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ECONOMIC APPLICATIONS OF OPTIMAL COMTROL

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1. INTRODUCTION

The purpose of this lectures is to give an *elementary* introduction to the mathematical theory of optimal control and then to apply it to a variety of different situations arising in economic applications. These applications involve the control of *dynamic systems*, i.e. systems that evolve over time.

1.1. The optimal control problem

We consider a *continuous time* system with the time variable t. Let x(t) be the *state variable* of the system at time t, i.e. the inventory level at time t. We assume that there is a way of controlling the state of the system. Let u(t) be the *control variable* at time t, i.e. the production rate at time t. Note that x(t) and u(t) are one-dimensional real variables.

The *system dynamics* is described by an ordinary differential equation (state equation)

$$\dot{x}(t) = f(x(t),u(t),t), \tag{1.1}$$

where $\dot{x}(t) = dx(t)/dt$ is the instantaneous rate of change in the state variable, and f is a given function of x,u,t.

If we know the *initial value*

$$x(0) = x_0 \tag{1.2}$$

and the *control trajectory* u(t) over the whole planning interval [0,T], then we can integrate (1.1) to get the *state trajectory*.

However, not all trajectories (x(t),u(t)) are equally desirable. Denote by F(x(t),u(t),t) the instantaneous utility rate at time t depending on x(t) and u(t). The *objective functional* is the discounted utility stream

$$J = \int_0^T e^{-rt}F(x(t),u(t),t)dt + e^{-rT}S(x(T),T). \tag{1.3}$$

In (1.3) r denotes the discount rate, and the function S gives the "salvage value" of the ending state x(T).

The optimal control problem is to *choose* the control trajectory so that together with the state trajectory obtained by (1.1,2) it *maximizes* the objective functional (1.3).

Usually in economic applications it is reasonable to assume the control u(t) as a piecewise continuous function[1]. Moreover, often u will be constrained:

$$u(t)\in\Omega \subseteq \mathbb{R}. \tag{1.4}$$

Frequently, the control variables has to satisfy a path restriction depending on the state variable:

$$g(x(t),u(t),t) \geq 0. \tag{1.5}$$

For the present a control variable is called *feasible* if it is piecewise continuous and satisfies (1.4) or (1.5), respectively. Problems with prescribed control regions (1.4) are dealt with in sections 2, 3 and 4. In section 5 path constraints (1.5) are taken into consideration. There the more difficult case of pure state restrictions is also treated.

1.2. Examples

Example 1.1: Maintenance - Production Planning for Deterministic Deterioration

The purpose of preventive maintenance is to slow down the continuous deterioration of machine's quality. Let x(t) denote the quality (resale value) of a machine at time t. t is the age of the machine and T its sale date time. If we denote the maintenance expenditures by u(t), the system dynamics is

$$\dot{x}(t) = -\gamma(t) - \delta(t)x(t) + g(u(t),t). \tag{1.6}$$

Here γ(t) measures the rate of technical obsolescence, δ(t) the rate of attrition, g(u(t),t) is the maintenance effectiveness function (measured in dollars added to the resale value per dollar spent on preventive maintenance).

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[1] Thus, x(t) satisfies the state equation (1.1) only at those times t at which u(t) is continuous. At discontinuity points of u(t) the differentiable parts of the state trajectory are pieced together such that x(t) becomes continuous.

It is assumed that γ and δ are nondecreasing in t, while $g(u,t)$ is concave in u and nonincreasing in age t. Moreover,

$$u(t) \in \Omega \in [0,\bar{u}] \quad \text{for all} \quad t \in [0,T]. \tag{1.7}$$

The present value of the machine is the sum of two terms, the discounted production stream during its life, plus the discounted resale value at T:

$$J = \int_0^T e^{-rt}[\pi x(t) - u(t)]dt + e^{-rT}x(T). \tag{1.8}$$

An interesting extension arises if we assume that the production rate π is a (second) control instrument v. In this case the state equation (1.6) is transformed to

$$\dot{x}(t) = -\gamma(t) - \delta(t)x(t) + g(u(t),t) - h(v(t),t). \tag{1.9}$$

Here $h(v,t)$ measures the deterioration of the "state" of the machine by the production intensity. It is plausible to assume that h is convex in v and nondecreasing in t.

Example 1.2: Inventory and Production Planning

We consider the production and inventory storage of a given good in order to meet an exogenously given demand $d(t)$. Denoting by $z(t)$ the inventory level at time t and by $v(t)$ the production rate the rate of change in the inventory level is governed by

$$\dot{z}(t) = v(t) - d(t). \tag{1.10}$$

The (pure) state constraint

$$x(t) \geq 0 \quad \text{for all} \quad t \in [0,T] \tag{1.11}$$

is imposed, if the demand is to satisfied for all t. Moreover, it seems reasonable to suppose that

$$0 \leq v(t) \leq \bar{v}. \tag{1.12}$$

Let $c(v,t)$ be the production costs and $h(z,t)$ the inventory holding costs. Then the task of the decision-maker is to minimize the total costs over the whole (given) planning interval, i.e. to maximize

$$J = -\int_0^T [c(v(t),t) + h(z(t),t)]dt. \tag{1.13}$$

Example 1.3: Optimal Capital Accumulation

The following neoclassical growth model characterizes economic growth in an aggregative closed economy. The economy produces a single homogeneous good using the capital K as single production factor. If we denote by $F(K)$ the production function (output) and by C the consumption rate, then $I = F(K) - C$ measures the gross investment rate. Assuming that the capital stock depreciates at a constant rate δ, the gross investment identity states that

$$\dot{K}(t) = I(t) - \delta K(t). \tag{1.14}$$

Thus, (net) capital accumulation is that part of investment not used to replace depreciated capital. The initial capital endowment is given as $K(0) = K_0$.

The problem of a central planner who has authority over the entire economy during the planning period $[0,T]$ is to choose a time path for consumption such that the discounted utility stream

$$J = \int_0^T e^{-rt}U(C(t))dt + e^{-rT}S(K(T)) \tag{1.15}$$

is maximized. The consumption rate is restricted by

$$0 \leq C(t) \leq F(K(t)) \tag{1.16}$$

for any $t \in [0,T]$.

1.3. Historical Remarks

Calculus of variations: Newton, Leibniz, Bernoullis, Euler, Lagrange, Hamilton etc., Valentine, McShane, Hestenes.

Optimal control theory: Pontryagin, Boltyanski, Gamkrelidze and Mischenko (1962), proof of the maximum principle (constrained control variables).

Dynamic programming: Bellman (1957).

Two-person zero sum differential games: Isaacs (1965).

Economic applications: Evans (1924), Ramsey (1929), Hotelling (1931), Connors and Teichroew (1967), Arrow and Kurz (1970), Bensoussan, Hurst and Näslund (1974), Cass and Shell (1976), Sethi and Thompson (1981), Kamien and Schwartz (1981).

