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NUMERICAL APPROXIMATION OF POROUS MEDIUM EQUATION

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Numerical approximation of porous medium equation

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1 - The porous medium equation in one dimension

Let us consider the Cauchy problem for the *porous medium equation*

$$u_t = (u^m)_{xx}, \quad (x,t) \in Q_T := R \times]0,T[, \quad T < \infty, \tag{1.1}$$

$$u(\cdot,0) = u_0(\cdot) \quad \text{in } R, \tag{1.2}$$

where $m > 1$ is a given constant and u_0 is a given (nonnegative) function.

The problem arises in the flow of an ideal gas in a *homogeneous porous medium* (which occupies the whole of R). Denoting by v the *velocity*, by p the *pressure* and by ρ the *density*, the flow of the fluid is governed by

Equation of state : $\quad p = p_0 \rho^\alpha \quad (\alpha > 0)$,

Equation of conservation of mass : $\quad \kappa \rho_t + (\rho v)_x = 0$

$(0 < \kappa < 1$: *porosity*$)$,

Darcy's law : $\quad \nu v = -\mu p_x$

(ν: *viscosity* of the gas; μ: *permeability* of the medium).

By eliminating p and v and rescaling, we get (1.1), where $m = 1 + \alpha$ $(m > 1)$ and u represent the (scale) *density* of the fluid. We call $p := \frac{m}{m-1} u^{m-1}$ the *pressure* and $v := -p_x$ the *velocity*.

We suppose that at time $t = 0$ the fluid is contained in the slab $a_1 \le x \le a_2$; namely u_0 is a nonnegative function, such that $u_0(x) > 0$ if $x \in]a_1, a_2[$ and $u_0(x) = 0$ elsewhere (we assume that u_0 is continuous in R).

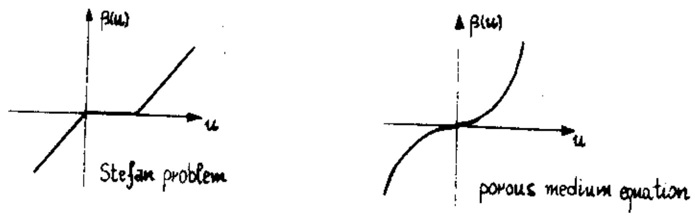
We can write $(u^m)_{xx} = (m\, u^{m-1} u_x)_x$. Because $m > 1$, the diffusion rate of (1.1), i.e., $m\, u^{m-1}$, vanishes where $u = 0$; hence $u(\cdot,t)$ is supported in a finite interval $[s_1(t), s_2(t)]$. This contrasts with the case of $m = 1$ and (1.1) is the heat equation. The functions $x = s_j(t)$ $(j = 1,2)$ represent the interfaces of the region occupied of gas.

In many applications we have the more general equation

$$u_t = (\beta(u))_{xx} + (b(\beta(u)))_x + f(\beta(u)), \tag{1.5}$$

where $\beta : R \to R$, $\beta(0) = 0$, $\beta'(0) = 0$, is a nondecreasing and Lipschitz continuous function, and $(b(\beta(u)))_x$ and $f(\beta(u))$ represent convection and source terms, respectively. Depending on the shape of β, the equation (1.5) models various physical processes with phase change.

The constitutive relation corresponding to the Stefan problem is shown in Figure 1.



This graph corresponds to processes of heat transfer involving melting (or solidification) when the liquid phase is allowed to move ($b \neq 0$) and an internal generation (or absorption) of heat is present ($f \neq 0$). Here u is the enthalpy whereas $\beta(u)$ is the temperature.

In any case, here we will deal with the simplest case (1.1).

The problem (1.1), (1.2) can be reformulated in terms of the pressure $p = \frac{m}{m-1} u^{m-1}$, which satisfies

$$p_t = [m-1] p\, p_{xx} + (p_x)^2 \quad \text{in } Q_T , \tag{1.3}$$

$$p(\cdot,0) = p_0(\cdot) := \frac{m}{m-1} u_0^{m-1}(\cdot) \quad \text{in } R . \tag{1.4}$$

The interface curves s_j ($j = 1,2$) and the pressure p are connected by the Rankine-Hugoniot jump condition [2, 26] (which are not part of the original problem (1.1)):

$$\begin{cases} \lim\limits_{x \uparrow s_2(t)} p_x(x,t) = - s_2'(t) \\ \lim\limits_{x \downarrow s_1(t)} p_x(x,t) = - s_1'(t) \end{cases} \quad \text{in }]0,T] , \tag{1.6}$$

$$s_2(0) = a_2 , \quad s_1(0) = a_1 .$$

Since the function $\beta(\xi) := \xi^m$, $\xi \in R^+$, is degenerate, the solutions of (1.1),(1.2) have a low regularity and must be understood in the sense of distributions, as proposed first in [37]. So, we recall the concept of *weak solution*.

(P1) *$u = u(x,t) : R \times [0,T] \to R^+$ is a weak solution of (1.1),(1.2) if*

$$u \in L^2(Q_T) , \quad (u^m)_x \in L^2(Q_T) ,$$

and for all $z \in H^1(Q_T)$ compactly supported in R, $z(\cdot,T) = 0$, the following equation holds

$$\int_{R \times]0,T[} [-u\, z_t + (u^m)_x z_x]\, dx\, dt = \int_R u_0(x)\, z(x,0)\, dx .$$

Existence and uniqueness of the weak solution was first proved in [37]. The equivalence of (P1) and the weak formulation of (1.3),(1.4) is given in [1].

The following regularity results are well known (see, e.g., [3, 18] and the references given therein). Assume the following hypotheses:

(H0) $\quad supp\, p_0 = [a_1, a_2] , \quad a_1, a_2 \in R$;

(H1) $\quad 0 \le p_0(x) \le M , \quad \forall x \in R$;

(H2) $\quad |p_0(x) - p_0(y)| \le L_p\, |x-y| , \quad \forall x,y \in R$.

Then the solution p satisfies

(R1) $\quad 0 \le p(x,t) \le M , \quad \forall (x,t) \in Q_T$ (*Maximum principle*) ;

(R2) $\quad \|p_x\|_{L^\infty(Q_T)} \le L_p$ (see [1]) ;

(R3) $\quad |p(x,t) - p(x,t')| \le C\, |t-t'|^{1/2}$ (see [19]) ;

(R4) $\quad p_{xx} \ge - \frac{1}{m+1} \frac{1}{t}$ (*semiconvexity*) (see[4]) ;

(R5) $\quad \|p_{xx}(\cdot,t)\|_{L^1(R)} , \|p_t(\cdot,t)\|_{L^1(R)} \le \frac{C}{t}$ (*regularizing effect*) (see [4]) .

