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SECOND WORKSHOP ON MATHEMATICS IN INDUSTRY

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INTRODUCTION INTO THE FUNDAMENTAL CONCEPTS OF LINEAR SYSTEM AND CONTROL THEORY

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A rather big part of the "Second Workshop on Mathematics in Industry", organized by ICTP in Trieste during February 1987 is dedicated to system and control theory; this expresses the conviction of the directors that this part of mathematics is especially appropriate also for applications in industry and economy of developping countries. To find a mathematical model for a system (the economy of a country, the system consisting of climate and agriculture, etc.), for which the internal details are not very well known, to control such a system is a typical problem of identification and control of dynamical systems. Many mathematical disciplines are included in this subject: The "normal theory" of ODE's, stochastic processes, rather special topics of linear algebra (especially for the kind of system theory which was developped by Kalman), concepts of algebraic geometry (especially for nonlinear systems and for identification). The applications, originally concentrated on electrical systems and economics, have now reached almost all fields which can be mathematized; there is even a philosophical view claiming that science is nothing else but identifying mathematical models which approximate measurements and can be used for prediction. We certainly do not want to go so far - but we believe that system and control theory is a topic of rather high mathematical level, which very strikingly can prove the usefulness of mathematics in almost all societies.

In contrast to the first workshop we have organized the system and control theory part of the workshop in 6 blocks, which are, besides the introductory ones, taught by specialists: For modern topics in linear control (Knobloch), for economic applications (Feichtiger), for nonlinear control (Isidori) and for identification (Hazewinkel). The introducing blocks are given by myself and by a former coworker of myself (Krüger) who owns now a small but quickly growing mathematical consulting company - we both believe to have now a rather wide experience about how "sophisticated" mathematics can be used in different fields of economical and mainly industrial life.

We all hope to be able to transfer our knowledge and experience to mathematicians who were mainly interested in the "pure" side of our science, up till now. I personally believe (and this is the main reason for me to codirect this workshop) that the knowledge how to use mathematics for practical problems in a proper and responsible way gives the only chance for mathematicians to help their societies in emancipating from any suppression from outside; staying in the ivory tower does not prevent scientists from becoming "guilty".

Literature:

There are many books on Control Theory - linear and nonlinear, mathematically more or less rigorous, etc. In the introductory block we concentrate on fundamental concepts for linear systems and use mainly the book of

H. W. Knobloch, H. Kwakernaak: Lineare Kontrolltheorie, Springer 1985 (KK)

Especially for the first lecture we refer also to a series of articles

"From Time Series to Linear Systems", Part I,II and III, by Jan Willems (JW), which appeared as a preprint of the University of Groningen, Dep. of Mathematics, P.O.B. 800., 9700 AV Groningen, The Netherlands (March 1986 - now at least in parts also in Automata)

Literature for subsequent blocks, for example for nonlinear aspects will be given in the corresponding lectures.

Chapter 1 : Dynamical Input/Output and Control Systems

(JW part I,(KK),p2-26)

Definition 1.1 A dynamical system is a triple $\Sigma := \{T, W, \mathcal{B}\}$, where $T \subset \mathbb{R}$ is the time set, W the signal space and $\mathcal{B} \subset W^T$ gives the behaviour of the system (JW, p.3).

Typically $T = [0,T]$ or $T = \mathbb{R}$ or for time discrete cases $T = \mathbb{Z}^+$ or $T = \mathbb{Z}$. W describes the nature of the signal and typically we have $W = \mathbb{R}^q$. $\mathcal{B}$ consists of those trajectories $\underline{w} : T \to W$, which are possible in the system.

Remark : There is no input/output concept in this definition; W may however consist of input- and output space and $\mathcal{B}$ may consist of the pairs of input- and output sequences. In many practical applications there is no natural distinction between input and output, since there is no natural decision what causes and what effects are. (Pollution problems, stockmarket)

Example (Leontief economy): N products, which have production rates $x_1,\ldots,x_N$; in order to produce one unit of the j-th product one needs at least a_{ij} units of the i-th product (a_{ij} : "technology coefficients")

$\Longrightarrow$ (LE) $x_i(t) > \sum_{j=1}^{n} a_{ij} x_j(t+1)$, $t \in \mathbb{Z}_+$ or $\mathbb{Z}$

$\Longrightarrow$ $(\mathbb{Z}, \mathbb{R}_+^N, \{\underline{x} : \mathbb{Z} \to \mathbb{R}_+^N \mid \text{(LE) is satisfied}\}) = \Sigma$ is a dynamical system

Special cases: 1.) Autoregressive Models (AR), where $T = \mathbb{Z}$ or $\mathbb{Z}_+$, $W = \mathbb{R}^q$ and $\mathcal{B}$ defined by $R_l\, w(t+l) + R_{l-1} w(t+l-1) + \ldots + R_o w_o(t) = 0$ ($R_o,\ldots,R_l$ are gxq- matrices) or shorter $R(z)w = 0$ with $R(z) = R_l r^l + \ldots + R_o z^o$ and $(zw)(t) = w(t+1)$ is the backward shift. The time continous analogue is

$$R(\frac{d}{dt})w = 0$$

i.e. a higher order ODE.

2.) Input/Output Systems (ARMA), $T = \mathbb{Z}$, $W = \mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^k = U \times Y$

$$P(z)y = Q(z)u \quad \text{(i/o)}$$

If $\det(P) \neq 0$, one may interprete u as cause and y as effect

3.) Linear Control System: $W = \mathbb{R}^m \times \mathbb{R}^k$, $w = (u,y)$ and $zx = Ax + Bu$, $y = Cx + Du$ where $x \in \mathbb{R}^n$, A(nxn), B(nxm), C(kxn) and D(kxm).

The continous analogue is the time independent linear system

(TLS) $\dot{x} = Ax + Bu$, $y = Cx + Du$

(but realize, that discretization of (TLS) gives a control system with different A,B, C,D)

4.) The most general concept in this direction is the concept of systems with auxiliary variables, where the behaviour $\mathcal{B}$ is defined as

$$\mathcal{B} = \{ w : T \to \mathbb{R}^q \mid \exists\, \varphi : T \to \mathbb{R}^r \text{ with } R'(z)w = R''(z)\varphi \}$$

φ is called "auxiliary variable".

This model is useful, if a clear distinction between input and output is not possible.

(Example: System = electrical circuit, w consists of the voltage / current pairs at the external parts, φ of the voltage / current pairs in the internal branches and the relation describes Kirchhoff's laws and constitutive equations of the electrical elements in these branches).

Remark : All models can be reformulated as (AR)- models. Moreover, every (AR) can be described as (ARMA), where w can be separated into input u and output y in many different ways (which means, that causality is also a question of representation) - but one can show, that the dimensions of the input- and the output space are uniquely defined.

State space models can be defined also for general dynamical systems

Definition 1.2 : A state space system is a quadruple $\check{\Sigma} := \left\{T, W, X, \check{B}\right\}$ where $T \subset \mathbb{R}$ is the time set, W is the (external) signal space, X the state space and $\check{B} \subset (W \times X)^T$ gives the (internal) behaviour; $\check{B}$ has to satisfy the axiom of state:
If $(w_1(\cdot),\ x_1(\cdot))$ and $(w_2(\cdot),\ x_2(\cdot))$ are in $\check{B}$ and for some $\dot{t}$ the equation $x_1(\dot{t}) = x_2(\dot{t})$ holds, then the two behaviours concentrated at $\dot{t}$ are also forming a behaviour
i.e. $(w(\cdot),\ x(\cdot))$ defined by

$$(w(t),\ x(t)) = \begin{cases} (w_1(t), x_1(t)) & \text{for } t < \dot{t} \\ (w_2(t), x_2(t)) & \text{for } t > \dot{t} \end{cases}$$

is also in $\check{B}$ (any path leading to a particular state will be compatible with any other path emanuating from the same state = the past and the future are conditionally independent given the present state).