2. THE MAXIMUM PRINCIPLE

In this chapter the maximum principle for an optimal control problem in standard form is introduced. First, the set of necessary optimality conditions of the maximum principle is stated and illustrated by some *simple* examples. Second, a heuristic proof using dynamic programming is given. Third, the optimality conditions are interpreted economically. Fourth, sufficient conditions for optimality are discussed. Fifth, the standard control problem is modified by including simple terminal conditions for the state variable. Sixth, we deal with infinite time horizon problems and stationarity. Finally, we sketch the solution for free terminal time problems.

2.1. Necessary Optimality Conditions for a Standard Control Problem

Motivated by the discussion in section 1 we consider the following optimal control problem in *standard form:*

$$\max_{u(t)} \{J = \int_0^T e^{-rt} F(x(t),u(t),t)dt + e^{-rT} S(x(T),T)\} \tag{2.1a}$$

$$\dot{x}(t) = f(x(t),u(t),t) \tag{2.1b}$$

$$x(0) = x_0 \tag{2.1c}$$

$$u(t) \in \Omega \subseteq \mathbb{R}^m, \tag{2.1d}$$

where $x(t) \in \mathbb{R}^n$ is the n-dimensional state vector and $u(t) \in \mathbb{R}^m$ is the m-dimensional control vector.

The right-hand side of the state equation (2.1b) $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function assumed to be continuously differentiable. The path $x(t)$, $t \in [0,T]$ is called a state trajectory, and $u(t)$, $t \in [0,T]$ is called a control trajectory.

Furthermore, the instantaneous utility (or profit) rate $F: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, and the salvage value function $S: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ has also this property.

Finally, the terminal time T is assumed to be given.

We define a feasible control to be a control trajectory $u(t)$, $t \in [0,T]$ which is piecewise continuous and satisfies (2.1d). The problem is to find a feasible control $u^*(t)$ which maximizes the objective functional (2.1a) subject to the state equation (2.1b), the initial condition (2.1c) and the control constraints (2.1d). The control $u^*(t)$ is called an *optimal control*, and $x^*(t)$, determined by means of the state equation, is called an *optimal trajectory* or an *optimal path*.

The optimal control problem (2.1) specified above is said to be in *Bolza form* because of the form of the objective function. It is said to be in the *Lagrange form* when $S(x,T) = 0$, and it is in the *Mayer form* when $F(x,u,t) = 0$. It can be shown that both the Bolza and the Lagrange form can be reduced to the Mayer form. The price paid for going from Bolza or Lagrange to Mayer is the addition of one state variable and its associated differential equation. Thus, all three forms of an optimal control problem an equivalent.

When no confusion arises, we will usually suppress the time notation t, e.g., $x(t)$ will be written simply as x.

Using the so-called *Hamilton function*

$$H(x,u,\lambda,t) = F(x,u,t) + \lambda f(x,u,t) \tag{2.2}$$

where $\lambda \in \mathbb{R}^n$ denotes the costate variable, the main theorem of optimal control theory (for the standard problem (2.1)) is as follows.

Theorem 2.1 (Maximum principle for the standard problem, necessary optimality conditions).

Let $u^*(t)$ be the optimal control for the problem (2.1) and $x^*(t)$ the corresponding state trajectory. Then there exists a continuous and piecewise continuously differentiable vector valued function

$$\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t)) \in \mathbb{R}^n \tag{2.3}$$

which is called costate or adjoint variable such that for all points $t \in [0,T]$, where $u^*(t)$ is continuous, the following necessary optimality conditions hold:

$$H(x^*(t), u^*(t), \lambda(t), t) = \max_{u \in \Omega} H(x^*(t), u, \lambda(t), t) \tag{2.4}$$

$$\dot{\lambda}(t) = r\,\lambda(t) - H_x(x^*(t), u^*(t), \lambda(t), t); \tag{2.5}$$

$$\lambda(T) = S_x(x^*(T), T). \tag{2.6}$$

Remarks: According to (2.4) $u^*(t)$ must provide a *global* maximum of the Hamiltonian $H(x^*,u,\lambda,t)$. For this reason the necessary conditions (2.4) - (2.6) are called the (a) maximum principle. Note that beside of the Hamiltonian maximizing condition (2.4) and the adjoint equation (2.5) the system dynamics must be satisfied by x^* and u^*:

$$x^*(t) = f(x^*(t),u^*(t),t) \quad (2.7)$$

$$x^*(0) = x_o. \quad (2.8)$$

Usually we conceive x, u, f as column vectors. The transposition is denoted by a prime. Let us define the derivative of a scalar function with respect to a column vector (the gradient) as a row vector. Furthermore, we agree that the differentiation of scalar function with respect to a row vector yields a column vector. Taking into consideration that x is a column vector and λ a row vector the state equation and the adjoint equation can be written as

$$\dot{x}^* = H_\lambda(x^*,u^*,\lambda,t) \quad (2.9a)$$

$$\dot{\lambda} = r\lambda - H_x(x^*,u^*,\lambda,t). \quad (2.9b)$$

(2.9) is a system of ordinary differential equations known as *canonical system* (also called *modified Hamiltonian system*).

(2.4) says that $u^*(t)$ must provide a *global* maximum of the Hamiltonian $H(x^*(t),u,\lambda(t),t)$ for each time t. For this reason the necessary conditions is called maximum principle. The Hamiltonian maximizing condition (2.4) provides

$$u^*(t) = u(x^*(t),\lambda(t),t) \quad (2.10)$$

which is substituted into the canonical system.

If the Hamiltonian is maximized in the *interior* of Ω, then a necessary first-order condition for a *local* maximum is

$$H_u = 0. \quad (2.4a)$$

If H_{uu} is negative semidefinite for $u\in\Omega$ (for m = 1 this means that $H_{uu} \le 0$), condition (2.4a) is also sufficient for the Hamiltonian maximizing condition (2.4).

With the control variable so obtained the n state equations for x depend on the initial value x_o, and the n adjoint equations for λ depend on the terminal value $\lambda(T)$. Thus we have 2n differential equations with 2n boundary conditions (n initial conditions (2.8) and n terminal conditions (2.6)). The general solution of such a *two-point boundary value problem* can be very difficult. In certain special cases, however, the solution is easy. One such is the case in which the adjoint equation (2.5) is independent of the state and the control variables; here we can solve the adjoint equation first, then obtain the optimal control, and finally solve the system dynamics to get the state trajectory. Generally, however, we have to determine m + 2n unknown functions *simultaneously*, namely $u^*(t)$, $x^*(t)$, and $\lambda(t)$. For the solution m+ 2n conditions are available, namely the maximizing condition (2.4) (or (2.10), respectively), the system dynamics (2.7), and the adjoint equation (2.5).

Essentially, the maximum principle permits to decouple the intertemporal decision problem into a series of problems (2.4) holding at each instant. Moreover, the state and the costate variables are connected by the canonical system.

Because an integral is uneffected by values of the integrand at a finite set of points, the optimal solution can be modified for a finite set of time arguments without changing the optimality (or feasibility) of the solution.