Moreover, if we assume

(H3) $\quad (p_0)_{xx} \ge -C$,

then p satisfies

(R6) $\quad p_{xx} \ge -C$.

Finally, if p_0 is concave, then so is $p(\cdot,t)$. In this case p_{xx} and p_t are bounded in $L^\infty(\Omega)$.

The conditions (1.6) are satisfied by the unique solution of (P1) and the interface curves $(t, s_1(t))$ and $(t, s_2(t))$ are Lipschitz continuous and monotone decreasing and increasing, respectively, [2]. It is proved [26] that there exist $t_j^* \ge 0$ ($j = 1,2$) such that

$$s_j(t) = a_j \quad \text{for } t \in [0, t_j^*] ; \quad s_j(\cdot) \text{ is strictly monotone for } t \in]t_j^*, \infty[.$$

The times t_j^* are called ***waiting times***. Various estimates for waiting times are known depending on the shape of initial pressure (see [26, 5, 10, 3]).

1.1 - Interface tracking algorithms

Because of the relevant physical meaning of the interfaces s_j ($j = 1,2$), their numerical approximation is frequently as important as the approximation of the density u (or pressure p).

Fixed domain methods discretize equation (1.1) and provide a sequence of discrete solutions convergent to the continuous density u (with a rate measured in some Sobolev space); then the discrete free boundaries have to be computed a posteriori as level curves of the discrete solutions. Unfortunatly, it is clear that not even a good approximation of u implies an approximation of the s_j's.

Only in some particular cases we can expect to find interfaces precisely as level curves of the solution [22, 33].

Nevertheless, for the problem in one space variable, 'many authors suggested front tracking schemes. These methods use a finite difference scheme to discretize pressure equation (1.3), which is modified by means of a discretized form of jump conditions (1.6) in order to track the interfaces of the support of pressure.

The idea of exploiting interface conditions for computational purposes was first proposed by Huber [23] for the one-phase Stefan problem (see also [17]).

Here we give a detailed description of two front tracking algorithms, proposed by Di Benedetto & Hoff [14] and Hoff [21] (we refer to [20, 30, 41] for other schemes tipically one dimensional).

Before doing that, let us introduce some notation.

Let h and $\tau := T/N$ (N integer) denote the space and time steps, respectively, and let

$$x_i := i\,h\,, \quad i \in Z\,; \qquad t^n := n\,\tau\,, \quad n = 0,\dots,N\,.$$

The approximations of $p(x_i, t^n)$, $s_1(t^n)$ and $s_2(t^n)$ will be denoted by p_i^n, s_1^n and s_2^n, respectively. Moreover, we set

$$p^{h,\tau}(x,t) := \sum_{n=1}^{N} \sum_{i \in Z} p_i^n \varphi_i(x) \chi^n(t)\,, \quad s_j^{h,\tau}(t) := \sum_{n=1}^{N} s_j^n \chi^n(t)\,, \quad (j = 1,2)\,,$$

where χ^n is the caracteristic function of the interval $]t^{n-1}, t^n]$ and φ_i is the hat function of the node x_i.

Di Benedetto & Hoff scheme (S_{DH})

Denoting by $\varepsilon > 0$ an $O(h)$ viscosity parameter, the algorithm (S_{DH}) reads as follows.

(S_{DH}) *Let*

$$p_i^0 := p_0(x_i)\,, \quad i \in Z\,; \quad s_1^0 := a_1\,, \quad s_2^0 := a_2\,. \tag{1.7}$$

For $n = 0,\dots,N-1$, *set*

$$I_1^n := \min\{ i : x_{i-1} \geq s_1^n \}\,, \quad I_2^n := \max\{ i : x_{i+1} \leq s_2^n \}\,, \tag{1.8}$$

and

$$d_1^n := x_{I_1^n} - s_1^n\,, \quad d_2^n := s_2^n - x_{I_2^n}\,; \tag{1.9}$$

in analogy with (1.6) *we compute* s_1^{n+1} *and* s_2^{n+1} *from the equations*

$$s_1^{n+1} := s_1^n + \tau\,\frac{p_{I_1^n}^n}{d_1^n}\,, \quad s_2^{n+1} := s_2^n + \tau\,\frac{p_{I_2^n}^n}{d_2^n}\,; \tag{1.10}$$

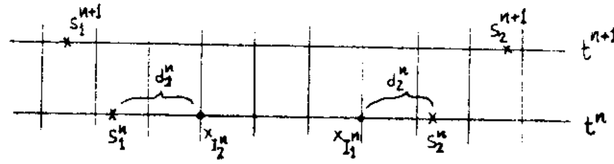
then we compute $\{p_i^{n+1}\}_{I_1^n \leq i \leq I_2^n}$ *by solving the finite difference equation*

$$\frac{p_i^{n+1} - p_i^n}{\tau} = [m-1][p_i^n + \varepsilon]\,\frac{p_{i-1}^n - 2p_i^n + p_{i+1}^n}{h^2} + \Big[\frac{p_{i+1}^n - p_{i-1}^n}{2h}\Big]^2\,; \tag{1.11}$$

finally for $s_1^{n+1} \leq x_i < x_{I_1^n}$ *and* $x_{I_2^n} < x_i \leq s_2^{n+1}$ *we compute* p_i^{n+1} *from the linear interpolation*

$$p_i^{n+1} := \frac{x_i - s_1^{n+1}}{x_{I_1^n} - s_1^{n+1}}\,p_{I_1^n}^{n+1} \quad \text{and} \quad p_i^{n+1} := \frac{s_2^{n+1} - x_i}{s_2^{n+1} - x_{I_2^n}}\,p_{I_2^n}^{n+1}\,, \tag{1.12}$$

respectively, and set $p_i^{n+1} := 0$ *for* $x_i \not\in [s_1^{n+1}, s_2^{n+1}]$.



Setting

$$\alpha := \tau / h^2\,, \quad A_h z_i := z_{i-1} - 2z_i + z_{i+1}\,, \tag{1.13}$$

(1.11) can be rewritten in the form

$$p_i^{n+1} := p_i^n + [m-1]\,\alpha\,[p_i^n + \varepsilon]\,A_h p_i^n + \alpha\Big[\frac{p_{i+1}^n - p_{i-1}^n}{2}\Big]^2\,. \tag{1.14}$$

Remark that the difference equation (1.11) is not enforced across the interfaces. The fact that $d_1^n, d_2^n \geq h$ guarantees numerical stability in the computation (1.10).

