{ij} for some i,j and because the (r,s)-th entry of $H_\beta(Y)$ is Y^β_{rs}, the formulae (4.1.5) provide us with certain expressions for the X^β_{ij} and $Y^\beta_{r,s}$ in terms of the X_{ij}, Y_{rs}, which by (4.1.6), (4.1.5) and (4.1.3) (and the usual formula for matrix inversion) can be written as polynomials in X^{α}_{ij}, Y^{α}_{rs}, $d_{\alpha\beta}(X)^{-1}$, say

(4.1.7) $$X^{\beta}_{ij} = \phi_{\alpha\beta}(i,j)(X^{\alpha}_{ij}, d_{\alpha\beta}(X)^{-1}),\quad Y^{\beta}_{rs} = \phi_{\alpha\beta}(r,s)(X^{\alpha}_{ij}, d_{\alpha\beta}(X)^{-1}, Y^{\alpha}_{rs})$$

Then

(4.1.8) $$\phi^{*}_{\alpha\beta} : X^{\beta}_{ij} \mapsto \phi_{\alpha\beta}(i,j)(X^{\alpha}),\quad Y^{\beta}_{rs} \mapsto \phi_{\alpha\beta}(r,s)(X^{\alpha}, Y^{\alpha})$$

defines an isomorphism of rings.

$$\mathbb{Z}[X^{\beta}_{ij}, Y^{\beta}_{rs}, d_{\beta\alpha}(X)^{-1}] \simeq \mathbb{Z}[X^{\alpha}_{ij}, Y^{\alpha}_{rs}, d_{\alpha\beta}(X)^{-1}]$$

It follows from 4.1.5 that (with the obvious notations)

(4.1.9) $$\phi^{*}_{\alpha\beta}R(F_{\beta}(X), G_{\beta}(X)) = S_{\alpha\beta}(X)^{-1}R(F_{\alpha}(X), G_{\alpha}(X))$$
$$\phi^{*}_{\alpha\beta}H_{\beta}(Y) = H_{\alpha}(Y)S_{\alpha\beta}(X)$$

and these formulae describe $\phi^{*}_{\alpha\beta}$ completely. It follows that

$$\phi^{*}_{\alpha\beta}d_{\beta\alpha}(X) = \phi^{*}_{\alpha\beta}\det(R(F_{\beta}(X), G_{\beta}(X))_{\alpha}) = \det(S_{\alpha\beta}(X))^{-1} = d_{\alpha\beta}(X)^{-1}$$

so that $\phi^{*}_{\alpha\beta}$ does indeed map $d_{\beta\alpha}(X)^{-1}$ into $\mathbb{Z}[X^{\alpha}_{ij}, Y^{\alpha}_{rs}, d_{\alpha\beta}(X)^{-1}]$.

The $\phi^{*}_{\alpha\beta}$ induce isomorphisms of open subschemes

(4.1.10) $$\phi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$$

and $M^{cr}_{m,n,p}$ is now the scheme obtained by glueing together the schemes V_{α}, for all nice selections α, by means of the isomorphisms $\phi_{\alpha\beta}$.

As in section 3 above one can now embed $M^{cr}_{m,n,p}$ into a product of a Grassmannian over $\mathbb{Z}$ and an affine space over $\mathbb{Z}$ to see that $M^{cr}_{m,n,p}$ is a separated scheme.

For each nice selection α let V^{co}_{α} be the open subscheme of V_{α} defined by

(4.1.11) $$V^{co}_{\alpha} = \bigcup_{\gamma} \operatorname{Spec}(\mathbb{Z}[X^{\alpha}_{ij}, Y^{\alpha}_{rs}, Q(F_{\alpha}(X), H_{\alpha}(Y))^{-1}_{\gamma}]$$

where γ runs through all the nice selections of the set of row indices $J_{p,n}$ of $Q(F_{\alpha}(X), H_{\alpha}(Y))$. Then the $\phi_{\alpha\beta}$ restrict to give isomorphisms

(4.1.12) $$\phi^{co}_{\alpha\beta} : V^{co}_{\alpha\beta} \to V^{co}_{\beta\alpha}$$

where $V^{co}_{\alpha\beta} = V^{co}_{\alpha} \cap V_{\alpha\beta}$. Glueing together the V^{co}_{α} by means of the $\phi^{co}_{\alpha\beta}$ we obtain the open subscheme $M^{cr,co}_{m,n,p}$ of $M^{cr}_{m,n,p}$.

To see how all these abstract formulas look in concrete consider the case $m = 2$, $n = 2$, $p = 1$. In this case, there are three nice selections α, β, γ $J_{2,2}$, viz.

(4.1.13) $$\alpha = \{(0,1), (0,2)\},\quad \beta = \{(0,1), (1,1)\},\quad \gamma = \{(0,1), (1,2)\}$$

We have

$$F_{\alpha}(X) = \begin{pmatrix} X^{\alpha}_{11} & X^{\alpha}_{12} \\ X^{\alpha}_{21} & X^{\alpha}_{22} \end{pmatrix},\quad G_{\alpha}(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad H_{\alpha}(Y) = (Y^{\alpha}_{1}, Y^{\alpha}_{2})$$

$$F_{\beta}(X) = \begin{pmatrix} 0 & X^{\beta}_{11} \\ 1 & X^{\beta}_{21} \end{pmatrix},\quad G_{\beta}(X) = \begin{pmatrix} 1 & X^{\beta}_{12} \\ 0 & X^{\beta}_{22} \end{pmatrix},\quad H_{\beta}(Y) = (Y^{\beta}_{1}, Y^{\beta}_{2})$$

$$F_{\gamma}(X) = \begin{pmatrix} 0 & X^{\gamma}_{12} \\ 1 & X^{\gamma}_{22} \end{pmatrix},\quad G_{\gamma}(X) = \begin{pmatrix} X^{\gamma}_{11} & 1 \\ X^{\gamma}_{21} & 0 \end{pmatrix},\quad H_{\gamma}(Y) = (Y^{\gamma}_{1}, Y^{\gamma}_{2})$$

Thus

$$d_{\alpha\beta}(X) = X^{\alpha}_{21},\quad d_{\alpha\gamma}(X) = -X^{\alpha}_{12},\quad d_{\beta\gamma}(X) = X^{\beta}_{12}X^{\beta}_{12} + X^{\beta}_{12}X^{\beta}_{21}X^{\beta}_{22} - X^{\beta}_{11}X^{\beta}_{22}X^{\beta}_{22}$$

$$d_{\beta\alpha}(X) = X^{\beta}_{22},\quad d_{\gamma\alpha}(X) = -X^{\gamma}_{21},\quad d_{\gamma\beta}(X) = X^{\gamma}_{11}X^{\gamma}_{11} + X^{\gamma}_{11}X^{\gamma}_{22}X^{\gamma}_{21} - X^{\gamma}_{21}X^{\gamma}_{21}X^{\gamma}_{12}$$

$$S_{\alpha\beta}(X) = \begin{pmatrix} 1 & X^{\alpha}_{11} \\ 0 & X^{\alpha}_{21} \end{pmatrix}, \quad S_{\beta\alpha}(X) = \begin{pmatrix} 1 & X^{\beta}_{12} \\ 0 & X^{\beta}_{22} \end{pmatrix},$$

$$S_{\alpha\gamma}(X) = \begin{pmatrix} 0 & X^{\alpha}_{21} \\ 1 & X^{\alpha}_{22} \end{pmatrix}, \quad S_{\gamma\alpha}(X) = \begin{pmatrix} X^{\gamma}_{11} & 1 \\ X^{\gamma}_{21} & 0 \end{pmatrix},$$

$$S_{\beta\gamma}(X) = \begin{pmatrix} X^{\beta}_{12} & X^{\beta}_{11}X^{\beta}_{22} \\ X^{\beta}_{22} & X^{\beta}_{12}+X^{\beta}_{21}X^{\beta}_{22} \end{pmatrix}, \quad S_{\gamma\beta}(X) = \begin{pmatrix} X^{\gamma}_{11} & X^{\gamma}_{12}X^{\gamma}_{21} \\ X^{\gamma}_{21} & X^{\gamma}_{11}+X^{\gamma}_{22}X^{\gamma}_{21} \end{pmatrix}$$

Thus for example the two isomorphisms $\phi^*_{\alpha\beta}$ and $\phi^*_{\beta\alpha}$ are given by

$$\phi^*_{\alpha\beta} : \mathbb{Z}[X^{\beta}_{ij}, Y^{\beta}_{r}, (X^{\beta}_{22})^{-1}] \to \mathbb{Z}[X^{\alpha}_{ij}, Y^{\alpha}_{r}, (X^{\alpha}_{21})^{-1}]$$

$$X^{\beta}_{12} \mapsto -(X^{\alpha}_{21})^{-1}X^{\alpha}_{11}, \quad X^{\beta}_{22} \mapsto (X^{\alpha}_{21})^{-1}$$

$$X^{\beta}_{11} \mapsto X^{\alpha}_{12}X^{\alpha}_{21} - X^{\alpha}_{11}X^{\alpha}_{22}, \quad X^{\beta}_{21} \mapsto X^{\alpha}_{11} + X^{\alpha}_{22}$$

$$Y^{\beta}_{1} \mapsto Y^{\alpha}_{1}, \quad Y^{\beta}_{2} \mapsto (X^{\alpha}_{21})^{-1}Y^{\alpha}_{2} - (X^{\alpha}_{21})^{-1}X^{\alpha}_{11}Y^{\alpha}_{1}$$

$$\phi^*_{\beta\alpha} : \mathbb{Z}[X^{\alpha}_{ij}, Y^{\alpha}_{r}, (X^{\alpha}_{21})^{-1}] \to \mathbb{Z}[X^{\beta}_{ij}, Y^{\beta}_{r}, (X^{\beta}_{22})^{-1}]$$

$$X^{\alpha}_{11} \mapsto -(X^{\beta}_{22})^{-1}X^{\beta}_{12}, \quad X^{\alpha}_{21} \mapsto (X^{\beta}_{22})^{-1}$$

$$X^{\alpha}_{12} \mapsto X^{\beta}_{11}X^{\beta}_{22} - X^{\beta}_{12}X^{\beta}_{21} - (X^{\beta}_{22})^{-1}X^{\beta}_{12}X^{\beta}_{12}$$

$$X^{\alpha}_{22} \mapsto (X^{\beta}_{22})^{-1}X^{\beta}_{12} + X^{\beta}_{21}$$

$$Y^{\alpha}_{1} \mapsto Y^{\beta}_{1}, \quad Y^{\alpha}_{2} \mapsto (X^{\beta}_{22})^{-1}Y^{\beta}_{2} - (X^{\beta}_{22})^{-1}Y^{\beta}_{1}X^{\beta}_{12}$$

and one checks without trouble that indeed $d_{\beta\alpha}(X)^{-1} = (X^{\beta}_{22})^{-1}$ gets mapped into $\mathbb{Z}[X^{\alpha}_{ij}, Y^{\alpha}_{r}, (X^{\alpha}_{21})^{-1}]$ and $d_{\alpha\beta}(X)^{-1} = (X^{\alpha}_{21})^{-1}$ into $\mathbb{Z}[X^{\beta}_{ij}, Y^{\beta}_{r}, (X^{\beta}_{22})^{-1}]$ and that indeed $\phi^*_{\alpha\beta} \circ \phi^*_{\beta\alpha} = \mathrm{id}$, $\phi^*_{\beta\alpha} \circ \phi^*_{\alpha\beta} = \mathrm{id}$. (The formulas are not always so simple; for instance the formulas for $\phi^*_{\beta\gamma}$ and $\phi^*_{\gamma\beta}$ are a good deal more complicated).

4.2. Small Intermezzo: Completely reachable systems over a ring.

A system $\Sigma=(F,G,H)$ over a ring R is said to be completely reachable if $R(F,G)\colon R^r \to R^n$, $r = (n+1)m$ is a surjective map, cf. e.g. [Sol] or [Rou]. This is equivalent to each element of the family $\Sigma(\mathfrak{p}) = (F(\mathfrak{p}),G(\mathfrak{p}),H(\mathfrak{p}))$, $\mathfrak{p} \in \mathrm{Spec}(R)$ being completely reachable. Indeed $R(F,G)\colon R^r \to R^m$ is surjective if it is surjective mod every maximal ideal [Bou, Ch.II, §3.3, Prop.11] and the statement follows.

4.3. The algebraic geometric version of the nice selection lemma.

The next thing to do is to discuss the algebraic-geometric version of the nice selection lemma, 3.2.3. Recall that this lemma says that if the system (F,G,H) over a field k is cr then there is a nice selection α such that $R(F,G)_{\alpha}$ is invertible. Now let (F,G,H) be a cr system over a ring R, which per definition means that $R(F,G)\colon R^r \to R^n$, $r = (n+1)m$, is surjective, which in turn is equivalent to condition that the systems $\Sigma(\mathfrak{p}) = (F(\mathfrak{p}),G(\mathfrak{p}),H(\mathfrak{p}))$ over $k(\mathfrak{p})$, the quotient field of $R/\mathfrak{p}$, are cr for all prime ideals $\mathfrak{p}$. Then of course one does not expect the existence of a nice selection α such that $R(F,G)_{\alpha}$ is an invertible matrix over R; after all $\Sigma = (F,G,H)$ should be interpreted as a family and not as a single system.

For a continuous topological family $\Sigma(\sigma)$ over a topological space M the nice selection lemma implies that there is a finite covering $M = \cup U_{\alpha}$ such that for all $\sigma \in U_{\alpha}$, $R(F(\sigma),G(\sigma))_{\alpha}$ is invertible. And this property generalizes nicely.

4.3.1. Lemma. Let $\Sigma = (F,G,H)$ be a cr system over a ring R. For each nice selection α let $d_{\alpha} = \det(R(F,G)_{\alpha})$. Then the ideal generated by the d_{α} is the whole ring R. (This means of course that the $U_{\alpha} = \mathrm{Spec}(R[d_{\alpha}^{-1}])$ cover all of $\mathrm{Spec}(R)$).

Proof. Let I be the ideal generated by the d_{α}, α nice. Suppose that $I \neq R$. Then there is a maximal ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. Consider $\Sigma(\mathfrak{m}) = (F(\mathfrak{m}),G(\mathfrak{m}),H(\mathfrak{m}))$. Then $\det(R(\Sigma(\mathfrak{m}))_{\alpha}) = 0$ in $R/\mathfrak{m}$ for all α, showing that $\Sigma(\mathfrak{m})$ is not cr(by the old nice selection lemma 3.2.3 over the field $R/\mathfrak{m}$) which contradicts the assumption that Σ was cr.

To state the more global version of this lemma we need a bit of notation Let Σ be a family of cr systems over a scheme V. For each nice selection α we define

(4.3.2) $$U_{\alpha} = \{v \in V \mid \det(R(\Sigma(v))_{\alpha}) \neq 0\}$$

This definition seems a bit ambiguous at first because $R(\Sigma(v))$ depends on what basis we choose in the state space of $\Sigma(v)$ and hence is only defined up to multiplication on the left by an $n \times n$ invertible matrix with coefficients in $k(v)$. This matrix being invertible, however, means that the whole symbol group $\det(R(\Sigma(v))_\alpha) \neq 0$ makes perfectly good sense so that U_α is welldefined. Of course U_α is an open subscheme of V.

4.3.3. Lemma. Let Σ be a family of cr systems over a scheme V. For each nice selection α let U_α be as in (4.3.2). Then $\bigcup_{\alpha \text{ nice}} U_\alpha = V$.

This follows immediately from lemma 4.3.1 because V can be covered with affine schemes $\mathrm{Spec}(R_i)$ (such that moreover the underlying bundle of Σ is trivial over each $\mathrm{Spec}(R_i)$).

4.4. The universal bundle E^u over $M^{cr}_{m,n,p}$. The universal bundle E^u over $M^{cr}_{m,n,p}$ is constructed just as in 3.4.11 above. Writing things out in relentness detail one obtains the following algebraic-geometric local pieces and patching data description.

For each nice selection α let

(4.4.1) $$E_\alpha = \mathrm{Spec}(\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}] \otimes \mathbb{Z}[Z^\alpha_1, \ldots, Z^\alpha_n]) = V_\alpha \times \mathbb{A}^n$$

where $\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}]$ is as in 4.1.1; i.e. $\mathrm{Spec}\ \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}] = V_\alpha$. Let

(4.4.2) $$\pi_\alpha : E_\alpha \to V_\alpha$$

be the projection induced by the natural inclusion $\pi^*_\alpha : \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{r,s}] \subset \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{r,s}, Z^\alpha_t]$.

Define for each pair of nice selections α, β.

(4.4.3) $$E_{\alpha\beta} = \mathrm{Spec}\ \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{r,s}, Z^\alpha_t, d_{\alpha\beta}(X)^{-1}]) = V_{\alpha\beta} \times \mathbb{A}^n$$

and let

(4.4.4) $$\varphi_{\alpha\beta} : E_{\alpha\beta} \to E_{\beta\alpha}$$

be the isomorphism given by the ring isomorphism

(4.4.5) $$\varphi^*_{\alpha\beta} : \mathbb{Z}[X^\beta_{ij}, Y^\beta_{rs}, Z^\beta_t, d_{\beta\alpha}(X)^{-1}] \to \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t, d_{\alpha\beta}(X)^{-1}]$$

given by

(4.4.6) $$X^\beta_{ij} \mapsto \phi_{\alpha\beta}(i,j)(X^\alpha), Y^\beta_{r,s} \mapsto \phi_{\alpha\beta}(r,s)(X^\alpha, Y^\beta),\ Z^\beta_t \mapsto \varphi_{\alpha\beta}(t)(X^\alpha, Z^\alpha)$$

where the $\varphi_{\alpha\beta}(t)(X^\alpha, Z^\alpha)$ are defined by the equality

(4.4.7) $$\begin{pmatrix} \varphi_{\alpha\beta}(1)(X^\alpha, Z^\alpha) \\ \vdots \\ \varphi_{\alpha\beta}(n)(X^\alpha, Z^\alpha) \end{pmatrix} = S_{\alpha\beta}(X)^{-1} \begin{pmatrix} Z^\alpha_1 \\ \vdots \\ Z^\alpha_n \end{pmatrix}$$

The $\varphi_{\alpha\beta}$ are compatible (by their definition) with the $\phi_{\alpha\beta}$ in that the following diagram commutes for each pair of nice selections α, β.

(4.4.8) $$\begin{array}{ccc} E_{\alpha\beta} & \xrightarrow[\varphi_{\alpha\beta}]{\sim} & E_{\beta\alpha} \\ \downarrow \pi_\alpha & & \downarrow \pi_\beta \\ V_{\alpha\beta} & \xrightarrow[\phi_{\alpha\beta}]{\sim} & V_{\beta\alpha} \end{array}$$

It follows that by glueing the E_α together by means of the $\varphi_{\alpha\beta}$ we obtain a vectorbundle E^u.

(4.4.9) $$\pi : E^u \to M^{cr}_{m,n,p}$$

4.5. The morphism into $M^{cr}_{m,n,p}$ associated to an algebraic family of cr systems.
We start with the case that the underlying vectorbundle E of the family Σ is trivial and that the parametrizing scheme V is affine. Σ is then described by a ring R, $V = \mathrm{Spec}(R)$, $E = \mathrm{Spec}(R[Z_1, \ldots, Z_n]$, $\pi : E \to V$ induced by the natural inclusion $R \to R[Z_1, \ldots, Z_n]$, and vectorbundle homomorphisms $F: E \to E$, $G: \mathrm{Spec}(R[U_1, \ldots, U_m]) \to E$, $H: E \to \mathrm{Spec}(R[Y_1, \ldots, Y_p])$. The fact that these morphisms are vectorbundle homomorphisms is reflected by the fact that the associated homomorphisms of rings $F^*: R[Z_1, \ldots, Z_n] \to R[Z_1, \ldots, Z_n]$, $G^*: R[Z_1, \ldots, Z_n] \to R[U_1, \ldots, U_m]$, $H^*: R[Y_1, \ldots, Y] \to R[Z_1, \ldots, Z_n]$ are firstly R-algebra homomorphisms and further of the form

(4.5.1) $$F^*(Z_i) = \sum_{j=1}^{n} f_{ij} Z_j,\ G^*(Z_i) = \sum_{j=1}^{m} g_{ij} U_j,\ H^*(Y_i) = \sum_{j=1}^{n} h_{ij} Z_j$$

where the f_{ij}, g_{ij}, h_{ij} are elements of R. This defines a triple of matrices $\bar{F} = (f_{ij})$, $\bar{G} = (g_{ij})$, $\bar{H} = (h_{ij})$. For each nice selection α let $S_\alpha = R(\bar{F},\bar{G})_\alpha$, $d_\alpha = \det(S_\alpha) \in R$, let $U_\alpha = \mathrm{Spec}(R[d_\alpha^{-1}])$, and let $V_\alpha = \mathrm{Spec}(\mathbb{Z}[X_{ij}^\alpha, Y_{rs}^\alpha])$ be "the nice-selection-α-piece of $M_{m,n,p}^{cr}$" of 4.1 above. Now define

$$\psi_\alpha : U_\alpha \to V_\alpha \tag{4.5.2}$$

by the morphism of rings

$$\psi_\alpha^* : \mathbb{Z}[X_{ij}^\alpha, Y_{rs}^\alpha] \to R[d_\alpha^{-1}] \tag{4.5.3}$$

given by

$$\begin{array}{l} X_{ij}^\alpha \mapsto \text{i-th entry of the column vector } S_\alpha^{-1}R(\bar{F},\bar{G})_{s(\alpha,j)} \\ Y_{rs}^\alpha \mapsto \text{r-th entry of the column s of the matrix } \bar{H}S_\alpha \end{array} \tag{4.5.4}$$

where $s(\alpha,j)$ is the j-th successor index of the nice selection α, cf. 3.2 above.
Or, using the obvious notation, ψ_α^* is defined by

$$\psi_\alpha^*(R(F_\alpha(X),G_\alpha(X)) = S_\alpha^{-1}R(\bar{F},\bar{G}),\ \psi_\alpha^*\, H_\alpha(Y) = \bar{H}S_\alpha \tag{4.5.5}$$

Now let β be a second nice selection. We claim that the ψ_α and ψ_β agree on $U_\alpha \cap U_\beta = \mathrm{Spec}(R[d_\alpha^{-1}, d_\beta^{-1}])$. In view of how the V_α, V_β are glued together to obtain $M_{m,n,p}^{cr}$ this means that we must prove the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{Z}[X_{ij}^\alpha, Y_{rs}^\alpha, d_{\alpha\beta}(X)^{-1}] & \xrightarrow{\psi_\alpha^*} & \\ \uparrow \phi_{\alpha\beta}^* & & R[d_\alpha^{-1}, d_\beta^{-1}] \\ \mathbb{Z}[X_{ij}^\beta, Y_{rs}^\beta, d_{\beta\alpha}(X)^{-1}] & \xrightarrow{\psi_\beta^*} & \end{array} \tag{4.5.6}$$

Note first that

$$\psi_\alpha^*(S_{\alpha\beta}(X)) = \psi_\alpha^*(R(F_\alpha(X),G_\alpha(Y)_\beta) = S_\alpha^{-1}R(\bar{F},\bar{G})_\beta = S_\alpha^{-1}S_\beta \tag{4.5.7}$$

so that ψ_α^* does indeed map $d_{\alpha\beta}(X)^{-1}$ into $R[d_\alpha^{-1}, d_\beta^{-1}]$. Now ψ_β^* is described by

$$\psi_\beta^*(R(F_\beta(X),G_\beta(X)) = S_\beta^{-1}R(\bar{F},\bar{G}),\ \psi_\beta^* H_\beta(Y) = \bar{H}S_\beta \tag{4.5.8}$$

and on the other hand

$$\begin{aligned} \psi_\alpha^*\phi_{\alpha\beta}^*\, R(F_\beta(X),G_\beta(X)) &= \psi_\alpha^*(S_{\alpha\beta}(X)^{-1}R(F_\alpha(X),G_\alpha(X)) && \text{(by (4.1.9))} \\ &= S_\beta^{-1}S_\alpha S_\alpha^{-1}R(\bar{F},\bar{G}) && \text{(by (4.5.7) and (4.5.5))} \\ &= S_\beta^{-1}R(\bar{F},\bar{G}) \end{aligned}$$

which fits perfectly with (4.5.8). Similarly $\psi_\alpha^*\phi_{\alpha\beta}^* H_\beta(X) = \psi_\alpha^* H_\alpha(Y) S_{\alpha\beta}(X)$ $= \bar{H}S_\alpha S_\alpha^{-1}S_\beta = \bar{H}S_\beta = \psi_\beta^* H_\beta(Y)$, so that (4.5.6) indeed commutative. Thus the $\psi_\alpha: U_\alpha \to V_\alpha$ are compatible, and because $\bigcup_{\alpha \text{ nice}} U_\alpha = \mathrm{Spec}(R)$ we obtain a morphism of schemes

$$\psi_\Sigma : V = \mathrm{Spec}(R) \to M_{m,n,p}^{cr}$$

4.5.9. <u>Lemma</u>. The morphism ψ_Σ depends only on the isomorphism class of Σ (so in particular ψ_Σ does not depend on how E is trivialized).

Proof. Let Σ' be a second family of cr systems over $V = \mathrm{Spec}(R)$ with trivial underlying vectorbundle $E' = \mathrm{Spec}(R[Z_1', Z_2', \ldots, Z_n'])$. Suppose Σ' is isomorphic to Σ and let the isomorphism be $\mu : E \to E'$. Because μ is a morphism of vectorbundles over $V = \mathrm{Spec}(R)$ its ring homomorphism

$$\mu^* : R[Z_1', Z_2', \ldots, Z_n'] \to R[Z_1, Z_2, \ldots, Z_n]$$

is an R-algebra homomorphism of the form

$$\mu^*(Z_i') = \sum_{j=1}^{n} s_{ij}Z_j, \quad s_{ij} \in R$$

Let S be the matrix (s_{ij}). Then S is invertible (over R) because μ is an isomorphism. Now because μ defines an isomorphism $\Sigma' \simeq \Sigma$ we have $F'\mu = \mu F$, $\mu G = G'$, $H = H'\mu$ which in terms of the matrices $\bar{F},\bar{G},\bar{H}$ associated to Σ (cf. (4.5.1) above) and the analogous matrices

$\bar F', \bar G', \bar H'$ of Σ' means that

$$S\bar F = \bar F' S,\quad S\bar G = \bar G',\quad \bar H = \bar H' S$$

It follows that if d'_α, S'_α, U'_α are defined analogously to d_α, S_α, U_α then $S'_\alpha = SS_\alpha$, $d'_\alpha = \det(S)d_\alpha$ so that $U'_\alpha = U_\alpha$ and $\psi'_\alpha = \psi_\alpha$ all because $SR(\bar F,\bar G) = R(\bar F',\bar G')$, $\bar H S^{-1} = \bar H'$, which proves the lemma.

4.5.10. <u>Construction of ψ_Σ for families whose underlying bundle is not necessarily trivial</u>.