In most management science and economic problems the objective functional is usually formulated in money and utility terms. The future streams of money or utility are usually discounted. The Hamiltonian defined in (2.2) is called *current-value* Hamiltonian. Defining the *present-value* Hamiltonian $\tilde{H}$ as

$$\tilde{H} = e^{-rt}F(x,u,t) + \tilde{\lambda}f(x,u,t) \quad (2.11a)$$

the necessary optimality conditions can be stated as follows:

$$u^* = \arg\max \tilde{H}, \ \dot{\tilde{\lambda}} = -\tilde{H}_x, \ \tilde{\lambda}(T) = e^{-rT}S_x(x^*(T),T). \quad (2.11b)$$

Using the transformation

$$\tilde{H} = H\exp(-rt), \ \tilde{\lambda} = \lambda\exp(-rt) \quad (2.11c)$$

it is easily seem that the optimality conditions (2.4-6) and (2.11b) are equivalent. One reason for using the current-value notation is that the omitting of the factor exp(-rt) generates an autonomous canonical system for autonom ous functions f, F and S. For such system the phase portrait analysis provides a powerful tool to get insight into the qualitative behaviour of the solutions.

2.2. Elementary Examples

In order to absorb the maximum principle the reader should study very carefully the examples in this section, all of which are problems having only one state and one control variable.

Example 2.1. Consider the problem

$$\text{Maximize } \{J = \int_0^1 (-x)dt\} \tag{2.12}$$

subject to the state equation

$$\dot{x} = u, \quad x(0) = 1 \tag{2.13}$$

and the control constraint

$$u \in \Omega = [-1,1]. \tag{2.14}$$

Note that $T = 1$, $F = -x$, $S = 0$, and $f = u$. Because of the minus sign in F we can interpret the problem as being that of minimizing the (signed) area under the curve $x(t)$ for $0 \le t \le 1$.

Solution. First we form the Hamiltonian

$$H = -x + \lambda u \tag{2.15}$$

and note that, because the Hamiltonian is linear in u, the form of the optimal control is

$$u^*(t) = \begin{cases} 1 & \text{if } \lambda > 0 \\ \text{undefined} & \text{if } \lambda = 0 \\ -1 & \text{if } \lambda < 0 \end{cases} \tag{2.16}$$

or, in another notation

$$u^*(t) = \text{bang}[-1,1;\lambda] \tag{2.17}$$

To find λ we write the adjoint equation

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 1, \quad \lambda(1) = S_x[x(T)] = 0 \tag{2.18}$$

Because this equation does not involve x and u we can easily solve it as

$$\lambda(t) = t-1. \tag{2.19}$$

It follows that $\lambda(t) = t-1 \leq 0$ for all $t \in [0,1]$ and since we can set $u^*(1) = -1$ (which defines u at the single point $t=1$), we have the optimal control

$$u^*(t) = -1 \quad \text{for } t \in [0,1].$$

Substituting this into the state equation (2.13) we have

$$\dot{x} = -1, \quad x(0) = 1 \tag{2.20}$$

whose solution is

$$x^*(t) = 1-t \quad \text{for } t \in [0,1]. \tag{2.21}$$

The graphs of the optimal state and adjoint trajectories appear in figure 2.1 Note that the optimum value of the objective function is $J^* = -\frac{1}{2}$.

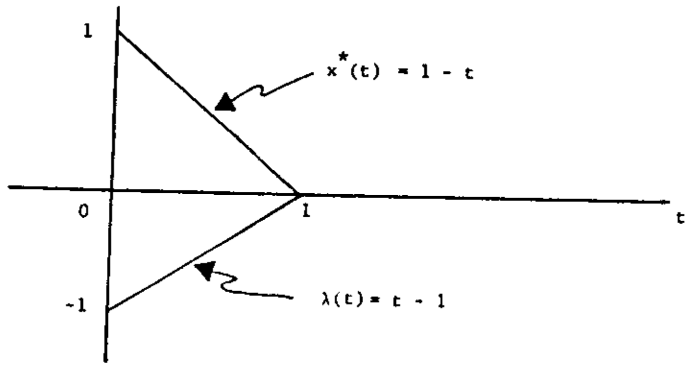


Fig. 2.1: Optimal state and costate trajectories of example 2.1

Example 2.2. Let us solve the same problem as in example 2.1 over the interval [0,2] so that the objective function is

$$\text{Maximize } \{J = \int_0^2 (-x)dt\} \tag{2.22}$$

The constraints are (2.13) and (2.14) as before. Here we want to minimize the signed area between the horizontal axis and the trajectory of $x(t)$ for $0 \leq t \leq 2$.

Solution. The Hamiltonian is (2.15) and the optimum control is (2.17) as before. The adjoint equation is

$$\dot{\lambda} = 1, \quad \lambda(2) = 0 \tag{2.23}$$

is same as (2.18) except $T = 2$ instead of $T = 1$. The solution of (2.23) is easily found to be

$$\lambda(t) = t-2 \text{ for } t\epsilon[0,2]. \tag{2.24}$$

Hence the state equation (2.20) and its solution (2.21) are exactly the same. The graphs of the optimal state and adjoint trajectories appear in figure 2.2. Note that the optimal value of the objective function here is $J^* = 0$.

Example 2.3. The next example has objective function

$$\text{Maximize } \{J = \int_0^1 (-\tfrac{1}{2}x^2)dt\} \tag{2.25}$$

subject to the same constraints as in example 2.1, namely

$$\dot{x} = u, \quad x(0) = 1 \tag{2.26}$$

$$u \in \Omega = [-1,1]$$

Here $F = -(1/2)x^2$ so that the interpretation of the objective function (2.25) is that we are trying to find the trajectory $x(t)$ so that the area under the curve $(1/2)x^2$ is minimized.

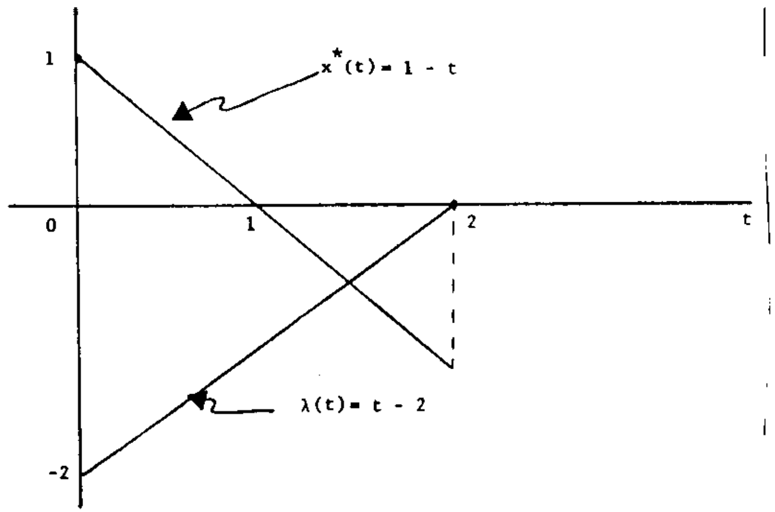


Fig. 2.2: Optimal paths of example 2.2

Solution. The Hamiltonian is

$$H = -\frac{1}{2}x^2 + \lambda u \tag{2.27}$$

which is linear in u so that the optimal policy is

$$u^*(t) = \text{bang } [-1,1;\lambda]. \tag{2.28}$$

The adjoint equation is

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = x, \quad \lambda(1) = 0 \tag{2.29}$$

Here the adjoint equation involves x so that we cannot solve it initially. Because the state equation (2.26) involves u, which depends on λ, we can't integrate it either!