Imposing the stability condition

$$\tau \leq C h^2 \quad (C \text{ depending on } m, M, L_p \text{ and } \varepsilon)\,, \tag{1.15}$$

Di Benedetto & Hoff proved that

$$p_i^n \geq 0\,, \quad \forall i \in \mathbf{Z}, \quad 0 \leq n \leq N\,,$$

so that by (1.10) we get

$$s_1^{n+1} \leq s_1^n \quad \text{and} \quad s_2^{n+1} \geq s_2^n\,.$$

Thus the support of the discrete solution increases monotonically in t. The fact that $x_{I_1^n} - s_1^{n+1}$, $s_2^{n+1} - x_{I_2^n} \geq h$ insures numerical stability in (1.12).

The main results proved by Di Benedetto & Hoff are the following error bounds for both pressure and interfaces and the convergence for the velocity:

$$\|p^{h,\tau} - p\|_{L^\infty(Q_T)} \leq C h^\gamma , \tag{1.16}$$

$$p_x^{h,\tau} \to p_x \quad in \quad L^p(Q_T) , \quad \forall p \in [1,\infty[, \tag{1.17}$$

$$\| s_j^{h,\tau} - s_j \|_{L^\infty(0,T)} \leq C h^{\gamma/2} , \quad (j=1,2) , \tag{1.18}$$

where γ depends suitably on m.

The crucial point in the analysis of the convergence of approximate solution and interfaces is that, owing to the presence of ε, Di Benedetto & Hoff are capable to reproduce for the computed solutions the finite difference analog of the basic estimates (R1)-(R6).

Hoff scheme (S_H)

Although very simple, the scheme (1.7)-(1.12) is of little practical importance because of the usual parabolic stability condition for explicit schemes (1.15).

Hoff [21] proposed a linearly implicit finite difference scheme for (1.3), (1.4) which is computationally appealing, since he avoided that stability condition.

Hoff's scheme differs from (S_{DH}) just in substituting (1.11) with the following implicit finite difference equation

$$p_i^{n+1} = p_i^n + [m-1]\alpha p_i^n A_h p_i^{n+1} + \alpha \varepsilon A_h p_i^n + \alpha [\frac{p_{i+1}^n - p_{i-1}^n}{2}]^2 . \tag{1.19}$$

This is a tridiagonal system of linear equations, and has a unique solution.

Under the rather mild mesh condition

$$\tau \leq C h , \tag{1.20}$$

Hoff proved the same convergence results (1.16)-(1.18) for both the solution and the interfaces.

1.2 - Fixed domain methods. Approximation of the solution

The previous front tracking methods are tipically one dimensional. Several difficulties appear when one attempts to apply them to problems in more than one space dimension. Here we describe two simple finite difference schemes based directly on (1.1), (1.2), which can be easily extended to the multidimensional case.

Denoting by u_i^n the approximation of $u(x_i, t^n)$, we set

$$u^{h,\tau}(x,t) := \sum_{n=1}^{N} \sum_{i \in \mathbb{Z}} u_i^n \varphi_i(x) \chi^n(t) .$$

Consider the following explicit finite difference scheme:

(S_e) *Let*

$$u_i^0 := u_0(x_i) , \quad i \in \mathbf{Z} ; \tag{1.21}$$

for $n = 0,\dots,N-1$ *compute:*

$$u_i^{n+1} - u_i^n = \alpha A_h((u_i^n)^m) , \quad i \in \mathbf{Z} . \tag{1.22}$$

Under the assumptions (H0)-(H3) and assuming the stability condition

$$\tau \leq C h^2 \quad (C \text{ depending on } m \text{ and } M) , \tag{1.23}$$

Hoff & Lucier [22] proved the following error bound for the solution

$$\|u - u^{h,\tau}\|_{L^\infty(Q_T)} \leq \bar{C} h^\gamma , \tag{1.24}$$

where γ depends suitably on m.

In order to introduce an implicit finite difference scheme, we write the problem (1.1), (1.2) in a bounded space domain with homogeneous Dirichlet conditions at the boundary. This is not rescrictive because the solution has compact support. Setting

$$\beta(\xi) := \xi^m , \quad \xi \in R^+ \quad (\text{or } \beta(\xi) := \xi |\xi|^{m-1} , \quad \xi \in R) ,$$

we look for the solution of the following problem

$$u_t = (\beta(u))_{xx} , \quad (x,t) \in]a,b[\times]0,T[=: Q_T , \tag{1.25}$$

$$u(\cdot,0) = u_0(\cdot) \quad in \ ]a,b[, \tag{1.26}$$

$$\beta(u)(a,\cdot) = \beta(u)(b,\cdot) = 0 \quad in \ ]0,T[. \tag{1.27}$$

Let $h := \frac{b-a}{I+1}$ and let $x_i := a + ih$, $i = 0,\dots,I+1$. The discrete problem reads as follows:

(S_i) *Let*

$$u_i^0 := u_0(x_i) , \quad 0 \leq i \leq I+1 ; \tag{1.28}$$

for $n = 0, 1, \dots, N-1$:

$$u_i^{n+1} - u_i^n = \alpha A_h \beta(u_i^{n+1}) , \quad 1 \leq i \leq I , \quad u_0^{n+1} = u_{I+1}^{n+1} := 0 . \tag{1.29}$$

It is easy to see that (1.29) is a system of nonlinear algebraic equations associated with an M-function. This guarantees the existence and uniqueness of the solution [38]. For computing the discrete solution we shall use the following nonlinear Gauss-Seidel method:

(GS) *For $n = 0,\dots,N-1$, we set $\xi_i^{n+1,0} := u_i^n$ and construct the sequence $\xi_i^{n+1,k}$, $k = 1,2,\dots$, defined by*

$$\xi_i^{n+1,k} + 2\alpha\beta(\xi_i^{n+1,k}) = u_i^n + \alpha[\beta(\xi_{i-1}^{n+1,k}) + \beta(\xi_{i+1}^{n+1,k-1})], \quad 1 \le i \le I. \tag{1.30}$$

It is easily seen that

$$\xi_i^{n+1,k} \underset{k\to\infty}{\to} u_i^{n+1};$$

moreover, the scheme converges as fast as the linear Gauss-Seidel method for the positive definite matrix

$$\begin{bmatrix} 1+2\alpha & -\alpha & \cdot & \cdot & 0 \\ -\alpha & 1+2\alpha & -\alpha & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & -\alpha & 1+2\alpha \end{bmatrix}$$

1.3 - Approximation of interfaces

The question now is how to construct discrete free boundaries $s_j^{h,\tau}$ which approximate the s_j's. The problem is harder than the approximation of density, because the convergence of the discrete solutions could say nothing about the behavior of level sets.

In [8], Brezzi & Caffarelli showed for the stationary obstacle problem that, if the continuous and discrete solutions satisfy a *non-degeneracy property*, the error between continuous and discrete (sought as boundary of the contact set) free boundaries can be bounded in terms of the square root of the L^∞-distance between the solutions.

Non-degeneracy property means roughly that the solution leaves the free boundary with a *minimum speed* or, in other words, that the free boundaries actually move.