Now let $\Sigma = (E;\ F,G,H)$ be a family of cr systems over a scheme V. We can cover V with affine pieces $U_i = \mathrm{Spec}(R_i)$ such that E is trivializable over U_i. By the construction above and lemma 4.5.9 this gives us morphisms (independant of the trivialization chosen)

$$\psi_i : U_i \to M^{cr}_{m,n,p}$$

Now on $U_i \cap U_j$ the ψ_i and ψ_j must agree, because by lemma 4.5.9 again ψ_i and ψ_j agree on all affine pieces $\mathrm{Spec}(R) \subset U_i \cap U_j$. Hence the ψ_i combine to define a morphism

$$\psi_\Sigma : V \to M^{cr}_{m,n,p}$$

which, again by lemma 4.5.9 depends only on the isomorphism class of Σ.

4.6. <u>The universal family Σ^u of cr systems over $M^{cr}_{m,n,p}$</u>. Let E^u be the vectorbundle over $M^{cr}_{m,n,p}$ constructed in 4.4 above. In this section I describe a (universal) family of cr systems over $M^{cr}_{m,n,p}$ whose underlying bundle is E^u. (That this family is indeed universal will be proved in 4.7 below).

Recall that E^u was constructed out of affine pieces $E_\alpha = \mathrm{Spec}(\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t])$ glued together by means of certain isomorphisms $\tilde\phi_{\alpha\beta}$, cf. 4.4. Let $\mathbb{A}^r = \mathrm{Spec}(\mathbb{Z}[U_1,\dots,U_r])$. To define $\Sigma^u = (E^u; F^u, G^u, H^u)$ it suffices to define vectorbundle homomorphisms

$$(4.6.1)\qquad F_\alpha : E_\alpha \to E_\alpha,\quad G_\alpha : V_\alpha \times \mathbb{A}^m \to E_\alpha,\quad H_\alpha : E_\alpha \to V_\alpha \times \mathbb{A}^p$$

which are compatible with the identifications

$\tilde\phi_{\alpha\beta} : E_{\alpha\beta} \to E_{\beta\alpha}$, $\phi_{\alpha\beta}\times \mathrm{id} : V_{\alpha\beta}\times \mathbb{A}^m \to V_{\beta\alpha}\times\mathbb{A}^m$, $\phi_{\alpha\beta}\times \mathrm{id} : V_{\alpha\beta}\times\mathbb{A}^p \to V_{\beta\alpha}\times\mathbb{A}^p$ in the sense that the following diagram must be commutative

$$(4.6.2)\qquad \begin{array}{ccccccc} V_{\alpha\beta}\times \mathbb{A}^m & \xrightarrow{G_\alpha} & E_{\alpha\beta} & \xrightarrow{F_\alpha} & E_{\alpha\beta} & \xrightarrow{H_\alpha} & V_{\alpha\beta}\times \mathbb{A}^p \\ \downarrow{\scriptstyle \phi_{\alpha\beta}\times \mathrm{id}} & & \downarrow{\scriptstyle \tilde\phi_{\alpha\beta}} & & \downarrow{\scriptstyle \tilde\phi_{\alpha\beta}} & & \downarrow{\scriptstyle \phi_{\alpha\beta}\times \mathrm{id}} \\ V_{\beta\alpha}\times \mathbb{A}^m & \xrightarrow{G_\beta} & E_{\beta\alpha} & \xrightarrow{F_\beta} & E_{\beta\alpha} & \xrightarrow{H_\beta} & V_{\beta\alpha}\times \mathbb{A}^p \end{array}$$

(cf. also (3.4.17)). We now describe $F_\alpha, G_\alpha, H_\alpha$ as those morphisms which on the ring level are given by the $\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}]$ - algebra homomorphisms

$$(4.6.3)\qquad F^*_\alpha : \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t] \to \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t],\quad Z^\alpha \mapsto F_\alpha(X)Z^\alpha$$

$$(4.6.4)\qquad G^*_\alpha : \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t] \to \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, U_1,\dots,U_m],\quad Z^\alpha \mapsto G_\alpha(X)U$$

$$(4.6.5)\qquad H^*_\alpha : \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, V_1,\dots,V_p] \to \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t],\quad V \mapsto H_\alpha(Y)Z^\alpha$$

where Z^α, U, V are respectively the column vectors $(Z^\alpha_1,\dots,Z^\alpha_n)^t$, $(U_1,\dots,U_m)^t$, $(V_1,\dots,V_p)^t$.

It remains to check that the diagram (4.6.2) is indeed commutative, which is done by checking that the dual diagram of rings homomorphisms is commutative.

This comes down to precisely the same calculations as in 3.4.14. As an example we check that the diagram

$$\begin{array}{ccc} \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t, d_{\alpha\beta}(X)^{-1}] & \xleftarrow{F^*_\alpha} & \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, Z^\alpha_t, d_{\alpha\beta}(X)^{-1}] \\ \uparrow{\scriptstyle \tilde\phi^*_{\alpha\beta}} & & \uparrow{\scriptstyle \tilde\phi^*_{\alpha\beta}} \\ \mathbb{Z}[X^\beta_{ij}, Y^\beta_{rs}, Z^\beta_t, d_{\beta\alpha}(X)^{-1}] & \xleftarrow{F^*_\beta} & \mathbb{Z}[X^\beta_{ij}, Y^\beta_{rs}, Z^\beta_t, d_{\beta\alpha}(X)^{-1}] \end{array}$$

is commutative. Because $\phi^*_{\alpha\beta}$ maps $\mathbb{Z}[X^\beta_{ij}, Y^\beta_{rs}, d_{\beta\alpha}(X)^{-1}]$ into $\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}, d_{\alpha\beta}(X)^{-1}]$ and because F^*_α and F^*_β are respectively $\mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}]$-algebra and $\mathbb{Z}[X^\beta_{ij}, Y^\beta_{rs}]$ - algebra homomorphisms it suffices to check that

$$\tilde{\phi}^*_{\alpha\beta} F^*_\beta(z^\beta_t) = F^*_\alpha \tilde{\phi}^*_{\alpha\beta}(z^\beta_t), \quad t = 1, \ldots, n$$

By the definitions (4.4.7), (4.6.3) and using the definition of $\phi^*_{\alpha\beta}$, cf. 4.1, we have

$$F^*_\alpha \tilde{\phi}^*_{\alpha\beta}(z^\beta) = F^*_\alpha(S_{\alpha\beta}(X)^{-1}z^\alpha) = S_{\alpha\beta}(X)^{-1}F_\alpha(X)z^\alpha$$

$$\tilde{\phi}^*_{\alpha\beta}F^*_\beta(z^\beta) = \tilde{\phi}^*_{\alpha\beta}(F_\beta(X)z^\beta) = \phi^*_{\alpha\beta}(F_\beta(X))S^{-1}_{\alpha\beta}z^\alpha$$

$$= S_{\alpha\beta}(X)^{-1}F_\alpha(X)S_{\alpha\beta}(X)S_{\alpha\beta}(X)^{-1}z^\alpha = S_{\alpha\beta}(X)^{-1}F_\alpha(X)z^\alpha$$

The remaining two squares of diagram (4.6.2) are similarly shown to be commutative.

4.7. A rigidity lemma.

The key to the proof of theorem 3.4.20 (the algebraic-geometric classifying theorem) is (as was remarked before) a rigidity property which in this context takes the following form.

4.7.1. Proposition. Let Σ, Σ' be two families of cr systems over a scheme V. Suppose that there is a covering by open subschemes (U_i) of V such that the two families Σ and Σ' restricted to U_i are isomorphic for all i. Then Σ and Σ' are isomorphic as algebraic families over V.

We note that no such proposition holds for arbitrary families of systems cf. [HP] for a counterexample.

Proof. We can assume that the underlying vectorbundles E and E' have been obtained by glueing together trivial pieces over affine subschemes of V. Refining the covering (U_i) if necessary (this does not change the validity of the hypothesis of the proposition) we can therefore assume that E and E' have been obtained by glueing together trivial bundles $U_i \times A^n$ over affine schemes U_i.

Our data are then as follows. We have for each i an affine scheme $U_i = \mathrm{Spec}(R_i)$ and for each i,j isomorphisms of (trivial) bundles

$$\phi_{ij}, \phi'_{ij} : (U_i \cap U_j) \times A^n \to (U_j \cap U_i) \times A^n$$

which respectively define the bundles E and E'. The remaining ingredients of the two families of systems Σ and Σ' are then given by vectorbundle homomorphisms

$$F_i, F'_i : U_i \times \mathbf{A}^n \to U_i \times \mathbf{A}^n, \quad G_i, G'_i : U_i \times \mathbf{A}^m \to U_i \times \mathbf{A}^n$$
(4.7.1)
$$H_i, H'_i : U_i \times \mathbf{A}^n \to U_i \times \mathbf{A}^p$$

such that the following diagrams are commutative for all i,j (where U_{ij} is short for $U_i \cap U_j$)

(4.7.2)
$$\begin{array}{ccccccc}
 & \overset{G_i,G'_i}{\nearrow} & U_{ij}\times\mathbf{A}^n & \xrightarrow{F_i,F'_i} & U_{ij}\times\mathbf{A}^n & \overset{H_i,H'_i}{\searrow} & \\
U_{ij}\times\mathbf{A}^m & & \downarrow \phi_{ij},\phi'_{ij} & & \downarrow \phi_{ij},\phi'_{ij} & & U_{ij}\times\mathbf{A}^p \\
 & \underset{G_j,G'_j}{\searrow} & U_{ij}\times\mathbf{A}^n & \xrightarrow{F_j,F'_j} & U_{ij}\times\mathbf{A}^n & \underset{H_j,H'_j}{\nearrow} &
\end{array}$$

Finally the fact that Σ and Σ' are isomorphic over each U_i means that there are vectorbundle isomorphisms $\phi_i : U_i \times \mathbf{A}^n \to U_i \times \mathbf{A}^n$ such that the following diagram is commutative for all i

(4.7.3)
$$\begin{array}{ccccccc}
 & \overset{G_i}{\nearrow} & U_i\times A^n & \xrightarrow{F_i} & U_i\times A^n & \overset{H_i}{\searrow} & \\
U_i\times A^m & & \downarrow \phi_i & & \downarrow \phi_i & & U_i\times A^p \\
 & \underset{G'_i}{\searrow} & U_i\times A^n & \xrightarrow{F'_i} & U_i\times A^n & \underset{H'_i}{\nearrow} &
\end{array}$$

We now claim that the ϕ_i are compatible and combine to define an isomorphism $\phi : E \to E'$ (it then follows, because this is locally true, that $\phi F = F'\phi$, $\phi G = G'$, $H'\phi = H$). To prove this we must show that for each $\mathrm{Spec}(R) = U \subset U_{ij} = U_i \cap U_j$ the following diagram commutes

(4.7.4)
$$\begin{array}{ccc}
U \times A^n & \xrightarrow{\phi_{ij}} & U \times A^n \\
\downarrow \phi_i & & \downarrow \phi_j \\
U \times A^n & \xrightarrow{\phi'_{ij}} & U \times A^n
\end{array}$$

Now vectorbundle homomorphisms of trivial vectorbundles over an affine scheme $U = \mathrm{Spec}(R)$ are given by matrices with coefficients in R as we explained en passant in the first few paragraphs of 4.5 above. Let $\bar{G}_i, \bar{G}'_i, \bar{F}_i, \bar{F}'_i, \bar{H}_i, \bar{H}'_i, S_{ij}, S'_{ij}, S_i, S_j$ be the matrices of the morphisms of vectorbundles $G_i, G'_i, F_i, F'_i, H_i, H'_i, \phi_{ij}, \phi'_{ij}, \phi_i, \phi_j$ restricted to U. The commutativity relations (4.7.2) and (4.7.3) then imply for these matrices with coefficients in R that

$$S_{ij}\bar{G}_i = \bar{G}_j,\ S_{ij}\bar{G}'_i = \bar{G}'_j,\ S_{ij}\bar{F}_i = \bar{F}_jS_{ij},\ S_{ij}\bar{F}'_i = \bar{F}'_jS'_{ij},$$

$$(4.7.5)\qquad S_{ij}\bar{H}_i = \bar{H}_j,\ S'_{ij}\bar{H}'_i = \bar{H}'_j,\ S_i\bar{G}_i = \bar{G}'_i,\ S_i\bar{F}_i = \bar{F}'_iS_i,\ \bar{H}'_iS_i = \bar{H}_i,$$

$$S_j\bar{G}_j = \bar{G}'_j,\ S_j\bar{F}_j = \bar{F}'_jS_j,\ \bar{H}'_jS_j = \bar{H}_j$$

and the matrices $S_i, S_j, S_{ij}, S'_{ij}$ are all invertible because they come from vectorbundle isomorphisms.
It follows that

$$(4.7.6)\qquad \begin{aligned} S_jS_{ij}R(\bar{F}_i,\bar{G}_i) &= S_jR(\bar{F}_j,\bar{G}_j) = R(\bar{F}'_j,\bar{G}'_j) \\ &= S'_{ij}R(\bar{F}'_i,\bar{G}'_i) = S'_{ij}S_iR(\bar{F}_i,\bar{G}_i) \end{aligned}$$

Now Σ is a family of cr systems and hence so is its restriction to $U = \mathrm{Spec}(R)$. It follows (cf. 4.2 above) that $R(\bar{F}_i,\bar{G}_i)\colon R^r \to R^n$, $r = (n+1)m$, is a surjective map. Hence (4.7.6) implies that $S_jS_{ij} = S'_{ij}S_i$ proving the commutativity of (4.7.4) and hence the proposition.

4.8. <u>On the pullback construction</u>. Let $\Sigma = (E; F,G,H)$ be a family of systems over a scheme M and let $\psi : V' \to M$ be a morphism of schemes. Assume that everything is given in terms of local affine pieces and patching data; i.e. Σ is given by trivial bundles $U_i \times \mathbb{A}^n \to U_i = \mathrm{Spec}(R_i)$ with vectorbundle isomorphisms $\phi_{ij}\colon U_{ij} \times \mathbb{A}^n \to U_{ij} \times \mathbb{A}^n$ and vector bundle morphisms $F_i\colon U_i \times \mathbb{A}^n \to U_i \times \mathbb{A}^n$, $G_i\colon U_i \times \mathbb{A}^m \to U_i \times \mathbb{A}^n$, $H_i\colon U_i \times \mathbb{A}^n \to U_i \times \mathbb{A}^p$ such the nonprime diagram (4.7.2) is commutative, and ψ is given by affine morphisms $\psi_i\colon U'_i \to U_i$, $U'_i = \mathrm{Spec}(R'_i)$. Let $\psi^*_i\colon R_i \to R'_i$ be the ring homomorphism of ψ_i. Let, as before, $\bar{F}_i, \bar{G}_i, \bar{H}_i$ be the matrices of the vectorbundle morphisms F_i, G_i, H_i.

Then the local pieces of the pullback family $\psi^!\Sigma = \Sigma'$ are: the trivial bundles $U'_i \times \mathbb{A}^n \to U'_i$ with the vectorbundle homomorphisms $F'_i\colon U'_i \times \mathbb{A}^n \to U'_i \times \mathbb{A}^n$, $G'_i\colon U'_i \times \mathbb{A}^m \to U'_i \times \mathbb{A}^n$, $H'_i\colon U'_i \times \mathbb{A}^n \to U'_i \times \mathbb{A}^p$ given by the matrices $\bar{F}'_i = \psi^*_iF_i$, $G'_i = \psi^*_iG_i$, $H'_i = \psi^*_iH_i$. The patching data are defined as follows. If $U' = \mathrm{Spec}(R') \subset U'_i \cap U'_j$ maps into $U = \mathrm{Spec}(R)$ $U_i \cap U_j$ under ψ and $\psi^*_1\colon R \to R'$ is the associated map homomorphism of rings, then over Spec(R') the isomorphism $\phi'_{ij}\colon U' \times \mathbb{A}^n \to U' \times \mathbb{A}^n$ is given by the matrix $S'_{ij} = \psi^*_1S_{ij}$ if S_{ij} is the matrix of $\phi_{ij}\colon U \times \mathbb{A}^n \to U \times \mathbb{A}^n$.

This can be taken as the definition of the pullback family $\psi^!\Sigma$. It agrees of course with the more informal description given in section 3 above.

4.9. <u>The classifying theorem for algebraic families of cr systems over schemes</u>.

($M^{cr}_{m,n,p}$ <u>is classifying over</u> $\mathbb{Z}$). We can now prove the algebraic-geometric classifying theorem for families of cr systems, i.e. theorem 3.4.20. Stated more precisely this theorem says

4.9.1. <u>Theorem</u>. Let Σ be an algebraic family of cr systems over a scheme V. Then there exists a unique morphism of schemes $\psi_\Sigma\colon V \to M^{cr}_{m,n,p}$ (defined in 4.5 above) such that $\psi^!_\Sigma\Sigma^u = \Sigma$ where Σ^u is the universal family constructed in section 4.6 above. That is the map $\Sigma \mapsto \psi_\Sigma$ and the map $\psi \mapsto \psi^!\Sigma^u$ (of 4.8 above) set up a bijective correspondence between the set of scheme morphisms $V \to M^{cr}_{m,n,p}$ and isomorphism classes of families of cr systems over V. Moreover this isomorphism is functorial.

Proof. First let $\psi\colon V \to M^{cr}_{m,n,p}$ be a morphism of schemes, let $\Sigma = \phi^!\Sigma^u$. Then we must show that $\psi_\Sigma = \Sigma$. To do this it suffices to show that ψ_Σ and ψ agree on all elements of some affine covering (U_i) of V. We can take this covering to be finer than the covering $(\psi^{-1}(V_\alpha), \alpha \text{ nice})$ where $V_\alpha \subset M^{cr}_{m,n,p}$ is the piece belonging to the nice selection α, cf. 4.1. Let therefore $U = \mathrm{Spec}(R)$ be such that $\psi(U) \subset V_\alpha$, and let

$$\psi^* : \mathbb{Z}[X^\alpha_{ij}, Y^\alpha_{rs}] \to R$$

be the associated ring homomorphism. Then according to 4.8 above and the definition of Σ^u, cf. 4.6, the family Σ over U is described by the three matrices

$$\bar{F} = \psi^* F_\alpha(X),\ \bar{G} = \psi^* G_\alpha(X),\ \bar{H} = \psi^* H_\alpha(Y) \tag{4.9.2}$$

By 4.5 above the morphism $\psi_\Sigma^* : \mathbb{Z}[X_{ij}^\alpha, Y_{rs}^\alpha] \to R$ associated to this family is characterized by

$$\psi_\Sigma^*(R(F_\alpha(X),\ G_\alpha(X)) = S_\alpha^{-1} R(\bar{F},\bar{G}),\ \psi_\alpha^* H_\alpha(Y) = \bar{H} S_\alpha \tag{4.9.3}$$

where $S_\alpha = R(\bar{F},\bar{G})_\alpha$. Because $R(F_\alpha(X),\ G_\alpha(X))_\alpha = I_n$, $S_\alpha = I_n$ in this case (cf. (4.9.2)) so that indeed (comparing (4.9.2) and (4.9.3)) $\psi_\Sigma^* = \psi^*$.

Now let Σ over V be a family of cr systems and let $\psi_\Sigma : V \to M^{cr}_{m,n,p}$ be the associated morphism as defined in 4.5. We have to show that $\psi_\Sigma^! \Sigma^u$ is isomorphic to Σ. By the rigidity result 4.7.1 it suffices to show that $\psi_\Sigma^! \Sigma^u$ and Σ are isomorphic over each element of some affine covering (U_i) of V, which we can take fine enough so that the underlying bundle E of Σ is trivial over each U_i. Let therefore $U = \mathrm{Spec}(R)$ be such that Σ over U is described by the triple of matrices $\bar{F},\bar{G},\bar{H}$.
Let $d_\alpha = \det(R(\bar{F},\bar{G})_\alpha)$ for each nice selection α. Then U in turn is covered by the $U_\alpha = \mathrm{Spec}(R[d_\alpha^{-1}])$ (by the nice selection lemma). So taking a still finer covering (if necessary) we can assume that $U = \mathrm{Spec}(R)$ is such that for a certain nice selection α we have that $S_\alpha = R(\bar{F},\bar{G})_\alpha$ is invertible over R. Then by 4.5 ψ_Σ is given on U by the ring homomorphism

$$\psi^* : \mathbb{Z}[X_{ij}^\alpha, Y_{r,s}^\alpha] \to R$$

characterized by

$$\psi^* R(F_\alpha(X), G_\alpha(X)) = S_\alpha^{-1} R(\bar{F},\bar{G}),\ \psi^* H_\alpha(Y) = \bar{H} S_\alpha \tag{4.9.4}$$

By 4.8 the family of cr systems $\psi_\Sigma^! \Sigma^u$ is defined by the matrices

$$\bar{F}' = \psi^* F_\alpha(X),\ \bar{G}' = \psi^* G_\alpha(X),\ \bar{H}' = \psi^* H_\alpha(X) \tag{4.9.5}$$

Comparing (4.9.4) and (4.9.5) we see that over U the families defined by $\bar{F},\bar{G},\bar{H}$ and by $\bar{F}',\bar{G}',\bar{H}'$ are indeed isomorphic with the isomorphism being defined by S_α (which is invertible over R). This concludes the proof of the theorem.

4.10. <u>On cr systems over rings</u>. The classsifying theorem 4.9.1 of course also applies to systems over rings R. Such a system (with finitely generated projective state module X) gives rise to a family of cr systems over R iff $R(F,G): R^r \to X$, $r = (n+1)m$, is surjective (cf. 4.2). If R is such that all finitely generated projective modules are free (which happens e.g. if R is a ring of polynomials over a field by the Quillen-Suslin theorem [Qu, Sus]), then theorem 4.9.1 says that the R-rational points of $M^{cr}_{m,n,p}$ are precisely the $GL_n(R)$ orbits in $L^{cr}_{m,n,p}(R)$, i.e.

$$M^{cr}_{m,n,p}(R) \xrightarrow{\sim} L^{cr}_{m,n,p}(R)/GL_n(R) \qquad \text{(if R is projective free)}$$

In general the theorem gives a canonical injection

$$L^{cr}_{m,n,p}(R)/GL_n(R) \hookrightarrow M^{cr}_{m,n,p}(R)$$

with the remaining points of $M^{cr}_{m,n,p}(R)$ corresponding to systems over R whose state module is projective but not free.

4.11. <u>A few final remarks</u>. There is a completely dual theory from the co instead of cr point of view. Also the open subscheme $M^{cr,co}_{m,n,p}$ is of course classifying for families of co and cr systems. This scheme is embeddable (over $\mathbb{Z}$) in an affine scheme $\mathbb{A}^{(n+1)mp}$ as a locally closed subscheme.

5. EXISTENCE AND NONEXISTENCE OF GLOBAL CONTINUOUS CANONICAL FORMS.

As a first application of the fine moduli spaces of section 3 and 4 above we discuss existence and nonexistence of global continuous canonical forms for linear dynamical systems.

5.1. The topological case.

Let L' be a $GL_n(\mathbb{R})$-invariant subspace of $L_{m,n,p}(\mathbb{R})$. A canonical form for $GL_n(\mathbb{R})$ acting on L' is a mapping $c: L' \to L'$ such that the following three properties hold

(5.1.1) $\qquad c(\Sigma^S) = c(\Sigma)$ for all $\Sigma \in L'$, $S \in GL_n(\mathbb{R})$

(5.1.2) $\qquad$ for all $\Sigma \in L'$ there is an $S \in GL_n(\mathbb{R})$ such that $c(\Sigma) = \Sigma^S$

(5.1.3) $\qquad c(\Sigma) = c(\Sigma') \Rightarrow S \in GL_n(\mathbb{R})$ such that $\Sigma' = \Sigma^S$

(Note that (5.1.3) is implied by (5.1.2).

Thus a canonical form selects precisely one element out of each orbit of $GL_n(\mathbb{R})$ acting on L'. We speak of a continuous canonical form if c is continuous.

Of course there exist (many) canonical forms. E.g. order the set of all nice selections α in $J_{n,m}$ in some way. For each $\Sigma \in L^{cr}_{m,n,p}(\mathbb{R})$ let $\alpha(\Sigma)$ be the first α such that $R(\Sigma)_\alpha$ is nonsingular. Then

(5.1.4) $$\Sigma \mapsto c_{\alpha(\Sigma)}(\Sigma) = \Sigma^S, \; S = R(\Sigma)_{\alpha(\Sigma)}$$

is a canonical form on $L^{cr}_{m,n,p}(\mathbb{R})$ (Luenberger canonical forms à la Bryson. This mapping is not continuous, however, except when m = 1 (in which case there is only one nice selection), which entails a number of drawbacks e.g. in numerical calculations and in identification procedures, cf. [GWi] for a discussion in the similar case of Jordan canonical forms.

5.1.5. Theorem. There is a continuous canonical form on $L^{cr,co}_{m,n,p}(\mathbb{R})$ if and only if p = 1 or m = 1.

Proof. If m = 1 let $\alpha \subset J_{1,n} = \{(0,1),(1,1), \ldots, (n,1)\}$ be the unique nice selection $\{(0,1), \ldots, (n-1,1)\}$. Then

(5.1.6) $$c_\alpha: \Sigma \mapsto c_\alpha(\Sigma) = \Sigma^S, \; S = R(\Sigma)_\alpha^{-1}$$

is a continuous canonical form, because $R(\Sigma)_\alpha$ is always invertible for Σ cr.

Similarly if p = 1, let $\beta \subset J_{n,1}$, be the unique nice row selection. Then $\Sigma \mapsto \Sigma^S$, $S = Q(\Sigma)_\beta^{-1}$ is a continuous canonical form because $Q(\Sigma)_\beta$ is invertible for all co Σ (if p = 1).

It remains to show that there cannot be a continuous canonical form c on all of $L^{cr,co}_{m,n,p}(\mathbb{R})$ if both m > 1, p > 1.

To do this we construct two families of linear dynamical systems as follows for all $a \in \mathbb{R}$, $b \in \mathbb{R}$ (We assume $n \geq 2$; if n = 1 the examples must be modified somewhat).

$$G_1(a) = \left(\begin{array}{cc|ccc} a & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \hline 2 & 1 & & & \\ \vdots & \vdots & & B & \\ 2 & 1 & & & \end{array}\right), \quad G_2(b) = \left(\begin{array}{cc|ccc} 1 & b & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \hline 2 & 1 & & & \\ \vdots & \vdots & & B & \\ 2 & 1 & & & \end{array}\right)$$

where B is some (constant) (n-2) x (m-2) matrix with coefficients in $\mathbb{R}$

$$F_1(a) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & n \end{pmatrix} = F_2(b)$$

$$H_1(a) = \left(\begin{array}{cc|ccc} y_1(a) & 1 & 2 & \cdots & 2 \\ y_2(a) & 1 & 1 & \cdots & 1 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & C & \\ 0 & 0 & & & \end{array}\right), \quad H_2(b) = \left(\begin{array}{cc|ccc} x_1(b) & 1 & 2 & \cdots & 2 \\ x_2(b) & 1 & 1 & \cdots & 1 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & C & \\ 0 & 0 & & & \end{array}\right)$$

where C is some (constant) real (p-2) x (n-2) matrix. Here the continuous functions.

$y_1(a)$, $y_2(a)$, $x_1(b)$, $x_2(b)$ are e.g. $y_1(a) = a$ for $|a| \leq 1$, $y_1(a) = a^{-1}$ for $|a| \geq 1$, $y_2(a) = \exp(-a^2)$, $x_1(b) = 1$ for $|b| \leq 1$,

$x_1(b) = b^{-2}$ for $|b| \geq 1$, $x_2(b) = b^{-1}\exp(-b^{-2})$ for $b \neq 0$, $x_2(0) = 0$.
The precise form of these functions is not important. What is important is that they are continuous, that $x_1(b) = b^{-1}y_1(b^{-1})$, $x_2(b) = b^{-1}y_2(b^{-1})$ for all $b \neq 0$ and that $y_2(a) \neq 0$ for all a and $x_1(b) \neq 0$ for all b.