Clearly: $\check{\Sigma}$ given, Σ defined by $\left\{T, W, B\right\}$ with

$$B := \left\{ w : T \to W \mid \exists\ x(\cdot) : T \to X \text{ such that } (w(\cdot), x(\cdot)) \in \check{B} \right\}$$

defines an "induced" dynamical system, where B is called the external behaviour.

If Σ is given and we find a $\check{\Sigma}$ which induces Σ, one calls $\check{\Sigma}$ state space representation or realization of Σ. There arises the natural question of minimal realizations; for linear systems this means: Minimal dimension of X. Willems has shown, that for linear systems with $T = Z$ or $\mathbb{R}$ all minimal realizations are equivalent.

All (AR) can be transformed into control systems: There exists a constant nonsingular matrix T and (A,B,C,D), such that $R(z)T^{-1}w = 0$ is equivalent to the control system with A,B,C,D and $w = (u,y)$. T may be chosen such that $D = 0$. Control systems with $D = 0$ are in this way as general as (AR)- models.

We have not yet defined linearity and similar notions in the general context of dynamical systems - we have just considered special cases, which will turn out not to be so special as they look like.
$\Sigma = \left\{T, W, B\right\}$ is called <u>linear</u>, if W is a vectorspace and B is a linear subspace of W^T; Σ is called <u>time invariant</u> if T is closed with respect to summation and if $w(\cdot) \in B$ implies $(z^\tau w)(\cdot) \in B$ for all $\tau \in T$ $\left((z^\tau w)(t) := w(t+\tau)\right.$, such that $\left.z^1 = z\right)$
Willems introduces moreover the concept of completeness which is important in his general frame work.

Definition 1.3 : $\Sigma = \left\{T, W, B\right\}$ is called complete if the following holds:
$w(\cdot)$ belongs to B if and only if each restriction of w to a <u>finite</u> interval $[t_o, t_1]$ belongs to the restriction of B on this interval
i.e.

$$w(\cdot) \in B \Longleftrightarrow$$
$$w(\cdot)/_{T \cap [t_o, t_1]} \in B/_{T \cap [t_o, t_1]}$$
$$\text{for all } -\infty < t_o < t_1 < \infty$$

Completeness means, that the behaviour at $t = \pm\infty$ does not influence the question whether a trajactory belongs to B or not. (Thus, of B would be defined to be L^2 if $T = \mathbb{R}$ or l^2 of $T = Z$, then Σ would not be complete !)
More restrictive than completeness is the concept of finite memory, if we consider the continous case $T = \mathbb{R}$ or $\mathbb{R}_+$:
Σ has finite memory, if there exists a $\Delta > 0$ such that

$w(\cdot) \in B \Longleftrightarrow w(\cdot)/_{[\tau,\tau+\Delta]} \in B/_{[\tau,\tau+\Delta]}$ for all τ.

Σ has local memory, if the equivalence given above holds for all Δ. Systems described by ODE'S are typical for dynamical systems with local memory.
One can show, that many linear, time invariant systems with local memory are really of the continous (AR)- form $R(\frac{d}{dt})w=0$ and consequently also control systems. As far as I can see the question concerning the relation between general state space systems and nonlinear first order ODE system is not completely settled. Before coming to this point I just state one of the main theorems of Willems for the time discrete case:

> If $T = \mathbb{Z}$ or $\mathbb{Z}_+$ and $\Sigma = \{T, \mathbb{R}^q, B\}$, then Σ is (AR) if and only if Σ is linear, time invariant and complete.

This tells us very precisely how general all the special cases we have considered so long are. Since any (AR) can be transformed in a control system, these are also describing all linear time invariant complete systems. Since discrete linear systems are widely used in practice - and since the former literature seems to indicate a kind of religious war between different groups using different representations - the theorem of Willems and the clearing of the relations between the models are certainly worthwhile also for somebody interested mainly in applications.

But now we have to come to controlsystems given by ODE'S of first order. We start with a state space model in the sense of Def. 1.2 and impose some assumption on B.

<u>Definition 1.4 :</u> A state space model with $T = \mathbb{R}$ (or an interval of $\mathbb{R}$) and $W = U \times Y$ is called a general control system if it has the following properties

1.) To any $t \in T$, $x \in X$ and $u \in U$ there exists only one $y \in Y$ such that

$$(x,u,y) \in B/_{\{t\}}$$

(We denote the so defined y by $y = r(t,x,u)$ and call the mapping $r : T \times X \times U \to Y$ the outputfunction).

2.) To any $\dot{t} \in T$, $\dot{x} \in X$ and $u(\cdot): T \to U$ there exists at most one $x(\cdot)/_{t > \dot{t}}$ such that $x(\dot{t}) = \dot{x}$ and $(x(\cdot), u(\cdot), r(\cdot, x(\cdot), u(\cdot))) \in B_{\{t > \dot{t}\}}$

(we denote the value $x(t)$ of this uniquely defined $x(\cdot)$ as

$$x(t) = s(t,\dot{t},\dot{x},u(\cdot))$$

and call s the state transition function; by definition, $s(\dot{t},\dot{t},\dot{x},u(\cdot)) = \dot{x}$).
Moreover s is supposed to have the following properties
(i) If $u(\cdot)/_{[t_o,t_1)} = v(\cdot)/_{[t_o,t_1)}$
for all $t_o, t_1 \in T$, $t_o < t_1$ then
$s(t_1,t_o,x,u(\cdot)) = s(t_1,t_o,x,v(\cdot))$ for all $x \in X$
(this property is called causality - Willems defines this notion only for discrete linear systems, see [JW,page 19]).

(ii) For any $t_o < t_1 < t_2$, all in T, any $\dot{x} \in X$ and $u(\cdot)$

$$s(t_2,t_1,s(t_1,t_o,\dot{x},u(\cdot))) = s(t_2,t_o,\dot{x},u(\cdot))$$

whenever $s(t_i,t_o,\dot{x},u(\cdot))$ is defined for $i=1,2$
(this property is called semigroup property - and is much weaker than time invariance).

We go a little bit further, assuming that $W = \mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^k = U \times Y$, $X = \mathbb{R}^n$. We call $\check{U}$ the set of possible inputfunctions, i.e.

$$\check{U} = \Big\{u(\cdot) : T \to U : \exists\, x(\cdot) \text{ such that } (x(\cdot), u(\cdot), r(\cdot, x(\cdot), u(\cdot))) \in B\Big\}$$

Then s is defined on $T \times T \times X \times \check{U}$ and we will assume furtheron, that $\check{U}$ consists of all piecewise continous functions on T.

<u>Definition 1.5 :</u> A control system is called differential, if for all $(\dot{t},\dot{x},u(\cdot)) \in T \times X \times \check{U}$ the mapping

$$t \to s(t,t,x,u(\cdot))$$

is differentiable at $t = \overset{\bullet}{t}$.
(then it follows by semigroup property, that is it differentiable at any point t).

Defining $\frac{ds}{dt}(t,\overset{\bullet}{t},\overset{\bullet}{x},u(\cdot))/_{t=\overset{\bullet}{t}} =: f(\overset{\bullet}{t},\overset{\bullet}{x},u(\cdot))$ we get $x(t) = s(t,\overset{\bullet}{t},\overset{\bullet}{x},u(\cdot))$ as solution of the initial value problem $\dot{x} = f(t,x,u(\cdot))$, $x(\overset{\bullet}{t}) = \overset{\bullet}{x}$.
We assume finally that $f(t,x,u(\cdot))$ depends only on $u(t)$ and that the initial value problem has a unique solution.
Our differential control system is then given by

$$\text{(S)}\qquad \begin{aligned} \dot{x} &= f(t,x,u(t)),\ x(\overset{\bullet}{t}) = \overset{\bullet}{x} \\ y(t) &= r(t,x(t),u(t)),\ \text{where } \overset{\bullet}{x} \in \mathbb{R}^n \end{aligned}$$

$u(\cdot)$ piecewise continous with values in $\mathbb{R}^m$, $y(t) \in \mathbb{R}^k$ and f has a property which guarantees uniqueness of the solution.