In [33], Nochetto proved error estimates for free boundaries, only assuming to know

1. an L^p-error bound for solutions ($p \le \infty$):

$$\|u - u^{h,\tau}\|_{L^p(Q_T)} \le C^* h^r; \tag{1.31}$$

2. a non-degeneracy property for continuous solution:

$$meas\,(\{0 < u < \varepsilon^s\} \cup F) \le C\varepsilon, \tag{1.32}$$

where $Q_+ := \{u > 0\}$ is the positivity set and and $F := \partial Q_+ \cap Q_T$ is the free boundary (hence, without using non-degeneracy properties of discrete solutions).

For the porous medium equation, the non-degeneracy of the continuous

solution is known under certain qualitative assumptions upon the data [5, 9, 10]. Nevertheless, we cannot expect to get discrete non-degeneracy properties. Besides, in more than one space dimension, L^p-error estimates for solutions are known, but for $p < \infty$. Hence is very important the fact of avoiding discrete non-degeneracy and L^∞-error estimates.

The crucial point in [33] is to ***shift the level*** in the definition of discrete free boundaries $F^{h,\tau}$, namely, to define $F^{h,\tau}$ as σ_h-level curves of the discrete solution, where

$$\sigma_h := C^* h^{rsp/[1+sp]} \quad (\sigma_h := C^* h^r \quad \text{if } p = \infty). \tag{1.33}$$

Namely, we define

$$Q_+^{h,\tau} := \{u^{h,\tau} > \sigma_h\}, \quad F^{h,\tau} := \partial Q_+^{h,\tau} \cap Q_T. \tag{1.34}$$

This idea was introduced in [6]. Notice that $F^{h,\tau}$ is easy to be computed a posteriori without additional computational costs. Hence the interface estimates (1.35), (1.37) are not based on front tracking, but on a trivial post-processing of a numerical solution that may have rapidly increasing support.

Under assumptions (1.31), (1.32), Nochetto proved the following *error estimates in measure* for the free boundaries:

$$meas\,(Q_+ \Delta Q_+^{h,\tau}) \le C h^{rp/[1+sp]} \quad (\le C h^{r/s} \quad \text{if } p = \infty). \tag{1.35}$$

If we require the stronger non-degeneracy property

3. $$\{0 < u < \varepsilon^s\} \subset S_{C\varepsilon}(F) := \{(x,t) \in Q_T : d((x,t),F) < C\varepsilon\}, \tag{1.36}$$

and suppose that an L^∞-error estimate for solutions holds, then we have the following *error estimate in distance* for the free boundaries:

$$F^{h,\tau} \subset S_{Ch^{r/s}}(F). \tag{1.37}$$

For porous medium equation in one space dimension, if we assume that

$$u_0(x) \ge C \min_{j=1,2} |x - a_j|^{1/[m-1]}, \quad \forall x \in]a_1, a_2[, \tag{1.38}$$

then there exists a positive constant C such that [5]

$$p(s_j(t) - (-1)^j z, t) \ge C z, \quad \forall t > 0, \ z > 0. \tag{1.39}$$

Clearly, (1.39) entails the following non-degeneracy properties:

$$meas\,(\{(x,t) \in Q_T : 0 < u(x,t) < \varepsilon^{1/[m-1]}\}) \le C\varepsilon, \tag{1.40}$$

and

$$\{x \in]a_1, a_2[: 0 < u(x,t) < \varepsilon^{1/[m-1]}\} \subset S_{C\varepsilon}(s_1(t)) \cup S_{C\varepsilon}(s_2(t)), \quad \forall t > 0. \tag{1.41}$$

Any error estimate for solutions gives rise to an error estimate for interfaces. For instance, for the scheme (Se) we know the error bound (1.24) for solutions; hence we can define

$$s_j^{h,\tau} := \partial\{u^{h,\tau} > \bar{C}h^\gamma\} \cap Q_T$$

and conclude that

$$\|s_j^{h,\tau} - s_j\|_{L^\infty(0,T)} \le C h^{\gamma/[m-1]} . \qquad (1.42)$$

Hoff & Lucier [22] obtained an estimate for interfaces similar to (1.42) by using different techniques. They avoid the non-degeneracy property, but it is not evident that their approach may be applicable to several space dimensions

2 - The multidimensional case

Let $\Omega \subset R^d$ $(d \ge 1)$ be a bounded domain with regular boundary and $0 < T < \infty$ be fixed.

Consider the problem

$$u_t = \Delta\beta(u) \quad in\ Q_T := \Omega \times]0,T[, \quad \left(\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}\right) \qquad (2.1)$$

$$u(\cdot,0) = u_0(\cdot) \quad in\ \Omega , \qquad (2.2)$$

$$\beta(u) = 0 \quad on\ \partial\Omega , \qquad (2.3)$$

where u_0 is a bounded (nonnegative) function.

Since β is degenerate, the solutions of (2.1)-(2.3) must be understood in the weak sense.

(Pd) *u is a weak solution of (2.1)-(2.3) if*

$$u \in L^2(Q_T) \cap H^1(0,T;H^{-1}(\Omega)) , \quad \beta(u) \in L^2(0,T;H_0^1(\Omega)) ,$$

$$u(\cdot,0) = u_0 ,$$

and for a.e. $t \in]0,T[$ and for all $z \in H_0^1(\Omega)$ the following equation hold

$$_{H^{-1}(\Omega)}\langle u_t , z \rangle_{H_0^1(\Omega)} + \int_\Omega \nabla\beta(u)\cdot\nabla z\, dx = 0 . \qquad (2.4)$$

Existence and uniqueness are well known for (Pd) [4, 25, 37] (see also [3, 27] for a detailed description of regularity results). In particular the maximum principle holds and we have the following global regularity:

(R7) if $\beta(u_0) \in H_0^1(\Omega)$, then $u_t \in L^\infty(0,T;H^{-1}(\Omega))$ and $\beta(u) \in L^\infty(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

The hölderianity of density u was proved in [10]. Some regularity properties of the interface F are also known. In particular [4], the set $\Omega_+(t) := \{x \in \Omega : u(x,t) > 0\}$ expands as t increases; this expansion is strict after a possible waiting time, which is related to the shape of u_0 near $F_0 := \partial\Omega_+(0)$. More precisely, assuming

$$F_0 \in C^2 , \quad u_0(x) \ge C\, d(x,F_0)^{\xi/[m-1]} , \quad \xi < 2 , \quad x \in \Omega_+(0) , \qquad (2.5)$$

then $\Omega_+(t)$ is strictly increasing at $t = 0$ (and for all $t > 0$, of course) [10, 26]. As a by product of the result in [10], we have the following non-degeneracy property (see also [33]):

$$meas\,(\{0 < u < \varepsilon^{\frac{2-\sigma}{\sigma}\frac{1}{m-1}}\} \cap \{t \ge t_0\}) \le C(t_0)\,\varepsilon , \quad \forall t_0 > 0 \qquad (2.6)$$

$(0 < \sigma < 1$: Hölder exponent [10, p.375]). In particular, for $d = \xi = 1$, the assumption (2.5) reduces to (1.38) and the non-degeneracy property (2.6) is replaced by the stronger one (1.39) [5]. These authors expect this property to hold even in more than one space dimension.