For all $b \neq 0$ let $T(b)$ be the matrix

$$(5.1.7) \qquad T(b) = \begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Let $\Sigma_1(a) = (F_1(a), G_1(a), H_1(a))$, $\Sigma_2(b) = (F_2(b), G_2(b), H_2(b))$. Then one easily checks that

$$(5.1.8) \qquad ab = 1 \Rightarrow \Sigma_1(a)^{T(b)} = \Sigma_2(b)$$

Note also that $\Sigma_1(a), \Sigma_2(b) \in L^{co,cr}_{m,n,p}(\mathbb{R})$ for all $a,b \in \mathbb{R}$; in fact

$$(5.1.9) \qquad \Sigma_1(a) \in U_\alpha,\ \alpha = ((0,2), (1,2), \ldots, (n-1,2)) \text{ for all } a \in \mathbb{R}$$

$$(5.1.10) \qquad \Sigma_2(b) \in U_\beta,\ \beta = ((0,1), (1,1), \ldots, (n-1,1)) \text{ for all } b \in \mathbb{R}$$

which proves the complete reachability. The complete observability is seen similarly.

Now suppose that c is a continuous canonical form on $L^{co,cr}_{m,n,p}(\mathbb{R})$. Let $c(\Sigma_1(a)) = (\bar{F}_1(a), \bar{G}_1(a), \bar{H}_1(a))$, $c(\Sigma_2(b)) = (\bar{F}_2(b), \bar{G}_2(b), \bar{H}_2(b))$. Let $S(a)$ be such that $c(\Sigma_1(a)) = \Sigma_1(a)^{S(a)}$ and let $\bar{S}(b)$ be such that $c(\Sigma_2(b)) = \Sigma_2(b)^{\bar{S}(b)}$.
It follows from (5.1.9) and (5.1.10) that

$$(5.1.11) \qquad \begin{aligned} S(a) &= R(\bar{F}_1(a), \bar{G}_1(a))_\alpha R(F_1(a), G_1(a))_\alpha^{-1} \\ \bar{S}(b) &= R(\bar{F}_2(b), \bar{G}_2(b))_\beta R(F_2(b), G_2(b))_\beta^{-1} \end{aligned}$$

Consequently $S(a)$ and $S(b)$ are (unique and are) continuous functions of a and b.

Now take $a = b = 1$. Then $ab = 1$ and $T(b) = I_n$ so that (cf. (5.1.7), (5.1.8) and (5.1.11) $S(1) = \bar{S}(1)$. It follows from this and the

continuity of $S(a)$ and $\bar{S}(b)$ that we must have

$$(5.1.12) \qquad \operatorname{sign}(\det S(a)) = \operatorname{sign}(\det \bar{S}(b)) \text{ for all } a,b \in \mathbb{R}$$

Now take $a = b = -1$. Then $ab = 1$ and we have, using (5.1.8),

$$\begin{aligned} \Sigma_1(-1)^{\bar{S}(-1)T(-1)} &= (\Sigma_1(-1)^{T(-1)})^{\bar{S}(-1)} \\ &= \Sigma_2(-1)^{\bar{S}(-1)} = c(\Sigma_2(-1)) \\ &= c(\Sigma_1(-1)) = \Sigma_1(-1)^{S(-1)} \end{aligned}$$

It follows that $S(-1) = \bar{S}(-1)T(-1)$, and hence by (5.1.7), that

$$\det(S(-1)) = -\det(\bar{S}(-1))$$

which contradicts (5.1.12). This proves that there does not exists a continuous canonical form on $L^{co,cr}_{m,n,p}(\mathbb{R})$ if $m \geq 2$, and $p \geq 2$.

5.1.13. <u>Remark</u>. By choosing the matrices B, C in $G_1(a)$, $G_2(b)$, $H_1(a)$, $H_2(b)$ judiciously we can also see to it that rank $G_1(a) = m =$ rank $G_2(b)$, rank $H_1(a) = p =$ rank $H_2(b)$ if $p < n$ and $m < n$. Note also that F in the example above has n distinct real eigenvalues so that a restriction like "F must be semi simple" also does not help much.

5.1.14. <u>Discussion of the proof of theorem 5.1.5.</u> The proof given above, though definitely a proof, is perhaps not very enlightening. What is behind it is the following. Consider the natural projection.

$$(5.1.15) \qquad \pi : L^{cr,co}_{m,n,p}(\mathbb{R}) \to M^{cr,co}_{m,n,p}(\mathbb{R})$$

Let c be a continuous canonical form. Because c is constant on all orbits c induces a continuous map $\tau: M^{cr,co}_{m,n,p}(\mathbb{R}) \to L^{cr,co}_{m,n,p}(\mathbb{R})$ which clearly is a section of π, (cf. (5.1.1) – (5.1.3)). Inversely if τ is a continuous section of π then $\tau\circ\pi : L^{co,cr}_{m,n,p}(\mathbb{R}) \to L^{co,cr}_{m,n,p}(\mathbb{R})$ is a continuous canonical form.

Now (5.1.15) is (fairly easily at this stage, cf. [Haz 1]), seen to be a principal $GL_n(\mathbb{R})$ fibre bundle. Such a bundle is trivial iff it admits a continuous section. The mappings

$a \mapsto \Sigma_1(a),\ b \mapsto \Sigma_2(b)$

of the proof above now combine to define a continuous map of $\mathbb{P}^1(\mathbb{R})$ = circle into $M^{cr,co}_{m,n,p}(\mathbb{R})$ such that the pullback of the fibre bundle (5.1.15) is nontrivial. In fact the associated determinant $GL_1(\mathbb{R})$ fibre bundle is the Möbius band (minus zero section) over the circle.

5.2. The algebraic-geometric case.

The result corresponding to theorem 5.1.5 in the algebraic geometric case is the following. For simplicity we state it for varieties (over algebraically closed fields).

5.2.1. Theorem. Let k be an algebraically closed field. Then there exists a canonical form c: $L^{cr,co}_{m,n,p}(k) \to L^{cr,co}_{m,n,p}(k)$ which is a morphism of algebraic varieties if and only if m = 1 or p = 1.

Here of course a canonical form is defined just as in 5.1 above; simply replace $\mathbb{R}$ with k everywhere in (5.1.1) - (5.1.3) and replace the word "continuous" with "morphism of algebraic varieties", which means that locally c is given by rational expressions in the coordinates.

The proof is rather similar to the one briefly indicated in 5.1.14 above. In this case $L^{cr,co}_{m,n,p}(k) \to M^{cr,co}_{m,n,p}(k)$ is an algebraic principal $GL_n(k)$ bundle and one again shows that it is trivial if and only if m = 1 or p = 1. The only difference is the example used to prove nontriviality. The map used in 5.1.14 is non-algebraic, nor is there an algebraic injective morphism $\mathbb{P}^1(k) \to M^{co,cr}_{m,n,p}(k)$. Instead one defines a three dimensional manifold much related to the families $\Sigma_1(a)$, $\Sigma_2(b)$ together with an injection into $M^{cr,co}_{m,n,p}(k)$ such that the pullback of this principal bundle is easily seen to be nontrivial. Cf. [Haz 2] for details.

6. REALIZATION WITH PARAMETERS AND REALIZING DELAY-DIFFERENTIAL SYSTEMS.

As a second application of the existence of fine moduli spaces for cr systems we discuss realization with parameters (cf. also [By]) and realization of delay-differential systems. A preliminary step for this is the following bit of realization theory.

6.1. Resumé of some realization theory.

Let T(s) be a proper rational matrix-valued function of s with the (formal) power series expansion (around s = ∞)

(6.1.1) $$T(s) = A_1 s^{-1} + A_2 s^{-2} + \dots,\quad A_i \in k^{p\times m}$$

One says that T(s) is realizable by a linear system of dimension ≤ n, if T(s) is the Laplace transform (resp. z-transform) of a linear differentiable (resp. difference) system $\Sigma = (F,G,H) \in L_{m,n,p}(k)$. This means that

(6.1.2) $$T(s) = H(sT_n - F)^{-1}G$$

or, equivalently

(6.1.3) $$A_i = HF^{i-1}G,\quad i = 1,2,3,\dots$$

A necessary and sufficient condition that T(s) be realizable by a system of dimension n is that the associated Hankel matrix h(A) of the sequence $\mathcal{A} = (A_1, A_2, A_3, \dots)$ be of rank ≤ n. Here $h(\mathcal{A})$ is the block Hankel matrix

$$h(\mathcal{A}) = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & & \cdots \\ A_3 & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \end{pmatrix}$$

More precisely we have the partial realization result which says that there exist $F,G,H \in L^{co,cr}_{m,n,p}(k)$ such that $A_i = HF^{i-1}G$ iff rank $h_n(\mathcal{A})$ = rank $h_{n+1}(\mathcal{A}) = n$, where $h_i(\mathcal{A})$ is the block matrix consisting of the first i block rows and the first i block-columns of $h(\mathcal{A})$.

Now suppose that rank $h(\mathcal{A})$ is precisely n, and let F,G,H realize $\mathcal{A}$.

We have

$$h(\mathcal{A}) = \begin{pmatrix} H \\ HF \\ HF^2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} (G \,\vdots\, FG \,\vdots\, F^2G \,\vdots\, \ldots)$$

and it follows by the Cayley-Hamilton theorem that R(F,G) and Q(F,H) are both of rank n so that $\Sigma = (F,G,H)$ is in this case both cr and co.

Finally we recall that if Σ and Σ' are both cr and co and both realize $\mathcal{A}$, then Σ and Σ' are isomorphic, i.e. there is an $S \in GL_n(k)$ such that $\Sigma' = \Sigma^S$.

For all these facts,cf. e.g. [KFA] or [Haz3].

6.2. A realization algorithm.

Now let $\mathcal{A}$ be such that rank $h(\mathcal{A}) = n$. We describe a method for calculating a $\Sigma = (F,G,H) \in L^{cr,co}_{m,n,p}(k)$ which realizes $\mathcal{A}$. By the above we know that there exist a nice selection $\alpha_c \subset J_{m,n}$ the set of column indices of

(6.2.1)

$$h_{n+1}(\mathcal{A}) = \begin{pmatrix} A_1 & A_2 & \cdots & A_{n+1} \\ A_2 & & & \\ \vdots & & & \vdots \\ A_{n+1} & & \cdots & A_{2n+1} \end{pmatrix}$$

and a nice selection $\alpha_r \subset J_{p,n}$, the set of row indices of $h_{n+1}(\mathcal{A})$, such that the n n matrix $h_{n+1}(\mathcal{A})_{\alpha_r,\alpha_c}$ has rank n. Here $h_{n+1}(\mathcal{A})_{\alpha_r,\alpha_c}$ is the matrix obtained from $h_{n+1}(\mathcal{A})$ by removing all rows whose index is not in α_r and all columns whose index is not in α_c. We now describe a method for finding a $\Sigma = (F,G,H) \in L^{cr,co}_{m,n,p}(k)$ such that Σ realizes $\mathcal{A}$ and such that $R(F,G)_{\alpha_c} = I_n$. (Such a Σ is unique).

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Let γ_r be the subset of $J_{p,n}$ of the first p row indices, so that $h_{n+1}(\mathcal{A})_{\gamma_r}$ consists of the first row of blocks in (6.2.1). Now let

(6.2.2) $$H = h_{n+1}(\mathcal{A})_{\gamma_r,\alpha_c}$$

Now let

(6.2.3) $$S = h_{n+1}(\mathcal{A})_{\alpha_r,\alpha_c}$$

and define $R' = S^{-1}(h_{n+1}(\mathcal{A})_{\alpha_r})$. Then $(R')_{\alpha_c} = I_n$ and we let F,G be the unique n×n and n×m matrices such that

(6.2.4) $$R(F,G) = R'$$

Recall, cf.3.2.7above that the columns of F and G can be simply read of from the columns of R', being equal to either a standard basis vector or equal to a column of R'.

For every field k and each pair of nice selections $\alpha_c \subset J_{m,n}$, $\alpha_r \subset J_{p,n}$ let $W(\alpha_r,\alpha_c)(k)$ be the space of all sequence of pxm matrices $\mathcal{A} = (A_1, \ldots, A_{2n+1})$ such that $\text{rank}(h_{n+1}(\mathcal{A})) = n$ and $\text{rank}(h_{n+1}(\mathcal{A})_{\alpha_r,\alpha_c}) = n$. Then the above defines a map

(6.2.5) $$\tau(\alpha_r,\alpha_c)\colon W(\alpha_r,\alpha_c)(k) \to L^{cr,co}_{m,n,p}(k)$$

6.2.6. Lemma. If $k = \mathbb{R}$ or $\mathbb{C}$ the map $\tau(\alpha_r,\alpha_c)$ is analytic, and algebraic-geometrically speaking the $\tau(\alpha_r,\alpha_c)$ define a morphism of schemes from the affine scheme $W(\alpha_r,\alpha_c)$ into the quasi affine scheme $L^{cr,co}_{m,n,p}$.

6.2.7. Lemma. Let W(k) be the space of all sequences of pxm matrices $\mathcal{A} = (A_1,A_2,\ldots,A_{2n+1})$ such that $\text{rank}(h_{n+1}(\mathcal{A})) = n = \text{rank } h_n(\mathcal{A})$. Let $h\colon L^{cr,co}_{m,n,p}(k) \to W(k)$ be the map $h(F,G,H) = (HG,HFG,\ldots,HF^{2n}G)$. Then $h \circ \tau(\alpha_r,\alpha_c)$ is equal to the natural embedding of $W(\alpha_r,\alpha_c)(k)$ in W(k). (I.e. $h \circ \tau(\alpha_r,\alpha_c)$ is the identity on $W(\alpha_r,\alpha_c)(k)$.

Proof. Let $\mathcal{A} \in W(\alpha_r,\alpha_c)(k)$. By partial realization theory (cf. 6.1 above) we know that $\mathcal{A}$ is realizable, say by $\Sigma' = (F',G',H')$. Then because $\mathcal{A} \in W(\alpha_r,\alpha_c)(k)$ we have that $S = R(F',G')_{\alpha_c}$ is invertible. Let

$\Sigma = (F,G,H) = \Sigma'^{S^{-1}} = (S^{-1}F'S,\ S^{-1}G', H'S)$. Then Σ also realizes $\mathcal{A}$ and $R(F,G)_{\alpha_c} = I_n$. Now observe that the realization algorithm described above simply recalculates precisely these F,G,H from $\mathcal{A}$.

6.2.8. Corollary. Let $k = \mathbb{R}$ or $\mathbb{C}$ and let h: $M^{cr,co}_{m,n,p}(k) \to W(k)$ be the map induced by h: $L_{m,n,p}(k) \to W(k)$. Then h is an isomorphism of analytic manifolds.

6.2.9. Corollary. More generally h: $L^{co,cr}_{m,n,p} \to W$ induces an isomorphism of schemes $M_{m,n,p} \to W$. In particular if k is an algebraically closed field then we have an isomorphism of the algebraic varieties $M_{m,n,p}(k)$ and $W(k)$.

6.3. Realization with parameters.

6.3.1. The topological case. Let $T_a(s)$, $a \in V$ be a family of transfer functions depending continuously on a parameter $a \in V$. For each $a \in V$ write $_a(s) = A_1(a)c^{-1} + A_2(a)s^{-2} + \ldots$ and for each a let n(a) be the rank of the block Hankel matrix of $\Sigma(a) = (A_1(a), A_2(a), \ldots)$. The question we ask is: does there exist a continuous family of systems $\Sigma(a) = (F(a),G(a),H(a))$ such that the transfer functionof $\Sigma(a)$ is $T_a(s)$ for all a. The answer to this is definitely yes provided n(a) is bounded as a function of a. Simply take a long enough chunk of the $\mathcal{A}(a)$ for all a and do the usual realization construction by means of block companion matrices and observe that this is continuous in the $A_i(a)$.*) The question becomes much more delicate if we ask for a continuous family of realizations which are all cr and co. This obviously requires that n(a) is constant and provided that the space V is such that all n = n(a) dimensional bundles are trivial this condition is also sufficient. Indeed if n(a) is constant then the $\mathcal{A}(a)$ determine a continuous map $V \to W(\mathbb{R})$ and hence by Corollary 5.2.8 a continuous map $V \to M^{cr,co}_{m,n,p}(\mathbb{R})$. Pulling back the universal family over $M^{cr,co}_{m,n,p}(\mathbb{R})$ to a family over V gives us a family $(E;F,G,H)$ over V such that the transfer function of the system over $a \in V$ is $T_a(s)$ for all a. The bundle E is trivial by hypothesis so there are continuous sections $e_1, \ldots, e_n$: $V \to E$ such that $\{e_1(a), \ldots, e_n(a)\}$ is a basis for $E(a)$ for all $a \in V$. Now write out the matrices of F,G,H with respect to these bases to find a continuous family $\Sigma(a)$, which realizes $T_a(s)$ and such that $\Sigma(a)$ is cr and co for all a.

*) True if V is paracompact and normal, one needs partitions of unity (in any case, I do) to find continuous $T_i(a)$ such that $B_{n+1} = T_1B_n + \ldots + T_nB_1$ where B_i is the i-th block column of $h(\mathcal{A})$.

6.3.2. The polynomial case. Let k be a field and $\bar{k}$ its algebraic closure, e.g. $k = \mathbb{R}$ and $\bar{k} = \mathbb{C}$. Let $T_x(s)$ be a transferfunction with coefficients in $k[x_1, \ldots, x_q]$, where $x_1, \ldots, x_q$ are indeterminates. We ask whether there exists a realization of T(s) over $k[x_1, \ldots, x_q]$, that is a triple of matrices (F,G,H) with coefficients in $k[x_1, \ldots, x_q]$ such that $T_x(s) = H(sI-F)^{-1}G$. Again the answer is obviously yes if we do not require any minimality conditions on the realization (provided $n(x_1, \ldots, x_q)$ the degree of the Hankel matrix of T(s) is bounded for all $(x_1, \ldots, x_q) \in \bar{k}^q$.

Now assume that $n(x_1, \ldots, x_q)$ is constant for all $(x_1, \ldots, x_q) \in \bar{k}^q$. Then $(x_1, \ldots, x_q) \mapsto \mathcal{A}(x_1, \ldots, x_q)$ defines a morphism of algebraic varieties $\bar{k}^q \to W(k)$ and hence by Corollary 6.2.9 a morphism $\bar{k}^q \to M^{cr,co}_{m,n,p}(k)$. Pulling back the universal family by means of this morphism we find a family $(E;F,G,H)$ over $\bar{k}^q$ which is defined over k because the morphism $\bar{k}^q \to W(k)$ and with the isomorphism $M^{cr,co}_{m,n,p}(k)$ are defined over k. Thus Σ is defined over k and by the Quillen-Suslin theorem E is trivializable over k. Taking the corresponding sections and writing out the matrices of F,G,H with respect to the resulting bases we find an F,G,H with coefficients in $k[x_1, \ldots, x_q]$ which realize $T_x(s)$ for all $x \in \bar{k}^q$, i.e. such that $T_x(s) = H(sI-F)^{-1}G$. Moreover this system (F,G,H) is cr over $k[x_1, \ldots, x_q]$ (meaning that $R(F,G)$: $k[x_1, \ldots, x_q]^{(n+1)m} \to k[x_1, \ldots, x_q]^n$ is surjective); it is also co and even stronger its dual system is also cr (i.e. (F,G,H) is split in the terminology of [So 3].

6.3.3. Realization by means of delay-differentiable systems.

Let $\Sigma = (F(\sigma_1, \ldots, \sigma_q), G(\sigma_1, \ldots, \sigma_q), H(\sigma_1, \ldots, \sigma_q))$ be a delay differential system with q incommensurable delays. Here σ_i stands for the delay operator $\sigma_i f(t) = f(t-a_i)$, cf. 2.3 above for this notation. The transfer function of Σ is then

$$(6.3.4) \quad T(s) = G(e^{-a_1 s}, \ldots, e^{-a_q s})(sI-F(e^{-a_1 s}, \ldots, e^{-a_q s}))^{-1}H(e^{-a_1 s}, \ldots, e^{-a_q s})$$

which is a rational function in s whose coefficients are polynomials in $e^{-a_1 s}, \ldots, e^{-a_q s}$.

Now inversely suppose we have a transfer function T(s) like (6.3.4) and we ask whether it can be realized by means of a delay-differential system $\Sigma(\sigma)$. Now if the a_i are incommensurable then the functions $s, e^{-a_1(s)}, \ldots, e^{-a_q s}$ are algebraically independant and there is precisely one transfer function $T(s;\sigma_1, \ldots, \sigma_q)$ whose coefficients are polynomials in $\sigma_1, \ldots, \sigma_q$ such that $T(s) = T'(s, e^{-a_1 s}, \ldots, e^{-a_q s})$. Thus the problem is mathematically identical with the one treated just above 6.3.2. In passing let us remark that complete reachability for delay-systems in the sense of that the associated system over the ring $\mathbb{R}[\sigma_1, \ldots, \sigma_q]$ is required to be cr seems often a reasonable requirement, e.g. in connection with pole placement, cf. [So.1] and [Mo].

7. THE "CANONICAL" COMPLETELY REACHABLE SUBSYSTEM.

7.1. Σ^{cr} <u>for systems over fields</u>. Let $\Sigma = (F,G,H)$ be a system over a field k. Let X^{cr} be the image of $R(F,G)$: $k^r \to k^n$, $r = m(n+1)$. Then obviously $F(X^{cr}) \subset X^{cr}$, $G(k^m) \subset X^{cr}$, so that there is an induced subsystem $\Sigma^{cr} = (X^{cr}; F', G', H')$ which is called the canonical cr subsystem of Σ In terms of matrices this means that there is an $S \in GL_n(k)$ such that Σ^S has the form

(7.1.1) $$\Sigma^S = \left(\begin{pmatrix} G_1 \\ 0 \end{pmatrix}, \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix}, (H_1 \quad H_2) \right)$$

with $(F_{11}, G_1, H_1) = \Sigma^{cr}$, the "canonical" cr subsystem. The words Kalman "decomposition" are also used in this context. There is a dual construction relating to co and combining these two constructions "decomposes" the system into four parts.

In this section we examine whether this construction can be globalized, i.e. we ask whether this construction is continuous, and we ask whether something similar can be done for time varying linear dynamical systems.

7.2. Σ^{cr} <u>for time varying systems</u>. Now let $\Sigma = (F,G,H)$ be a time varying system, i.e. the coefficients of the matrices F,G,H are allowed to vary, say ~~continuously~~ differentiably, with time. For time varying systems the controlability matrix $R(\Sigma) = R(F,G)$ must be redefined as follows

(7.2.1) $$R(F,G) = (G(0) \mid G(1) \mid \ldots \mid G(n))$$

where

(7.2.2) $$G(0) = G;\; G(i) = FG(i-1) - \dot{G}(i-1)$$

where the $\dot{}$ denotes differentiation with respect to time, as usual. Note that this gives back the old $R(F,G)$ if F,G do not depend on time. The system is said to be cr if this matrix $R(\Sigma)$ has full rank. These seem to be the appropriate notions for time varying systems; cf. e.g. [We, Haz5] for some supporting results for this claim.

A time variable base change $x' = Sx$ (with $S = S(t)$ invertible for all t) changes Σ to Σ^S with

(7.2.3) $$\Sigma^S = (SFS^{-1} + \dot{S}S^{-1}, SG, HS^{-1})$$

Note that $R(\Sigma)$ hence transforms as

(7.2.4) $$R(\Sigma^S) = SR(\Sigma)$$

7.2.5. <u>Theorem</u>. Let Σ be a time varying system with continuously varying parameters. Suppose that rank $R(\Sigma)$ is constant as a function of t. Then there exists a continuous time varying matrix S, invertible for all t, such that Σ^S has the form (7.1.1) with (F_{11}, G_1, H_1) cr.

Proof. Consider the subbundle of the trivial $(n+1)m$ dimensional bundle over the real line generated by the rows of $R(\Sigma)$. This is a vectorbundle because of the rank assumption. This bundle is trivial. It follows that there exist r sections of the bundle, where $r = \text{rank } R(\Sigma)$, which are linearly independant everywhere. The continuous sections of the bundle are of the form $\Sigma a_i(t) z_i(t)$, where $z_1(t), \ldots, z_n(t)$ are the rows of $R(\Sigma)$ and the $a_i(t)$ are continuous functions of t. Let $b_1(t), \ldots, b_r(t)$ be the r everywhere linearly independant sections and let $b_j(t) = \Sigma a_{ji}(t) z_i(t)$, $j = 1, \ldots, r$; $i = 1, \ldots, n$.

Let E' be the r dimensional subbundle of the trivial bundle E of dimension n over the real line generated by the r row vectors $a_j(t) = (a_{j1}(t), \ldots, a_{jn}(t))$. Because the quotient bundle E/E' is trivial we can complete the r vectors $a_1(t), \ldots, a_r(t)$ to a system set of n vectors $a_1(t), \ldots, a_n(t)$ such that the determinant of the matrix formed by these vectors is nonzero for all t. Let $S_1(t)$ be the matrix formed by these vectors, then $S_1 R(\Sigma)$ has the property that for all t its first r rows are linearly independant and that it is of rank r for all t. It follows that there are unique continuous functions $c_{ki}(t)$, $k = r+1, \ldots, n$; $i = 1, \ldots, r$

such that $z'_k(t) = \Sigma c_{ki}(t) z'_i(t)$, where $z'_j(t)$ is the j-th row of $S_1R(\Sigma)$. Now let

$$S_2(t) = \begin{pmatrix} I_r & 0 \\ -C(t) & I_{n-r} \end{pmatrix}$$

Then $S(t) = S_2(t)S_1(t)$ is the desired transformation matrix (as follows from the transformation formula (7.2.4)).

Virtually the same arguments give a smoothly varying S(t) if the coefficients of Σ vary smoothly in time, and give a polynomial S(t) if the coefficients of Σ are polynomials in t (where in the latter case we need the constancy of the rank also for all complex values of t and use that projective modules over a principal ideal ring are free).

7.3. Σ^{cr} for families. For families of systems these techniques give

7.3.1. Theorem. Let Σ be a continuous family parametrized by a contractible topological space (resp. a differentiable family parametrized by a contractible manifold; resp. a polynomial family). Suppose that the rank of $R(\Sigma)$ is constant as a function of the parameters. Then there exists a continuous (resp. differentiable; resp. polynomial) family of invertible matrices S such that Σ^S has the form (7.1.1) with (F_{11}, G_1, H_1) a family of cr systems.

The proof is virtually the same as the one given above of theorem 7.2.5; in the polynomial case one of course relies on the Quillen-Suslin theorem [Qu; Sus] to conclude that the appropriate bundles are trivial. Note also that, inversely, the existence of an S as in the theorem implies that the rank of $R(\Sigma)$ is constant.

For delay-differential systems this gives a "Kalman decomposition" provided the relevant, obviously necessary, rank condition is met.

Another way of proving theorem 7.3.1 for systems over certain rings rests on the following lemma which is also a basic tool in the study of isomorphisms of families in [HP] and which implies a generalization of the main lemma of [OS] concerning the solvability of sets of linear equations over rings.