Since we are dealing furtheron mainly with differential control systems, we shall skip the word "differential". The only alternative will be time discrete linear systems.
Leaving the paper of Willems does not mean that it is not of interest for our subject. In fact the opposite is true:
It is dealing with what is normally called "identification", what he calls modelling: Given observations, i.e. $w(\cdot)$, find the dynamical system which (exactly or approximately) explains the observations. But these parts of Willems paper are much more than introductory lectures can cover - and will, at least in some respects, be taken up by Hazewinkel in the third block.
We go ahead with (KK).
We have already generally defined what time invariant, what linear means (completeness is automatically given for differential systems!) :
Time invariance means : f and r are independent of t, i.e. (S) reads as

$$\text{(TS)}\quad \dot{x} = f(x,u),\quad y = r(x,u)$$

Linearity means: f and r are linear in x and u, i.e.

$$f(t,x,u) = F(t)\begin{bmatrix} x \\ u \end{bmatrix},\quad r(t,x,u) = G(t)\begin{bmatrix} x \\ u \end{bmatrix}$$

where F is a (n,n+m)-matrix, G a (k,n+m).

Splitting F into appropride parts, we end up with

$$\text{(LS)}\quad \begin{aligned} \dot{x} &= A(t)x + B(t)u(t) \\ y &= C(t)x + D(t)u \end{aligned}$$

or, in the time invariant case, with

$$\text{(TLS)}\quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

which we had already (and where D could be set to zero)
Now all our basic knowledge about ODE'S can be applied - for example for (LS) (all stuff here can be found in any book about ODE'S - we just remind you):
We have to construct the transition matrix $\Phi(t,\tau)$, which is a solution of

$$\frac{\delta\Phi}{\delta t}(t,\tau) = A(t)\Phi(t,\tau),\ \Phi(t,t) = E_n\ (=\text{n-dim. unit matrix})$$

Then we get the solution of (LS) by

$$x(t) = \Phi(t,\overset{\bullet}{t})\overset{\bullet}{x} + \int_{\overset{\bullet}{t}}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau =: s(t,\overset{\bullet}{t},\overset{\bullet}{x},u(\cdot))$$

If we have (TLS), then $A(t) = A$,

$$\Phi(t,\tau) = e^{A(t-\tau)} \quad \text{and}$$

$$s(t,\overset{\bullet}{t},\overset{\bullet}{x},u(\cdot)) = e^{A(t-\overset{\bullet}{t})}\,\overset{\bullet}{x} + \int_{\overset{\bullet}{t}}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.$$

(LS) is as usually called stable, if the transition matrix $\Phi(t,\overset{\bullet}{t})$ is bounded for $t \in [\overset{\bullet}{t},\infty)$. It is called asymtotically stable, if $\lim_{t\to\infty} ||\Phi(t,\overset{\bullet}{t})|| = 0$. There are many equivalent definitions: $\Phi(t,\overset{\bullet}{t})$ may be substituted by any fundamental matrix $\Psi(t)$ and it is enough that the boundedness holds "at infinity" i.e. in <u>some</u> intervall $[t_1,\infty)$.
Asymtotic stability for example implies $\lim_{t\to\infty} ||y_1(t)-y_2(t)||=0$ if $||C(t)||$ is uniformly bounded, the input is the same but the initial conditions are different: The output forgets slowly, how the system is started.
For (TLS) everything is simpler:

(TLS) asymptotically stable $\Longleftrightarrow$ all eigenvalues of A have negative real part

(TLS) stable $\Longleftrightarrow$ all eigenvalues of A have non-positive real part and the Jordan blocks of all eigenvalues having vanishing real part are one-dimensional (i.e. for these eigenvalues algebraic and geometric multiplicity are the same)

We need a little bit more: If -for (LS)- $||\Phi(t,\dot{t})|| < \alpha e^{-\beta(t-\dot{t})}$ for all $\dot{t}, t > \dot{t}$ and some α, β only depending on A and if $||B(t)u(t)|| < K < \infty$ for all t then $\dot{x} = A(t)x + B(t)u(t)$ has exactly one bounded solution $x_o(\cdot)$, against which any other solution converges exponentially.

(This solution is $x_o(t) = \int_{-\infty}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$)

We end this first chapter up with some elementary notions of classical system theory - all for (TLS):

We consider here only $\dot{x} = Ax + Bu$, $y = Cx$

We choose as input a periodic function: $u(t) = e^{i\omega t}u_o$ (u_o constant).

A bounded solution of the system then is given by

$$x(t) = (i\omega E_n - A)^{-1}Be^{i\omega t}u_o \quad \text{(check!)}.$$

Assume, that (TLS) is asymptotically stable, i.e. all eigenvalues of A have negative real part; then $||\Phi(t,\dot{t})|| = ||e^{A(t-\dot{t})}|| < \alpha e^{-\beta(t-\dot{t})}$ and the bounded solution is uniquely defined

$$x(t) = \left(\int_{-\infty}^{t} e^{A(t-\tau)} Be^{i\omega\tau}d\tau \right) u_o$$

Therefore we get, using this bounding solution, which is "transient", since all others approach it

$$y(t) = C(i\omega E_n - A)^{-1} Be^{i\omega t} = \left(\int_{-\infty}^{t} Ce^{A(t-\tau)}Be^{i\omega\tau}d\tau \right)u_o$$

Using the abbreviation

$$H(s) := C(s\,E_n - A)^{-1}B \qquad \text{(transfermatrix)}$$

and
$$K(t) := \begin{cases} Ce^{At}B & \text{for } t>0 \\ 0 & \text{for } t<0 \end{cases} \qquad \text{(impulse response)}$$

then
$$y(t) = H(i\omega)\, e^{i\omega t}\, u_o = \int_{-\infty}^{\infty} K(t-\tau)e^{i\omega\tau}u_o d\tau.$$

Transfer matrix or impulse response give the transient answer to the input signal. In a certain sense, $H(i\omega)$ is the Fourier transform of K (which exists, since all eigenvalues of A have negative real part), such that the convolution with K is transformed into multiplication with $H(i\omega)$.

The elements of H are rational functions of s, the poles of which are the eigenvalues of A. $s \to H(s)$ represents therefore a meromorphic operatorvalued function, a subject, which is very often studied in functional analysis nowadays (see for example the books/papers of Gochberg).

Chapter 2 : Controllability

Problem: Which initial values $(t_0,\mathring{x})$ can be steered into $x = 0$ at time t_1, if the control system is given by

$$(S) \quad \dot{x} = f(t,x,u) \ ?$$

Search for

$$\mathcal{L}(t_o,t_1) := \left\{ \mathring{x} \in \mathbb{R}^n \mid \exists\, \breve{u} \in U : T_{t_1,t_o}[\breve{u}]\,\mathring{x} = 0 \right\}$$

$\mathring{x}$ is called "controllable" to 0, if $\mathring{x} \in \bigcup_{t_1>t_o} \mathcal{L}(t_o,t_1) = C_{t_o}$.

If $C_{t_o} = \mathbb{R}^n \ \forall\, t_o$, the system is called (completely) controllable.

Remarks concerning the nonlinear, time invariant case

The system is $\dot{x} = f(x,u)$ and we assume $f(0,0) = 0$

Then one may always choose $t_o = 0$ and we write $C_{t_o} = C_o = C$.

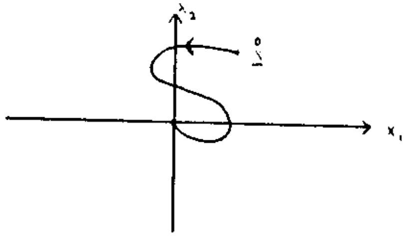
We have $0 \in C$.

Moreover: If $\mathring{x} \in C$ i.e.

$0 = T_{t_1}[\breve{u}]\,\mathring{x}$, then each $x^* = T_{t*}[\breve{u}]\,\mathring{x} \in C \quad (0<t^*<t_1)$

(Use $\overset{*}{u}(t) = \breve{u}(t+t^*)$ as sucessful control)

Consequence: C is (arcwise) connected



Moreover: C open $\Longleftrightarrow$ $0 \in$ interior of C.