2.1 - An implicit nonlinear method

Since we shall use finite element methods for discretizing problem (Pd), we recall some notation and well known finite element properties.

Let $\mathbf{S}_h$ be a decomposition of Ω into closed d-simplices of diameter bounded by h (so h stands for the mesh size).

Let us set $\Omega_h := \bigcup_{S \in \mathbf{S}_h} S$ and suppose for simplicity that $\bar{\Omega} = \Omega_h$.

We assume that

the family $\{\mathbf{S}_h\}_h$ is regular [11, *p.* 132];

namely there exists a positive constant c_1 independent of h such that

$$h_S \le c_1 \sigma_S , \quad \forall S \in \mathbf{S}_h , \quad \forall h > 0 ,$$

where h_S and σ_S denote the diameter of the simplex S and the radius of the largest ball contained in S, respectively.

We introduce the space of the discrete functions we shall use in the sequel.

$$M_h := \{\psi : \psi_{|S} \text{ is constant for all } S \in \mathbf{S}_h\} ; \qquad (2.7)$$

$$V_h := \{\chi \in C^0(\Omega_h) : \chi_{|S} \text{ is linear for all } S \in \mathbf{S}_h\} , \qquad (2.8)$$

$$V_h^0 := \{\chi \in V_h : \chi = 0 \text{ on } \partial\Omega\} . \qquad (2.9)$$

We denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\Omega)$. The corresponding discrete inner product is defined by

$$\langle\chi,\varphi\rangle_h := \sum_{S \in \mathbf{S}_h} \int_S \Pi_h(\chi\varphi)\, dx , \qquad (2.10)$$

for any piecewise uniformly continuous functions χ and φ, where Π_h stands for the local linear interpolant operator. Notice that the previous integral can be evaluated easily by means of the vertex quadrature rule, which is exact for piecewise linear functions [11, p.182].

It is well known that $<\cdot,\cdot>_h$ is an inner product in V_h which satisfies [39, p.260]:

$$\|\chi\|^2_{L^2(\Omega)} \le <\chi,\chi>_h \le C\|\chi\|^2_{L^2(\Omega)}, \quad \forall \chi \in V_h, \tag{2.11}$$

where $C \ge 1$ is a constant independent of h.

Approximation properties of discrete spaces (2.7)-(2.9) are well known [11]. The following error bounds takes into account the effect of numerical integration:

$$|<\chi,\varphi> - <\chi,\varphi>_h| \le C h \|\nabla\chi\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}, \quad \forall \chi,\varphi \in V_h, \tag{2.12}$$

$$|<\chi,\varphi> - <\chi,\varphi>_h| \le C h^2 \|\nabla\chi\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}, \quad \forall \chi,\varphi \in V_h. \tag{2.13}$$

We now introduce the discrete projection operators:

L^2–projection $P_{M_h} : L^2(\Omega) \to M_h : \quad \forall z \in L^2(\Omega), \ <P_{M_h} z, \psi> = <z,\psi>, \quad \forall \psi \in M_h$;

L^2–projection $P_{V_h} : L^2(\Omega) \to V_h : \quad \forall z \in L^2(\Omega), \ <P_{V_h} z, \chi> = <z,\chi>, \quad \forall \chi \in V_h$;

H^1–projection $E_h : H^1_0(\Omega) \to V^0_h : \quad \forall z \in H^1_0(\Omega), \ <\nabla E_h z, \nabla\chi> = <\nabla z, \nabla\chi>, \quad \forall \chi \in V^0_h$.

The following approximation properties hold:

$$\|z - P_{M_h} z\|_{H^{-s}(\Omega)} \le C h^{r+s}\|z\|_{H^r(\Omega)}, \quad 0 \le s, r \le 1, \quad \forall z \in H^r(\Omega); \tag{2.14}$$

$$\|z - P_{V_h} z\|_{H^{-s}(\Omega)} \le C h^{r+s}\|z\|_{H^r(\Omega)}, \quad 0 \le s, r \le 1, \quad \forall z \in H^r(\Omega); \tag{2.15}$$

$$\|z - E_h z\|_{H^s(\Omega)} \le C h^{r-s}\|z\|_{H^r(\Omega)}, \quad 0 \le s \le r \le 2, \ r \ge 1, \quad \forall z \in H^r(\Omega) \cap H^1_0(\Omega). \tag{2.16}$$

We shall use only continuous piecewise linear functions and piecewise constant functions, because the low regularity of the solution of the porous medium equation does not justify the use of higher order elements.

In general, we do not require *quasi-uniformity* of the family $\{S_h\}_h$, namely that there exists a positive constant c_2 independent of h, such that

$$h \le c_2 h_S, \quad \forall S \in S_h, \quad \forall h > 0.$$

Thus local refinements of the mesh are allowed.

If we assume that S_h is of *acute type*, namely:

the projection of the vertices of any d-simplex $S \in S_h$ onto the hyperplane containing the opposite face lies in the closure of this face,

then the following *discrete maximum principle* for the Laplace operator [12] holds:

let $\chi \in V_h$ attain its maximum at the internal node x_j and let $\chi_j \in V_h$ be the corresponding basis function. Then

$$\int_\Omega \nabla\chi \cdot \nabla\chi_j \ge 0.$$

In 2–D the acuteness of the mesh means that internal angles of any triangle of the decomposition do not exceed $\pi/2$.

Let I and K be the dimension of V^0_h and M_h, respectively; denoting by $\{\chi_i\}^I_i$ and by $\{\psi_k\}^K_{k=1}$ the canonical bases of V^0_h and M_h, respectively, we define the following matrices

$$\begin{aligned} \mathbf{M} &:= \{<\chi_i,\chi_j>_h\}^I_{i,j=1}, \quad (\textit{diagonalized mass matrix}); \\ \mathbf{K} &:= \{<\nabla\chi_i,\nabla\chi_j>\}^I_{i,j=1}, \quad (\textit{stiffness matrix}); \\ \mathbf{P} &:= \{<\chi_i,\psi_k>\}^{I \ \ K}_{i=1,k=1}. \end{aligned} \tag{2.17}$$

Moreover, given $\chi \in V_h$ and $\psi \in M_h$, we denote by $\boldsymbol{\chi}$ and $\boldsymbol{\psi}$ the vector of the nodal values of χ and the vector of the barycentral values of ψ, respectively.