7.3.2. Lemma. Let R be a reduced ring (i.e. there are no nilpotents $\neq 0$) and let A be a matrix over R. Suppose that the rank of $A(\mathfrak{p})$ over the quotient field of $R/\mathfrak{p}$ is constant as a function of $\mathfrak{p}$ for all prime ideals $\mathfrak{p}$. Then Im(A) and Coker(A) are projective modules.

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Now let Σ over R be such that rank $R(\Sigma(\mathfrak{p}))$ is constant, and let R be projective free (i.e. all finitely generated projective modules over R are free). Then Im $R(\Sigma) \subset R^n$ is projective and hence free. Taking a basis of Im $R(\Sigma)$ and extending it to a basis of all of R^n, which can be done because $R^n/\text{Im } R(\Sigma)$ = Coker $R(\Sigma)$ is projective and hence free, now gives the desired matrix S.

There is a complete set of dual theorems concerning co.

7.4. Σ^{cr} for delay differential systems. Now let $\Sigma(\sigma) = (F(\sigma), G(\sigma), H(\sigma))$ be a delay differential system. Then of course we can interpret Σ as a polynomial system over $\mathbb{R}[\sigma] = \mathbb{R}[\sigma_1, \ldots, \sigma_r]$ and apply theorem 7.3.1. The hypothesis that rank $R(\Sigma(\sigma))$ be constant as a function of $\sigma_1, \ldots, \sigma_r$ (including complex and negative values of the delays) is rather strong though.

Now if we assume that all functions involved in

$$\dot{x}(t) = F(\sigma)x(t) + G(\sigma)u(t),\ y(t) = H(\sigma)x(t) \tag{7.4.1}$$

are zero sufficiently far in the past, an assumption which is not unreasonable and even customary in this context, then it makes perfect sense to talk about base changes of the form

$$x' = S(\sigma)x \tag{7.4.2}$$

where $S(\sigma)$ is a matrix whose coefficients are power series in the delays $\sigma_1, \ldots, \sigma_r$ and which is invertible over the ring of power series $\mathbb{R}[[\sigma_1, \ldots, \sigma_r]]$. Indeed if $\sigma_1\alpha(t) = \alpha(t-a_1)$, $a_1 > 0$ and the function $\beta(t)$ is zero for $t < -N\alpha_1$ then

$$\left(\sum_{i=0}^{\infty} b_i\sigma_1^i\right)\beta(t) = \sum_{i=0}^{N+N'} b_i\beta(t-ia_i)$$

where N' is such that $t < N'a_1$.

Allowing such basis changes one has

7.4.3. Theorem. Let $\Sigma(\sigma)$ be a delay-differential system. Suppose that rank $R(\Sigma(\sigma))$ considered as a matrix over the quotient field $k(\sigma_1, \ldots, \sigma_r)$ is equal to rank $R(\Sigma(0))$ (over $\mathbb{R}$) where $\Sigma(0)$ is the system obtained from $\Sigma(\sigma)$ by setting all σ_i equal to zero. Then there exists a power series base change matrix $S \in GL_n(\mathbb{R}[[\sigma]])$ such that Σ^S has the form (7.1.1) with (F_{11},G_1,H_1) a cr system (over $\mathbb{R}[[\sigma]]$).

The proof is again similar where now of course one uses that a projective module over a local ring is free.

Note that $\Sigma(0)$ is not the system obtained from $\Sigma(\sigma)$ by setting all delays equal to zero. For example if $\Sigma(\sigma)$ is the one dimensional, one delay system $\dot{x}(t) = x(t) + 2x(t-1) + u(t) + u(t-2)$, $y(t) = 2x(t) - x(t-1)$, then $\Sigma(0)$ is the system $x(t) = x(t) + u(t)$, $y(t) = 2x(t)$ obtained by removing all delay terms.

8. CONCLUDING REMARKS ON FAMILIES OF SYSTEMS AS OPPOSED TO SINGLE SYSTEMS.

8.1. <u>Non extendability of the moduli spaces</u> $M^{cr}_{m,n,p}$ <u>and</u> $M^{co}_{m,n,p}$. One aspect of the study of families of systems rather than single systems is the systematic investigation of which of the many constructions and algorithms of systems and control theory are continuous in the system parameters (or more precisely to determine, so to speak, the domains of continuity of these constructions). This is obviously important if one wants e.g. to execute these algorithms numerically.

Intimately (and obviously) related to this continuity problem is the question of how a given single system can sit in a family of systems (deformation (perturbation) theory). The fine moduli spaces $M^{cr}_{m,n,p}$ and $M^{co}_{m,n,p}$ answer precisely this question (for a system which is cr or co): for a given cr (resp. co) system the local structure of $M^{cr}_{m,n,p}(\mathbb{R})$ (resp. $M^{co}_{m,n,p}(\mathbb{R})$) around the point represented by the given system describe exactly the most complicated family in which the given system can occur (all other families can up to isomorphism be uniquely obtained from this one by a change of parameters). Thus one may well be interested to see whether these moduli spaces can be extended a bit. In particular one could expect that $M^{cr}_{m,n,p}(\mathbb{R})$ and $M^{co}_{m,n,p}(\mathbb{R})$ could be combined in some way to give a moduli space for all systems which are cr or co. The following example shows that this is a bit optimistic.

8.1.1. <u>Example</u>. Let Σ and Σ' be the two families over $\mathbb{C}$ (or $\mathbb{R}$) given by the triples of matrices

$$\Sigma = \left(\begin{pmatrix} 1 & 1 \\ \sigma & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1,0) \right), \quad \Sigma' = \left(\begin{pmatrix} 1 & \sigma \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1,0) \right)$$

Σ is co everywhere and cr everywhere but in $\sigma = 0$, and Σ' is cr everywhere and co everywhere but in $\sigma = 0$. The systems $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ are isomorphic for all $\sigma \neq 0$, but $\Sigma(0)$ and $\Sigma'(0)$ are definitely not isomorphic. This kills all chances of having a fine moduli space for families which consist of systems which are co or cr. There cannot even be a coarse moduli space for such families.

Indeed let $\mathcal{F}$ be the functor which assigns to every space the set of all isomorphism classes of families of cr or co systems. Then a course moduli space for $\mathcal{F}$ (cf. [Mu] for a precise definition) consists of a space M together with a functor transformation $\mathcal{F}(-) \to \mathrm{Mor}(-,M)$ which is an isomorphism if $- = pt$ and which also enjoys an additional universality property. Now consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(\mathbb{C} \setminus \{0\}) & \to & \mathrm{Mor}(\mathbb{C} \setminus \{0\},M) \\ \uparrow & & \uparrow \\ \mathcal{F}(\mathbb{C}) & \xrightarrow{\alpha} & \mathrm{Mor}(\mathbb{C},M) \\ \downarrow & & \downarrow \\ \mathcal{F}(\{0\}) & \xrightarrow{\simeq} & \mathrm{Mor}(\{0\},M) \end{array}$$

Consider the elements of $\mathcal{F}(\mathbb{C})$ represented by Σ and Σ'. Because Σ and Σ' are isomorphic as families restricted to $\mathbb{C} \setminus \{0\}$ we see by continuity (of the elements of $\mathrm{Mor}(\mathbb{C},M)$) that $\alpha(\Sigma) = \alpha(\Sigma')$. Because $\Sigma(0)$ and $\Sigma'(0)$ are not isomorphic this gives a contradiction with the injectivity of $\mathcal{F}(\{0\}) \to \mathrm{Mor}(\{0\},M)$.

Coarse moduli spaces represent one possible weakening of the fine moduli space property. Another, better adapted to the idea of studying families by studying a maximally complicated example, is that of a <u>versal</u> deformation. Roughly a versal holomorphic deformation of a system Σ over $\mathbb{C}$ is a family of systems $\Sigma(\sigma)$ over a small neighbourhood U of 0 (in some parameter space) such that $\Sigma(0) = \Sigma$ and such that for every family Σ' over V such that $\Sigma'(0) = \Sigma$ there is some (not necessarily unique) holomorphic map ϕ (i.e. a holomorphic change in parameters) such that $\phi^! \Sigma \simeq \Sigma'$ is a neighbourhood of 0.

For square matrices depending holomorphically a parameters (with similarity as isomorphism) Arnol'd, [Ar], has constructed versal deformations and the same ideas work for systems (in any case for pairs of matrices (F,G), cf. [Ta 2]).

8.2. On the geometry of $M^{cr,co}_{m,n,p}$. From the identification of systems point of view not only the local structure of $M^{co,cr}_{m,n,p}(\mathbb{R})$ is important but also its global structure cf. also [BrK] and [Haz 8]. Thus for example if $m = 1 = p$, $M^{co,cr}_{1,n,1}(\mathbb{R}) = \mathrm{Rat}(n)$ decomposes into (n+1) components, and some of these components are of rather complicated topological type, [Br], which argues ill for the linearization tricks which are at the back of many identification procedures. One way to view identification is as finding a sequence of points in $M^{co,cr}_{m,n,p}(\mathbb{R})$ as more and more data come in. Ideally this sequence of points will then converge to something. Thus the question comes up of whether $M^{co,cr}_{m,n,p}(\mathbb{R})$ is compact, or compactifiable in such a way that the extra points can be interpreted as some kind of systems. Now $M^{co,cr}_{m,n,p}(\mathbb{R})$ is never compact. As to the compactification question. There does exist a partial compactification $\bar{M}_{m,n,p}$ such that the extra points, i.e. the points of $\bar{M}_{m,n,p} \setminus M^{cr,co}_{m,n,p}$ correspond to systems of the form

$$\dot{x} = Fx + Gu,\ y = Hx + J(D)u \tag{8.2.1}$$

where D is the differentiation operator and J is a polynomial in D. This seems to give still more motivation for studying systems more general than $\dot{x} = Fx + Gu$, $y = Hx$ [Ros]. This partial compactification is also maximal in the sense that if a family of systems converges in the sense that the associated family of input/output operators converges (in the weak topology) then the limit input/output operator is the input/output operator of a system of the form (8.2.1). Cf. [Haz 4] for details.

8.3. Pointwise-local-global isomorphism theorems. One perennial question which always turns up when one studies families rather than single objects is: to what extent does the pointwise or local structure of a family determine its global properties. Thus for square matrices one has e.g. the question studied by Wasov [Wa], cf. also [OS]; given two families of matrices A(z), A'(z) depending holomorphically on some parameters z. Suppose that for each separate value of z, A(z) and A'(z) are similar; does it follow that A(z) and A'(z) are similar as holomorphic families.

For families of systems the corresponding question is: let $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ be two families of systems and suppose that $\Sigma(\sigma)$ and $\Sigma'(\sigma)$ are isomorphic for all values of σ. Does it follow that Σ and Σ' are isomorphic as families (globally or locally in a neighbourhood of every parameter value σ).

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Here there are (exactly as in the holomorphic-matrices-under-similarity-case) positive results provided the dimension of the stabilization subgroups $\{S \in GL_n(\mathbb{R}) \mid \Sigma(\sigma)^S = \Sigma(\sigma)\}$ is constant as a function of σ, cf. [HP].

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ECONOMETRIC INSTITUTE

ON FAMILIES OF LINEAR SYSTEMS: DEGENERATING PHENOMENA

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ON FAMILIES OF LINEAR SYSTEMS: DEGENERATION PHENOMENA

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ABSTRACT

In this paper we study families of linear dynamical systems $\dot{x} = Fx + Gu$, $y = Hx + Ju$, where the matrices F,G,H,J depend on a parameter c. Let V_c be the associated input/output operator. Then this paper contains results about what operators can arise as limits of the V_c as $c \to \infty$.

CONTENTS

ON FAMILIES OF LINEAR SYSTEMS: DEGENERATION PHENOMENA

by

Michiel Hazewinkel

1. INTRODUCTION.

This paper is concerned with an aspect of the theory of families of linear dynamical systems rather than single systems, viz. degeneration phenomena. As such it is part of a general program (briefly discussed in [Haz 3]) which consists of trying to carry through for families of systems (and hence systems over rings) all the nice results and constructions which one has for single systems over fields (or finding out how and why these results and constructions break down in this more general setting). This includes a systematic investigation of which constructions are continuous in the system parameters; that is, which constructions and calculations are stable (more or less) with respect to small perturbations or errors in the system parameters, a topic which obviously deserves at least some attention in a world full of uncertain measurements. And in turn this topic includes trying to find out what may happen to systems and associated objects when certain parameters go to zero (or infinity, or ...), which is the topic of this paper.

Still more motivation for studying families rather than single systems can be found in [Haz 3] and some results concerning other aspects of the theory of families (then the degeneration phenomena discussed below) can be found in [Haz 4] (fine moduli spaces, continuous canonical forms) and [HP] (pointwise-local-global isomorphism problems).

Here we discuss degeneration phenomena. That is, suppose there is given a family of systems

$$(1.1) \qquad \Sigma(c)\colon \dot{x} = Fx + Gu,\ y = Hx + Ju$$

where the matrices F, C, H, J depend on a parameter c. What can be said about the limit as $c \to \infty$. For example let V_c be the input/output operator of $\Sigma(c)$

$$(1.2) \qquad V_c\colon u(t) \mapsto y(t) = \int_0^t He^{F(t-\tau)}Gu(\tau)d\tau$$

and suppose that as $c \to \infty$ the operators V_c converge (in some suitable sense) to some operator V. What can be said about V? E.g. can V still be viewed as the input/output operator of some sort of processing device?

There are a number of reasons for being interested in such degeneration phenomena, some of which can be characterized by the key words or phrases: identification, high-gain feedback, almost F mod G invariant subspaces (and almost disturbance decoupling), dynamic observers (and invertability).

1.3. Identification. Suppose we have given some sort of input/output device which is to be modelled "as best as possible" by means of a linear dynamical system (1.1) of dimension n. Now if $S \in GL_n(\mathbb{R})$, then a system $\Sigma = (F,G,H,J)$ and $\Sigma^S = (SFS^{-1},SG,HS^{-1},J)$ have the same input/output operator. Let M be the space of orbits of this action of $GL_n(\mathbb{R})$ on the space L of all n-dimensional systems (with a given number of inputs and outputs). The best we can do on the basis of input/output data alone is to identify the orbit of Σ (and even that is not true if Σ is not completely observable and completely reachable, a fact which can be expected to cause a fair amount of extra trouble). Thus we are trying to identify a point of M and we can picture identification as finding (or guessing at) a sequence of points in M representing better and better identifications as more and more data come in. From this point of view the question naturally arises. Does a "converging" sequence of points in M necessarily have a limit in M. The answer is no. It is perfectly possible for a sequence of linear dynamical systems (1.1) to have a limiting input/output behaviour which is not the input/output behaviour of any system like (1.1) as the following example shows

$$(1.4) \qquad \Sigma(c): \dot{x} = \begin{pmatrix} -c & -c \\ 0 & -c \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \quad y = (c^2, 0)x$$

(one input/one output, dimension 2). Let U be a smooth bounded function on $\mathbb{R}$ with compact support in $(0,\infty)$, then if $y_c = V_c u$ a little partial integration shows that $\lim_{c\to\infty} y_c(t) = \frac{d}{dt}u(t)$, uniformly in t on bounded t intervals, and $\frac{d}{dt}$ cannot possibly be the input/output operator of a system (1.1), (e.g. because $\frac{d}{dt}$ is not bounded on smooth bounded functions in $[0,1]$ while all the V_Σ are bounded operators).

The presence of these holes is by no means the only difficulty in identification caused by the nontrivial topology and geometry of M. For some more remarks concerning this topic cf. [Haz. 2] (though the point of view I took there is still a good deal too optimistic) and also [BK].

1.5. High-gain feedback. Consider a system with output feedback loop

$$(1.6) \qquad \dot{x} = Fx + Gu, \; y = Hx, \; u = Ly$$

What happens when L or certain entries of L go to infinity? For instance in [YKU] it is shown in the case of a large scalar gain factor $L = g$ and under some additional hypothesis the system (1.6) can be transformed into the standard singular perturbation framework

$$(1.7) \qquad \dot{x}_1 = F_{11}x_1 + F_{12}x_2, \; \mu\dot{x}_2 = F_{21}x_1 + F_{22}x_2, \; \mu \to g^{-1}$$

(with $F_{21} = 0$ in the case considered in [YKU], so that there is a separation of slow and fast modes; more precisely there is a fast subsystem which in the setting of [YKU] is asymptotically stable (if μ is small enough) feeding into a slow system). Of course setting $\mu = 0$ in (1.7) yields little information about (1.7) for small μ and the idea is rather to study (1.7) and (1.6) as perturbations of the limit behaviour as μ goes to zero or various coefficients of L go to infinity. In the setting of [YKU] the limit input/output operator is the zero operator, but in general this need not be the case, and one may hope that on the basis of some knowledge about what limit operators can arise it will prove possible to obtain some results on the lines of [YKU] and related papers in more general situations.

For some motivation for studying (very) high gain feedback cf. [YKU] and some of the references therein, cf. also below in 1.8.

1.8. Almost F mod G invariant subspaces and almost disturbance decoupling. An F mod G invariant subspace for $\dot{x} = Fx + Gu$ is a subspace V of the state space such that one is in it one can stay in it. As is wellknown (cf. [Won]) these subspaces "solve" the disturbance decoupling problem. An almost F mod G invariant subspace is one such that once one is in it one can stay arbitrarily close to it, and these spaces "solve" an almost disturbance decoupling problem, which turns out to be important especially when the disturbances (partly) come in on the same channels as the inputs (cf. [Wil1, Wil2]).

A subspace V of dimension r is almost F and G invariant if and only if there is for every $\varepsilon > 0$ a feedback matrix K_ε such that $(F+GK_\varepsilon)$ V is within ε of V (in a suitable sense), and if V is almost F mod G invariant but not F mod G invariant, K_ε will not remain finite as $\varepsilon \to 0$. Thus implementing a decoupling by means of an almost F mod G invariant subspace will give rise to a family of systems.

(1.9) $$\dot{x} = (F+GK_\varepsilon)x + Gu + G'v,\ y = Hx$$

where K_ε does not necessarily remain finite as $\varepsilon \to 0$.

1.10. Dynamic observers. In [BM1], [BM2] Basile and Marro consider the problem of constructing observers for the state of a system (1.1) when the inputs are unkown. For this it is advantagous to have differential operators (cf. loc. cit) and these, as is suggested by the example (1.4), may be approximated by systems (1.1) (of comparable rank), thus giving us arbitrarily good approximate observers of the form (1.1).

1.11. As we shall see the limit operators as $c \to \infty$ of the input/output operators V_c of a family of systems $\Sigma(c)$ are necessarily of the form $V_\Sigma + L(D)$, where Σ is a system (1.2) (and V_Σ its input/output operator) and where L(D) is a polynomial matrix (with constant coefficients) in the differentiation operator $D = \frac{d}{dt}$. I.e. the possible limit operators are the input/output operators of systems of the type

(1.12) $$\dot{x} = Fx + Gu,\ y = Hx + J(D)u$$

where J(s) is a matrix of polynomials, argueing that this wider class of systems is in some ways a more natural class to study then the class of systems (1.1), cf. also [Ros1,Ros2].

2. STATEMENT OF THE THEOREMS.

The first thing to do is to specify in what sense we shall understand the phrase "the family of input/output operators L_c converges to the operator L as $c \to \infty$". And, in turn, this means that we must describe the spaces of functions between which these operators act.

2.1. The spaces $\mathcal{F}^{(c)}(\mathbb{R}^r)$ and $\mathcal{F}_o(\mathbb{R}^r)$. The elements of $\mathcal{F}^{(o)}(\mathbb{R}^r)$ are all smooth functions $z: \mathbb{R} \to \mathbb{R}^r$ with support in $(0,\infty)$ and of no more than exponential growth. Here the support of a function z is as usual defined as the closure of the set of all $t \in \mathbb{R}$ where $z(t) \neq 0$. Thus $z \in \mathcal{F}^{(o)}(\mathbb{R}^r)$ iff there are an $\varepsilon > 0$, an $M > 0$, and $b \geq 0$ such that $z(t) = 0$ for $t \leq \varepsilon$ and

(2.2) $$||e^{-bt}z(t)|| \leq M \text{ for all } t$$

(Both ε and b may depend on the function z). This class of functions includes the smooth functions of slow growth with support in $(0,\infty)$ (cf. [Ze, chapter IV]), which space in turn contains the subspace $\mathcal{F}_o(\mathbb{R}^r)$ of smooth functions with compact support in $(0,\infty)$.

A sequence of funtions $z_c \in \mathcal{F}^{(o)}(\mathbb{R}^r)$ is said to converge to $z \in \mathcal{F}^{(o)}(\mathbb{R}^r)$ if there is a b such that

(2.3) $$\lim_{c\to\infty} \sup_t ||e^{-bt}(z_c(t)-z(t))|| = 0$$

Note that (2.3) in any case implies that the functions $z_c(t)$ converge to z(t) uniformly in t on bounded t intervals.

This defines a topology on $\mathcal{F}^{(o)}(\mathbb{R}^r)$, which is in fact the inductive limit topology defined by the inductive system of normal topological vector spaces

(2.4) $$\mathcal{F}_b^{(o)}(\mathbb{R}^r),\ i_{b,b'}: \mathcal{F}_b^{(o)}(\mathbb{R}^r) \to \mathcal{F}_{b'}^{(o)}(\mathbb{R}^r),\ b' \geq b$$

where for a given $b \in \mathbb{R}$

(2.5) $$\mathcal{F}_b^{(o)}(\mathbb{R}^r) = \{z \in \mathcal{F}^{(o)}(\mathbb{R}^r) |\ \sup_t ||e^{-bt}z(t)|| =: ||z||_b < \infty\}$$

with the norm $||z||_b$, and where $i_{b,b'}$ is defined by $z(t) \mapsto e^{(b'-b)t}z(t)$.

The space $\mathcal{F}^{(o)}(\mathbb{R}^r)$ tries hard to be complete in the sense of the following lemma.

2.6. Lemma. Let $\eta > 0$ and let $z_c \in \mathcal{F}^{(o)}(\mathbb{R}^r)$ be a sequence of functions with support in $[\eta,\infty)$ for all c. Suppose that there is a $b \in \mathbb{R}$ such that for all $\varepsilon > 0$ there is a c_o such that

$$(2.7)\qquad \sup_t \|e^{-bt}(z_c(t)-z_{c'}(t))\| < \varepsilon \quad \text{for all } c,c' \geq c_o$$

Then the z_c converge to a function $z \in \mathfrak{F}^{(o)}(\mathbb{R}^r)$ with support in $[\eta,\infty)$ as $c \to \infty$ (where the convergence is in the sense of (2.3)).

Proof. Let $z(t)$ be the pointwise limit of $z_c(t)$ as $t \to \infty$ (which clearly exists by (2.7). Then supp $z(t) \subset [\eta,\infty)$ and $z_c(t)$ converges to $z(t)$ uniformly on bounded t intervals (again by (2.7)). It follows that $z(t)$ is smooth. Take $\varepsilon = 1$ and let c_1 be such that (2.7) holds for this ε with $c_o = c_1$. Let $z_{c_1}(t) \in \mathfrak{F}^{(o)}_{b_1}(\mathbb{R}^r)$. We can assume $b_1 \geq b$. Then, using $b_1 \geq b$,

$$e^{-b_1 t}\|z(t)\| \leq e^{-b_1 t}\|z_{c_1}(t)\| + e^{-bt}\|z_{c_1}(t)-z_{c'}(t)\| + e^{-bt}\|z_{c'}(t)-z(t)\|$$

Choosing c' depending on t such that $\|z_{c'}(t)-z(t)\| < 1$ it follows that $z(t) \in \mathfrak{F}^{(o)}_{b_1}(\mathbb{R}^r) \subset \mathfrak{F}^{(o)}(\mathbb{R}^r)$, proving the lemma.

Just what $b \in \mathbb{R}$ is used in (2.3) is largely irrelevant. Firstly if (2.3) holds for a given b then it still holds with b replaced by $b' \geq b$. Secondly if (2.3) holds and $z \in \mathfrak{F}^{(o)}_{b'}(\mathbb{R}^r)$ then $z_c \in \mathfrak{F}^{(o)}_{b''}(\mathbb{R}^r)$ for all large enough c where $b'' = \max(b,b')$. The converse of this: "if $z_c(t) \in \mathfrak{F}^{(o)}_{b'}(\mathbb{R}^r)$ for all large enough c then $z(t) \in \mathfrak{F}^{(o)}_{b''}(\mathbb{R}^r)$ with $b'' = \max(b,b')$" follows as in the lemma. Thirdly and lastly it does not really matter if one uses "too big a b" in (2.3). Indeed, $z(t)$ as the pointwise limit of the $z_c(t)$ is of course independant of b. What (2.3) does is to require a certain mild uniformity about the way the limit is approached. (It is, incidentally, perfectly possible for a sequence of functions $z_c(t) \in \mathfrak{F}^{(o)}_b(\mathbb{R}^r)$ to converge to zero when considered as elements of $\mathfrak{F}^{(o)}_{b'}(\mathbb{R}^r)$ for $b' > b$ while not converging when considered as a sequence in $\mathfrak{F}^{(o)}_b(\mathbb{R}^r)$; take for example $z_c(t) = 0$ for $t \leq c$, $z_c(t) = e^{bt} - e^{bc}$ for $t \geq c$, suitably smoothed.)

2.8. <u>The spaces</u> $\mathfrak{F}(\mathbb{R}^r)$. For the purposes below the spaces $\mathfrak{F}^{(o)}(\mathbb{R}^r)$ are still too big to be suitable as input spaces (essentially because we shall want differentiation" to be a continuous operator). On the other hand $\mathfrak{F}_o(\mathbb{R}^r)$, while eminently suitable as an input function space is not large enough to accommodate output functions. As we shall need to be able to use the outputs of one dynamical system as the inputs of another, we need an intermediate space. A suitable one is

$$(2.9)\qquad \mathfrak{F}(\mathbb{R}^r) = \{z \in \mathfrak{F}^{(o)}(\mathbb{R}^r) \mid z^{(k)} \in \mathfrak{F}^{(o)}(\mathbb{R}^r) \text{ for all } k = 0,1,2,\dots\}$$

where $z^{(k)}$ denotes the k-th derivative of z. We give $\mathfrak{F}(\mathbb{R}^r)$ the topology determined by $z_c \to z$ as $c \to \infty$ iff $z_c^{(k)} \to z^{(k)}$ for all $k = 0,1,2,\dots$ in $\mathfrak{F}^{(o)}(\mathbb{R}^r)$. Thus the family z_c converges to z as $c \to \infty$ iff there are real numbers $b_o, b_1, \dots$ such that for all k

$$\lim_{c\to\infty} \sup_t e^{-b_k t}\|z_c^{(k)}(t) - z^{(k)}(t)\| = 0$$

When dealing with systems of dimension $\leq n$ only, one can also work with $\mathfrak{F}^{(n)}(\mathbb{R}^r) = \{z \in \mathfrak{F}^{(o)}(\mathbb{R}) \mid z^{(k)} \in \mathfrak{F}^{(o)}(\mathbb{R}),\ k = 0,1,\dots,n+1\}$.