Linear Systems

$$(LS) \quad \dot{x} = A(t)x + B(t)u(t)$$

Let $\Phi(t,\tau)$ be the transition matrix for $\dot{x} = A(t)x$.

$$\mathring{x} \in \mathcal{L}(t_o,t_1) \Longleftrightarrow 0 = \Phi(t_1,t_o)\mathring{x} + \int_{t_o}^{t_1} \Phi(t_1,t)B(t)u(t)dt$$

$$\Longleftrightarrow \quad -\mathring{x} = \int_{t_o}^{t_1} \Phi(t_o,t)B(t)u(t)dt$$

has a solution u.

$$W(t_o t_1) := \int_{t_o}^{t_1} \Phi(t_o,t)B(t)B(t)^T\Phi(t_o,t)^T dt \text{ pos. semidef.}$$

(Controllability granian)

Theorem 2.1 (i) $\mathcal{L}(t_o,t_1) = \text{Image of } W(t_o,t_1)$

(ii) kernel of $W(t_o,t_1) =$

$$\left\{ x \in \mathbb{R}^n \mid x^T\Phi(t_o,t)B(t) = 0 \ \forall\, t \in [t_o,t_1] \right\}$$

Proof Write $E(t) := \Phi(t_o,t)B(t)$ (nxm). We have to show: given y, there exists a $u \in U$ s.t.

$$y = \int_{t_o}^{t_1} G(t)u(t)dt$$

$$\Longleftrightarrow y \in \text{Image } (W(t_o,t_1)) = \text{Image } \int_{t_o}^{t_1} G(t)G(t)^T dt$$

$$x^T W(t_o,t_1)x = \int_{t_o}^{t_1} ||G(t)^T x||^2 dt = 0 \quad \Longleftrightarrow$$

$$G(t)^T x = 0 \ \forall\, t \in [t_o,t_1] \quad \Longleftrightarrow \quad W(t_o,t_1)x = 0 \ \text{(ii)}$$

Now, let $\mathcal{L} = \left\{ y \in \mathbb{R}^n \mid \exists\, u \in U : y = \int_{t_o}^{t_1} G(t)u(t)dt \right\}$

Then: Image $W(t_o,t_1) \subset \mathcal{L}$: Choose $u(t) = G(t)^T z \Longrightarrow$

$$y = \int_{t_o}^{t_1} G(t)\, G^T(t) z dt = W(t_o,t_1)z \in \mathcal{L} \ !$$

The statement is, that Image $W(t_o,t_1) = L$. If not, there must exist an element $x \in$ kernel $W(t_o,t_1) \cap L$, i.e. a $x \in \mathbb{R}^n$ with

$$G(t)^T x = 0 \text{ and } x = \int_{t_o}^{t_1} G(t)\check{u}(t)dt.$$

But then

$$||x||^2 = x^T \cdot x = \int_{t_o}^{t_1} x^T G(t)\check{u}(t)dt$$

$$= \int_{t_o}^{t_1} (G(t)^T x)^T \check{u}(t)dt = 0$$

$$\Longrightarrow x = 0 \Longrightarrow \text{Image } W = L$$

In the following theorem the solution of the homogenous, adjoint equation $\dot{x} = -A(t)^T x$ (h.a.eq.) with the transition matrix $\Psi(t,\tau) = \Phi(\tau,t)^T$ plays an important role

Theorem 2.2 (LS) is complete controllable $\Longleftrightarrow$
$t \to z^T \Phi(\tau,t)B(t)$ doesn't vanish identically on any intervall $[t_o,\infty)$ for all $z \neq 0$.
(i.e. $y(t)^T B(t)$ has this property for any nontrivial solution y of the h.a.eq.)

Proof: $\Longrightarrow$ is simple: Assume that there exists a solution y and a t_o, such that $y(t)^T B(t) = 0$ for all $t > t_o$. Let $x(\cdot)$ be an arbitrary solution of (S) for arbitrary $u \in U$. Then

$$\frac{d}{dt}(x(t)^T y(t)) = 0 \quad \text{for } t > t_o, \text{ i.e.}$$

$$<x(t),y(t)> = <\mathring{x},\mathring{y}> \neq 0 \text{ for all } t, \text{ if} <\mathring{x},\mathring{y}> \neq 0$$

$\Longrightarrow x(t) \neq 0$ for all t $\Longrightarrow \mathring{x}$ cannot be steered into 0 $\Longrightarrow$ (LS) is not controllable.

$\Longleftarrow$ is more troublesome. One shows (KK,p.33)

1.) To any t_o there exists a t_1, such that for any nontrivial solution y of the h.a.eq. the function $t \to y(t)^T B(t)$, $t \in [t_o,t_1]$ is not the zerofunction.

2.) In $[t_o,t_1]$ the positive semidefinite matrix $W(t_o,t_1)$ is even positive, i.e. nonsingular (follows easily from theorem 1,(ii))

3.) Now, each $\mathring{x} \in \mathbb{R}^n$ is in $L(t_o,t_1)$: Use the control $u(t) = B(t)^T \Phi(t_o,t)^T c$, such that

$$-\mathring{x} = \int_{t_o}^{t_1} \Phi(t_o,t)B(t)B(t)^T \Phi(t_o,t)^T c \, dt$$

$$= W(t_o,t_1)\, c \quad \text{is fulfilled, if } c := W(t_o,t_1)^{-1}\mathring{x}.$$

Corollary: (LS) is controllable $\Longleftrightarrow$ For every t_o there exists a t_1, such that $W(t_o,t_1)$ is not singular.

Time invariant linear systems

$$\dot{x} = Ax + Bu(t) \quad \text{(TLS)}$$

Theorem 2.3 For (TLS), the space $L(t_o,t_1)$ is independent of (t_o,t_1) and is given as the span of the columns of the matrix $K = (B,AB,\dots,A^{n-1}B)$
i.e. $L(t_o,t_1) = \text{Image } K$

Proof:
$L(t_o,t_1) = \text{Image } W(t_o,t_1)$ (theorem 2.1)
$\Longrightarrow L^\perp = \text{kernel } W(t_o,t_1)$
We have therefore to show
$W(t_o,t_1)x = 0 \Longleftrightarrow x \in (\text{Image } K)^\perp$, i.e. $x^T K = 0$
Now $W(t_o,t_1) = 0 \Longleftrightarrow$ (theorem 2.1)
$x^T \Phi(t_o,t)B = 0 \; \forall \; t \in [t_o,t_1]$
Since $\Phi(t_o,t) = e^{A(t_o-t)}$, this is equivalent to
$x^T e^{A\tau} B = 0$ for $t_o - t_1 < \tau < 0 \Longleftrightarrow$
$x^T A^\nu B = 0$, $\nu = 0,1,2,\dots \Longleftrightarrow$ (Cayley-Hamilton)
$x^T A^\nu B = 0$, $\nu = 0,1,2,\dots,n-1 \Longleftrightarrow$
$x^T K = 0$

Corollary : (TLS) is (completely) controllable $\Longleftrightarrow$
rank $K = n \Longleftrightarrow$ For any eigenvector p of A^T holds $p^T B \neq 0$

Proof (for the 2nd equivalence): "$\Longrightarrow$":
Assume that there exists an eigenvector p with $p^T B = 0$, i.e. $A^T p = \alpha p$ and $p^T B = 0 \Longrightarrow$

$p^TA = \alpha p^T$ and $p^TB = 0 \implies p^TAB = 0$ and successively $p^TA^\nu B = 0 \quad \nu = 0,1,2,\dots$
$\implies p^TK = 0 \implies$ rank $K < n \implies$ TLS is not controllable.

"$\Longleftarrow$": We show : If rank $K < n$ so that $x^TK = 0$ has nontrivial solutions, then there exists an eigenvector p of A^T with $p^TB = 0$.

$x^TK = 0 \implies x^TA^\nu B = 0, \ \nu = 0,\dots \implies x^TAK = 0$ i.e. with x also A^Tx is a solution. The subspace $\left\{ x \in \mathbb{R}^n \mid x^TK = 0 \right\}$ is not empty and invariant under A^T; it contains therefore an eigenvector p of A^T.