Now, let us introduce the discrete problem. The usual technique consists in discretizing problem (Pd) by means of backward differences in time and C^0-piecewise linear finite elements in space.

$(\mathrm{P}'_{h,\tau})$ *Let*

$$U^0 := P_{V_h} u_0;$$

for $n = 1,\dots,N$, *find* $U^n \in V_h$, *such that* $\beta(U^n) \in H^1_0(\Omega)$ *and*

$$< \frac{U^n - U^{n-1}}{\tau}, \chi> + <\nabla\beta(U^n), \nabla\chi> = 0, \quad \forall \chi \in V^0_h. \tag{2.18}$$

Other choices of U^0 are possible provided u_0 is smooth enough; for instance, if $u_0 \in C^0(\overline{\Omega})$, then we can choose $U^0 := I_h u_0$ (I_h stands for the linear interpolant operator in V_h).

Because of the nonlinearity of β, it is difficult to compute the integral $<\nabla\beta(U^n), \nabla\chi>$ in (2.18) exactly. Function $\beta(U^n)$ is then replaced by its interpolant $I_h\beta(U^n)$. Moreover, a lumping mass is usually emploied in order to diagonalize the mass matrix.

Thus we formulate the following discrete problem.

$(\mathrm{P}_{h,\tau})$ *Let*

$$U^0 := P_{V_h} u_0;$$

for $n = 1,\dots,N$, *find* $(U^n, B^n) \in V_h \times V^0_h$, *such that*

$$B^n := I_h \beta(U^n) \tag{2.19}$$

and

$$< \frac{U^n - U^{n-1}}{\tau}, \chi>_h + <\nabla B^n, \nabla\chi> = 0, \quad \forall \chi \in V^0_h. \tag{2.20}$$

Equation (2.20) can be written in matrix form as follows:

$$\mathbf{M}\mathbf{U}^n + \tau \mathbf{K}\mathbf{B}^n = \mathbf{M}\mathbf{U}^{n-1} . \tag{2.21}$$

It is easy to see that (2.21) is a system of nonlinear algebraic equation associated with a continuous and uniformly monotone operator; this guarantees existence and uniqueness of the solution [38]. We can use a nonlinear Gauss-Seidel method for computing the discrete solution.

Stability (in particular, maximum priciple, assuming the triangulation to be of acute type) and several error estimates for the solution have been proved by many authors for algorithms of that sort (see, e.g., [40, 24, 31, 34, 35, 16, 43]).

We point out that the use of numerical integration makes the scheme easy to be implemented on a computer and does not deteriorate its convergence properties (see [35, 16, 43]).

Setting

$$e_b(t) := \beta(u(t)) - B^n , \quad e_u(t) := u(t) - U^n , \quad \text{for } t \in]t^{n-1}, t^n], \quad 1 \le n \le N , \tag{2.22}$$

for algorithm ($P_{h,\tau}$) it is known that [43]:

$$\|e_b\|_{L^2(Q_T)} \le C h^{1/2} , \qquad \|e_u\|_{L^{m+1}(Q_T)} \le C h^{1/[m+1]} \tag{2.23}$$

(choosing $\tau = C h$).

Due to the low regularity of the problem, the numerical approximation of (Pd) is usually carried out by using a preliminar regularization procedure which consists in replacing β by a function with minimal slope $\varepsilon > 0$ (ε: ***regularization parameter***), namely:

$$\beta_\varepsilon(\xi) := \max(\varepsilon\,|\xi|, |\xi|^m)\, sign(\xi) , \quad \xi \in R . \tag{2.24}$$

Then the regularized problem is discretized in space and time as in ($P_{h,\tau}$).

For the approximate solutions of this scheme it is known that [35]:

$$\|e_b\|_{L^2(Q_T)} \le C h^{[m+1]/2m} , \qquad \|e_u\|_{L^{m+1}(Q_T)} \le C h^{1/m} \tag{2.25}$$

(choosing $\tau = C_1 h^{[m+1]/m}$, $\varepsilon = C_2 h^{[m-1]/m}$), which are slightly better than (2.23).

Moreover, if non-degeneracy properties for continuous solutions are known, the previous estimate (2.25) can be improved. For instance, in the one dimensional case, assuming condition (1.38) for the initial density u_0, the non-degeneracy property (1.40) holds; hence we have the improved estimates:

$$\|e_b\|_{L^2(Q_T)} \le C h^{3m/[3m-1]} , \qquad \|e_u\|_{L^{m+1}(Q_T)} \le C h^{4m/\{[m+1][3m-1]\}} \tag{2.26}$$

(choosing $\tau = C_1 h^{4m/[3m-1]}$, $\varepsilon = C_2 h^{2[m-1]/[3m-1]}$).

Clearly, the considerations of section 1.3 provide the following error estimate for free boundaries:

$$meas\,(Q_+ \Delta Q_+^{h,\tau}) \le C h^{2[m-1]/[3m-1]} . \tag{2.27}$$

In more space dimensions, recalling the non-degeneracy property (2.6) we get the following error estimate in measure for free boundaries:

$$meas\,((Q_+ \Delta Q_+^{h,\tau}) \cap \{t > t_0\}) \le C(t_0) h^\gamma , \tag{2.28}$$

with γ depending on m and σ.

2.2 - Linear schemes

In section 2.1 we presented the usual technique to approximate problem (Pd), which amounts at each time step to solve a system of nonlinear equations.

In order to avoid this nonlinearity, it is not recommended to discretize equation (2.1) explicitly in time, namely:

$$< \frac{U^n - U^{n-1}}{\tau}, \chi > + < \nabla\beta(U^{n-1}), \nabla\chi > = 0 , \tag{2.29}$$

because of the well known restrictive parabolic stability condition $\tau \le C h^2$, which forces in using too small time steps (this is too expensive, expecially in many space dimensions).

However, when dealing with nonlinear problems, one usually tries to linearize them so as to take advanntage of efficient linear solvers. Standard techniques consist, for instance, in rewriting (2.1) as follows:

$$u_t = \nabla \cdot (\beta'(u) \nabla u) \tag{2.30}$$

and discretizing it by means of the scheme

$$< \frac{U^n - U^{n-1}}{\tau}, \chi > + < \beta'(U^{n-1}) \nabla U^n, \nabla\chi > = 0 . \tag{2.31}$$

Alternatively, by introducing the functions $\gamma := \beta^{-1}$ (β possibly regularized as in (2.24)) and $b := \beta(u)$, equation (2.1) can be rewritten as:

$$\gamma'(b)\, b_t = \Delta b \tag{2.32}$$

and can be discretized as follows:

$$< \gamma'(B^{n-1}) \frac{B^n - B^{n-1}}{\tau}, \chi > + < \nabla B^n, \nabla\chi > = 0 . \tag{2.33}$$

The success of such tricks relies on the smoothness of the solutions u and $b = \beta(u)$. Consequently, it is not a priori obvious that these standard techniques for mildly nonlinear parabolic equations apply in our context, because (Pd) is just a low regularity problem.