2.10. <u>Convergence of input/output operators</u>. Now let $\Sigma = (F,G,H,J)$ be a linear dynamical system with direct feed through term

$$(2.11)\qquad \dot{x} = Fx + Gu,\ y = Hx + Ju,\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^p,\ u \in \mathbb{R}^m$$

where F, G, H, J are real matrices of the appropriate dimensions (independent of t). Then the associated input/output operator is defined by

$$(2.12)\qquad V_\Sigma : u(t) \mapsto y(t) = Ju(t) + \int_0^t He^{F(t-\tau)}Gu(\tau)d\tau$$

Let $\mathcal{U} = \mathfrak{F}(\mathbb{R}^m)$, $\mathcal{Y} = \mathfrak{F}(\mathbb{R}^p)$, $\mathcal{U}_o = \mathfrak{F}_o(\mathbb{R}^m)$, $\mathcal{Y}_o = \mathfrak{F}_o(\mathbb{R}^p)$. Then V_Σ is a continuous linear operator $\mathcal{U} \to \mathcal{Y}$. Indeed if $u \in U$ is such that $\|u\|_b < \infty$ and if $b' > \max\{\mathrm{Re}\lambda, 0\}$ where λ runs through the eigenvalues of F then $\|V_\Sigma(u)\|_{b+b'} < \infty$.

Thus for every $b \geq 0$ there is a $b' \geq 0$, usually necessarily larger than b, such that V_Σ maps $\mathfrak{F}^{(o)}_b(\mathbb{R}^m)$ into $\mathfrak{F}^{(o)}_{b'}(\mathbb{R}^p)$, with b' depending on Σ. Thus, when dealing with families of systems one is practically forced to use the union of the all the $\mathfrak{F}^{(o)}_b(\mathbb{R}^p)$, i.e. $\mathfrak{F}^{(o)}(\mathbb{R}^p)$, and if one would like differential operators to be continuous one is almost obliged to work with $\mathfrak{F}(\mathbb{R}^p)$ and $\mathfrak{F}(\mathbb{R}^m)$. From now on we fix the dimensions m,n,p of the systems (2.11) which we are considering. Let L denote the space of all systems (2.11). I.e. L is the space of all real quadruples of matrices (F,G,H,J) of the dimensions $n \times n$, $n \times m$, $p \times n$, $p \times m$ respectively.

We shall use L^{co}, L^{cr}, $L^{co,cr}$ to denote the subspaces of completely observable, (abbreviated co), resp. completely reachable (cr), resp. completely reachable and completely observable systems.

We now define

2.13. Definition. The family of systems $\Sigma(c)_c \subset L$ converges in input/output behaviour to an operator V iff for all $u \in U$ the functions $V_{\Sigma(c)}u$ converge to Vu in $\mathcal{Y}$ as $c \to \infty$.

Let $\text{supp}(u) \subset [\eta,\infty)$ (such an η necessarily exists because $\text{supp}(u) \subset (0,\infty)$ and supp(u) is closed by definition). Then $\text{supp}\, V_{\Sigma(c)}(u) \subset [\eta,\infty)$. It follows by lemma 2.6 that one can decide whether the family $(\Sigma(c))$ converges without mentioning (or knowing) the limit operator V. The family $(\Sigma(c))_c$ converges in input/output behaviour iff there are for every $u \in U$ a sequence of numbers $b_o, b_1, b_2, \ldots$ such that for every $\varepsilon > 0, k = 0,1,2,\ldots$ there is a $c(\varepsilon,k)$ such that

$$\sup_t (e^{-b_k t} \| D^k V_{\Sigma(c)} - D^k V_{\Sigma(c')} \|) < \varepsilon \quad \text{if } c,c' \geq c(k,\varepsilon) \tag{2.14}$$

where D is the differentiation operator $D = \frac{d}{dt}$. Thus if $(\Sigma(c))$ converges in input/output behaviour (in the sense that (2.14) holds) then there is a well-defined limit operator V. (This uses of course (cf. (2.14)) that D is a continuous operator $\mathcal{U} \to \mathcal{U}$). Whether this limit operator V is continuous is unclear at this stage. (It is though, as will be shown below in section 5).

2.15. Differential operators. Let $\mathcal{U}$ and $\mathcal{Y}$ be as above. Then a (matrix) differential operator (in this paper) is an operator of the form

$$V(D): u(t) \mapsto y(t),\ y_j(t) = \sum_{i=1}^{m} v_{ji}(D) u_i(t)$$

where $v_{ji}(D)$ is a polynomial with constant real coefficients in $D = \frac{d}{dt}$. Every polynomial V(s) (of size p x m) thus defines a continuous linear operator $\mathcal{U} \to \mathcal{Y}$.

2.16. The scalar case. If $m = 1 = p$, i.e. if we are dealing with one input and one output the main theorem of this paper says that

2.17. Theorem. Let $(\Sigma(c))$ be a family of one input/one output linear dynamical systems (2.11) of dimension $\leq n$ converging in input/output behaviour to the operator $V: \mathcal{U} \to \mathcal{Y}$. Then there exist a system Σ and a polynomial L(s) such that $V = V_\Sigma + L(D)$, where moreover

$\dim(\Sigma) + \text{degree } L(s) \leq n$. It follows in particular that the limit operator V is continuous. Inversely, if V is an operator of the form $V = V_\Sigma + L(D)$ where L(s) is a polynomial of degree $\leq n - \dim(\Sigma)$, then there exists a family $(\Sigma(c)) \subset L^{co,cr}$ such that $\Sigma(c)$ converges in input/output behaviour to V.

In case one wants to restrict oneself to systems (2.11) with $J = 0$ the theorem remains essentially the same except that the essential inequality $\dim(\Sigma) + \text{degree}(L(s)) \leq n$ gets replaced by $\dim(\Sigma) + \text{degree}(L(s)) \leq n-1$ (where by definition $\text{degree}(0) = -1$). This is stated and proved (more or less) in [Haz 1] and the proof readily adapts to a proof of the present theorem. In section 5 below a different proof of theorem 2.17 is given which also covers the multivariable case.

2.18. Degree of a matrix polynomial (differential operator).

Obviously if $\Sigma(c)$ is a family of systems of dimension $\leq n$ which converges to the p x m matrix differential operator L(D) then all the entries of L(s) have degree $\leq n$ (by the result in the scalar case). One might think that inversely every such operator arises as a limit of systems $\leq n$. This ,however, is not the case as the example.

$$L(D) = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \tag{2.19}$$

shows. One shows readily by explicit calculation that the operator (2.19) cannot arise as a limit of one-dimensional systems. A more sensitive definition of "degree" is needed.

2.19. Definition. Let L(s) be a matrix polynomial. Then we define

$$\deg L(s) = \max_M(\text{degree}(M)) \tag{2.20}$$

where M runs over all the minors of L. This agrees with the MacMIllan degree of a polynomial matrix, (lemma 4.10, or cf. [AV], section 3.6, properties 5 and 10).

2.21. The multivariable case. In the case of more inputs more outputs the main theorem now is precisely analogous to theorem 2.17. I.e.

2.22. Theorem. Let $\Sigma(c)$ be a family of n dimensional systems with m inputs and p outputs. Suppose that $\Sigma(c)$ converges in input/output behaviour to the operator $V: \mathcal{U} \to \mathcal{Y}$ as $c \to \infty$. Then there exist a system Σ and a p x m matrix polynomial $L(s)$ such that $V = V_\Sigma + L(D)$ (so that V is continuous) and moreover $\dim(\Sigma)$ + degree $L(s) \leq n$. Inversely if V is an operator of the form $V_\Sigma + L(D)$ with $\dim(\Sigma)$ + degree $L(s) \leq n$, then there exists a family of completely observable and completely reachable systems $\Sigma(c)$ of dimension $\leq n$ which converges in input/output behaviour to V as $c \to \infty$.

The proof of the first half of the theorem uses the continuity (in this case) of the Laplace transforms and the upper semicontinuity of the MacMIllan degree (theorem 4.16) and thus gives us (besides lemma 4.10) yet another characterization of the MacMillan degree of a matrix of rational functions.

2.23. Theorem. Let $L(s)$ be a matrix of rational functions. Then the MacMillan degree of $L(s)$ is $\leq n$ iff there exists a sequence $L_c(s)$ of proper rational function matrices such that $L_c(s)$ converges to $L(s)$ for $c \to \infty$ pointwise in s for infinitely many values of s. Moreover one can see to it that the poles of $L_c(s)$ fall into two sets one equal (together with multiplicities) to the set of poles $\neq \infty$ of $L(s)$ while the remaining poles of $L_c(s)$ all go to $-\infty$ as $c \to \infty$.

It is not true, however, that one can always obtain $L(s)$ as a limit of the $L_c(s)$ in the sense of the mappings on the Riemann sphere that these matrices of rational functions define. This in fact only happens when $L(s)$ is itself proper.

To prove theorem 2.23 without the extra requirement that the remaining poles of $L_c(s)$ go to $-\infty$ as c goes to ∞ is quite easy (Proposition 4.18). The extra requirement complicates things considerably and I know of no direct proof except for certain special, albeit generic, cases. (Like "the matrix of coefficients of maximal powers of s in each row is of maximal rank"). Another corollary of the proof of the second half of theorem 2.22 is

2.24. Corollary. Let $L(s)$ be a polynomial matrix of size p x m. Then $L(s)$ has degree $\leq n$ if and only if it can be obtained from the zero matrix by means of the operations.

(i) addition of a matrix of constants

(ii) multiplication on the left by a nonsingular polynomial p x p matrix of degree 1

(iii) multiplication on the right by a nonsingular m x m matrix of constants

where one uses at most n times an operation of type (ii).

There is of course an analogous statement with right instead of left in (ii) and left instead of right in (iii), and also an analogous statement where in both (ii) and (iii) multiplications on both sides are allowed.

3. ON LIMITS OF RATIONAL FUNCTIONS.

The degree of a rational function $T(s) = q(s)^{-1}p(s)$, $p(s), q(s) \in k[s]$ with no common factors is equal to $\delta(T) = \max(\delta(p),\delta(q))$ where the degree of a polynomial is defined as usual. We shall need the following intuitively obvious fact.

3.1. Proposition. Let $T_c(s)$ be a sequence of rational functions of degree $\leq n$. Suppose that $\lim_{c\to\infty} T_c(s)$ exists (and is finite) for infinitely many s. Then there exists a rational function $T(s)$ of degree $\leq n$ such that $\lim_{c\to\infty} T_c(s) = T(s)$ for all but finitely many s (and if the $T_c(s)$ and $T(s)$ are interpreted as functions $\mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ then $T_c(s)$ converges to $T(s)$ in the compact open topology).

Proof. Write

$$(3.2) \qquad T_c(s) = \frac{p_c(s)}{q_c(s)} = \frac{a_n(c)s^n + a_{n-1}(c)s^{n-1} + \ldots + a_1(c)s + a_o(c)}{b_n(c)s^n + b_{n-1}(c)s^{n-1} + \ldots + b_1(c)s + b_o(c)}$$

and associate to $T_c(s)$ the point $\psi(c) \in \mathbb{P}^{2n+1}(\mathbb{C})$ with the homogeneous coordinates $(a_n,\ldots,a_o,b_n,\ldots,b_o)$. Note that this is well defined because the coefficients of $p_c(s)$ and $q_c(s)$ are well defined up to a common scalar factor. (This map is not continuous if the space of all rational functions of degree $\leq n$ is given the compact open topology of maps $\mathbb{C} \to \mathbb{P}^1(\mathbb{C})$; but it is continuous on the open subspace of function of degree n, and on the subspaces of functions of fixed degree i).

Let $M \subset \mathbb{P}^{2n+1}(\mathbb{C})$ be the subspace of all points $(x_n,\ldots,x_o,y_n,\ldots,y_o) \in \mathbb{P}^{2n+1}(\mathbb{C})$ such that at least one y_i is unequal to zero. Because $\mathbb{P}^{2n+1}(\mathbb{C})$ is compact the sequence $\{\psi(c)\}$ has limit points.

3.3. Lemma. If $\lim_{c\to\infty} T_c(s)$ exists for infinitely many s then all limit points of the sequence $\{\psi(c)\}$ are in M.

Proof. Suppose that $\lim_{c\to\infty} T_c(s) = T(s) \in \mathbb{C}$, and suppose that $\{\psi(c)\}$ has a limit point in $\mathbb{P}^{2n+1}(\mathbb{C}) \setminus M$. Let this limit point be $x = (a_n, \ldots, a_{i+1}, 1, 0, \ldots, 0)$. Taking a subsequence we can assume that $\{\psi(c)\}$ converges to x. For large enough c we then have $a_i(c) \neq 0$ and multiplying both $p_c(s)$ and $q_c(s)$ with $a_i(c)^{-1}$ we can assume that $a_i(c) = 1$ for all c. We then have for all c

$$(3.4) \qquad a_n(c)s^n + \ldots + a_{i+1}(c)s^{i+1} + s^i + a_{i-1}(c)s^{i-1} + \ldots + a_o(c) = \\ = T_c(s)(b_n(c)s^n + \ldots + b_o(c))$$

with

$$(3.5) \qquad \begin{aligned} &\lim_{c\to\infty} b_j(c) = 0, \ j = 0, \ldots, n \\ &\lim_{c\to\infty} a_j(c) = 0, \ j = 0, \ldots, i-1 \\ &\lim_{c\to\infty} a_j(c) = a_j, \ \ j = i+1, \ldots, n \end{aligned}$$

Taking the limit as $c \to \infty$ in (3.4) and using the relations (3.5) one finds because $\lim_{c\to\infty} T_c(s) = T(s) \neq \infty$

$$(3.6) \qquad a_n s^n + \ldots + a_{i+1}s^{i+1} + s^i = 0$$

and there are only finitely many s for which this can hold. Thus there can be no limit points of $\{\psi(c)\}$ in $\mathbb{P}^{2n+1}(\mathbb{C}) \setminus M$ if $\lim_{c\to\infty} T_c(s)$ exists (and is finite) for infinitely many s.

The proof of proposition now continues as follows. Let $x \in M \subset \mathbb{P}^{2n+1}(\mathbb{C})$ $x = (x_n, \ldots, x_o, y_n, \ldots, y_o)$. Because at least one of the $y_i \neq 0$ the expression

$$(3.7) \qquad T_x(s) = \frac{x_n s^n + \ldots + x_1 s + x_o}{y_n s^n + \ldots + y_1 s + y_o}$$

is well-defined for all but finitely many s. Now let $x \in M$ be a limit point of $\{\psi(c)\}$. Let i be the largest index such that $y_i \neq 0$. Multiplying all coordinates with y_i^{-1} if necessary, we can assume $y_i = 1$. Take a subsequence of $\{\psi(c)\}$ which converges to x. For large enough c we then have $b_i(c) \neq 0$. Multiplying both $p_c(s)$ and $q_c(s)$ with $b_i(c)^{-1}$ we then obtain sequence of rational functions.

$$(3.8) \qquad T_c(s) = \frac{a_n(c)^s s^n + \ldots + a_1(c)s + a_o(c)}{b_n(c)s^n + \ldots + s^i + \ldots + b_1(c)s + b_o(c)}$$

such that as $c \to \infty$.

$$(3.9) \qquad a_j(c) \to x_j, \ b_j(c) \to y_j, \ j = 0, 1, \ldots, n$$

It follows that $\lim_{c\to\infty} T_c(s) = T_x(s)$ for all but fnitely many s, where the limit is a priori over the subsequence. In turn this says that $\lim_{c\to\infty} T_c(s) = T_x(s)$ for all but finitely many s of the infinitely many s for which $\lim T_c(s)$ was assumed to exist.

This holds for all limit points of $\{\psi(c)\}$, hence if x' is a second limit point of $\{\psi(c)\}$ then $T_x(s) = T_{x'}(s)$ for infinitely many s so that $T_x(s) = T_{x'}(s)$ if both x,x' are limit points of $\{\psi(c)\}$, and this in turn says that $\lim_{c\to\infty} T_c(s) = T_x(s)$ for all but finitely many s, where now we are dealing with the original sequence $\{T_c(s)\}$. This concludes the proof of the proposition (except for the last statement between brackets which is easy because by the above the convergence $T_c(s) \to T_x(s)$ really means that the coefficients, suitably normalized, converge).

3.10. Corollary. (of the proof). Let $T_c(s) \to T(s)$ as $c \to \infty$ and let $T_c(s) = q_c(s)^{-1}p_c(s)$, $T(s) = q(s)^{-1}p(s)$ with no common factors. Suppose that degree $p_c(s) \leq n'$ for all c. Then degree $p(s) \leq n'$. This follows immediately because (using the notations of the proof) after a suitable normalization and for c large enough the coefficients of $p_c(s)$ converge to the coefficients of $p_x(s)$ where $p_x(s)$ is the numerator of (3.7), and because $q(s)^{-1}p(s) = T(s) = T_x(s) = q_x(s)^{-1}p_x(s)$ where $q_x(s)$ is the denominator of (3.7). So degree $p_x(s) \leq$ degree $p_c(s)$ for all large enough c. (Of course $p_x(s)$ and $q_x(s)$ may have common factors so that degree p(s) may be smaller than $\lim_{c\to\infty}\inf(\text{degree}(p_c(s)))$.

4. ON THE DEGREE OF RATIONAL MATRICES.

Recall that the MacMillan degree $\delta(T)$ of a matrix of rational functions T(s) can be defined in a variety of ways ([Ka], [AV, section 3.6], [Ros, section 3.4]). First let T(s) be proper, i.e. $\lim_{s\to\infty} T(s)$ exists, then $\delta(T) = \nu(T)$, which is by definition the minimal dimension of a realization (F,G,H,J) of T(s). If T(s) is not proper write

(4.1) $$T(s) = T_-(s) + T_1 s + T_2 s^2 + \dots + T_r s^r,\quad V(s) = T_1 s^{-1} + \dots + T_r s^{-r}$$

where $T_-(s)$ is the proper part of T(s). Then V(s) is also proper (in fact strictly proper, meaning that $\lim_{s\to\infty} V(s) = 0$) and we define

(4.2) $$\delta(T) = \nu(T_-) + \nu(V)$$

This definition shows that if $T(s) = T_-(s) + T_+(s)$, where $T_-(s)$ is proper and $T_+(s)$ is polynomial then

(4.3) $$\delta(T) = \delta(T_+) + \delta(T_-)$$

(It does not matter how the "constant part" of T(s) is split up between T_- and T_+).

Another way to obtain $\delta(T)$ goes as follows. (cf. [Kal]). Let T(s) be a p x m matrix of rational functions. For each m x p matrix of constants K write

(4.4) $$\det(I_m + KT(s)) = b_K(s)^{-1} a_K(s)$$

where I_m is the m x m identity matrix and $a_K(s)$, $b_K(s)$ are polynomials without common factors. Let

(4.5) $$\delta_K(T) = \text{degree}(a_K(s))$$

Then one has the proposition (cf. [Kal])

(4.6) $$\delta(T) = \max_K \delta_K(T)$$

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We shall need a few elementary properties of $\delta(T)$. If A and B are matrices of constants such that AT(s)B is defined then (cf. [AV, (3.6.6)]

(4.7) $$\delta(ATB) \le \delta(T)$$

(which is also immediately obvious from definition 4.2.

Now let T'(s) be obtained from T(s) by augmenting T(s) with some rows and columns of constants. Then

(4.8) $$\delta(T') = \delta(T)$$

This is seen as follows. Let T(s) and V(s) be as in (3.1) and let $T'_-(s)$ and V'(s) be the analogous matrices for T'(s). Then if (F,G,H,J) realizes $T_-(s)$ a realization for $T'_-(s)$ is obtained by adding some zero columns to G, some zero rows to H and by augmenting J with the same rows and columns of constants as were used to obtain T'(s) from T(s). Similarly a realization (F_1,G_1,H_1,J_1) for V(s) can be changed in a realization of the same dimension for V'(s) by augmenting G_1 with zero columns, H_1 with zero rows and J_1 with both zero rows and zero columns. This shows that $\delta(T') \le \delta(T)$. The opposite inequality follows from (4.7) because T(s) is a submatrix of T'(s).

A third result we need is. Let T(s) be square such that $\det(T(s)) \not\equiv 0$. Then (cf. e.g. [Ros1, theorem 7.2 on page 135)].

(4.9) $$\delta(T^{-1}) = \delta(T)$$

As an application of (4.8) and (4.9) we show (using a few tricks which will also be useful further on).

4.10. <u>Lemma</u>. Let T(s) be a matrix of polynomials. Then

(4.11) $$\delta(T) = \max_{M(s)} \{\text{degree}(\det(M(s))\}$$

where M(s) runs through all square submatrices of T(s).

Proof. Define $\delta'(T)$ as being equal to the right hand side of (4.11).

Then we have to prove that $\delta(T) = \delta'(T)$. Then the analogues of (4.7) and (4.8) also hold for δ', i.e.

(4.12) $$\delta'(ATB) \le \delta'(T), \quad \delta'(T') = \delta'(T)$$

To see this recall that a minor of a product of matrices is a sum of products of minors (of the same size) of the factors (cf. e.g. [Ros1, thm 1.3, page 5) and that a minor of a matrix T' obtained by adding a row of constants or column of constants to T is either a minor of T or a sum of minors (of one size smaller) of T with constant coefficients. This proves (4.12).

It follows that if A and B are invertible then $\delta'(ATB) = \delta'(T)$. So by taking A and B to be suitable permutation matrices we can assume that T is of the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

with $\deg(\det(T_{11})) = \delta'(T)$. Let the dimensions of T_{11}, T_{12}, T_{21}, T_{22} be respectively $r \times r$, $r \times (m-r)$, $(p-r) \times r$, $(p-r) \times (m-r)$. Let $T'(s)$ be the matrix

$$T'(s) = \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & I \\ 0 & I' & 0 \end{pmatrix}$$

where I is the $(p-r) \times (p-r)$ unit matrix and I' the $(m-r) \times (m-r)$ unit matrix. Then by (4.12)

(4.13) $$\delta'(T') = \delta'(T)$$

Also $\det(T') = \det(T_{11})$ so that degree $\det(T') \ge$ degree(M) for all minors M of T'. It follows that $T'(s)^{-1}$ is proper so that

(4.14) $$\delta(T'(s)^{-1}) = \nu(T'(s)^{-1})$$

At this stage we need one more property of the degree function which is essentially proved in [Ros1], cf. thm 4.3 on page 115, cf. also [MH, section 2]. Viz.

4.15. Lemma. Let T(s) be a $p \times m$ proper matrix of rational functions. Then there are polynomial matrices N(s), D(s), of sizes $p \times m$, $m \times m$ such that

(i) $T(s) = N(s)D(s)^{-1}$

(ii) N(s) and D(s) are right coprime, which means that there are polynomial matrices X(s), Y(s) such that $X(s)N(s) + Y(s)D(s) = I_m$.

Moreover N(s) and D(s) are unique up to a common unimodular right factor and $\nu(T(s)) = \deg(\det D(s))$.

(The last statement of the lemma is more usually stated for strictly proper T(s), i.e. matrices of rational functions T(s) such that $\lim_{s\to\infty} T(s) = 0$; the slight extension is immediate; indeed if T(s) is proper and $T(s) = J + \bar{T}(s)$, with $\bar{T}(s)$ strictly proper, $\bar{T}(s) = \bar{N}(s)\bar{D}(s)^{-1}$. Then $T(s) = N(s)D(s)^{-1}$ with $N(s) = J\bar{D}(s) + \bar{N}(s)$, $D(s) = \bar{D}(s)$, and if $\bar{X}(s)\bar{N}(s) + \bar{Y}(s)\bar{D}(s) = I_m$, then $X(s)N(s) + Y(s)D(s) = I_m$, with $X(s) = \bar{X}(s)$, $Y(s) = \bar{Y}(s) - \bar{X}(s)J$).

Continuing with the proof of lemma 4.10. Applying lemma 4.15 to T'(s) we find

(4.16) $$\nu(T'(s)^{-1}) = \text{degree}(\det(T'(s)))$$

So combining (4.8), 4.9), (4.12) - (4,14), (4.15) we have

$$\delta(T) = \delta(T') = \delta((T')^{-1}) = \nu((T')^{-1})$$

$$= \text{degree}(\det(T')) = \text{degree}(\det(T_{11}))$$

$$= \delta'(T)$$

which concludes the proof of lemma 4.10.

4.17. Theorem. (upper semicontinuity of $\delta(T)$). Let $T_c(s)$ be a sequence of matrices of rational functions of s. Suppose that the sequence converges to matrix of rational functions T(s) as $c \to \infty$ and suppose that $\delta(T_c(s)) \le n$ for all large enough c. Then $\delta(T) \le n$.

Here a sequence of matrices of rational functions is said to converge iff the sequences of entries converge in the sense of section 3 above; i.e. $T_c(s)$ converges as $c \to \infty$ iff $\lim_{c\to\infty} T_c(s)$ exists for infinitely many s and then the limit is necessarily a matrix of rational functions T(s) and $\lim_{c\to\infty} T_c(s) = T(s)$ for all but finitely many s.

The proof of the theorem is easy. We have for each m x n matrix of constants K that

$$\lim_{c\to\infty} \det(I_m+KT_c(s)) = \det(I_m+KT(s))$$

Hence using proposition 3.1 (which among other things contains the scalar case of theorem (4.16)), or rather using corollary 3.10, and using the second definition of the degree of a rational matrix discussed above (cf. (4.4) - (4.6), we have for large enough c (which may depend on K)

$$\delta_K(T) = \text{degree}(a_K(s)) \leq \text{degree}(a_{K,c}(s)) = \delta_K(T_c) \leq n$$

where

$$\frac{a_K(s)}{b_K(s)} = \det(I_m+KT(s)),\ \frac{a_{K,c}(s)}{b_{K,c}(s)} = \det(I_m+KT_c(s))$$

(without common factors). It follows that $\delta(T) = \max_K\{\delta_K(T)\} \leq n$.

It is now not difficult to prove theorem 2.23 without the extra requirement that the poles of $L_c(s)$ unequal to the finite poles of L(s) go to $-\infty$ as $c \to \infty$. Indeed the upper semicontinuity property of theorem 4.17 takes care of the "if" part. So let L(s) be of degree n. Write L(s) = A(s) + T(s), where T(s) is proper and A(s) is polynomial. Then $\delta(L) = \delta(T) + \delta(A)$. So if $A(s) = \lim_{n\to\infty} T_n(s)$, with $T_n(s)$ proper and $\delta(T_n(s)) \leq \delta(A(s))$ we will be done.

4.18. Proposition. Let A(s) be a polynomial matrix of degree δ. Then there exist a sequence of proper rational matrices $T_n(s)$ of degree $\leq \delta$ such that $\lim_{n\to\infty} T_n(s) = A(s)$.

Proof. By multiplying A(s) on the left and on the right with suitable invertible matrices we can assume that A is of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with $\deg(\det(A_{11})) = \delta$. As above let

$$A' = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I \\ 0 & I & 0 \end{pmatrix}$$

Then $\delta = \delta(A') = \text{degree} \det(A')$. Now let

$$T_n'(s) = nA'(nI+A')^{-1}$$

(Note that $(nI+A'(s))^{-1}$ exists if we assume, as we can, that $\delta > 0$). Then claerly for a fixed s, $\lim_{n\to\infty} T_n'(s) = A'(s)$. We claim that $T_n'(s)$ is proper for all but finitely many n. Indeed for a fixed n

(4.19) $$T_n' = nA'(nI+A')^{-1} = nA'(A')^{-1}(nA'^{-1}+I)^{-1} = ((A')^{-1}+n^{-1}I)^{-1}.$$

Now because $\delta(A') = \deg(\det(A'))$ we know that $(A')^{-1}$ is proper. Let $J = \lim_{s\to\infty} (A')^{-1}$. Then if $-n^{-1}$ is not an eigenvalue of J it follows from (4.19) that $\lim_{s\to\infty} T_n'(s)$ exists, proving that $T_n'(s)$ is proper for all but finitely many n.