The way out of this dilemma is to use some intuition. Since we want to minimize the area under $(1/2)x^2$, and since $x(0) = 1$, it is clear that we want to decrease x as quickly as possible. Let us therefore temporarily *assume* that λ is non-positive in the interval $[0,1]$ so that from (2.28) we have $u = -1$ through out the interval. With this assumption we can solve (2.26) as

$$x(t) = 1-t \tag{2.30}$$

Substituting this into (2.29) gives

$$\dot{\lambda} = 1-t$$

Integrating both sides of this equation from t to 1 gives

$$\int_t^1 \dot{\lambda}(\tau)d\tau = \int_t^1 (1-\tau)d\tau$$

$$\lambda(1) - \lambda(t) = (\tau - \frac{1}{2}\tau^2)\Big|_t^1$$

which, using $\lambda(1) = 0$, yields

$$\lambda(t) = -\frac{1}{2}t^2 + t - \frac{1}{2} \tag{2.31}$$

The reader may now verify that $\lambda(t)$ is non-positive in the interval $[0,1]$, verifying our original assumption. Hence (2.30) and (2.31) satisfy the necessary conditions. Finally you are asked to show that they satisfy sufficient conditions so that they are, in fact, optimal.

Figure 2.3 shows the graphs of the optimal trajectories.

Example 2.4. Let us rework Example 2.3 with $T = 2$, that is, with objective function

$$\text{Maximize } \{J = \int_0^2 (-\frac{1}{2}x^2)dt\} \tag{2.32}$$

with constraints (2.26).

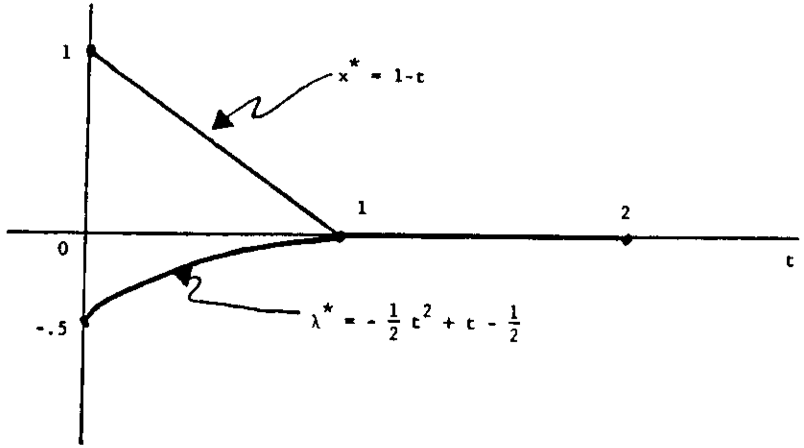


Fig. 2.3: Optimal trajectories of example 2.3

Solution. The Hamiltonian is still (2.27) and the form of the optimal policy remains (2.28). The adjoint equation is

$$\dot{\lambda} = x, \quad \lambda(2) = 0$$

which is the same as (2.29) except $T = 2$ instead of 1. Let us try to extend the solution of the previous example. Note from (2.31) that $\lambda(1) = 0$. If we recall from the definition of the bang function that bang $[-1,1;0]$ is not defined, so that we can choose u in (2.28) arbitrarily when $\lambda = 0$. This is an instance of singular control. Let us see if we can maintain singular control. To do this we choose $u = 0$ when $\lambda = 0$. Since $\lambda(1) = 0$ we set $u(1) = 0$ so that from (2.26) $\dot{x}(1) = 0$. Now note that if we set $u(t) = 0$ for $t > 1$ then by integrating these equations forward from 1 to 2 we see that $x(t) = 0$ and $\lambda(t) = 0$ for $1 \leq t \leq 2$; in other words $u(t) = 0$ maintains singular control in the interval. Intuitively, this is the correct answer since once we get $x = 0$ we should keep it at 0 in order to maximize the objective function J in (2.32).

In figure 2.3 we can get the singular solution by extending the graphs shown to the right making $\dot{x}^*(t) = 0$, $u^*(t) = 0$ for $1 \leq t \leq 2$.

Example 2.5. Our last example is slightly more complicated and the optimal control is not bang-bang. The problem is

$$\text{Maximize } \{J = \int_0^2 (2x - 3u - u^2)dt \tag{2.33}$$

subject to

$$\dot{x} = x + u, \quad x(0) = 5 \tag{2.34}$$

and the control constraint

$$u \in \Omega = [0,2] \tag{2.35}$$

Solution. Here $T = 2$, $F = 2x - 3u - u^2$, $S = 0$ and $f = x + u$. The Hamiltonian is

$$\begin{aligned} H &= (2x - 3u - u^2) + \lambda(x + u) \\ &= (2 + \lambda)x - (u^2 + 3u - \lambda u) \end{aligned} \tag{2.36}$$

Let us find the optimal control policy by differentiating (2.36) with respect to u;

$$\frac{\partial H}{\partial u} = -2u - 3 + \lambda = 0$$

so that the form of the optimal control is

$$u(t) = \frac{\lambda(t) - 3}{2} \tag{2.37}$$

provided this expression stays with in the limits of $\Omega = [0,2]$. (Note that the second derivative of H with respect to u is $\partial^2 H/\partial u^2 = -2 < 0$ so that (2.37) satisfies the second order condition for the maximum of a function.

We next derive the adjoint equation as

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2 - \lambda, \quad \lambda(2) = 0 \tag{2.38}$$

which can be rewritten as

$$\dot{\lambda} + \lambda = -2, \quad \lambda(2) = 0.$$

This solution of this linear differential equation

$$\lambda(t) = 2\left(e^{2-t} - 1\right). \tag{2.39}$$

If we substitute this into (2.37) and impose the Ω constraints, we see that the optimal control is

$$u^*(t) = \begin{cases} 2 & \text{if } e^{2-t} - 2.5 > 2 \\ e^{2-t} - 2.5 & \text{if } 0 \leq e^{2-t} - 2.5 \leq 2 \\ 0 & \text{if } e^{2-t} - 2.5 < 0 \end{cases}$$

The graph of u^* appears in figure 2.4.

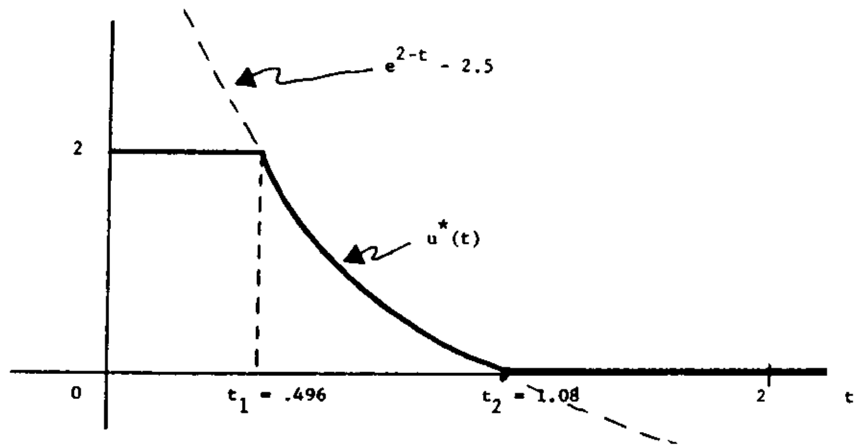


Fig. 2.4: Optimal control of example 2.5

In the figure t_1 is the solution of $e^{2-t} - 2.5 = 2$ which is $t_1 = .496$ while t_2 solves $e^{2-t} - 2.5 = 0$ which is $t_2 = 1.08$.