Since $p^TK = 0$, $p^TB = 0$ holds.

Remarks :

1.) We have considered only unrestricted controls, i.e. we assume only piecewise continuity for u. If we accept only restricted controls and assume that u has to be in
$U_b := \left\{ u(\cdot) \mid |u_i(t)| < 1 \text{ for } i = 1,\dots,m \text{ and all } t \right\}$
then controllability with respect to U_b is a stronger property.
The following theorem can be proved (see for example MS,p.34)
(TLS) is controllable with respect to $U_b \Longleftrightarrow$ rank $K = n$ and Re $\lambda < 0$ for each eigenvalue of A.

2.) The corollary may be interpreted as follows:
(TLS) is not controllable, if there exists an eigenvector p of A^T with $p^TB = 0$. If we take $V = p^\perp := \left\{ x \in \mathbb{R}^n \mid p^Tx = 0 \right\}$, then V is invariant under A ($p^TAx = x^TA^Tp = \alpha x^Tp = 0$) and moreover range $B \subset V$ ($p^TBy = 0 \ \forall \ y \in \mathbb{R}^n$)
One can easily verify, that (TLS) is not controllable, if there exists an A-invariant subspace of dimension less than n with range $B \subset V$:
If one adapts a basis of $\mathbb{R}^n$ to V and $V^\perp$, A gets a matrix representation

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \} \dim V$$

and B gets a representation

$$B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \} \dim V$$

(TLS) reduces to

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u$$
$$\dot{x}_2 = A_{22}x_2$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \} \dim V$

Since the equation for x_2 is not depending on u and x_1 we get a solution

$$x_1(t) = e^{A_{11}t}\overset{\circ}{x}_1 + \int_0^t e^{A_{11}(t-\tau)} A_{12}e^{A_{22}\tau}d\tau \ \overset{\circ}{x}_2$$
$$+ \int_0^t e^{A_{11}(t-\tau)}B_1u(\tau)dt \quad \text{and}$$

$$x_2(t) = e^{A_{22}t}\overset{\circ}{x}_2.$$

The control doesn't influence $x_2(t)$ at all; therefore $L(0,t)$ can at most consist of elements $\begin{bmatrix} \overset{\circ}{x}_1 \\ \overset{\circ}{x}_2 \end{bmatrix}$ with $\overset{\circ}{x}_2 = 0$.

Turned around: points, that can be reached starting at $\overset{\circ}{x}$ can be written as
$$x(t) = x_o(t) + v$$
where $x_o(t)$ is the point reached with zero control ($u = 0$) i.e. $x_o(t) = T_{t,o}[0]\overset{\circ}{x}$ and $v \in V$.

It is clear, that $V = \sum_{\nu=0}^{n-1}$ Image $(A^\nu B)$ has all properties; if rank $K = n$, then dimension $V = n$ and we have controllability, since V is the smallest subspace which has these properties.
If dim $V < n$, one can show that really any point of the form $T_{t,o}[o]\overset{\circ}{x} + v$ with $v \in V$ can be reached from $\overset{\circ}{x}$. The affin linear manifold

$$x_o(t) + V$$

is the reachable set - a concept which will be generalized to the nonlinear case in the lectures of Isodori.

3.) There is a different characterization of controllability, using a linear feedback, i.e. in assuming

$$u = - Fx, \quad F \ (mxn)$$

(TLS) $\Longrightarrow$ $\dot{x} = (A - BF)x$

How can the dynamics of this system be influenced by the choice of F?

Theorem 2.4 (TLS) controllable $\Longleftrightarrow$ To any real normalized polynomial of degree n there exists a (mxn) matrix F such that this polynomial is the characteristic polynomial of A-BF.

Proof: "$\Longleftarrow$" Assume, (TLS) is not controllable, then there exists an eigenvector p of A^T with $p^TB = 0$

$\Longrightarrow$ $(A-BF)^Tp = A^Tp - F^T(p^TB)^T = A^Tp = \alpha p$

$\Longrightarrow$ α is eigenvector of A-BF for all F $\Longrightarrow$ the characteristic polynomial has the zero α for any F.

"$\Longrightarrow$" uses the theory of normalforms (chapter 3).

Chapter 3 : Normalforms (KK, p. 41-54)

Consider only (TLS) $\dot{x} = Ax + Bu$ describing a mapping

$$u \longrightarrow x$$

We may transform the input u and the state vector x in order to get a simpler structure for the matrices A,B.

Transformation of the state (here:= output) space $x = Px'$ $\Longrightarrow$

$$\dot{x}' = P^{-1}APx' + P^{-1}Bu$$

Transformation of the input space

$$u = - Rx + Qv \Longrightarrow$$
$$\dot{x} = (A - BR)x + BQv$$

Remark : In an input-output system of the form

$$\dot{x} = Ax + Bu, \quad y = Cx$$

we may change the matrices A,B,C without transforming the input- and the output space but only by transforming the state space $x = Px'$; we then get

$$\dot{x}' = P^{-1}APx' + P^{-1}Bu$$
$$y = CPx'$$

and the same input-output behaviour is given by $(P^{-1}AP, P^{-1}B, CP)$ as by the original triple (A,B,C). This will be used in the lectures of Hazewinkel.

In the context of this lecture, where we allow transformation of input and output, we accept transformations

$$(A',B') = P^{-1}(A,B) \begin{bmatrix} P & O \\ -R & Q \end{bmatrix}$$

Since we allow rather many transformations, the set of equivalence classes is relatively small, and each equivalence class is characterized only by few numbers. This is theoretically valuable. For practical purposes, the restricted transformations (Q = I, R = 0), where the input and output remains unchanged, are more useful; we denote the most simple forms with respect to this equivalence "Identification normalforms".

Easy: If (A,B) is controllable, then also (A',B')

Problem: Find P,Q,R such that A',B' are as simple as possible.

Theorem 3.1 A controllable system given by (A,B) with A nxn, B nxm and rank B = m (m<n) is equivalent to (A',B') with the following structure:
There exist numbers $n_1<n_2<\ldots<n_m$, $n_1+\ldots+n_m = n$ such that

$$A' = \begin{pmatrix} Jn_1 & & 0 \\ & \ddots & \\ 0 & & Jn_m \end{pmatrix} \text{ and } B' = \left(\begin{array}{ccc} 0 & 0 & \vdots \\ 1 & \vdots & \vdots \\ 0 & 0 & \vdots \\ \vdots & 1 & \vdots \\ \vdots & 0 & \vdots \\ \vdots & \vdots & 0 \\ 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \leftarrow n_1 \\ \\ \leftarrow n_1+n_2 \\ \\ \\ \leftarrow n_1+..+n_m \end{array}$$

$$\text{with } Jn_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} n_i x n_i.$$

Remark : Only the numbers $n_1,\ldots,n_m$ are therefore characteristic for a certain equivalence class; they define the "structure of the system". (A',B') are called "general normal forms"

Proof for m = 1 : B consists only of one column b and $K = (b, Ab, \ldots A^{n-1}b)$ is nxn and regular (controllability).
We first use only the transformation of the state space : We shall find a regular nxn-matrix P with

$$P^{-1}AP = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -\alpha_0 & \cdots & & -\alpha_{n-1} \end{pmatrix} \text{ and}$$

$$P^{-1}B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$ where $\alpha_0,\ldots\alpha_{n-1}$ are the coefficients of the characteristic polynomial $\lambda^n+\alpha_{n-1}\lambda^{n-1}+\ldots+\alpha_0$ of A.