Finally, by lemma (4.15), if $T_n'(s)$ is proper,

(4.20) $$\nu(T_n'(s)) \leq \deg(\det(nI+A'))$$

Now $\det(nI+A')$ is a polynomial in n whose coefficients are sums of minors of A'. Hence $\deg(\det(nI+A')) \leq \max_M \deg(M) = \delta(A') = \delta$ where M runs through the minors of A'.

Now let $T_n(s)$ be obtained from $T_n'(s)$ be removing the appropriate columns and rows. Then $\lim_{n\to\infty} T_n(s) = A(s)$, $T_n(s)$ is proper if $T_n'(s)$ is proper and $\delta(T_n) \leq \delta(T_n')$ proving proposition 4.18.

5. PROOF OF THE MAIN THEOREM.

5.1. First half of the proof of theorem 2.22. Let $\Sigma(c) \subset L$ be a family of systems of dimension n and suppose they converge in input/output behaviour. This means (cf. 2.10) that for every $u \in U$ the sequence of functions

(5.2) $$(V_{\Sigma(c)}u) \subset Y$$

converges. In turn this means (as in the proof of lemma 2.6) that there is a b such that for all sufficiently large c

(5.3) $$V_{\Sigma(c)}u \in \mathcal{F}_b^{(o)}(\mathbb{R}^p)$$

If $z \in \mathcal{F}_b^{(o)}(\mathbb{R}^p)$, then $\sup_t ||e^{-bt}z(t)|| < \infty$ so that

$$\int_0^\infty ||e^{-(b+1)t}z(t)||dt < \infty$$

which implies (cf. [Doe] or [Zem]) that z(t) is Laplace transformable and that $(\mathcal{L}z)(s)$ is defined for $Re(s) \geq b+1$.

Applying this to the $V_{\Sigma(c)}u$ we see that their Laplace transforms are well defined for $s \geq b+1$. This gives us a sequence of functions

(5.4) $$Y_c(s) = T_c(s)U(s)$$

where $Y_c(s)$ is the Laplace transform of $V_{\Sigma(c)}u$, $T_c(s)$ is the transfer function of $\Sigma(c)$ and $U(s)$ is the Laplace transform of $u(t)$.

The Laplace transform $\mathcal{L}$ is continuous when considered as an operator on the normed space $\mathcal{F}'_{b+1}(\mathbb{R}^p)$ consisting of all locally integrable functions such that

(5.5) $$\int_0^\infty ||e^{-(b+1)t}z(t)||dt < \infty$$

equipped with the norm defined by the integral (5.5), cf [Doe, Kap.III, §8]. As $\mathcal{F}_b^{(o)}(\mathbb{R}^p) \subset \mathcal{F}'_{b+1}(\mathbb{R}^p)$ is a continuous embedding it follows that the sequence (5.4) converges for $Re(s) \geq b+1$ as $c \to \infty$. Choosing various $u \in U$ judiciously this implies that the family of rational matrix functions $T_c(s)$ converges for infinitely many values of s. According

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to section 4 above this means that there is a rational matrix function T(s) such that

(5.6) $$\lim_{c\to\infty} T_c(s) = T(s)$$

and moreover $\delta(T) \leq n$ by the upper semicontinuity theorem 4.17. Write

(5.7) $$T(s) = T'(s) + L(s)$$

where T'(s) is proper and where L(s) is polynomial. Let Σ be a co and cr realization of T'(s). Consider the operator

(5.8) $$V = V_\Sigma + L(D)$$

Applying this operator to a $u \in U$ and taking the Laplace transform of the result (which can be done because $Vu \in \mathcal{Y}$ and all functions in $\mathcal{Y}$ are Laplace transformable) we find (for $Re(s) \geq b'+1$, for some $b' \geq b$)

$$(\mathcal{L}Vu)(s) = T'(s)U(s) + L(s)U(s) = \lim_{c\to\infty} T_c(s)U(s) = \lim_{c\to\infty} Y_c(s) =$$

$$= (\mathcal{L}(\lim_{c\to\infty} y_c))(s)$$

where $y_c = V_{\Sigma(c)}u$, and where we have again used the same continuity property of the Laplace transform. The Laplace transform being injective on the space of functions under consideration it follows that

$$Vu = \lim_{c\to\infty} V_{\Sigma(c)}u$$

for all $u \in U$. Thus the limit operator is indeed of the form $V = V_\Sigma + L(D)$ with $\dim(\Sigma) + \text{degree } L(s) = \delta(T) \leq n$, which finishes the proof of the first half of theorem 2.22.

To prove the second half we need some lemma's. If A is any matrix we use the following notation for its various minors:

$$A\begin{matrix} i_1,\dots,i_r \\ j_1,\dots,j_r \end{matrix}$$

denotes the determinant of the submatrix of A obtained by removing all rows except those with the indices $i_1, \dots, i_r$ and all columns except those with the indices $j_1, \dots, j_r$. Recall that the minors of a product matrix are given by

$$\text{(5.9)} \qquad (AB)^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum_{k_1,\dots,k_r} A^{i_1,\dots,i_r}_{k_1,\dots,k_r} \, B^{k_1,\dots,k_r}_{j_1,\dots,j_r}$$

5.10. <u>Lemma</u>. Let L(s) be a polynomial matrix of size p x m. Suppose that for a certain $1 \le r \le \min(p,m)$

$$\text{(5.11)} \qquad \deg L(s)^{1,\dots,r}_{1,\dots,r} \ge \deg L(s)^{2,\dots,r,j}_{1,\dots,r} \quad , \; j = r+1, \dots, p$$

Then there exists an invertible p x p matrix of constants A such that

$$\text{(5.12)} \qquad \deg\,(AL(s))^{1,\dots,r}_{1,\dots,r} > \deg\,(AL(s))^{2,\dots,r,j}_{1,\dots,r} \; , \; j = r+1,\dots,p$$

Proof. Let $E_j(c) = E$, $j \in \{r+1,\dots,p\}$ be the matrix with 1's on the diagonal, a c in spot (j,1) and zero's elsewhere. Then as is easily checked

$$E^{1,\dots,r}_{i_1,\dots,i_r} = \begin{cases} 1 & \text{if } \{i_1,\dots,i_r\} = \{1,\dots,r\} \\ 0 & \text{otherwise} \end{cases}$$

and for $k \ne j$, $k \in \{r+1,\dots,p\}$

$$E^{2,\dots,r,k}_{i_1,\dots,i_r} = \begin{cases} 1 & \text{if } \{i_1,\dots,i_r\} = \{2,\dots,r,k\} \\ 0 & \text{otherwise} \end{cases}$$

while

$$E^{2,\dots,r,j}_{i_1,\dots,i_r} = \begin{cases} (-1)^r c & \text{if } \{i_1,\dots,i_r\} = \{1,\dots,r\} \\ 1 & \text{if } \{i_1,\dots,i_r\} = \{2,\dots,r,j\} \\ 0 & \text{otherwise} \end{cases}$$

It now follows from the minor product rule (5.9) that

$$\text{(EL)} \qquad (EL)^{2,\dots,r,k}_{1,\dots,r} = \begin{cases} L^{1,\dots,r}_{1,\dots,r} & \text{if } k = 1 \\ L^{2,\dots,r,k}_{1,\dots,r} & \text{if } k \in \{r+1,\dots,p\} \setminus \{j\} \\ L^{2,\dots,r,j}_{1,\dots,r} + (-1)^r c \; L^{1,\dots,r}_{1,\dots,r} & \text{if } k = j \end{cases}$$

It follows that (5.12) holds if we take for A a suitable product of matrices $E_j(c)$.

5.13. <u>Lemma</u>. Let L(s) be a polynomial p x m matrix without constant terms of degree n. Suppose that for a certain r all minors of size < r have degree < n and that

$$\text{(5.14)} \qquad \deg(L^{1,\dots,r}_{1,\dots,r}) = n > \deg(L^{2,\dots,r,j}_{1,\dots,r}), \; j = r+1, \dots, p$$

Let d(s) be the diagonal matrix with diagonal entries $(s,1,\dots,1)$ and let $L'(s) = d(s)^{-1}L(s)$. Then L'(s) is polynomial (because the first row of L(s) has no constant terms) and $\deg(L'(s)) = n-1$.

Proof. Because $\deg(d(s)) = 1$ and $\deg L(s) \le \deg(d(s)) + \deg(L'(s))$ we must have $\deg(L'(s)) \ge n-1$. It remains to show that $\deg(L'(s)) \le n-1$. Let $\bar{L}(s)$ be the square matrix

$$\bar{L} = \begin{pmatrix} L_{11} & L_{12} & 0 \\ L_{21} & L_{22} & I \\ 0 & I & 0 \end{pmatrix}$$

where $L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$, where L_{11} is the top-left r x r submatrix of $\bar{L}$, and where the I's are the appropriate unit matrices. Then

$$\text{(5.15)} \qquad \deg(L) = \deg(\bar{L}) = \deg(\det(L_{11})) = \deg(\det(\bar{L})) = n$$

which implies that $\bar{L}^{-1}$ is proper. We claim that the first column of $\bar{L}^{-1}$ consists of strictly proper rational functions. Indeed the entries of the first column are the functions

$$(5.16)\qquad \det(\bar{L})^{-1}\bar{L}_j^{!}\ ,\ j = 1,\ \ldots,\ m+p-r$$

Now if $j = 1, \ldots, r$, $\bar{L}_j^{!}$ is the determinant of a $(r-1) \times (r-1)$ submatrix of L_{11} and hence $\deg(\bar{L}_j^{!}) < n$ by hypothesis. If $j = r+1, \ldots, m$ then $\bar{L}_j^{!} = 0$ and finally if $j = m + k$, $k = 1, \ldots, p-r$ then

$$\bar{L}_j^{!} = L\begin{matrix}2,\ldots,r,r+k\\ 1,\ldots,r\end{matrix}\ ,\ j = m + k$$

which by hypothesis is of degree $< n = \deg(\det(L)^{-1}$. This proves the claim.

Now let $d'(s)$ be the $(m+p-r) \times (m+p-r)$ diagonal matrix with entries $(s,1,\ldots,1)$, and let $\bar{L}' = d'(s)^{-1}\bar{L}$. Then L' is the $p \times m$ top left submatrix of $\bar{L}'$ and hence

$$(5.17)\qquad \text{degree}(L') \leq \text{degree}(\bar{L}')$$

On the other hand $(\bar{L}')^{-1} = (\bar{L})^{-1}d'(s)$ is still proper because the first column of $\bar{L}^{-1}$ consists of strictly proper rational functions. Hence (cf. lemma 4.15)

$$(5.18)\qquad \begin{aligned}\deg(\bar{L}') &= \deg((\bar{L}')^{-1}) \leq \deg(\det(\bar{L}')) =\\ &= \deg(\det(d'(s))^{-1}\det(\bar{L}))\\ &= \deg(s^{-1}\det(L_{11})) = n - 1\end{aligned}$$

because L_{11} has no constants. Combining (5.18) and (5.17) we see that indeed $\deg(L') \leq n-1$, proving the lemma. (NB it is not true as a rule that $(L')^{-1}$ is proper).

Note that lemma 5.13 and 5.10 combine to give a proof of corollary 2.24.

5.19. <u>Proposition</u>. Let $L(s)$ be a polynomial matrix of degree n. Then there exists a family of n-dimensional systems $\Sigma(c)$ such that the $\Sigma(c)$ converge in input/output behaviour to $L(D): \mathcal{U} \to \mathcal{Y}$ as $c \to \infty$ and such that moreover the poles of (the transfer functions of) the $\Sigma(c)$ all go to $-\infty$ as $c \to \infty$.

Proof. This is proved by induction, the case $n = 0$ being trivial because $L(s)$ has degree zero iff it is a matrix of constants. The first thing to do next is to obtain the scalar operator $D: \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ as a limit of input/output operators of one dimensional systems. To this end let $\Sigma(c)$, $c = 1,2,\ldots$ (or $c \in \mathbb{R}$) be the family of systems

$$(5.20)\qquad \Sigma(c) = (F_c,G_c,H_c,J_c),\ J_c = c,\ F_c = -c,\ H_c = c,\ G_c = -c$$

The associated input/output operator of $\Sigma(c)$ is $V_c: \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$

$$(5.21)\qquad V_c: u(t) \mapsto y_c(t) = cu(t) + \int_0^t -c^2e^{-c(t-\tau)}u(\tau)d\tau$$

By partial integration (twice) we see that

$$(5.22)\qquad y_c(t) = u^{(1)}(t) - \int_0^t e^{-c(t-\tau)}u^{(2)}(\tau)d\tau$$

Let b be such that $u^{(2)} \in \mathcal{F}_b^{(o)}(\mathbb{R})$ (i.e. $\sup_t e^{-bt}|u^{(2)}(t)| < \infty$). Then if $M = ||u^{(2)}||_b$, we have

$$(5.23)\qquad \left|\int_0^t e^{-c(t-\tau)}u^{(2)}(\tau)d\tau\right| \leq \int_0^t e^{-c(t-\tau)}e^{b\tau}M \leq (b+c)^{-1}Me^{bt}$$

and it follows that the $y_c(t)$ converge to $u^{(1)}(t)$ in $(\mathbb{R})$. More precisely if b is such $u^{(1)}$, $u^{(2)}$ are both in $\mathcal{F}_b^{(o)}(\mathbb{R})$ then $y_c(t) \in \mathcal{F}_b^{(o)}(\mathbb{R})$ and the $y_c(t)$ converge to $u^{(1)}(t)$ in $\mathcal{F}_b^{(o)}(\mathbb{R})$.

Now suppose with induction that the proposition has been proved for all polynomial matrices of degree $\leq n-1$.

Let $L(s)$ be a polynomial matrix of degree n. First note that if P,Q are invertible matrices of constants then $L(D)$ is the limit of a family as in the statement of the theorem if and only if $PL(D)Q$ is. Also adding a matrix of constants makes no difference. Removing the constants and multiplying $L(s)$ on the left and on the right with suitable invertible matrices of constants we can therefore assume that for a certain minimal $r \in \mathbb{N}$ the topleft $r \times r$ minor of $L(s)$ is of degree n. Using lemma 5.10 and lemma 5.13 we see that after a further multiplication on the left by an invertible matrix of constants $L(s)$ factorizes as

$$L(s) = \begin{pmatrix} s & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix} L'(s)$$

with $L'(s)$ polynomial of degree $n-1$. By induction we have that there exists a family of $(n-1)$-dimensional systems $\Sigma'(c) = (F'_c, G'_c, H'_c, J'_c)$ such that the poles of $\Sigma'(c)$ go to $-\infty$ as $c \to \infty$ (if $n-1 > 0$, if $n = 1$, $L'(s)$ is constant and one takes $\Sigma'(c) = (0,0,0,L')$) and such that $V_{\Sigma'(c)}$ converges in input/output behaviour to $L'(s)$.

Now let $\Sigma(c)$ be the composed system

$$(5.24) \qquad \longrightarrow \boxed{\Sigma''(c)} \longrightarrow \boxed{\Sigma'(c)} \longrightarrow$$

where $\Sigma''(c)$ is the m input/m output one dimensional system given by the matrices

$$F''_c = -c,\ G''_c = (-c,0,\ldots,0),\ H''_c = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ J''_c = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

I.e. if $\Sigma'(c) = (F'_c, G'_c, H'_c, J'_c)$ then $\Sigma(c)$ is given by the matrices

$$(5.25) \quad F_c = \begin{pmatrix} F''_c & 0 \\ G'_c H''_c & F'_c \end{pmatrix},\ G_c = \begin{pmatrix} G''_c \\ G'_c J''_c \end{pmatrix},\ H_c = (J'_c H''_c \quad H'_c),\ J_c = J'_c J''_c$$

(if $n > 1$; if $n = 1$, $F_c = -c$, $G_c = (-c,0,\ldots,0)$, $H_c = L'H''_c$, $J_c = L'H''_c$). Then the $\Sigma(c)$ converge in input/output behaviour to $L(D)$. Moreover (as follows from (5.25)) the poles of $\Sigma(c)$ go to $-\infty$ as $c \to \infty$ because $F''_c = -c$ and because the poles of $\Sigma'(c)$ go to $-\infty$ as $c \to \infty$ if $n > 1$. This proves the proposition.

We can be somewhat more precise about how well the $\Sigma(c)$ converge in input/output behaviour to $L(D)$. Indeed one has

5.26. Corollary. Let $L(D)$ and $(\Sigma(c))_c$ be as above in the proof of proposition 5.19. Let $b \geq 0$ be such that $u, u^{(1)}, \ldots, u^{(n+1)} \in \mathcal{F}_b^{(o)}(\mathbb{R}^m)$. Then there is a constant M such that

$$(5.27) \qquad ||V_{\Sigma(c)}u - L(D)u|| \leq c^{-1}Me^{bt}$$

In particular if $u \in \mathcal{U}$ is of compact support or, more generally if $u, u^{(1)}, \ldots, u^{(n+1)}$ are all bounded, we can take $b = 0$ and for such input functions u, $V_{\Sigma(c)}u$ converges uniformly in t to $L(D)u$.

This follows readily by induction from the proof of proposition 5.19 above, (5.22), and the estimate (5.23), because $L'(D)u$ is a vector of linear combination of the $u, u^{(1)}, \ldots, u^{(n-1)}$.

5.28. Proof of the second half of theorem 2.22. Now let $V: \mathcal{U} \to \mathcal{Y}$ be an operator of the form $V = L(D) + V_\Sigma$ with $\dim(\Sigma) + \deg(L(s)) \leq n$. Let $\Sigma(c)$ be a sequence of $\deg(L(s))$-dimensional systems converging to $L(D)$ in input/output behaviour as in proposition 5.19. Then if $\Sigma'(c)$ is the sum system of $\Sigma(c)$ and Σ, the family $\Sigma'(c)$ converges in input/output behaviour to V. More precisely if $\Sigma = (F,G,H,J)$, $\Sigma(c) = (F_c, G_c, H_c, J_c)$ then $\Sigma'(c)$ is given by the matrices

$$F'_c = \begin{pmatrix} F & 0 \\ 0 & F_c \end{pmatrix},\ G'_c = \begin{pmatrix} G \\ G_c \end{pmatrix},\ H'_c = (H \quad H_c),\ J'_c = J + J_c$$

Because the co and cr systems are open and dense in L we can perturb each $\Sigma'(c)$ slightly to a $\Sigma''(c)$ which is co,cr such that $\Sigma''(c)$ still converges to V in input/output behaviour as $c \to \infty$, and such that the behaviour of the poles of the $\Sigma''(c)$ as $c \to \infty$ is like that of the $\Sigma'(c)$ as $c \to \infty$. This finishes the proof of theorem 2.22.

5.29. Remark. One has of course in the setting of 5.28 above also an estimate like (5.27) for $||V_{\Sigma'(c)}u - Vu||$.

5.30. Remark. If $\Sigma(c)$ is e.g. the family of (5.20) above, the Markov parameters of the family J_c, H_cG_c, $H_cF_cG_c$, $H_cF_c^2G_c$, ... definitely do not converge as $c \to \infty$.

One can of course examine what the possible limits are of families of systems $\Sigma(c)$ of dimension n which converge in input/output operators and such that moreover the Markov parameters converge as well (or more generally such that the Markov parameters remain bounded) as $c \to \infty$. The answer is simple: the limit operator is then necessarily of the form V_Σ where Σ is a possibly lower dimensional system. Inversely every V_Σ with $\dim(\Sigma) \leq n$ can arise a limit of input/output operators of co and cr systems of dimension n, cf. [Haz 2].

5.31. Approximation by systems with $J = 0$. Let $T(s)$ be a matrix of rational functions. Write

(5.32) $$T(s) = T_-(s) + L(s)$$

with $T_-(s)$ strictly proper and $L(s)$ polynomial. Define

(5.33)
$$\begin{aligned} n_r(T) &= \text{dim of the } \mathbb{R}\text{-vectorspace spanned by the rows of } L(s) \\ n_c(T) &= \text{dim of the } \mathbb{R}\text{-vectorspace spanned by the columns of } L(s) \\ q(T) &= \min\{n_r(T), n_c(T)\}. \end{aligned}$$

E.g. if $T(s) = L(s) = \begin{pmatrix} s^2 & s & s^3 \\ 1 & s & 1 \end{pmatrix}$, then $n_r(T) = 2$, $n_c(T) = 3$ and if $T(s) = L(s) = \begin{pmatrix} s^2 & s^2 \\ s & s \end{pmatrix}$, $n_r(T) = 2$, $n_c(T) = 1$.

Let Σ realize $T_-(s)$. Then the operator $V_\Sigma + L(D)$ is the limit in input/output behaviour of a family of $(\deg(T(s)) + q(T(s)))$ - dimensional systems.

This can be seen as follows. Because $T_-(s)$ is strictly proper it suffices to see that $L(D)$ can be obtained as the limit of the input/output operators of a family of $\deg(L(s)) + q(L(s))$ dimensional systems. Assume for definitiveness that $q(T) = n_c(T)$. Then we can factorize $L(s)$ as

$$L(s) = (L'(s) \quad 0)Q$$

where Q is a square invertible matrix of constrnts and $L'(s)$ has $q(T)$ columns. It now clearly suffices to obtain $L'(D)$ as a limit of $\deg(L) + q(L)$ dimensional systems. To this end let $\Sigma(c)$ be a family of systems converging to $L(D)$ of dimension $\deg(L)$ and let $\Sigma'(c)$ be a $q = q(L)$-dimensional family of systems with $J_c = 0$ for all c with limit input/output operator equal to I, the $q \times q$ identity matrix. Such a family is e.g. given by the matrices

$$F_c = \begin{pmatrix} -c & & 0 \\ & \ddots & \\ 0 & & -c \end{pmatrix}, \quad G_c = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad H_c = \begin{pmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{pmatrix}, \quad J_c = 0.$$

Let $T'_c(s)$ be the transfer function matrix of $\Sigma'(c)$ and $T_c(s)$ that of $\Sigma(c)$. Then the $(q+\deg(L))$-dimensional system $\Sigma''(c)$ obtained by applying first $\Sigma'(c)$ and then $\Sigma(c)$ has transfer function matrix $T_c(s)T'_c(s)$, which is strictly proper, and the $\Sigma''(c)$ converge in input/output behaviour to $L'(s)$.

This result is optimal if $p = 1$ or $m = 1$, but, though definitely generically best possible (meaning that for almost all $T(s)$ with given $q(T) = q$, $\deg(T) + q$ is the best one can do), it is not best possible for every particular $T(s)$. E.g. the factorization

$$L(s) = \begin{pmatrix} s^2 & s^3 \\ s & s^2 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s & s^2 \\ 0 & 0 \end{pmatrix}$$

shows that this $L(s)$ can be obtained as the input/output limit of a family of four dimensional systems with $J = 0$, although $\deg(L) = 3$ and $q(L) = 2$.

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Lie algebraic method in filtering and identification

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These lectures concern (nonlinear) filtering. Very roughly the art of obtaining best estimates for some stochastic time-varying variable x on the basis of observations of another process y. The more concrete object under consideration being a stochastic dynamical system $dx = f(x)dt + G(x)dw$, where w is Wiener noise, with observations $dy = h(x)dt + dv$, corrupted by further noise. The subject as presented here involves ideas and techniques from Lie algebra theory, stochastics, differential topology, approximation theory and partial differential equations and has relations with quantum theory and stochastic physics. The lectures are adressed to practitioners in any one of these areas assuming that as a rule they are not experts in the other ones.

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1. Introduction

Filtering is concerned with making estimates of quantities associated with a stochastic process $\{x_t\}$ on the basis of information gleaned from a related process $\{y_t\}$. The process $\{x_t\}$ is called the *signal* or *state* process and $\{y_t\}$ is the *observation* process. In this paper the following more concrete realization will be considered

$$dx_t = f(x_t)dt + G(x_t)dw_t, \quad x_t \in \mathbf{R}^n, \ w_t \in \mathbf{R}^m \tag{1.1}$$

$$dy_t = h(x_t)dt + dv_t, \ y_t \in \mathbf{R}^p, \ v_t \in \mathbf{R}^p \tag{1.2}$$

Here f is a function $\mathbf{R}^n \to \mathbf{R}^n$; G is an $n \times m$ matrix valued function on $\mathbf{R}^n$, h is a function $\mathbf{R}^n \to \mathbf{R}^p$ and w_t and v_t are Wiener processes, assumed independent of each other and also independent of the initial random variable x_0. More precisely these equations can be written

$$x_t = x_0 + \int_0^t f(x_s)ds + \int_0^t G(x_s)dw_s \tag{1.3}$$

$$y_t = \int_0^t h(x_s)ds + v_t \tag{1.4}$$

where the last term of (1.3) is a stochastic integral in the sense of *Ito*.

Much more loosely one can look at equations (1.1) and (1.2) as

$$\dot{x} = f(x) + G(x)\dot{w} \tag{1.5}$$

$$\dot{y} = h(x) + \dot{v} \tag{1.6}$$

with $\dot{w}$ and $\dot{v}$ white noise. Thus we have a differential equation $\dot{x} = f(x)$ on $\mathbf{R}^n$ which is subject to continuous random shocks whose intensity and direction (distribution) is state dependant and as observations we have an integral of some function of x and these observations are corrupted by more noise.

The general filtering problem for the state process $\{x_t\}$ with observation process $\{y_t\}$ is now to

calculate for (interesting) functions ϕ of the state the conditional expectation

$$E[\phi(x_t)|y_s, 0 \le s \le t] = \widehat{\phi(x_t)}, \tag{1.7}$$

i.e. the best (least squares) estimate of $\phi(x_t)$ given the observations y_s up to time t. That is we are interested in calculation procedures for $\widehat{\phi(x_t)}$. In many (engineering) applications the data come in sequentially and one does not really want a calculating procedure which needs all the data y_s, $0 \le s \le t$, every time t that it is desired to find $\widehat{\phi(x_t)}$; rather we would like to have a procedure which uses a statistic m_t which can be updated using only the new observations y_s, $t \le s \le t'$ to its value $m_{t'}$ i.e.

$$m_{t'} = a(m_t, t', t, \{y_s : t \le s \le t'\}) \tag{1.8}$$

and from which the desired conditional expectation can be calculated directly, i.e.

$$\widehat{\phi(x_t)} = E[\phi(x_t)|y_s, 0 \le s \le t] = b(t, y_t, m_t). \tag{1.9}$$

Finally to actually implement the filter it would be nice if m_t were a finite dimensional quantity. All this leads to the (ideal) notion of a *finite dimensional recursive filter*. By definition such a filter is a system

$$d\xi_t = \alpha(\xi_t)dt + \sum_{i=1}^{p} \beta_i(\xi_t)dy_{it} \tag{1.10}$$

driven by the observations y_{it}; y_{it} is the i-th component of y_t, $i = 1, \dots, p$; together with an output map

$$\widehat{\phi(x_t)} = \gamma(\xi_t) \tag{1.11}$$

More precisely formulated our problem is now the following: given a system (1.1)-(1.2) and a function ϕ on $\mathbf{R}^n$, how can we decide whether for these data there exists a finite dimensional recursive filter (1.10)-(1.11) which calculates $\widehat{\phi(x_t)}$, the best least squares estimate, and how do we find the functions (vectorfields) $\alpha, \beta_1, \cdots, \beta_p, \gamma$ of (1.10)-(1.11).