As exercise you will be asked to compute the optimal trajectory $x^*(t)$ by piecing together the solutions of three different differential equations.

2.3. A "Proof" by Dynamic Programming

We shall now derive the maximum principle by using a dynamic programming approach. The proof is intuitive in nature and is not intended to be mathematically rigorous.

2.3.1. The Hamilton-Jacobi-Bellman Equation

Suppose $V(x,t): \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^1$ is a function whose value is the maximum value of the objective function of the control problem given that we start it at time t at state x. We initially assume that the *value function* V(x,t) exists for all x and t in the relevant ranges; later we will make additional assumptions about V.

Bellman (1957) in his book on dynamic programming states the *principle of optimality* as follows:

> "An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the outcome resulting from the first decision."

Intuitively this principle is obvious, for if we were to start in state x at time t and did not follow an optimal path from thereon, then there would exist (by assumption) a better path from t to T, hence we could improve by following the better path from time t on. We will use the principle of optimality to derive conditions on the value of function V(x,t).

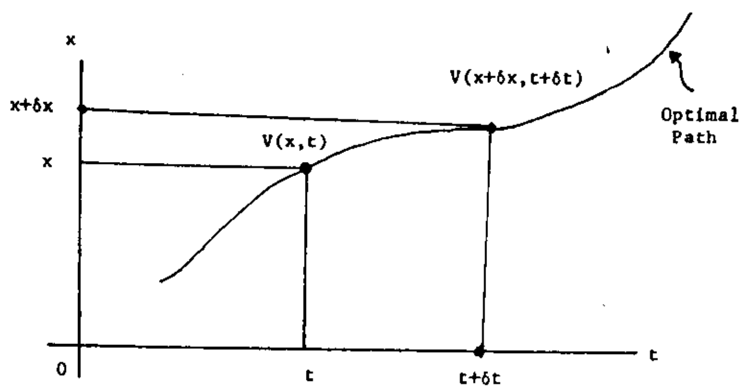


Fig. 2.5: Illustration of the principle of optimality

Figure 2.5 is a schematic picture of the optimal path in state-time space, and two nearby points (x,t) and (x+h,t+h) where h is a small increment of time. The value function changes from V(x,t) to V(x+h,t+h) between these two points. By the principle of optimality, the change in the objective function is made up of two parts: first, the incremental change in J from t to t+h which is given by the integral of F(x,u,t) from t to t+h; second, the value function V(x+h,t+h) at time t+h). The current control action should be chosen within constraints Ω to maximize the sum of these two terms. In equation form this is

$$V(x,t) = \max_{\substack{u(\tau)\in\Omega \\ \tau\in[t,t+h]}} \left\{ \int_t^{t+h} F[x(\tau),u(\tau),\tau]d\tau + V[x(t+h),t+h] \right\} \qquad (2.40)$$

where h represents a small increment in t. Note that we assume for simplicity r = 0.

Since F is continuous the integral in (2.40) is approximately F(x,u,t)h so that we can rewrite (2.40) as

$$V(x,t) = \max_{u\in\Omega} \{F(x,u,t)h + V[x(t+h),t+h]\} + o(h) \qquad (2.41)$$

where o(h) denotes a collection of higher order terms in h. (By definition o(h) is a function such that $\lim_{h\to 0} \frac{o(h)}{h} = 0$.)

We now make an assumption of which we will talk more later. We assume that the return function or payoff function V is a continuously differentiable function of its arguments. This allows us to use the Taylor's series expansion of V with respect to h to obtain

$$V[x(t+h),t+h] = V(x,t) + [V_x(x,t)\dot{x} + V_t(x,t)]h + o(h). \qquad (2.42)$$

where V_x and V_t are partial derivatives of $V(x,t)$ with respect to x and t, respectively.

Substituting for $\dot{x}$ from (2.1) in the above equation and then using it in (2.41), we obtain

$$V(x,t) = \max_{u\epsilon\Omega} \;[F(x,u,t)h + V(x,t) + V_x(x,t)f(x,u,t)h + V_t(x,t)h] + o(h). \tag{2.43}$$

Cancelling $V(x,t)$ on both sides and then dividing by h we get

$$0 = \max_{u\epsilon\Omega} \;[F(x,u,t) + V_x(x,t)f(x,u,t) + V_t(x,t)] + \frac{o(h)}{h} \tag{2.44}$$

Now we let $h \to 0$ and obtain the following equation

$$0 = \max_{u\epsilon\Omega} \;[F(x,u,t) + V_x(x,t)f(x,u,t) + V_t(x,t)] \tag{2.45}$$

with the boundary condition

$$V(x(T),T) = S(x(T),T). \tag{2.46}$$

This boundary condition is obvious from the fact that if we are at $t = T$ then the value function is simply the salvage value function.

Note that the components of the vector $V_x(x,t)$ can be interpreted as the marginal contributions of the state variables x to the objective function being maximized. We denote this marginal return vector by an adjoint (row) vector $\lambda(t)$, E^n, i.e.,

$$\lambda(t) = V_x(x,t). \tag{2.47}$$

From the preceding remark, we can also interpret λ to be the per unit change in the objective function for small changes in x. Next we introduce the Hamiltonian

$$H(x,u,V_x,t) = F(x,u,t) + V_x(x,t)f(x,u,t) \tag{2.48}$$

or, simply

$$H(x,u,\lambda,t) = F + \lambda f. \tag{2.49}$$

we can rewrite equation (2.45) as the following equation, called the *Hamilton-Jacobi-Bellman equation*

$$0 = \max_{u\epsilon\Omega} \;(H + V_t)\;. \tag{2.50}$$

Note that it is possible to take V_t out of the maximizing operation since it does not depend on u.

The Hamiltonian maximizing condition of the maximum principle can be obtained from (2.50) by observing that if $x^*(t)$ and $\lambda(t)$ are optimal values of state and adjoint variables at time t, then the optimal control $u^*(t)$ must satisfy (2.50), i.e., for all $u(t)\epsilon\Omega$,

$$H[x^*(t),u^*(t),\lambda(t),t] + V_t(x^*(t),t) \geq H[x^*(t),u(t),\lambda(t),t] + V_t(x^*(t),t). \tag{2.51}$$

Cancelling the term V_t on both sides, we obtain

$$H[x^*(t),u^*(t),\lambda(t),t] \geq H[x^*(t),u(t),\lambda(t),t] \tag{2.52}$$

for all $u(t)\epsilon\Omega$.