So the question is how to linearize (Pd) properly.

In [15], Douglas & Dupont introduced the following Laplace-modified forward Galerkin method:

$$< \frac{U^n - U^{n-1}}{\tau}, \chi> + <a(U^{n-1}) \nabla U^{n-1}, \nabla \chi> + \lambda <\nabla[U^n - U^{n-1}], \nabla \chi> = 0 , \quad (2.34)$$

where the *stabilization parameter* λ is chosen so that

$$\lambda > \frac{1}{2} max(a) . \quad (2.35)$$

Douglas & Dupont proved the stability and convergence properties of this *linearized* scheme for regular nonlinear problems $u_t = \nabla \cdot (a(u) \nabla u)$, $a(u) \geq a_{max} > 0$.

In the same spirit of the Laplace-modified forward Galerkin method, we consider the following discrete-time linear scheme:

$$< \frac{U^n - U^{n-1}}{\tau}, \chi> + <\nabla \beta(U^{n-1}), \nabla \chi> + \lambda <\nabla[U^n - U^{n-1}], \nabla \chi> = 0 , \quad (2.36)$$

which can be equivalently written in the form:

(C_τ) *Set*

$$U^0 := u_0 ; \quad (2.37)$$

for $n = 1,\ldots,N$ *solve*

$$<\Theta^n, z> + \lambda \tau <\nabla \Theta^n, \nabla z> = <\beta(U^{n-1}), z> , \quad \forall z \in H_0^1(\Omega) \quad (2.38)$$

and compute

$$U^n = U^{n-1} + \frac{\Theta^n - \beta(U^{n-1})}{\lambda} . \quad (2.39)$$

This scheme was also suggested by semigroup theory ((C_τ) is a particular nonlinear Chernoff formula) and was first used in numerical analysis of singular parabolic problems by Berger, Brézis & Rogers [7] (see also [42, 28, 36]). It is also connected to the discrete phase relaxation scheme proposed in [44].

The algorithm (C_τ) gives rise actually to an effective and simple numerical scheme after discretizing in space; namely, the variable Θ^n is approximated by continuous piecewise linear functions and the variable U^n by piecewise constant functions:

$(C_{h,\tau})$ *Set*

$$U^0 := P_{M_h} u_0 ; \quad (2.40)$$

for $n = 1,\ldots,N$, *find* $(U^n, \Theta^n) \in M_h \times V_h^0$ *such that*

$$<\Theta^n, \chi>_h + \lambda \tau <\nabla \Theta^n, \nabla \chi> = <\beta(U^{n-1}), \chi> , \quad \forall \chi \in V_h^0 , \quad (2.41)$$

and

$$U^n = U^{n-1} + \frac{P_{M_h} \Theta^n - \beta(U^{n-1})}{\lambda} . \quad (2.42)$$

By setting $B^n := \sum_{k=1}^{K} \beta(U_k^n) \psi_k$ and noting that $P_{M_h} \chi = \sum_{k=1}^{K} \chi(y_k) \psi_k$ for $\chi \in V_h$, the discrete problem $(C_{h,\tau})$ can be equivalently written in matrix form as follows:

$$(\mathbf{M} + \lambda \tau \mathbf{K}) \boldsymbol{\Theta}^n = \mathbf{P} \mathbf{B}^{n-1} , \quad (2.43)$$

$$U_k^n = U_k^{n-1} + \frac{\Theta^n(y_k) - \beta(U_k^{n-1})}{\lambda} , \quad k = 1,\ldots,K . \quad (2.44)$$

Since $\mathbf{A} := (\mathbf{M} + \lambda \tau \mathbf{K})$ is a symmetric and positive definite matrix, the *linear system* (2.43) has a unique solution. (2.44) may be regarded as an element-by-element correction, which takes the nonlinearity into account.

The crucial point in the performance of the method is the efficient resolution of the linear system (2.43). A Cholesky factorization of the matrix $\mathbf{A}$ is recommended whenever the band-width is *small*. Indeed, since the matrix $\mathbf{A}$ remains unchanged at each time step, the factorization is made only once at the beginning. Nevertheless, when the use of an automatic grid code is required, for instance in decomposing a general domain, the band-width may be *large*. Then iterative methods seem to perform better than direct ones. In particular, since $\mathbf{A}$ is strictly diagonally dominant, the incomplete Cholesky factorization for preconditioning matrix A combined with conjugate gradient iterations is recommended for solving linear system (2.43) [29].

In [36] (see also [44]) the stability of the scheme $(C_{h,\tau})$ in reproducing the basic estimates of solution of problem (Pd) is shown under the following condition on the *relaxation parameter* λ:

$$\lambda \geq L_\beta , \quad (2.45)$$

where L_β is the Lipschitz constant of β (restricted to the interval $[-\|u_0\|_{L^\infty(\Omega)}, \|u_0\|_{L^\infty(\Omega)}]$). In particular, the *maximum principle* holds assuming the acuteness of the finite element decomposition.

Several error estimates in energy norms for both variable u and $b = \beta(u)$ were proved. In particular, setting $e_b(t) := \beta(u(t)) - \Theta^n$, $t \in]t^{n-1}, t^n]$, $1 \leq n \leq N$, the estimate

$$\|e_b\|_{L^2(Q_T)} \leq C h^{1/2} \quad (2.46)$$

holds with a relationship between h and τ depending on the regularity of initial data.

Remark - The algorithms discussed in sections 2.1 and 2.2 can be applied to very general nonlinear (possibly singular) parabolic problems; two-phase Stefan problems are included.

References

[1] D. G. ARONSON, *Regularity properties of flows through porous media*, SIAM J. Appl. Math., 17 (1969), 461-467.

[2] D. G. ARONSON, *Regularity properties of flows through porous media: the interface*, Arch. Rational Mech. Anal., 37 (1970), 1-10.

[3] D. G. ARONSON, *The porous medium equation*, International Mathematical Summer Center (Fondazione CIME), June 1985 (to be published by Pitman).

[4] D. G. ARONSON & Ph. BENILAN, *Régularité des solutions de l'équation de millieux poreaux dans* $\mathbf{R}^N$, C. R. Acad. Sci. Paris, 288 (1979), pp. 103-105.

[5] D. G. ARONSON, L. A. CAFFARELLI & S. KAMIN, *How an initially stationary interface begins to move in porous medium flow*, SIAM J. Math. Anal., 14 (1983), pp. 639-658.