Now this may of course be a totally unreasonable question to ask. It could be that such nice filters virtually never exist. That is not the case though. In the case of linear systems

$$dx_t = Ax_t dt + Bdw_t \tag{1.12}$$

$$dy_t = Cx_t dt + dv_t \tag{1.13}$$

where now A, B, C are matrices of the appropriate sizes (which may be time varying), the well known Kalman-Bucy filter is precisely such a filter as (1.10)-(1.11). The equations are as follows. The statistic ξ_t is a pair (m_t, P_t) consisting of an n-vector and a symmetric $n \times n$ matrix P_t. These evolve according to

$$dP_t = (AP_t + P_t A^T + BB^T - P_t C^T C P_t)dt \tag{1.14}$$

$$dm_t = Am_t dt + P_t C^T (dy_t - Cm_t dt). \tag{1.15}$$

Here X^T denotes the transpose of a matrix X. This filter was discovered in 1961 and it is hard to overestimate its importance: whole books are devoted to its applications into single specialized fields and substantial companies can make a good living doing little more than Kalman-Bucy filtering. Naturally, efforts immediately started to find similar filters for more general systems than (1.12)-(1.13). This turned out to be unexpectedly difficult and this is still the case though there exists hosts of approximate filters of various kinds which (seem to) work well in a variety of situations; there is very little systematically known about how to construct approximate filters or about how to predict that a given one or class will work well when applied to a given collection of systems.

The approach based on Lie-algebraic considerations which I will try to discuss and explain below seems to hold great promise both in understanding the difficulties involved and in providing some kind of systematic foothold in the area of constructing approximate filters. For, as will become clear below, the existence of finite dimensional recursive filters for a nontrivial statistic will be a rare event.

Let me pause at this point to point out that identification problems can easily be construed as filtering problems. By way of illustrating this point consider again a linear system

$$dx_t = Ax_t + Bdw_t, \quad dy_t = Cx_t dt + dv_t \tag{1.16}$$

where now the matrices A, B, C are (partially) unknown. By adding to (1.16) the stochastic equations

$$da_{ij} = 0, \quad db_{kl} = 0, \quad dc_{qr} = 0 \tag{1.17}$$

for all unknown a_{ij}, b_{kl}, c_{qr}, one obtains a system (1.16)-(1.17) (of much larger state space dimension). And solving the filtering problem for the functions which project the vector $(x, (a_{ij}), (b_{kl}), (c_{qr}))$ onto a suitable component means identifying that particular coefficient.

2. The DMZ-equation and the estimation algebra

Let $\{x_t\}$ be a diffusion process as in (1.1)-(1.2) above. Given sufficient regularity of f, G, h the conditional expectation $\hat{x}_t$ will have a density $\pi(x,t)$.

THEOREM 2.1. Under appropriate regularity conditions there exists an unnormalized version $\rho(x,t)$ of $\pi(x,t)$ which satisfies an equation

$$d\rho = \mathcal{L}\rho dt + \sum_{i=1}^{p} h_i(x)\rho dy_{it} \tag{2.2}$$

where $\mathcal{L}$ is the second order differential operator given by

$$(\mathcal{L}\psi) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}((GG^T)_{ij}\psi) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f_i\psi) - \frac{1}{2}\sum_{j=1}^{p} h_j^2\psi \tag{2.3}$$

Here $(GG^T)_{ij}$ is the (i,j)-th component of the $n \times n$ matrix $G(x)G(x)^T$ and f_i, h_j are the i-th and j-th component respectively of f and h.

Several comments are in order. First of all equation (2.2) is in Fisk-Stratonovic form. The corresponding Ito equation looks the same with $\mathcal{L}$ changed by removing the $-\frac{1}{2}\sum h_j^2\psi$ term. The word "unnormalized" means that $\rho(x,t) = \sigma(t)\pi(x,t)$ where $\sigma(t)$ is an unknown function of time. Under appropriate reachability conditions on (1.1) $\rho(x,t)$ is a positive function. That ρ is unnormalized does not hurt much as $\rho(x,t)$ still suffices to calculate such things as $\widehat{\phi(x_t)}$. Indeed

$$\widehat{\phi(x_t)} = \left(\int \rho(x,t)dx\right)^{-1} \int \rho(x,t)\phi(x)dx \tag{2.4}$$

Theorem 2.1 was proved by Duncan [13], Mortensen [28] and Zakai [36] and the corresponding equation 2.1 is often refered to as the Duncan-Mortensen-Zakai or DMZ equation.

It is a stochastic partial differential equation being driven by the stochastic processes $y_1, \ldots, y_p$.

It is important to note (Brockett [5]), that equations (2.2), (2.4) together constitute in fact a recursive filter in the sense of (1.10)-(1.11). The role of ξ_t is played by $\rho(x,t)$ so that instead of a point ξ evolving on a finite dimensional M we have an evolving density, i.e. a point ρ in an infinite dimensional space of positive functions evolving with time.

The simplest nontrivial example of a system (1.1)-(1.2) is

$$dx = dw, \quad dy = xdt + dv \tag{2.5}$$

i.e. one dimensional Wiener noise linearly observed corrupted by further noise. In this case the DMZ-equation becomes

$$d\rho = \left(\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1}{2}x^2\right)\rho dt + x\rho dy_t, \quad \left(\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2} - \frac{1}{2}x^2\rho + x\rho\dot{y}\right) \tag{2.6}$$

i.e. we are dealing with the Euclidean Schrödinger equation with an extra forcing term. This is not an

accident but part of a general pattern of which we shall see a further manifestation below in section 8, cf. also Mitter [26,27] for other remarks on this theme. I do not know wether the use Bismut makes of the filtering equations when dealing with a stochastic approach to index theorems and the Dirac operator can also be fitted into this framework.

3. Robustness and numerical matters

As it stands equation (2.2) is not a very useful object for applications. It is a stochastic partial differential equation (with as probability space a space of paths $\{y\}$) and as such a solution is in principle only defined apart from a set of measure zero. On the other hand actual observations will always consist of piecewise smooth $y(t)$ and the class of all such is of measure zero. Thus there arises the question whether there exist a version of (2.2) which can be interpreted pathwise for all $y(t)$ and for which the solutions of (2.2) for piecewise smooth ($y(t)$ carry (approximative) information, cf. Clark [9] and Davis [11]. Fortunatedly the time dependent gauge transformation

$$\tilde{\rho}(x,t) = \exp(-h_1(x)y_1(t) - \cdots - h_p(x)y_p(t))\rho(t,x) \tag{3.1}$$

transforms (2.2) into an equation

$$\frac{\partial \tilde{\rho}}{\partial t} = \mathcal{L}\tilde{\rho} - \sum_{i=1}^{p} y_i(t)\mathcal{L}_i\tilde{\rho} + \sum_{i,j=1}^{p} y_i(t)y_j(t)\mathcal{L}_{ij}\tilde{\rho} \tag{3.2}$$

where $\mathcal{L}_i = [h_i, \mathcal{L}] := h_i\mathcal{L} - \mathcal{L}h_i$ and $\mathcal{L}_{ij} = \mathcal{L}_{ji} = \frac{1}{2}[h_i, [h_j, \mathcal{L}]]$, and this equation, which does not anymore involve derivatives of y, can simply be interpreted as a family of partial differential equations parametrized by the possible observation paths $y(t)$.

Equation (3.2) can of course be verified directly (remembering that (2.2) is a Fisk-Stratonovic integral so that the ordinary rules of calculus apply; removing the term $-\frac{1}{2}\sum h_i^2$ from $\mathcal{L}$ gives the corresponding Ito equation and then Ito calculus of course also gives (3.2). An easier way of obtaining (3.2) is to observe that (3.1) in (2.2) gives $d\tilde{\rho} = \exp(-\sum h_i y_i)\mathcal{L}\exp(\sum h_i y_i)\tilde{\rho}$ and to use the version of the Baker-Campbell-Hausdorff formula which says

$$\exp(-rA)B\exp(rA) = \sum_{k=0}^{\infty}(-1)^k \frac{r^k}{k!} ad_A^k(B) \tag{3.3}$$

where $ad_A(B) = [A,B] = AB - BA$, $ad_A^k(B) = ad_A(ad_A^{k-1}(B))$ for linear operators A, B. In our case the contributions of (3.3) for $k \geqslant 2$ disappear because then A is a function, $B = \mathcal{L}$ is a second order differential operator, so $[A,B]$ is first order, $[A,[A,B]]$ is a function and $[A,[A,[A,B]]] = 0$.

Also of course there still remains the question of how to use equation (3.2) or (2.2) effectively to calculate certain desired conditional expectations. A direct numerical discretization approach is out of the question. Typically x is a fairly large dimensional object; for example around 27 for certain problems involving helicopters. Taking three data points per coordinate axis (which is ridiculous) then gives $3^{27} \approx 2.10^{14}$ space grid points! So other methods must be tried. It seems likely that the Lie-algebraic considerations to be discussed below will help. Other promising work into the numerics of the nonlinear filtering equations has been started by Pardoux-Talay [29].

4. Wei-Norman theory

It is important to note that the filtering equation (3.2) (or (2.2)) is of the general form

$$\dot{x} = (A_1x)u_1 + \cdots + (A_kx)u_k \tag{4.1}$$

where the A_i are linear operators and the u_i known functions of time. Of course in (3.2) the role of x is played by ρ, an infinite dimensional object. Here for the moment lets consider (4.1) as a finite dimensional object. Let us also assume that the $A_1, \ldots, A_k$ who are now, say, $n \times n$ matrices, form the basis of a Lie algebra. (By adding a few more terms with corresponding u_i equal to zero this can of course always be assured.) Let us look for solutions of the form (Wei-Norman [35]).

$$x(t) = e^{g_1 A_1 g_2 A_2} \dots e^{g_k A_k} x(0) \tag{4.2}$$

Differentiating this gives

$$\dot{x} = \dot{g}_1 A_1 e^{g_1 A_1} e^{g_2 A_2} \dots e^{g_k A_k} x(0) + e^{g_1 A_1} \dot{g}_2 A_2 e^{g_2 A_2} \dots e^{g_k A_k} x(0) + \dots \tag{4.3}$$

and inserting

$$e^{-g_1 A_1} e^{-g_2 A_2} \dots e^{-g_i A_i} e^{g_i A_i} \dots e^{g_1 A_1}$$

just after $\dot{g}_i A_i$ in the i-th term equation (1.1) can be rewritten

$$\dot{x} = \sum_{i=1}^{k} \dot{g}_i \Big(A_i + \sum_{r=1}^{i-1} \sum_{\substack{j_1, \dots, j_{i-1} \\ j_1 + \dots + j_{i-1} = r}} \frac{g_1^{j_1} \cdots g_{i-1}^{j_{i-1}}}{j_1! \cdots j_{i-1}!} ad_{A_1}^{j_1} \dots ad_{A_{i-1}}^{j_{i-1}} (A_i)\Big) x \tag{4.4}$$

$$= \sum_{i=1}^{k} \dot{g}_i (A_i + h_{ij}(g_1, \dots, g_k) A_j)$$

with $h_{ij}(0, \dots, 0) = 0$, where, again, the Campbell-Baker-Hausdorff formula (3.3) has been used. Note that h_{ij} are universal functions which only depend on the Lie algebra and the chosen basis. Thus it remains to solve (equating the coefficients of the basic elements A_i in (4.4) and (4.1))

$$\begin{aligned} \dot{g}_1 + \dot{g}_2 h_{11}(g_1, \dots, g_k) + \dot{g}_2 h_{21}(g_1, \dots, g_k) + \dots + \dot{g}_k h_{k1}(g_1, \dots, g_k) &= u_1 \\ \dot{g}_2 + \dot{g}_1 h_{12}(g_1, \dots, g_k) + \dot{g}_2 h_{22}(g_1, \dots, g_k) + \dots + \dot{g}_k h_{k2}(g_1, \dots, g_k) &= u_2 \\ &\dots \\ \dot{g}_k + \dot{g}_1 h_{1k}(g_1, \dots, g_k) + \dot{g}_2 h_{2k}(g_1, \dots, g_k) + \dots + \dot{g}_k h_{kk}(g_1, \dots, g_k) &= u_k \end{aligned} \tag{4.5}$$

which can be done for small t and $g_1(0) = \dots = g_k(0) = 0$ because $h_{ij}(0, \dots, 0) = 0$. In general a representation (4.2) for the solution is only possible for small t. However things change if the Lie-algebra in question is solvable, then ([35]) there is such a representation for all t. More precisely there is a suitable basis such that there is such a representation for all t. How this comes about is easy to see in the case that the Lie algebra L is nilpotent. Indeed let

$$L \underset{\neq}{\supset} L^{(1)} = [L, L] \underset{\neq}{\supset} L^{(2)} = [L, L^{(1)}] \underset{\neq}{\supset} \dots \underset{\neq}{\supset} L^{(m)} = [L, L^{(m-1)}] = 0 \tag{4.6}$$

be a basis such that $A_1, \dots, A_{k_1}, A_{k_1+1}, \dots, A_{k_2}, \dots, A_{k_{m-1}+1}, \dots, A_{k_m} = A_k$, $k_1 < k_2 < \dots < k_m$ such that $A_{k_i+1}, \dots, A_{k_m}$ is a basis for $L^{(i)}$, $i = 0, \dots, m-1$ $(k_0 = 1, k_m = k)$. Then it immediately follows from (4.4) that $h_{ij} = 0$ for $j \leq i$ and the set of equations (4.5) gets a nice triangular structure. Moreover $h_{ij}(g_1, \dots, g_k)$ involves only $g_1, \dots, g_{i-1}$ (this is always the case, cf.(4.5), so the h_{1j} in (4.5) are always all zero) and the resulting equations (4.5) for the nilpotent case are therefore of the form

$$\begin{aligned} &\dot{g}_1 = u_1, \dots, \dot{g}_{k_1} = u_{k_1} \\ &\dot{g}_{k_1+1} = u_{k_1+1} + \alpha_{k_1+1}(u_1, \dots, u_{k_1}; g_1, \dots, g_{k_1}), \dots, \dot{g}_{k_2} = u_{k_2} + \alpha_{k_2}(u_1, \dots, u_{k_1}; g_1, \dots, g_{k_1}) \\ &\dot{g}_{k_2+1} = u_{k_2+1} + \alpha_{k_2+1}(u_1, \dots, u_{k_2}; g_1, \dots, g_{k_2}), \dots, \dot{g}_{k_3} = u_{k_3} + \alpha_{k_3}(u_1, \dots, u_{k_2}; g_1, \dots, g_{k_2}) \\ &\dots \end{aligned} \tag{4.7}$$

where the α_j are known (universal) functions of the u's and g's.

These considerations are not limited to Lie-algebras of matrices. Indeed the left hand sides of equations (4.5) only depend on the abstract structure of the Lie algebra in question and the choice of basis. Thus all this equally applies to Lie-algebras of say differential operators (given suitable

definitions of $\exp(tA)$), though in order to have a finite set of equations (4.5) one needs of course a finite dimensional algebra. It also follows from (4.4) that the Wei-Norman equations are compatible with homomorphisms of Lie algebras, more precisely quotients. Indeed if $\mathfrak{A} \subset L$ is an ideal and $A_1, \dots, A_{k_1}, A_{k_1+1}, \dots, A_k$ is a basis of L such that $A_{k_1+1}, \dots, A_k$ is a basis of $\mathfrak{A}$ then the h_{ij} are zero for $j \in \{1, \dots, k_1\}$ and $i \in \{k_1 + 1, \dots, k\}$. So in the case of a topologically nilpotent algebra L, or more generally one with a chain of ideals $\mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \mathfrak{A}_3 \supset \dots$ such that $\cap \mathfrak{A}_i = 0$ and $L / \mathfrak{A}_i$ finite dimensional for all i one can in principle still do Wei-Norman theory with now infinite ordered product expressions $x = e^{g_1 A_1} e^{g_2 A_2} \dots e^{g_k A_k} \dots x_0$ in the sense that the equations for the g_i belonging to a quotient $L / \mathfrak{A}_j$ involve only those same g_i. Of course now questions of convergence arise.

5. The estimation Lie algebra

The considerations of the previous section already make it clear that the Lie algebra generated by the operators which occur in equation (2.2) or (3.2) contains important information concerning the filtering problem. One therefore defines the estimation Lie algebra $EL(\Sigma)$ of a system Σ given by (1.1)-(1.2) as the Lie algebra of differential operators generated by the 2-nd order differential operator $\mathfrak{L}$ and the multiplication operators $h_1, \dots, h_p$.

$$EL(\Sigma) = Lie(\mathfrak{L}, h_1, \dots, h_p). \tag{5.1}$$

Note that the Lie algebra generated by the operators which occur in (3.2) is in any case a subalgebra of $EL(\Sigma)$. Often it is equal.

EXAMPLE 5.2. Consider again the simplest nonzero linear system (2.5). Then $p = 1$ and $\mathfrak{L} = \frac{1}{2} d^2 / dx^2 - \frac{1}{2} x^2$. So we have in this case the Lie algebra $Lie(\frac{1}{2} d^2 / dx^2 - \frac{1}{2} x^2, x)$. Now $[\frac{1}{2} d^2 / dx^2 - \frac{1}{2} x^2, x] = d / dx$ (as operators on functions), $[\frac{1}{2} d^2 / dx^2 - \frac{1}{2} x^2, \frac{d}{dx}] = x$, $[\frac{d}{dx}, x] = 1$ and $[?, 1] = 0$. So in this case we obtain the well-known oscillator Lie algebra, which is four dimensional with basis $\frac{1}{2} d^2 / dx^2 - \frac{1}{2} x^2$, x, d / dx, 1. It is solvable (but not nilpotent) with as derived algebra the nilpotent Heisenberg algebra with basis x, d / dx, 1.

$EL(\Sigma)$ is (of course) an invariant of Σ meaning that a change of coordinates in Σ (a diffeomorphism $x \to x'$ taking Σ to Σ') will yield isomorphic estimation Lie algebras. The algebra also has a gauge transformation invariance. A gauge transformation $\rho(x,t) \to \psi(x) \rho(x,t)$, where $\psi(x) \neq 0$ for all x, transforms the DMZ-equation in such a way that the operators in the new equation generate an isomorphic Lie algebra.

The new equation may again have the form of a DMZ-equation, and in this way systems which are definitely not equivalent as systems may have equivalent filtering problems associated to them. An example are the 1-dimensional Benes systems (cf. various contributions in [19]).

In a way which will (hopefully) become clearer below the estimation Lie algebra $EL(\Sigma)$ encodes information about how difficult the filtering problem for Σ is. For example if it is finite dimensional (a very rare case) Wei-Norman theory does the job for small time; if it is also solvable one thus gets a filter. If it is infinite dimensional but solvable things become more difficult but asymptotic expansions are possible, cf. below; etc.

The BC principle

Let me now describe a second reason why the Lie algebra $EL(\Sigma)$ of a system Σ is important for filtering problems. I like to call it the *BC principle*, not because it is very old, though it could have been maybe, nor is it named after Johny Hart's chartoon character; the BC stand for Brockett and Clark [6] who first enunciated it.

Suppose we have a filter (1.10)-(1.11) on a finite dimensional manifold M for a statistic $\widehat{\phi(x_t)}$. We may as well assume that it is minimal, i.e. has minimal $\dim(M)$. The α and $\beta_1, \dots, \beta_p$ in (1.10) are vectorfields on M. Let $V(M)$ denote the Lie algebra of smooth vectorfields on M. Then the BC principle states the following

5.1. BC principle.
If (1.10)-(1.11) is a minimal filter for a statistic then $\mathcal{L} \mapsto \alpha$, $h_1 \mapsto \beta_1, \ldots, h_p \mapsto \beta_p$ defines an antihomomorphism of Lie algebras from $EL(\Sigma)$ into $V(M)$.

Here "anti" means the following: if $\phi: L_1 \to L_2$ is a map of vectorspaces from the Lie-algebra L_1 to the Lie-algebra L_2, it is called an antihomomorphism of Lie-algebras if $\phi([A,B]) = -[\phi(A),\phi(B)]$ for all $A,B \in L_1$.

EXAMPLE 6.2. Consider again the simplest nonzero linear system (2.5). It is linear so there is the Kalman-Bucy filter for the conditional state $\hat{x}$. This filter is

$$dP_t = (1-P_t^2)dt, \quad dm_t = P_t(dy_t - m_t dt). \tag{6.3}$$

So the two vectorfields α and β of the filter are respectively

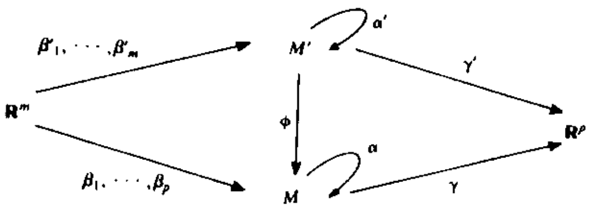
$$\alpha = (1-P^2)\frac{\partial}{\partial P} - Pm\frac{\partial}{\partial m}, \quad \beta = P\frac{\partial}{\partial m}. \tag{6.4}$$

A simple calculation shows $[\alpha,\beta] = \frac{\partial}{\partial m}$, and it is now indeed a simple exercise to show that $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2 \mapsto \alpha$, $x \mapsto \beta$, induces an antimorphism of Lie-algebras. (It also induces a homomorphism, but that is an accident which happens for linear systems (1.12)-(1.13) if the drift term Ax is absent).

A feeling of why the BC principle should be true can be generated as follows. Think for the moment of two automata with given initial state and with outputs (Moore automata), which, when fed the same string of input data, produce exactly the same string of output data. Suppose the second automaton is minimal. Then it is wellknown (and easy to prove by constructing the minimal automaton from the input-output data) that there is a homomorphism of the subautomaton of the first consisting of the states reachable from the initial state to the second automaton; this homomorphism so to speak makes visible that the two machines do the same job. A similar theorem holds for initialized finite dimensional systems [Sussmann [34]), in particular for systems of the form

$$\dot{x} = \alpha(x) + \sum_{i=1}^{m}\beta_i(x)u_i, \quad y = \gamma(x) \tag{6.5}$$

Here the picture produced by theorem is the following commutative diagram



(The theorem asserts the existence of a differentiable map ϕ defined on the reachable from x'_0 subset

of M' which makes the diagram commutative. This in particular implies that $d\phi$ takes the vectorfields $\alpha', \beta'_1, \ldots, \beta'_m$ into $\alpha, \beta_1, \ldots, \beta_m$ respectively, and, ϕ being a differentiable map, $d\phi$ induces a homomorphism from the Lie algebra generated by $\alpha', \beta'_1, \ldots, \beta'_m$ to $V(M)$.

In the case of the BC principle we also have two "machines" which do the same job: one is the postulated minimal filter, the other is the infinite dimensional machinge given by the DMZ-equation (2.2) and the ouput map (2.4). So we are in a similar situation as above but with M' infinite dimensional. A proof in this case follows from considerations of Hijab [20].

The fact that in the case of the BC-principle we get an antihomomorphism arises from the following. Given a linear space V and an operator A on it we can define a (linear) vectorfield on V by assigning to $v \in V$ the tangent vector Av. (So we are considering the equation $\dot{v} = Av$.) This defines an anti-isomorphism of the Lie algebra of operators on V to the Lie algebra of linear vectorfields on V.

What about a converse to the BC principle? I.e. suppose that we have given an antihomomorphism of Lie-algebras $EL(\Sigma) \to V(M)$ into the vectorfields of some finite dimensional manifold. Does there correspond a filter for some statistic of Σ. Just having the homomorphism is clearly insufficient. There are also explicit counterexamples. This is understandable for in any case we completely ignored the output aspect when making the BC-principle plausible. This is not trivial contrary to what the diagram above may suggest. It is not true that given ϕ and any γ one can take $\gamma' = \gamma \circ \phi$. The problem is that γ' as a function on M' = space of unnormalized densities is of a very specific type cf. (2.4).

Even apart from that things are not guaranteed. What we need of course is a ϕ making the left half of the diagram above commutative. Then, if $m' \in M'$ is going to the mapped on $m \in M$, obviously the isotropy subalgebra of $EL(\Sigma)$ at m' will go into the isotropy subalgebra of $V(M)$ at m.

For the case of finite dynamical systems there are positive results of Krener [21] stating that in such a case this extra condition is also sufficient to guarantee the existence of ϕ locally.

The whole clearly relates to seeing to what extend a manifold can be recovered from its Lie algebra of vectorfields (via its maximal subalgebras of finite codimension) and whether differentiable maps can be recovered from the map between Lie-algebras they induce. This question has been examined by Pursell-Shanks [30].

A more representation theoretic way of looking at things is as follows. Both $EL(\Sigma)$ and $V(M)$ come with a natural representation on the space of functionals on M' and the space of functions on M respectively. If there were a ϕ as in the diagram above ϕ would also induce a map between these representation spaces compatible with the homomorphism of Lie algebras. That therefore is clearly a necessary condition. This way of looking at things contains the isotropy subalgebra condition and also contains output function aspects. Thus the total picture regarding a converse to the BC-principle is not unpromising but nothing is established.

Except for one quite positive aspect. If $EL(\Sigma)$ is finite dimensional, the Wei-Norman equations practically define the filter, for small time in the general case, for arbitrary time in the solvable case.

6. Examples of estimation algebras

7.1. The cubic sensor
This is the one dimensional system

$$dx_t = dw_t, \quad dy_t = x_t^3 dt + dv_t \tag{7.2}$$

and it is about the simplext nonlinear system imaginable. Its estimation Lie algebra is generated by $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^6$, x^3.

THEOREM 7.3. (Hazewinkel-Marcus [17]). EL(cubic sensor) = W_1, where $W_1 = \mathbf{R}\langle x, \frac{d}{dx}\rangle$ is the Lie algebra of the differential operators (any order, zero included) with polynomial coefficients.

EXAMPLE 7.4. $dx_1 = dw, dx_2 = x_1^2 dt; dy_1 = x_1 dt + dv_1, dy_2 = x_2 dt + dv_2$. In this case the estimation Lie algebra is $W_2 = \mathbf{R}<x_1, x_2, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}>$, the Lie algebra of all differential operators in two variables with polynomial coefficients.

EXAMPLE 7.5. $dx_1 = dw, dx_2 = x_1^2 dt, dy = x_1 dt + dv_1$. In this case the estimation Lie algebra has a basis A, B_i, C_i, D_i $i = 1,2,\ldots$ with the commutation relations $[A,B_i] = C_i$, $[A,C_i] = B_1 + 2B_{i+1}$, $[B_i,C_j] = -D_{i+j}$ and all other commutation relations between basis elements are zero. Note that in this case the Lie algebra is infinite dimensional but has many ideals $\mathfrak{A}_i$ such that $L / \mathfrak{A}_i$ is finite dimensional.

EXAMPLE 7.6. $dx = dw_1 + xdw_2$, $dy = xdt + dv$. Here again $EL = W_1$.

It has become clear that as a rule estimation algebras tend to be infinite dimensional (except in the linear case: then EL(linear system) has dimension $2n+2$ if the linear system is completely reachable and observable); it has also become noticeable that the Weyl-Heisenberg algebras or Weyl algebras W_n have a tendency to appear very often.

CONJECTURE 7.7. Consider systems (1.1)-(1.2) with polynomial f, G, h. Then generically, i.e. for almost all f, G, h, the estimation algebra will be W_n.