In order to complete the statement of the maximum principle we must still obtain the adjoint equations.

2.3.2. Derivation of the Adjoint Equation

The derivation of the adjoint equation proceeds from the Hamilton-Jacobi-Bellman equation (2.50). Note that, given the optimal path x^*, the optimal control u^* maximizes the right side of (2.50), and its maximum value is zero. We now consider small perturbations of the values of the state variables in a neighborhood of the optimal path x^*. Thus, let

$$x(t) = x^*(t) + \delta x(t), \quad (2.53)$$

where $\|\delta x(t)\| < \epsilon$, be such a perturbation.

We now consider a 'fixed' time instant t. We can then write (2.50) as

$$H[x^*(t),u^*(t),V_x(x^*(t),t),t] + V_t(x^*(t),t) \geq H[x(t),u^*(t),V_x(x(t),t),t] + V_t(x(t),t). \quad (2.54)$$

To explain, we note that the left hand side of (2.54) equals zero. The right hand side can attain the value zero only if $u^*(t)$ is also an optimal control for $x(t)$. In general for $x(t) \neq x^*(t)$, i.e., $\delta x(t) \neq 0$, this will not be so. From this observation, it follows that the expression on the right side of (2.54) attains its maximum (of zero) at $x(t) = x^*(t)$. Furthermore, $x(t)$ is not explicitly constrained. In other words $x^*(t)$ is an unconstrained local maximum of the right hand side of (2.54), so that the derivative of this expression with respect to x must vanish at $x^*(t)$; i.e.,

$$H_x[x(t),u^*(t),V_x(x(t),t),t] + V_{tx}(x(t),t) = 0 \quad (2.55)$$

In order to take the derivative as we did in (2.55) we must further assume that V is a *twice differentiable* function of its arguments. Using the definition of the Hamiltonian in (2.48) we obtain

$$F_x + V_x f_x + (V_{xx} f)' + V_{tx} = 0 \quad (2.56)$$

where the prime denotes the transpose of a vector.

The derivation of the necessary condition (2.56) is crux of the reasoning in the derivation of the adjoint equation. It is easy to obtain the so called adjoint equation from it. We begin by taking the time derivative of $V_x(x,t)$; thus

$$\frac{dV_x}{dt} = (V_{xx}\,\dot{x})' + V_{xt} = (V_{xx}\,f)' + V_{tx} \quad (2.57)$$

Since the terms on the right of (2.57) are the same as the last two terms in (2.56), we see that (2.57) becomes

$$\frac{dV_x}{dt} = -F_x - V_x f_x \quad (2.58)$$

Because λ was defined in (2.15) to be V_x, we can rewrite (2.58) as

$$\dot{\lambda} = -F_x - \lambda f_x.$$

To see that the right hand side of this equation can be written simply as $-H_x$, we need to go back to the definition of H in (2.49) and recognize that when taking the partial derivative of H with respect to x, the adjoint variables λ are considered to be independent of x. We note further that along the optimal path, λ is a function of t only. Thus,

$$\dot{\lambda} = -H_x \quad (2.59)$$

(note that we have assumed for simplicity that $r = 0$). Also, from the definition of λ in (2.47) and the boundary condition (2.46), we have the terminal boundary condition, which is also called the transversality condition:

$$\lambda(T) = \left.\frac{\partial S(x,t)}{\partial x}\right|_{x=x(T)} = S_x(x(T),T). \quad (2.60)$$

The adjoint equation (2.59) together with its boundary condition (2.60) determine the adjoint variables.

2.4. Economic Interpretations of the Maximum Principle

Recall that the objective function (2.3) is

$$J = \int_0^T e^{-rt} F(x,u,t)dt + S(x(T),T)$$

where r is the discount rate, F will be considered to be the instantanecus profit rate measured in dollars per unit of time, and $S(x(T),T)$ is the salvage value in dollars of the firm at time T when the terminal state is $x(T)$. For purposes of discussion it will be convenient to consider the (one-dimensional) state $x(t)$ as the stock of capital at time t.

In (2.47) we interpreted $\lambda(t)$ to be the per unit change in the value function $V(x,t)$ for small changes in capital stock x. In other words $\lambda(t)$ is the marginal value per unit capital at time t, and it is also referred to as the "price" or "shadow price" of a unit of capital. In particular, the value of $\lambda(0)$ is the marginal rate of change of the maximum value of J, (the objective function) with respect to the change in the initial capital stock, x_0.

The interpretation of the Hamiltonian function in (2.49) can now be derived. Multiplying (2.49) by dt gives

$$\begin{aligned} Hdt &= Fdt + \lambda f dt \\ &= Fdt + \lambda \dot{x} dt \\ &= Fdt + \lambda dx \end{aligned}$$

where we made use of the state equation (2.1b). The first term $F(x,u,t)dt$ represents the direct contribution to J in dollars from time t to t + dt if we are in state x and we apply control u. The differential $dx = f(x,u,t)dt$ represents the change in capital stock from t to t + dt,

when we are in state x and control u is applied. Therefore, the second term λdx represents the value in dollars of the incremental capital stock, dx, and hence can be considered as the *indirect contribution* to J in dollars. Thus Hdt can be interpreted as the *total contribution* to J from the interval t to t + dt when $x(t) = x$ and $u(t) = u$.

With this interpretation of the Hamiltonian it is easy to see why the Hamiltonian must be maximized at each time t. If we were just to maximize F at each time t, we would not be maximizing J, because we would ignore the effect of control in changing the capital stock, which give rise to indirect contributions to J. The maximimum principle derives prices, the adjoint variables λ, in such a way that $\lambda(t)dx$ is the correct valuation for the indirect contribution from the interval t to t + dt. As a consequence the Hamiltonian maximizing problem can be treated as a static problem at each instant t. In other words the maximum principle "decouples" the dynamic maximization problem (2.1) in the interval [0,T] into a series of static maximization problems at each instant t in [0,T]. Therefore the Hamiltonian can be interpreted as a surrogate profit rate to be maximized at each time t.

The value of λ to be used in the maximum principle is given by (2.59) and (2.60), i.e.

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial x} = -\frac{\partial F}{\partial x} - \lambda\left(r + \frac{\partial f}{\partial x}\right); \quad \lambda(T) = S_x(x(T),T).$$

Rewriting the equation as

$$r(\lambda dt) - d\lambda = H_x dt = (F_x dt) + \lambda(f_x dt)$$

we can observe that along the optimal path the interest rate of a capital unit evaluated by the shadow price λ plus the decrease in the price of capital from t to t + dt, which can be considered as the *marginal cost of holding* *that capital*, equals the *marginal revenue of investing the capital* as is evident from the interpretation of the Hamiltonian given above. The marginal revenue, $H_x dt$, consists of the sum of direct marginal contribution, $F_x dt$, and the indirect marginal contribution, $\lambda f_x dt$. Thus the adjoint equations become the familiar economic equilibrium relation: *marginal cost equals marginal revenue*.