(We see, that P does already the full job - almost: Q,R must work to let $\alpha_0,\ldots,\alpha_{n-1}$ become 0; this normal form is important for the case, where only the state space is transformed and will appear again in the Hazewinkel lectures)

We construct $P = (p_1,\ldots,p_n)$ with

$$P \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -\alpha_0 & \cdots & & -\alpha_{n-1} \end{pmatrix} = AP, \quad P \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = b$$

i.e with $p_n = b$, $-\alpha_0 p_n = Ap_1$, $p_1-\alpha_1 p_n = Ap_2, \ldots\ldots$

$$p_j-\alpha_j p_n = Ap_{j+1},\ j=1,\ldots,n-1$$

$\Longrightarrow p_n = b$, $p_{n-1} = (\alpha_{n-1}I + A)b$, $p_{n-2} = \alpha_{n-2}b + Ap_{n-1}$

$$p_1 = \alpha_1 b + Ap_2$$

The last equation $Ap_1 + \alpha_0 b = 0$ is a consequence of the others:

$$Ap_1 + \alpha_0 b = A(\alpha_1 b + Ap_2) + \alpha_0 b = \ldots.$$
$$= A^n b + \alpha_{n-1}A^{n-1}b + \ldots + \alpha_0 Ib = p(A)b = 0$$

according to Cayley-Hamilton.

For m > 1, P is not uniquely determined and one needs a more general construction. We sketch only the main step in order to show where the characteristic numbers $n_1,\ldots,n_m$ come in.
Let $B = (b_1,\ldots,b_m)$ and $\mathcal{B} := [b_1,\ldots,b_m] \subset \mathbb{R}^n$ be the m-dimensional subspace of $\mathbb{R}^n$ spanned by the columns of B.
The smallest subspace $\mathcal{B}'$ of $\mathbb{R}^n$, invariant under A and containing $\mathcal{B}$ is the whole $\mathbb{R}^n$, since (A,B) is controllable and rank K = n. Now the numbers $n_1,\ldots,n_m$ enter the game:
There exists a basis $c_1,\ldots,c_m$ of $\mathcal{B}$ and numbers $n_1<n_2<\ldots<n_m$ with $n_1+\ldots+n_m = n$ such that $\left\{ A^j c_i,\ 0<j<n_i-1 \right\}$ form a basis of $\mathcal{B}' = \mathbb{R}^n$.
One gets this basis by starting with an arbitrary one, for example with $b_1,\ldots,b_m$. Some of the $Ab_1,\ldots,Ab_m$ may be elements of $\mathcal{B}$, some are independent; by changing the indices of $b_1,\ldots,b_m$ one gets a basis

$$(b_1,\ldots,b_m, Ab_k,\ldots,Ab_m)$$

for $\mathcal{B} + A\mathcal{B} = \mathcal{B}_1$. In changing the numeration of $b_k,\ldots,b_m$ one gets a basis

$$(b_1,\ldots,b_m,\ Ab_k,\ldots,Ab_m,A^2b_1,\ldots,A^2b_m)$$

of $\mathcal{B} + A\mathcal{B} + A^2\mathcal{B} = \mathcal{B}_2$.

In this way one may continue, constructing bases for $\mathcal{B}_3,\ldots$ until $\mathcal{B}_r = \mathbb{R}^n$ and everything is finished. One gets $c_1,\ldots,c_m$ just as a reordering of $b_1,\ldots,b_m$ and the numbers $n_1,\ldots n_m$ just by counting the elements A^jc_i.

It is clear, that each $A^{n_i}c_i$ is not an element of the basis, but a linear combination of all basis elements. One can construct a new basis $c'_1,\ldots,c'_m$ (by inverting an upper triangle matrix) such that again $\left\{ A^jc'_i,\ 1\leq j\leq n_i,\ i=1,\ldots,m \right\}$ is a basis of $\mathbb{R}^n$ and moreover $A^{n_i}c'_i$ is a linear combination of all basis elements $A^jc'_i$ with $j\leq n_i-1$,

$$A^{n_i}c'_i = - \sum_{\nu=1}^{n_i} A^{n_i-\nu}\, z_{i,\nu} \quad \text{with } z_{i,\nu} \in \mathcal{B},\ i=1,\ldots,m$$

To go from $b_1,\ldots,b_m$ to $c'_1,\ldots,c'_m$ means only a transformation $u = Qv$ of the inputspace, i.e. BQ has the columns $c'_1,\ldots,c'_m$. Therefore we write b_i instead of c'_i again.
But we are not ready: We want to find a final basis of
$\mathcal{B}' = \mathbb{R}^n$: $p_{1,1},\ldots,p_{1n_1},\ldots,p_{m1},\ldots,p_{mn_m}$
with $p_{i,n_i} = b_i$, $p_{ij} - Ap_{i,j+1} \in \mathcal{B}$ and $Ap_{i,1} \in \mathcal{B}$ for
$i = 1,\ldots,m$, $j = 1,\ldots,n_{i-1}$.
This basis is easily constructed, if we use

$$A^{n_i}b_i = - \sum_{\nu=1}^{n_i} A^{n_i-\nu}\, z_{i,\nu} :$$

We define $p_{i,n_i} := b_i$ and $p_{i,j} := A^{n_i-j}b_i + \sum_{\nu=1}^{n_i-j-\nu} A^{n_i-j-\nu}z_{i,\nu}$

The matrix $P = (\, p_{1,1},\ldots,p_{1,n_1},\ldots,p_{m,1},p_{m,n_m} \,)$ is nxn and not-singular.
Defining $A'' = P^{-1}AP$ and $B' = P^{-1}B$ $(Q = I,\ R = 0)$ one gets an equivalent system and the "identification normal form"

$$A'' = \begin{array}{c|c|c|c|c}
n_1 & n_2 & & n_m & \\
\begin{matrix}0\;1\;0..0\\ \cdots\cdots\\ 0\;..\;0\;1\\ *\;*\;*\;*\;*\end{matrix} & \begin{matrix}0\\ *\;*\;*\;*\;*\end{matrix} & \begin{matrix}0\\ *\;*\;*\;*\end{matrix} & \begin{matrix}0\\ *\;*\;*\;*\;*\end{matrix} & n_1 \\ \hline
\begin{matrix}0\\ *\;*\;*\;*\;*\end{matrix} & \begin{matrix}0\;1\;0..0\\ \cdots\cdots\\ 0\;..\;0\;1\\ *\;*\;*\;*\end{matrix} & \begin{matrix}0\\ *\;*\;*\;*\end{matrix} & \begin{matrix}0\\ *\;*\;*\;*\;*\end{matrix} & n_2 \\ \hline
 & & & & \\ \hline
\begin{matrix}0\\ *\;*\;*\;*\;*\end{matrix} & \begin{matrix}0\\ *\;*\;*\;*\;*\end{matrix} & \begin{matrix}0\\ *\;*\;*\;*\end{matrix} & \begin{matrix}0\;1\;0..0\\ \cdots\cdots\\ 0\;..\;0\;1\\ *\;*\;*\;*\;*\end{matrix} & n_m
\end{array}$$

B' as in theorem 3.1.

The asterisks in A'' denote arbitrary numbers, which, besides $n_1,\ldots,n_m$, are characterizing the identification normal form. If we denote by a_i^T the i-th row consisting of asterisks, i.e. the $n_1+\ldots+n_i$-th row vector of A'' then a simple final equivalence transformation

$$u = - a_i^T\cdot x + v$$

gives the general normal form (A',B') defined in theorem 3.1. This proves the theorem.

Remarks:
1.) It is enough to prove theorem 2.4 for general normal forms - which is almost trivial.

2.) One has the assumption rank $B = m$, which is not really a restriction: If rank $B = m'<m$, then $B = B'Z$ with B': nxm', Z: $m'xm$ and rank B' = rank $Z = m'$. Our system is controllable if and only if $\dot{x} = Ax + B'u'$ is controllable - the reduction to the case rank $B' = m'$ is therefore easy.

3.) (3.1) has moreover the assumption, that (A,B) is already controllable. There are also normal forms for noncontrollable systems.

Corollary : A general linear system can be transformed by $x = Px'$, $u = Qv$ into a system

$$\dot{x}' = \left(\begin{array}{c|c} A_{11} & * \\ \hline 0 & A_{22} \end{array} \right) x' + \left(\begin{array}{c} B_{11} \\ 0 \end{array} \right) v$$

where - (A_{11},B_{11}) has identification normal form
- the number of rows of A_{11} and B_{11} is equal to rank K
- the system $\dot{x}_1 = A_{11}x_1 + B_{11}u$ is controllable.
- Image $K = \left\{ (x_1,x_2)^T \mid x_2 = 0 \right\}$

The proof of this corollary is simple, if we start with $\mathcal{B}$ = Image B.