7. The Segal-Shale-Weil representation and all Kalman-Bucy filters

8.1. The linear systems Lie-algebra ls_n.

Consider all differential operators in n indeterminates with polynomial coefficients

$$D = \Sigma c_{\alpha\beta} x^{\alpha} \frac{\partial^{\beta}}{\partial x^{\beta}} \tag{8.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ are multiindices $\alpha_i, \beta_j \in \mathbf{N} \cup \{0\}$. Consider those D which are of total degree $\leqslant 2$; i.e. such that $|\alpha| + |\beta| > 2 \Rightarrow c_{\alpha\beta} = 0$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. As is readily verified these form a finite dimensional Lie algebra (under the commutator product $[D_1, D_2] = D_1 D_2 - D_2 D_1$) of dimension $2n^2 + 3n + 1$. A basis is

$$1; x_1, \ldots, x_n; \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}; \frac{\partial}{\partial x_i \partial x_j}, i,j = 1, \ldots, n; x_i x_j, i,j = 1, \ldots, n; x_i \frac{\partial}{\partial x_j}, i,j = 1, \ldots, n. \tag{8.3}$$

The operators of total degree $\leqslant 1$ form a subalgebra h_n (basis: $1; x_1, \ldots, x_n; \partial / \partial x_1, \ldots, \partial / \partial x_n$) which is in fact an ideal. The quotient is isomorphic to the symplectic algebra sp_n of all real $2n \times 2n$ matrices M such that

$$MJ + JM^T = 0, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{8.4}$$

The isomorphism is given by

$$E_{i,n+j} + E_{j,n+i} \mapsto x_i x_j; \; E_{n+i,j} - E_{n+j,i} \mapsto \frac{\partial^2}{\partial x_i \partial x_j}; \tag{8.5}$$

$$E_{i,j} - E_{n+j,n+i} \mapsto x_i \frac{\partial}{\partial x_j} + \tfrac{1}{2}\delta_{ij}; \; i,j = 1, \ldots, n.$$

Here $E_{i,j}$ is the matrix with a 1 at spot (i,j) and 0 everywhere else; these linear combinations of the $E_{i,j}$ form a basis of sp_n; this isomorphism exhibits sp_n as a subalgebra complementary to h_n; i.e. as a Levi-factor for the short exact sequence $0 \to h_n \to ls_n \to sp_n \to 0$).

8.2. The oscillator representation.

There is a famous representation of sp_n which occurs in the framework of symmetries of boson fields (Shale, Segal), in algebraic number theory (Weil), and a multitude of other places, known variously as the Segal-Shale-Weil representation or the oscillator representation. One way to obtain it is as follows. Let H_n denote the Heisenberg group, $H_n = \mathbf{R}^n \times \mathbf{R}^n \times S^1$ with multiplication

$$(x,y,z)(x',y',z') = (x + x', y + y', e^{-2\pi i <x,y'>} zz') \tag{8.7}$$

where $<\, ,\, >$ denotes the standard scalar product on $\mathbf{R}^n$. The Lie algebra of H_n is of course $h_n = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$. And the Lie-bracket of h_n can be interpreted as giving (and given by) a bilinear form $\mathbf{R}^{2n} \times \mathbf{R}^{2n} \to \mathbf{R}$ defined by the matrix J, cf. (8.4) above. Thus the Lie group Sp_n of sp_n can be seen as a group of automorphism of h_n and H_n which is moreover the identity on the centre $S^1 \subset H_n$. Let ρ be the standard Schrödinger representation of H_n in $L^2(\mathbf{R}^n)$

$$\begin{aligned} &(x,0,0) \to M_x, \; (M_x f)(x') = e^{2\pi i <x,x'>} f(x'), \; f \in L^2(\mathbf{R}^n) \\ &(0,y,0) \to T_y, \; (T_y f(x') = f(x' - y), \; f \in L^2(\mathbf{R}^n) \\ &(0,0,z) \to S_z, \; (S_z f)(x') = z f(x'), \; f \in L^2(\mathbf{R}^n) \end{aligned} \tag{8.8}$$

Now let $g \in Sp_n$ be seen as a group of automorphisms of H_n. Then $h \mapsto \rho(g(h))$ is another irreducible representation of H_n with the same central character. So by the Stone-von Neumann theorem there is an $\omega(g)$ intertwining them, i.e. such that

$$\omega(g)\rho(h)\omega(g)^{-1} = \rho(g(h)). \tag{8.9}$$

These $\omega(g)$ are unique up to scalar factors and therefore define a projective representation of Sp_n. The factors can be fixed up to define a representation of the two-fold covering $\tilde{Sp}_n$ of Sp_n. This is the Segal-Shale-Weil representation.

8.3. All Kalman-Bucy filters.

Now consider something apparently totally unrelated, namely the DMZ-filtering-equation (2.2) for a linear dynamical system

$$dx = Axdt + Bdw, \; dy = Cxdt + dv, \; x \in \mathbf{R}^n, \; w \in \mathbf{R}^m, \; y, v \in \mathbf{R}^p. \tag{8.11}$$

It is a trivial remark that the operators occurring in (2.2) are all in ls_n in this case. And in fact the Lie algebra generated by them will consist of the second order operator $\mathcal{L}$ and a subalgebra of h_n stable under $\mathcal{L}$. In most cases, to be precise in the case that the system (A,B,C) is completely reachable and completely observable, this will be all of h_n, giving us generically an estimation algebra of dimension $2n+2$ which is a subalgebra of ls_n, which has dimension $2n^2 + 3n + 1$.

The Kalman-Bucy filter defines by BC principle an antihomomorphism of this Estimation Lie algebra $EL(A,B,C)$ into the vector Lie algebra of vector fields $V(\mathbf{R}^N)$, $N = n + \frac{1}{2}n(n+1)$.

THEOREM [16] 8.12. For varying (A,B,C) these anti-homomorphisms fit together to define a antihomomorphism of all of ls_n into $V(\mathbf{R}^N)$ with as kernel the centre $\mathbf{R}1$. This representation can be lifted to one on $V(\mathbf{R}^{N+1})$ which is faithful.

The explicit formulas are as follows. Interpret a point $x \in \mathbf{R}^{N+1}$ as a triple $x = (c,m,P)$ consisting of a scalar c, an n-vector m, and a symmetric $n \times n$ matrix P. The antihomomorphism is then given by

$$1 \to \frac{\partial}{\partial c} \tag{8.13}$$

$$x \to m_i \frac{\partial}{\partial c} + \sum_i P_{ii} \frac{\partial}{\partial m_i} \tag{8.14}$$

$$\frac{\partial}{\partial x_i} \to -\frac{\partial}{\partial x_i} \tag{8.15}$$

$$x_ix_j \to (m_im_j + P_{ij})\frac{\partial}{\partial c} + \sum_l (m_iP_{jl} + m_jP_{il})\frac{\partial}{\partial m_l} \tag{8.16}$$

$$+ \sum_{s,t} P_{is}P_{jt}\frac{\partial}{\partial P_{st}} + \sum_l P_{il}P_{jl}\frac{\partial}{\partial P_{ll}}$$

$$x_i\frac{\partial}{\partial x_j} \to -m_i\frac{\partial}{\partial m_j} - \delta_{ij}\frac{\partial}{\partial c} - P_{ij}\frac{\partial}{\partial P_{jj}} - \sum_l P_{il}\frac{\partial}{\partial P_{jl}} \tag{8.17}$$

$$\frac{\partial^2}{\partial x_i\partial x_j} \to \frac{\partial}{\partial P_{ij}} \text{ if } i\neq j, \quad \frac{\partial^2}{\partial x_i^2} \to 2\frac{\partial}{\partial P_{ii}}. \tag{8.18}$$

Inversely these formulas can be checked directly to give an antihomomorphism of Lie algebras and this thus verifies the BC principle for linear dynamical systems and also for families of such depending on a parameter.

Changing all the minus signs in (8.17) and (8.15) into plus signs gives a faithful homomorphism of ls_n into $V(\mathbf{R}^{N+1})$.

Restricting this homomorphism to sp_n as given by (8.5) then defines a homomorphism of sp_n into $V(\mathbf{R}^{N+1})$.

The final remark is that this realization of sp_n as a Lie algebra of vector fields on $\mathbf{R}^{N+1}$ has much to do with the Segal-Shale-Weil representation. The precise statement is that the mapping

$$(c,m,p) \to \exp(c + <2\pi i m,x> - 2\pi^2 P(x)) \in L^2(\mathbf{R}^n) \tag{8.19}$$

(where $P(x)$ is the quadratic form defined by the symmetric matrix P; note that apart from a scaling factor this is the normal distribution with mean m and covariance P) linearizes the vectorfields in the image of) ls_n in $V(\mathbf{R}^{N+1})$ and switching from a linear vectorfield to the operator which defines it then defines a representation of $ls_n \supset sp_n$. This is another real form of the Segal-Shale-Weil representation meaning that after tensoring with $\mathbf{C}$ (= extending scalars to the complexes), they become isomorphic.

9. W_n and $V(M)$

We have seen that the Weyl-Heisenberg algebra $W_n = \mathbf{R}<x_1,...,x_n; \partial/\partial x_1,...,\partial/\partial x_n>$ of all differential operators with polynomial coefficients often occurs in filtering problems, i.e. as an Estimation Lie algebra. Given the BC-principle it is therefore of interest to know something about its relations with another class of infinite dimensional Lie algebras, viz the Lie algebras $V(M)$ of smooth vectorfields on a finite dimensional manifold. The algebra W_n has a one-dimensional centre $\mathbf{R}\cdot 1$ consisting of the scalar multiples of the identity operator.

THEOREM 9.1. (Hazewinkel-Marcus [17]). Let $\alpha: W_n \to V(M)$ or $W_n/\mathbf{R}.1 \to V(M)$ be a homomorphism or antihomomorphism of Lie algebras, where M is a finite dimensional manifold. Then $\alpha = 0$.

The original proof of this result ([17]) was long and computational. Another much shorter proof based on the nonexistence of finite dimensional representations of h_n for which 1 gets mapped onto the unit operator has more recently been given by Toby Stafford.

10. The cubic sensor.

Consider again the cubic sensor, i.e. the one-dimensional system

$$dx = dw, \quad dy = x^3dt + dv \tag{10.1}$$

consisting of Wiener noise, cubically observed with further independent noise corrupting the observations. As noted before (theorem 7.3)

$$EL(\text{cubic sensor}) = W_1. \tag{10.2}$$

Now suppose that we have a finite dimensional filter for some conditional statistic $\widehat{\phi(x_t)}$ of the cubic sensor. By the BC-principle (6.1) if follows that there is an antihomomorphism of Lie-algebras

$W_1 = EL$ (cubic sensor) $\overset{*}{\to} V(M)$. By theorem 9.1 it follows that $\alpha = 0$ and from this it is not hard to see that the only statistics of the cubic sensor for which there exists a finite dimensional exact recursive filter are the constants.

A direct proof of this, which sort of proves the BC-principle in this particular case along the way, is contained in Hazewinkel-Marcus-Sussmann [18].

11. Perturbations and approximations

Let us start with an example. Consider the weak cubic sensor

$$dx = dw, \quad dy = xdt + \epsilon x^3dt + dv \quad (\Sigma_\epsilon) \tag{11.1}$$

If $\epsilon = 0$, this the simplest nontrivial linear system for which there is the Kalman filter. For $\epsilon\neq 0$ one can prove that $EL(\Sigma_\epsilon) = W_1$ again, [15]. So for all $\epsilon\neq 0$ there is no recursive exact filter for any non-constant statistic. Yet it is hard to believe that for small ϵ the Kalman-Bucy filter would not do a good job of first approximation. A question thus arises whether the estimation Lie algebra also has things to say about approximate filters. In this section and the following ones I shall argue that it does.

The first observation is as follows. If one actually commutes the two generators $\frac{1}{2}d^2/dx^2 - \frac{1}{2}(x+\epsilon x^3)^2$, $(x+\epsilon x^3)$ repeatedly of course eventually all the basis elements of W_1 appear. But they appear with higher and higher powers of ϵ and the ϵ-degree grows faster than the degree in (α) and (β) of the $x^\alpha\partial^\beta/\partial x^\beta$.

A precise version of this is as follows. Consider the two generators just listed as operators over the ring $\mathbf{R}[\epsilon]$ (or $\mathbf{R}[[\epsilon]]$), i.e. consider ϵ as an extra variable. Then it makes sense to consider

$$EL(\Sigma_\epsilon)\otimes_{\mathbf{R}[\epsilon]}\mathbf{R}[\epsilon]/\epsilon^n =: EL(\Sigma_\epsilon)\text{mod}(\epsilon^n). \tag{11.2}$$

This simply amounts to setting $\epsilon^m = 0$ for $m \geq n$ whenever it appears. The set of all $\epsilon^i x^j d^k/dx^k$ with $i \geq n$ form an ideal in $\mathbf{R}[\epsilon]<x, d/dx>$, so this makes sense. Now observe

PROPOSITION 11.3. [15]. The Lie-algebras $EL(\Sigma_\epsilon)$ mod ϵ^n are finite-dimensional for all n.

As an example $EL(\Sigma_\epsilon)$ mod ϵ^2 turns out to be 14 dimensional with basis

$$\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2 - \epsilon x^4,\ x,\ \epsilon x^3,\ \frac{d}{dx},\ 1,\ \epsilon,\ \epsilon x^2\frac{d}{dx},\ \epsilon x,$$

$$\epsilon x\frac{d}{dx},\ \epsilon\frac{d^2}{dx^2},\ \epsilon\frac{d}{dx},\ \epsilon\frac{d^3}{dx^3},\ \epsilon x\frac{d^2}{dx^2},\ \epsilon x^2.$$

This is a general phenomenon.

THEOREM 11.4. [15]. Let Σ_ϵ be a system of the form

$$dx = (Ax + \epsilon P_A(x))dt + (B + \epsilon P_B(x))dw, \quad dy = (Cx + \epsilon P_C(x))dt + dv \tag{11.5}$$

where P_A, P_B, P_C are polynomial vector and matrix valued functions of the approximate dimensions. Then $EL(\Sigma_\epsilon)$ mod (ϵ^n) is finite dimensional for all n. It is also solvable.

In [15] this is proved for the case $P_B = P_A = 0$. The proof generalizes immediately (simply give ϵ a negative enough degree to make degree decreasing all terms in the generators of $EL(\Sigma_\epsilon)$ in which ϵ appears (both $x_i, \partial/\partial x_j$ are given degree 1 in this argument).)

The next obvious question is: do these "finite dimensional quotients of $EL(\Sigma_\epsilon)$" actually compute anything, do they correspond to filters for some statistic? In the case of the weak cubic sensor this is (11.1) easy to answer. Consider the unnormalized conditional density $\rho(x,t,\epsilon)$ and (formally) expand it as a power series in ϵ

$$\rho(x,t,\epsilon) = \rho_0(x,t) + \epsilon\rho_1(x,t) + \epsilon^2\rho_2(x,t) + \ldots \tag{11.6}$$

Then $EL(\Sigma_\epsilon)$ mod (ϵ^n) corresponds to the first n coefficients $\rho_0(x,t),\ldots,\rho_{n-1}(x,t)$, and via Wei-Norman theory actually computes them. This is generally true, also in the setting of theorem 11.4. In the case of the weak cubic sensor (11.6) actually converges (for small ϵ). That, it appears, is not generally true. But it is still true that (11.6) gives an asymptotic expansion (Blankenschip-Liu-Marcus [4]). The Lie algebras being solvable one can of course implement these approximate filters, using the Wei-Norman technique. This was done in [4] and also the results were compared with the extended Kalman filter (EKF). The zero-th order approximation (of course) performed worse than EKF but the first order approximation performed better!

These Lie algebras $EL(\Sigma_\epsilon)$ tend to become large rapidly and to actually produce the, say FORTRAN, code is a long, but mechanical, job, prone to errors. Even the simplest nontrivial case needs several pages of densely written code. It is thus natural to try to let the computer do the job itself and in this way these ideas and techniques are being implemented in an expert system which is a joint effort of INRIA and the Department of Electrical Engineering of the University of Maryland (cf. Blankenschip [3]; the system also contains many other facets of stochastic control, filtering and optimization).

From the point of view developed in section 4 above the fact that calculating $\rho_0(x,t),\ldots,\rho_{n-1}(x,t)$ corresponds to $EL(\Sigma_\epsilon)$ mod (ϵ^n) can be understood as follows. Choosing a basis suitably the remarks made in section 4 about the compatibility of Wei-Norman theory with quotients say that $\rho(x,t,\epsilon)$ admits an "expansion"

$$\rho(x,t,\epsilon) = e^{g_1(t)A_1}e^{g_2(t)A_2}\ldots e^{g_i(t)A_i}\ldots\rho_0(x) \tag{11.7}$$

with $g_1,\ldots,g_{m(n)}$ where $m(n)=\dim(EL(\Sigma_\epsilon)$ mod $(\epsilon^n))$ depending only on $EL(\Sigma_\epsilon)$ mod (ϵ^n). The operators A_i in (11.7) involve higher and higher powers of ϵ. Writing out the exponentials one recovers (11.6). (And this point of view also strongly suggests (because also higher derivatives appear in the A_i) that the best one can hope for in general is an asymptotic expansion.)

12. The profinite dimensional case.

A Lie-algebra L is said to be profinite dimensional if there is a sequence of ideals $L\supset\mathfrak{A}_1\supset\mathfrak{A}_2\supset\mathfrak{A}_3\supset\ldots$ such that

$$\dim L/\mathfrak{A}_i < \infty \text{ for all } i \tag{12.1}$$

$$\bigcap_i \mathfrak{A}_i = \{0\}. \tag{12.2}$$

Suppose the estimation Lie algebra $EL(\Sigma)$ has this property. Then again, as in the previous section, one can write an expansion

$$\rho(x,t) = e^{g_1A_1}e^{g_2A_2}\ldots e^{g_nA_n}\ldots\rho_0(x) \tag{12.3}$$

and consider the possible approximants

$$\rho^{(n)}(x,t) = e^{g_1A_1}e^{g_2A_2}\ldots e^{g_nA_n}\rho_0(x). \tag{12.4}$$

Using, again, that the equations for $g_1,\ldots,g_n$; $n=n(m)=\dim L/\mathfrak{A}_m$ do not depend on $g_{n+1},\ldots$. Abstractly, there is no immediate reason to expect the higher $e^{g_kA_k}$ to be small, though one would expect this to the case in the majority of the interesting cases, even in more general cases than this, as I shall argue below in section 14.

Profinite dimensional estimation Lie algebras occur frequently. Consider systems

$$dx = f(x)dt + G(x)dw_t,\ \ dy = h(x)dt + dv \tag{12.5}$$

with the additional assumptions that f,G and h are analytic (totally around zero) and that $f(0)=G(0)=0$.

THEOREM 12.6. [17] Under the assumptions made immediately above $EL(\Sigma)$ is profinite dimensional.

If one adds the condition that $h(0)=0$ (which surely does no harm; removing a known constant from the observation equation is a triviality) the resulting estimation Lie algebra is even solvable (meaning that all the quotients $L/\mathfrak{A}_i$ are solvable).

Another case of a profinite dimensional estimation Lie algebra (different from the class of theorem 12.6, the identification case to be treated below, and the perturbation case of section 11 above) is example 7.5. As a rule one should probably not expect that "the statistic calculated by $L/\mathfrak{A}_i$" of a system whose estimation Lie algebra happens to be profinite dimensional, is easily interpretable (recognizable) as the statistic of an interesting quantity. In the case of example 7.5 this is however the case, (Liu-Marcus [23]).

13. Identification of linear dynamical systems

Suppose now that we are faced with a somewhat different problem. Namely suppose one has reason to believe, or simply does not know anything better to do, that a given phenomenon, say a time series, is modeled by a linear dynamical system

$$dx = Axdt + Bdw,\ \ dy = Cxdt + dv \tag{13.1}$$

Now, however, the coefficients in A,B,C are unknown and also have to be estimated from the observation $y(t)$. That is the system (13.1) has to be identified. It is easy to turn this into a filtering problm by adding the (stochastic) equations

$$dA = 0,\ \ dB = 0,\ \ dC = 0 \tag{13.2}$$

(or just $dr_{ij}=0$ whether the r_{ij} run through the coefficients which are unknown, if A,B,C are partly known; for example because of structural considerations). The resulting filtering problem is nonlinear.

13.1. Observation

The estimation Lie algebra of the system (13.1)-(13.2) is a sub-Lie-algebra of the current Lie algebra $ls_n\otimes\mathbf{R}[A,B,C]$ where $\mathbf{R}[A,B,C]$ stands for the ring of polynomials in the indeterminates a_{ij},b_{kl},c_{rs}.

A corollary is that these estimation algebras are profinite dimensional. And looking a bit more closely at them, they are solvable [37]. Thus the ideas and considerations of the previous two sections can be brought into play and one can try to do infinite dimensional Wei-Norman theory etc. This is attempted in Krishnaprasad-Marcus-Hazewinkel [37]. In this rather special case it turns out that the higher approximations (the zero-th approximation is simply the family of Kalman-Bucy filters parametrized by A,B,C also discussed in section 8 above) have to do with sensitivity equations: sensitivities of the ouput $y(t)$ with respect to changes in the parameters A,B,C.

As stated above, though, the problem is degenerate and likely to cause all kind of difficulties. The problem is that the conditional density $\rho(x,A,B,C,t)$ will be degenerate because the A,B,C are not uniquely determined by the observations. Indeed if S is an invertible $n\times n$ matrix then the system (13.1) given by the matrices SAS^{-1},SB,CS^{-1} instead of A,B,C gives exactly the same input-output behaviour. Thus we should really be considering this problem on a suitable quotient space $\{(A,B,C)\}/GL_n$. These quotient spaces as a rule are not diffeomorphic to open sets in some $\mathbf{R}^n$. This is one way in which stochastic systems like (1.1)-(1.2) on nontrivial manifolds naturally arise and it leads to the necessity of finding a DMZ-equation in this more general context. Work in this direction has been done by Ji Dunmu and T.E. Duncan.

Let me add one observation. For the filters giving $\hat{x},\hat{A},\hat{B},\hat{C}$ for problem (13.1)-(13.2) one expects $\hat{x}$ to move fast relative $\hat{A},\hat{B},\hat{C}$. Thus it would make sense to consider a system

$$dx = (A_0+\epsilon A_1)xdt + (B_0+\epsilon B_1)dw,\ \ dy = (C_0+\epsilon C_1)dt + dv \tag{13.4}$$

$$dA_1 = 0,\ \ dB_1 = 0,\ \ dC_1 = 0$$

(where A_0,B_0,C_0 are assumed known) and apply the ideas of section 11 above to find optimal directions of change (i.e. the A_1,B_1,C_1).

14. Asymptotic expansions and approximate homomorphisms.

The ideas to be outlined below in this section are still speculative but there are quite a number of positive signs.

First however let me point out that the procedures based on Wei-Norman techniques as described in sections 11 and 12 above clearly indicate that existence, uniqueness and regularity results for solutions of the DMZ-equation have a lot to do with the existence of asymptotic expansions ([2,4]). For regularity results etc. cf. e.g. work of D. Michel, J.-M. Bismut, E. Pardoux, M. Chaleyat-Maurel, D. Ocone, Th. Kurtz, W.E. Hopkins Jr., H.J. Sussmann a.o. ([25,8,22,2] and references in these papers).

Let us consider a control system of the form

$$\dot{x} = f(x) + \Sigma u_i g_i(x) \tag{14.1}$$

where the f and g_i are vectorfields. To make thinking easier assume that 0 is a stable and asymptotically stable equilibrium for the unforced equation. A system like (14.1) is intended as a model of something and as such one can argue that say the values of $f(x), g_i(x)$ are relatively well known, the values of their (partial) derivatives (w.r.t. the x_i) will be less well known, the second partial derivations are still less well determined etc..

Thus, intuitively, for systems which represent or model real (stable) things one would expect that in many cases the behaviour of (14.1) will depend primarily on the first few terms which appear in the Lie algebra generated by f and the g_i. The higher brackets should matter less and less.

That means that instead of looking at $Lie\{f, g_1, \ldots, g_m\}$, the Lie algebra generated by $f, g_1, \ldots, g_m$ as a Lie algebra without further structure, we should look at it as a Lie algebra with a given set of generators and sort of keep track of how often these generators are used to generate further elements of the algebra. (For each time a bracket is taken a differentiation is applied, and thus the higher brackets of the $f, g_1, \ldots, g_m$ depend only on the deeper parts of the Taylor expansions of $f, g_1, \ldots, g_m$.)

Personally I would also say that having noises rather than precise deterministic controls u_i would enhance this type of (structural?) stability.

A precise way to keep track of how often the generators are used is to introduce one extra counting indeterminate z and to consider instead of $L = Lie\{f, g_1, \ldots, g_m\}$ the Lie algebra generated by the vectorfields $\{zf, zg_1, \ldots, zg_m\}$. This Lie algebra L_z is topologically nilpotent, i.e. if $L_z^{(n)} = [L_z, L_z^{(n-1)}]$, $L_z^{(0)} = L_z$, then $\cap L_z^{(m)} = \{0\}$. And a homomorphism $L_z \to V(M)$ into the vectorfields on M with kernel $L_z^{(n)}$ precisely means "respecting the structure of the Lie algebra L up to brackets of order n". All this is very much related to the ideas of nilpotent approximation as introduced by Stein, Rothschild, Goodman and Rockland, [32,14,31] in the study of hypoellipticity and taken up by Crouch in system theory [10].

Thus in filtering theory it would seem natural to look at the Lie algebra of operators $EL_z(\Sigma)$ generated by the operators

$$z_0 \mathcal{L},\ z_1 h_1, \ldots, z_p h_p$$

where the $z_0, z_1, \ldots, z_p$ are additional variables (so as to give, if desired, certain observations more weight than others and to be able to set certain of them, especially z_0, equal to 1). The idea would be then to study the filters produced by Wei-Norman type techniques for the various finite dimensional quotients and to see whether this produces viable expansions.

15. Removing outliers

A final idea in much the same spirit as before is the following. Suppose we are again dealing with a system

$$dx = f(x)dt + G(x)dw, \quad dy = h(x)dt + dv. \tag{15.1}$$

Suppose also to make thinking easier that the thing is more or less stable so that x tends to remain in some bounded partion of $\mathbf{R}^n$ (f asymptotically stable) and maybe suppose also that h is proper, so that large y observations are exceedingly rare and should probably be discounted. Suppose that $e^{-\|x\|^2}$

is differential algebraically independent of f, G, h. This is for example this case if f, G, h are polynomial and also if they are of compact support. In other cases other functions with similar properties can presumably be found. Now instead of (15.1) consider the modified system

$$dx = f(x)dt + G(x)dw, \quad dy = e^{-a\|x\|^2} h(x)dt + dv \tag{15.2}$$

where $a > 0$ is a small parameter. Note that the only thing which (15.2) does with respect to (15.1) is to discount large y observations.

Now consider the estimation Lie algebra of the sytem (15.2).

THEOREM 15.3. If $e^{-a\|x\|^2}$ is differentially algebraically independent of f, G, h then the estimation Lie algebra of (15.2) is pro-finite dimensional and solvable. To be more precise it is finite dimensional and solvable mod $(a^i e^{-ja\|x\|^2}, i + j \geq n)$ for all n.

Thus the yoga of the previous sections can again be applied and the behaviour of the resulting filters as a goes to zero could be studied.

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