Further insight can be obtained for $r = 0$ by integrating the above adjoint equation from t to T as follows:

$$\lambda(t) = \lambda(T) + \int_t^T H_x(x(\tau),u(\tau),\tau)d\tau$$

$$= S_x(x(T),T) + \int_t^T H_x d\tau \qquad (2.61)$$

Note that the price $\lambda(T)$ of a unit of capital at time T is its marginal salvage value, $S_x(x(T),T)$. The price $\lambda(t)$ of a unit of capital at time t is the sum of its terminal price, $\lambda(T)$, plus the integral of the marginal surrogate profit rate, H_x, from t to T.

The above interpretations show that the adjoint variables behave in much the same way as dual variables in linear (and nonlinear) programming. The differences being that here the adjoint variables are time dependent and satisfy derived differential equations.

2.5. Sufficient Conditions

We next prove a theorem that gives conditions under which the maximum principle conditions are sufficient (as well as necessary). This theorem is important from our point of view since the models derived from many management science applications will satisfy conditions required for the sufficiency result. As remarked earlier, our technique for proving existence will be to display for any given model a solution that satisfies both necessary and sufficient conditions.

We first define a function $H^0: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ called the *maximized Hamiltonian* as follows:

$$H^0(x,\lambda,t) = \underset{u\in\Omega}{\text{Maximum}}\ H(x,u,\lambda,t) \qquad (2.62)$$

We assume that from this equation a function $u = u^*(x,\lambda,t)$ is implicitly and uniquely defined. Given these assumptions we have by definition,

$$H^0(x,\lambda,t) = H(x,u^*,\lambda,t). \qquad (2.63)$$

It is also possible to show that

$$H^0_x(x,\lambda,t) = H_x(x,u^*,\lambda,t) \qquad (2.64)$$

for all $u\in\Omega$. To see this for the case of differentiable u^*, let us differentiate (2.63) with respect to x:

$$H^0_x(x,\lambda,t) = H_x(x,u^*,\lambda,t) + H_u(x,u^*,\lambda,t)\frac{\partial u^*}{\partial x}. \qquad (2.65)$$

Let us look at the second term on the right hand side of (2.65). We must show

$$H_u(x,u^*,\lambda,t)\frac{\partial u^*}{\partial x} = 0$$

for all x. There are two cases to consider: (i) Case 1; the unconstrained global maximum of H occurs in the interior or on the boundary of $\Omega(t)$. Here $H_u(x,u^*,\lambda,t)=0$. (ii) Case 2; the unconstrained global maximum of H occurs outside of Ω. Here $\frac{\partial u^*}{\partial x} = 0$ because changing x does not influence the optimum value of u. Thus (2.66) and therefore (2.64) hold.

We have shown the result in (2.64) for cases where u^0 is a differentiable function of x. It holds more generally provided Ω is appropriately qualified.

Theorem 2.2 (Sufficient optimality conditions for the standard problem). *Let $u^*(t)$ and the corresponding $x^*(t)$, $\lambda(t)$ satisfy the maximum principle necessary condition (2.4-7) for all $t \in [0,T]$; then u^* is an optimal control if $H^0(x,\lambda,t)$ is concave in x for each λ, t, and S(x,T) is concave in x*

Proof. By definition

$$H(x(t),u(t),\lambda(t),t) \leq H^0(x(t),\ \lambda(t),t). \tag{2.67}$$

From concavity of H^0,

$$H^0(x(t),\ \lambda(t),t) \leq H^0(x^*(t),\ \lambda(t),t) + H^0_x(x^*(t),\ \lambda(t),t)(x(t) - x^*(t)); \tag{2.68}$$

thus from (2.63), (2.67) and (2.68)

$$H(x(t),u(t),\lambda(t),t) \leq H(x^*(t),u^*(t),\lambda(t),t) + H^0_x(x^*(t),\lambda(t),t)(x(t) - x^*(t)). \tag{2.69}$$

By definition of H in (2.49) and the adjoint equation

$$F(x(t),u(t),t) + \lambda(t)f(x(t),u(t),t) \leq F(x^*(t),u^*(t),t)$$

$$+ \lambda(t)f(x^*(t),u^*(t),t) - \dot{\lambda}(t)(x(t)-x^*(t)). \tag{2.70}$$

Using the state equation, transposing and regrouping,

$$F(x^*(t),u^*(t),t) - F(x(t),u(t),t) \geq \dot{\lambda}(t)(x(t)-x^*(t))$$

$$+ \lambda(t)(\dot{x}(t) - \dot{x}^*(t)). \tag{2.71}$$

Furthermore, since S(x(T),T) is a concave function, we have

$$S(x(T),T) \leq S(x^*(T),T) + S_x(x^*(T),T)(x(T) - x^*(T)), \tag{2.72}$$

or

$$S(x^*(T),T) - S(x(T),T) + S_x(x^*(T),T)[x(T)-x^*(T)] \geq 0. \tag{2.73}$$

Integrating both sides of (2.71) from 0 to T and adding (2.73), we have

$$J(u^*) - J(u) + S_x[x^*(T),T][x(T)-x^*(T)]$$

$$\geq \lambda(T)[x(T) - x^*(T)] - \lambda(0)[x(0)-x^*(0)]. \tag{2.74}$$

where J(u) is the value of the objective function associated with a control u. Since $x^*(0) = x(0) = x_0$ (the initial condition) and since $\lambda(T) = S_x(x^*(T),T)$ we have

$$J(u^*) \geq J(u), \tag{2.75}$$

thus, u^* is an optimal control. This completes the proof of the theorem. □

The following conditions are easiear to check than concavity of H^0. If the Hamiltonian H is concave in (x,u) or if F and f are concave in (x,u) and if $\lambda \geq 0$, then the maximized Hamiltonian H^0 is concave in x so that the sufficiency conditions of theorem 2.2 holds.

Example 2.6. Let us show that the problem in Examples 2.1 and 2.2 satisfy the sufficient conditions. We have from (2.15) and (2.62)

$$H^0 = - x + \lambda u^0$$

where u^0 is given by (2.16). Since u^0 is a function of λ only, H^0 is certainly concave in x for each t and λ (and in particular for $\lambda = \lambda$). Since $S(x,T) = 0$ the sufficient conditions hold.

Remark 2.1. It can be shown that the concavity of the maximized Hamiltonian H^O follows from the *negative semidefiniteness* of the matrix

$$D^2H = \frac{\partial^2 H}{\partial (x,u)^2} = \begin{pmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{pmatrix}$$

for all (x,u,λ) with $H_u = 0$.

If D^2H is *globally* negative semidefinite then this is equivalent with the *concavity* of H in (x,u).

For $\lambda \geq 0$ the concavity of H in (x,u) follows from those of F and f.

Note that the negative semidefiniteness can be checked by the criterion of principal minors.