Stabilization of (LS)

Definition : We call (LS) "stabilizable" if for each $(t_o,\dot{x})$ there exists a control u such that the solution of $\dot{x} = A(t)x + B(t)u(t)$ with $x(t_o) = \dot{x}$ tends to zero with $t \to \infty$.

Remark: A controllable system is always stabilizable: Steer it to zero during some finite time and continue with zero control. Stabilizable is therefore weaker than controllable.

Theorem 3.2 : (LS) is stabilizable ⟹ If y(t) is a nontrivial solution of $\dot{y} = -A(t)^T y$, which is bounded, then $y(t)^T B(t) = 0$ from some t_o on is impossible.

Proof : Assume, there exists such a solution y with $y(t)^T B(t) = 0$ for $t > t_0$. Then, as in the proof of theorem 2.2 we would have

$$< x(t),y(t) > = < \dot{x},y(t_0) > \text{ for } t > t_0$$

Since $x(t) \to 0$ and $||y(t)|| < \infty$, there exist (t_ν) $\nu \in N$ such that

$$< x(t_i),y(t_i) > \to 0$$

$\Longrightarrow < \dot{x},y(t_o) > = 0 \Longrightarrow$ Not all $\dot{x}$ can be steered into zero ⟹ (LS) is not stabilizable.

For (TLS) also sufficient conditions are possible.

Theorem 3.3 : (TLS) is stabilizable ⟺ If λ is an eigenvalue of A with Re $\lambda > 0$ and p a corresponding eigenvector of A^T, then $p^T B \neq 0$.
See (KK) page 56.

Chapter 4 : Observability (KK p.57-72)

We consider again

(LS) $\dot{x} = A(t)x + B(t)u, \quad y = C(t)x,$

but now consider also the output relation $y = C(t)x$ and assume, that the output is known; that means: We have information about x in form of $C(t)x$. Since C is (kxn) and generally $k \ll n$ we have a rather incomplete information about x, which we want to control or stabilize.
Can we guess x from y? Can we even observe it indirectly, at least in the limit $t \to \infty$? We call a method, which "reconstructs" x from y, a "dynamical observer" and we shall describe this concept in this chapter.
If we cannot reconstruct the whole x, i.e. $x_1,\dots,x_n$, then maybe at least some components or functions like $\langle c,x \rangle$ for certain vectors $c \in \mathbb{R}^n$; we then speak of "reduced observers".

Definition 4.1 : (LS) is called reconstructable, if for all t_o and all $u \in U$ two solutions x, x' of $\dot{x} = A(t)x + B(t)u$ coincide for $t < t_o$ if $C(t)x(t) = C(t)x'(t)$ for $t < t_o$.

Remark : (LS) is reconstructable, iff $C(t)x(t) = 0$ for $t < t_o$ and $\dot{x} = A(t)x$ implies $x(t) = 0$ in $t < t_o$. Therefore the control (and B) doesn't influence reconstructability.

Theorem 4.1 (LS) is reconstructable $\Longleftrightarrow$ $\dot{x} = A(-t)^T x + C(-t)^T u$ is controllable.

Please realize, that (LS) is reconstructable, iff $C(t)x(t)$ can not vanish identically in any intervall $(-\infty,t_o)$, if $x(t)$ is a nontrivial solution of $\dot{x} = A(t)x$; that means, that $C(-t)x(-t)$ may not vanish in any $[t_o,\infty)$. But $x(-t)$ is solution of $\dot{x} = -A(-t)x$; this is the homogenous adjoint equation to $\dot{x} = A(-t)^T + C(-t)^T u$ which is controllable according to theorem 2.2 iff $x^T(-t)C(-t)^T = (C(-t)x(-t))^T$ does not vanish in any $[t_o,\infty)$.

The theorem establishes a duality between reconstructability and controllability and we can transfer theorems for controllability in those about reconstructability.
The corollary of 2.2 tells you for example, that (LS) is reconstructable, iff for every t_o there exists a t_1 such that

$\int_{t_o}^{t_1} \check{\Phi}(t_o,t)C(-t)^T C(-t)\check{\Phi}(t_o,t)^T dt$ is really positiv definit where $\check{\Phi}$ is the transitionmatrix of $\dot{x} = A(-t)^T x$, such that $\check{\Phi}(t_o,t) = -\Phi(t,t_o)^T$.

This is equivalent to $(-t_o \to t_1, -t_1 \to t_o)$

$$W^*(t_o,t_1) := \int_{t_o}^{t_1} \Phi(t,t_1)^T C(t)^T C(t)\Phi(t,t_1)dt > 0$$

and we get $x(t_1)$ from y and y as solution of

$$W^*(t_o,t_1)x(t_1) = \int_{t_o}^{t_1} \Phi(t,t_1)C(t)^T y(t)dt -$$

$$\int_{t_o}^{t_1} \Phi(t,t_1)^T C(t)^T C(t) \left[\int_{t_1}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \right] dt.$$

In the same way we get reconstructability criteria for (TLS) $\dot{x} = Ax + Bu, \ y = Cx$

(TLS) is reconstructable $\Longleftrightarrow$ rank $K^* =$ rank $(C^T, A^T C^T, \dots, (A^{n-1})^T C^T) = n$

$\Longleftrightarrow$ for any eigenvector p of A there holds $Cp \neq 0$.

In the general case, we can find a basis in $\mathbb{R}^n$, such that with respect to that basis

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad C = (C_{11},0)$$

where $\dot{x}_1 = A_{11}x_1$, $y = C_{11}x_1$ is reconstructable. The dimension of x_1 is $r =$ rank K^*.

Remark : If (TLS) is not reconstructable, there exist $x \neq 0$ with

$$\left[\int_{t_o-t_1}^{0} e^{A^T t} C^T C e^{At} dt \right] x = 0$$

which is equivalent to $x^T K^* = 0 \Longleftrightarrow Ce^{At}x = 0 \ \forall \ t$
These x form the non-reconstructable subspace;

one gets the state of the system by observing input-output up to an element of this subspace.

We introduce now a weaker concept

<u>Definition 4.2 :</u> (TLS) is called discoverable, if for all x in the nonreconstructable subspace the relation $\lim_{t\to\infty} e^{At}x = 0$ holds.

<u>Remark :</u> Discoverability means that the uncertainty about the state dies out:
Let $x(t)$ and $\check{x}(t)$ be two solutions belonging to the same observation y, then $x(t)-\check{x}(t) \to 0$ for $t \to \infty$.

<u>Theorem 4.2</u> (TLS) is discoverable $\Longleftrightarrow$ Any solution of $\dot{x} = Ax$ with $Cx(t) = 0 \ \forall\ t$ tends to zero with $t \to \infty$ $\Longleftrightarrow$ For any eigenvector p belonging to an eigenvalue α with $\mathrm{Re}\alpha > 0$, $Cp \neq 0$.

<u>Remark :</u> The very often used term "observatibility" is defined in the same way as reconstructability in Def.4.1 - besides the fact that you look into the future and not into the past; i.e. equal in- and output for $t > t_o$ implies equal states for $t > t_o$ - and this for all t_o. So one way convert the one to the other by transforming $t \to -t$; for (TLS) both concepts are equivalent.

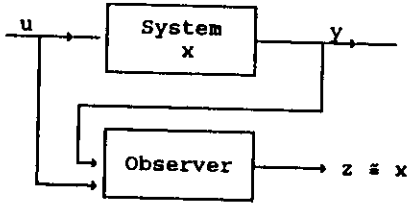
We now come to what is called a "dynamical observer" - and we restrict ourselves to (TLS), where observability and reconstructability means the same - namely the controllability of the system $\dot{x} = A^Tx + C^Tu$. We use now the ideas of theorem 2.4 feedback.