[6] C. BAIOCCHI & G. POZZI, *Error estimates and free-boundary convergence for a finite difference discretization of a parabolic variational inequality*, RAIRO Anal. Numér., 11 (1977), 315-340.

[7] A. E. BERGER, H. BREZIS & J. C. W. ROGERS, *A numerical method for solving the problem* $u_t - \Delta f(u) = 0$, RAIRO Anal. Numér., 13 (1979), pp. 297-312.

[8] F. BREZZI & L. A. CAFFARELLI, *Convergence of the discrete free boundaries for finite element approximations*, RAIRO Anal. Numér., 17 (1983), 385-395.

[9] L. A. CAFFARELLI & A. FRIEDMAN, *Regularity of the free boundary for the one dimensional flow of gas in a porous medium*, Amer. J. Math., (1979), 1193-1218.

[10] L. A. CAFFARELLI & A. FRIEDMAN, *Regularity of the free boundary for a gas in an n-dimensional porous medium*, Indiana Univ. Math. J., 29 (1980), pp. 361-391.

[11] P. G. CIARLET, *The finite element method for elliptic problems*, North-Holland, Amsterdam (1978).

[12] P. G. CIARLET & P. A. RAVIART, *Maximum principle and uniform convergence for the finite element method*, Comp. Meth. Appl. Engnr., 2 (1973), 17-31.

[13] E. DI BENEDETTO, *Continuity of weak solutions to a general porous medium equation*, Indiana Univ. Math. J., 32 (1983), pp. 83-118.

[14] E. DI BENEDETTO & D. HOFF, *An interface tracking algorithm for the porous medium equation*, Trans. Amer. Math. Soc., 284 (1984), 463-500.

[15] J. DOUGLAS, Jr. & T. DUPONT, *Alternating-direction Galerkin methods on rectangles*, in *Numerical solutions of partial differential equations*, vol. II (B. Hubbard Ed.), Academic Press, New York (1971), 133-214.

[16] C. M. ELLIOTT, *Error analysis of the enthalpy method for the Stefan problem*, preprint (1985).

[17] A. FASANO & M. PRIMICERIO, *Convergence of Huber's method's for heat conduction problems with change of phase*, Z. Angew. Math. Mech., 53 (1973), 341-348.

[18] A. FRIEDMAN, *Variational principles and free boundary problems*, John Wiley & Sons, New York (1982).

[19] B. H. GILDING, *Holder continuity of solutions of parabolic equations*, J. London Math. Soc., 13 (1976), 103-106.

[20] J. L. GRAVELEAU & P. JAMET, *A finite difference approach to some degenerate nonlinear parabolic equations*, SIAM J. Appl. Math., 20 (1971), 199-223.

[21] D. HOFF, *A linearly implicit finite-difference scheme for the one-dimensional porous medium equation*, Math. Comp., 45 (1985), 23-33.

[22] D. HOFF & B. J. LUCIER, *Numerical methods with interface estimates for the porous medium equation*, to appear in M^2AN.

[23] A. HUBER, *Hauptaufsatze uber das fortschreiten der schmelzgrenze in einem linearen leiter*, Z. Angew. Math. Mech., 19 (1939), 1-21.

[24] J. W. JEROME & M. ROSE, *Error estimates for the multidimensional two-phase Stefan problem*, Math. Comp., 39 (1982), pp. 377-414.

[25] S. L. KAMENOMOSTSKAJA, *On the Stefan problem*, Naucn. Dokl. Vyss. Skoly, 1 (1958), pp. 60-62; Mat. Sb., 53 (1961), pp. 489-514.

[26] B. KNERR, *The porous medium equation in one dimension*, Trans. Amer. Math. Soc., 234 (1977), pp. 381-415.

[27] E. MAGENES, *Problemi di Stefan bifase in più variabili spaziali*, V SAFA, Catania, Le Matematiche, XXXVI (1981), pp. 65-108.

[28] E. MAGENES, R. H. NOCHETTO & C. VERDI, *Energy error estimates for a linear scheme to approximate nonlinear parabolic problems*, to appear in M^2AN.

[29] T. A. MANTEUFFEL, *The shifted incomplete Cholesky factorization*, Report Sandia Laboratories 78-8226 (1978).

[30] M. MIMURA, T. NAKAKI & K. TOMOEDA, *A numerical approach to interface curves for some nonlinear diffusion equations*, Japan J. Appl. Math., 1 (1984), 93-139.

[31] R. H. NOCHETTO, *Error estimates for two-phase Stefan problems in several space variables, I: linear boundary conditions*, Calcolo, XXII (1985), 457-499.

[32] R. H. NOCHETTO, *A class of non-degenerate two-phase Stefan problems in several space variables*, to appear in Comm. Partial Differential Equations.

[33] R. H. NOCHETTO, *A note on the approximation of free boundaries by finite elements methods*, M²AN, 20 (1986), 355-368.

[34] R. H. NOCHETTO, *Error estimates for multidimensional singular parabolic problems*, to appear in Japan J. Appl. Math.

[35] R. H. NOCHETTO & C. VERDI, *Approximation of degenerate parabolic problems using numerical integration*, submitted to SIAM J. Numer. Anal.

[36] R. H. NOCHETTO & C. VERDI, *An efficient linear scheme to approximate parabolic free boundary problems: error estimates and implementation*, submitted to Math. Comp.

[37] O. A. OLEINIK, A. S. KALASHINIKOV & CZHOU YUI-LIN, *The Cauchy-problem and boundary problems for equations of the type of nonstationary filtration*, Izv. Akad. Nauk. SSSR Ser. Mat., 22 (1958), pp. 667-704.

[38] J. M. ORTEGA & W. C. RHEINBOLDT, *Iterative solution of non-linear equations in several variables*, Academic Press, New York (1970).

[39] P. A. RAVIART, *The use of numerical integration in finite element methods for solving parabolic equations*, in *Topics in numerical analysis* (J.J.H. MILLER ed.), Academic Press, London (1973), pp. 233-264.

[40] M. ROSE, *Numerical methods for flows through porous media, I*, Math. Comp., 40 (1983), pp. 435-467.

[41] K. TOMOEDA & M. MIMURA, *Numerical approximations to interface curves for a porous medium equation*, Hiroshima Math. J., 13 (1983), 273-294.

[42] C. VERDI, *On the numerical approach to a two-phase Stefan problem with non-linear flux*, Calcolo, 22 (1985), 351-381.

[43] C. VERDI, *Optimal error estimates for an approximation of degenerat parabolic problems*, submitted to Numer. Funct. Anal. Optim.

[44] C. VERDI & A. VISINTIN, *Error estimates for a semi-explicit numerical scheme for Stefan-type problems*, submitted to Numer. Math.