<u>Theorem 4.3</u> (TLS) is discoverable $\Longleftrightarrow$ $\exists$ (nxk)-matrix F such that all eigenvalues of A-FC have negative real part.
(TLS) is observable $\Longleftrightarrow$ To any real normalized polynomial of degree n there exists a (nxk)-matrix F such that this polynomial is the characteristic polynomial of A-FC.

<u>Definition 4.3 :</u> A dynamical observer is a (TLS) with input (u,y) and dim X = n, defined by

$$\dot{z} = Az + Bu + F(y - Cz)$$

where F is an arbitrary (nxk)-matrix.



Try to find F such that z approaches x (independent of initial conditions)
z is called "estimator" for x. F amplifies the output error y-Cz.
The estimation error $e(t) := x(t) - z(t)$ fullfills the equation

$$\dot{e} = (A - FC)e$$

Corollary: (TLS) discoverable $\Longleftrightarrow$ $\exists$ (nxk) F such that $e(t)\to 0$ for all initial values of $x(\cdot)$ and $z(\cdot)$.

We come to the final point: Reduced observers for (TLS). z doesn't estimate x itself but only some linear functions <c,x>. Without any further assumption about discoverability etc: Which functions can be estimated? There exists a recursive scheme: It constructs a sequence (C_i) of linear subspaces of X, $C_o \subset C_1 \subset \ldots$, such that for all $c \in C_i$ the functionals <c,x> can be assymtotically reconstructed.

<u>Theorem 4.4</u> Let C_o = image (C^T) and $C_i = C_{i-1} +$ $[\, c^{(i,1)},\ldots,c^{(i,r_i)}]$ where the vectors $c^{(i,j)}$ are constructed in such a way that there exist $\alpha_{ij} \in \mathbb{C}$, Re $\alpha_{ij} < 0$ with $(A^T - \alpha_{ij} E_n)\, c^{(i,j)} \in C_{i-1}$, $i=1,\ldots,p, j=1,\ldots,r_i$

Then there exists an observer who reconstructs asymtotically the functionals

$$x \to < c^{(i,j)}, x >, \; x \in X$$

Proof : Elements in C_{i-1} are linear combinations of vectors $c^{(\nu,\mu)}$, $1<\nu<i-1$, $1<\mu<r_\nu$ and of vectors of C_o, i.e. of the row vectors of C. Therefore

$$A^T c^{(i,j)} = \alpha_{i,j}\, c^{(i,j)} + \sum_{\nu=1}^{i-1} \sum_{\mu=1}^{r_\nu} \beta_{i,j,\nu,\mu}\, c^{(\nu,\mu)} +$$

$$\sum_{\tau=1}^{k} l_{i,j,\tau} c^{(\tau)T} \quad \text{with} \quad C = \begin{bmatrix} c^{(1)} \\ \vdots \\ c^{(k)} \end{bmatrix}$$

or

$$(*) \; c^{(i,j)T} A = \alpha_{ij} c^{(i,j)T} + \sum_{\nu=1}^{i-1} \sum_{\mu=1}^{r_\nu} \beta_{i,j,\nu,\mu}$$

$$c^{(\nu,\mu)T} + l_{(i,j)}{}^T C$$

$$\text{with } l_{(i,j)} = \begin{bmatrix} l_{i,j,1} \\ \vdots \\ l_{i,j,k} \end{bmatrix} \in \mathbb{R}^k \quad \text{and} \quad \alpha_{i,j}, \beta_{i,j,\nu,\mu} \in \mathbb{C}$$

The observer is then defined by

$$(**) \quad \dot{z}_{(i,j)} = \alpha_{ij}\, z_{(i,j)} + \sum_{\nu=1}^{i-1} \sum_{\mu=1}^{r_\nu} \beta_{i,j,\nu,\mu}\, z_{(\nu,\mu)}$$

$$+ \, l^T_{(i,j)}\, y + c^{(i,j)T} Bu$$

or, by ordering the scalar function $z_{(i,j)}$ into a vector z

$$\dot{z} = \hat{A}z + \hat{L}y + \hat{B}u$$

Where $\hat{A}$ is a lower triangular matrix with α_{ij} as diagonal elements.

We have to show: For arbitrary $u(\cdot) \in U$, $x(\cdot)$ solution of $\dot{x} = Ax + Bu(t)$ and $z(\cdot)$ solution of $\dot{z} = \hat{A}z + \hat{L}y(t) + \hat{B}u(t)$ with $y(t) = Cx(t)$ the difference $\delta_{(i,j)}(t) = z_{(i,j)}(t) - <c^{(i,j)}, x>$ $i=1,\dots,p,$ $j=1,\dots,r_i$

decays exponentially for $t \to \infty$.

If $\Delta(\cdot) = (\delta_{(1,1)}(\cdot), \dots, \delta_{(1,r_1)}(\cdot), \dots, \delta_{(p,1)}(\cdot), \dots,$ $\delta_{(p,r_p)}(\cdot))$, then

$$\dot{\Delta} = \hat{A}\Delta$$

(Realize, that $\zeta_{(i,j)}(t) := <c^{(i,j)}, x(t)>$ solves (**)

$$\frac{d}{dt} < c^{(i,j)}, x(t) > = \frac{d}{dt}\, c^{(i,j)T} \cdot x(t) = c^{(i,j)T} \dot{x}(t) =$$

$$c^{(i,j)T}(Ax(t) + Bu(t)) \overset{(*)}{=} \alpha_{ij} c^{(i,j)T} x(t) +$$

$$\sum_{\nu=1}^{i-1} \sum_{\mu=1}^{r_\nu} \beta_{i,j,\nu,\mu} c^{(\nu,\mu)T} x(t) + l^T_{(i,j)} Cx(t) +$$

$$c^{(i,j)T} Bu(t),$$ such that for the vector ζ gathering $<c^{(i,j)}, x(t)>$ in an appropriate order the equation

$$\dot{\zeta} = \hat{A}\zeta + \hat{L}Cx + \hat{B}u = \hat{A}\zeta + \hat{L}y + \hat{B}u \text{ holds;}$$

$\Delta = z - \zeta$ gives the equation $\hat{\Delta} = \hat{A}\Delta$)

$\hat{A}$ has the eigenvalues $\alpha_{i,j}$ which have negative real part $\Longrightarrow$ $\Delta(t)$ tends exponentially to zero with $t \to \infty$. This is the statement.

The theorem is constructive but does not characterize all functionals discoverable by observers. For this purpose it is better to use the normalform given above with

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad C = (C_{11}, 0),$$ where the (1,1)-system is reconstructable. Since the control is not of importance for the reconstructability, we put $u = 0$.

Moreover, A_{22} is assumed to be a diagonalmatrix $\operatorname{diag}(A^-_{22}, A^+_{22})$, where A^-_{22} consists of all eigenvalues with negative real part:

$$\dot{x} = A_{11}\, x_1, \quad y = C_{11}\, x_1$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A^-_{22} & 0 \\ 0 & A^+_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + A_{21}\, x_1, \; \dim x_1 = \operatorname{rank} K^*.$$

We can write $<c,x>$ as $<c_1,x_1> + <c_2,x_2> + <c_3,x_3>$.

We want to describe all c with

$$y(t) = 0 \ \forall \ t \Longrightarrow \lim_{t \to \infty} \langle c, x(t) \rangle = 0$$

Since the (1,1) system is reconstructable, $y \equiv 0$ implies $x_1 \equiv 0$; we have to care only for

$$\lim_{t \to \infty} (\langle c_2, x_2(t) \rangle + \langle c_3, x_3(t) \rangle) = 0$$

with $\dot{x}_2 = A^{-}_{22} x_2$, $\dot{x}_3 = A^{+}_{22} x_3$. For this $c_3 = 0$ is a necessary condition. One concludes, that $c_3 = 0$ is necessary for the asymtotic constructability of $\langle c, x \rangle$.
This functionals are "caught" by the algorithm described in the proof of theorem 4.4.

This ends the introductory block on control theory. Examples are provided in special tutorials.