2.6. Terminal Conditions

In our standard optimal control problem no condition for the terminal state x(T) is imposed (free end state). In some applications (especially in the natural sciences) the terminal state must lie in a target set. Thus, we modify the optimal control problem (2.1) by adding the following terminal state condition:

$$x_j(T) \text{ free} \quad \text{for } j = 1, \ldots, n' \qquad (2.76a)$$

$$x_j(T) = x_j^T \quad \text{for } j = n' + 1, \ldots, n'' \qquad (2.76b)$$

$$x_j(T) \geq x_j^T \quad \text{for } j = n'' + 1, \ldots, n. \qquad (2.76c)$$

To state the necessary optimality conditions for the extended optimal control problem (2.1, 76) we have to define a (new) Hamiltonian as

$$H(x,u,\lambda,\lambda_o,t) = \lambda_o F(x,u,t) + \lambda f(x,u,t),$$

where λ_o is a nonnegative constant. Note that the Hamiltonian (2.2) of the standard problem (2.1) arises a special case by setting $\lambda_o = 1$ ($\lambda_o = 0$ leads to a contraction; see (2.77)).

The case $\lambda_o = 0$ is denoted as *abnormal* case. Note that for $\lambda_o \neq 0$ the Hamiltonian case be normed such that $\lambda_o = 1$. This case is called *normal*. It turns out that if x(T) is not free, then the abnormal case cannot excluded. In principle,

for each (economic) problem should be checked whether $\lambda_o = 1$ or not. (If it turns out that $\lambda_o = 0$, then the problem is badly specified and a reformulation is necessary.)

Note that a control is now called *feasible*, if it is not only piecewise continuous and $u \in \Omega$, but it also generates by the state equation and the initial condition a terminal state satisfying (2.76).

Theorem 2.3 (Maximum principle for the standard problem enlarged by terminal constraints). Let u and x* be an optimal solution of the control problem (2.1, 76). Then there is a constant $\lambda_o \geq 0$ and a continuous and piecewise continuously differentiable costate function $\lambda(t) \in \mathbb{R}^n$, such that*

$$(\lambda_o, \lambda(t)) \neq 0 \quad \text{for each } t \in [0,T] \qquad (2.77)$$

and, besides of (2.7), the following necessary optimality conditions are valid for all points of continuity of u(t):*

$$H(x^*(t),u^*(t),\lambda_o,\lambda(t),t) = \max_{u \in \Omega} H(x^*(t),u,\lambda_o,\lambda(t),t) \qquad (2.78)$$

$$\dot{\lambda}(t) = r\lambda(t) - H_x(x^*(t),u^*(t),\lambda_o,\lambda(t),t). \qquad (2.79)$$

In extension to (2.6) the following transversality conditions hold true

$$\lambda_j(T) = \lambda_o S_{x_j}(x^*(T),T) \quad \text{for } j = 1, \ldots, n' \qquad (2.80a)$$

$$\lambda_j(T) \text{ free} \quad \text{for } j = n' + 1, \ldots, n'' \qquad (2.80b)$$

$$\lambda_j(T) \geq \lambda_o S_{x_j}(x^*(T),T) \quad \text{with} \qquad (2.80c)$$

$$[\lambda_j(T) - \lambda_o S_{x_j}(x^*(T),T)][x_j^*(T) - x_j^T] = 0 \quad \text{for } j = n'' + 1, \ldots, n.$$

Example 2.7. Solve the linear control problem

$$\max_u \int_0^4 (-x)\,dt \qquad (2.81a)$$

$$\dot{x} = u, \; x(0) = 1 \qquad (2.81b)$$

$$u \in \Omega = [-1,1] \qquad (2.81c)$$

alternatively for the following end conditions

(a) $x(4) = 1$, (b) $x(4) \geq 1$, (c) $x(4) \leq 1$, (d) $x(4)$ free. (2.82)

The Hamiltonian is

$$H = -\lambda_o x + \lambda_u.$$

The necessary optimality conditions are

$$u = \arg\max_{u\in[-1,1]} H(x,u,\lambda)$$

i.e.

$$u = \begin{Bmatrix} -1 \\ \text{undetermined} \\ 1 \end{Bmatrix} \text{ for } \lambda \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \qquad (2.83)$$

$$\dot\lambda = -H_x = \lambda_o. \qquad (2.84)$$

(a) We first show that $\lambda_o = 1$. Assume on the contrary $\lambda_o = 0$. Then from (2.84) follows λ = const. For $\lambda > 0$ we obtain from (2.83) $u = 1$ for all t.

This together with (2.81b) yields $x(4) = 5$ which contradicts (2.82a). For $\lambda = 0$ we get a contradiction to the non-degeneracy assumption (2.77). Finally, from $\lambda < 0$ we conclude $x(4) = -3$. Hence, $\lambda_o = 1$. In case (a) from the transversality condition (2.80b) we get no information on $\lambda(4)$ ($\lambda(4)$ is free).

From the adjoint equation $\dot\lambda = 1$ follows

$$\lambda(t) = t + c, \qquad (2.85)$$

where the constant $c\in\mathbb{R}$ is still open. For $c \le -4$ (2.85) implies $\lambda(t) < 0$ for $t\in[0,4)$, i.e. $u^* = -1$. Thus, from (2.81b) we get according to (2.83), $x(4) = -3$ contradicting (2.82a).

Analogously, for $c \ge 0$ we obtain a contradiction, since $x(4) = 5$.

For the remaining values c (-4,0) (2.85) and (2.83) yield

$$u = \begin{cases} -1 & \text{for} \quad 0 \le t < |c| \\ +1 & \phantom{\text{for}} \quad |c| \le t \le 4. \end{cases}$$

This implies $x(4) = 5 + 2c$. Then from (2.82a) follows $c = -2$. This procedure determines λ and u according to (2.85) and (2.83), respectively, and by (2.81b) the state path x:

time interval	4	x	λ
[0,2]	-1	1-t	t-2
[2,4]	1	t-3	t-2

For the following cases it also holds that $\lambda_o = 1$.

(b) From (2.80c) we obtain

$$\lambda(4) \ge 0,\ x(4) \ge 1,\ \lambda(4)[x(4) - 1] = 0. \qquad (2.86)$$

Suppose that $x(4) > 1$. Then from the complementary slackness condition $\lambda(4) = 0$. This together with (2.84), i.e. $\dot\lambda = 1$, yields $\lambda(t) < 0$ for $t\in[0,4)$. Then from (2.83) follows $u = -1$ and (as above) $x(4) = -3$ which contradicts (2.82b). Thus, $x(4) = 1$, and the solution described in (a) is optimal.

(c) $\lambda(4) \le 0$, $1 - x(4) \ge 0$, $\lambda(4)[1 - x(4)] = 0$. If $x(4) < 1$, then $\lambda(4) = 0$. Thus, $\dot\lambda = 1$, $\lambda < 0$, $u = -1$, $x = 1 - t$, and $x(4) = -3$. Hence, $\lambda(4) = 0$. The resulting optimal solution is

time interval	4	x	λ
[0,4]	-1	1-t	t-4

2.7. Infinite Horizon and Stationarity

In many economic problems there is no finite planning interval. Thus, we consider the control problem (2.1) for $T = \infty$. It is assumed that the improper integral converge for any feasible solution.

In many cases it turns out that the optimal solution of an autonomous infinite-time control problem $(x^*(t),u^*(t))$ converges for $t\to\infty$ to a stationary point $(\hat x,\hat u)$. For finite but "long" time horizons T a similar observation can often be made: the optimal solution lies "most of the time" "near" to the equilibrium.

This equilibrium is defined by the requirement that all motion ceases, i.e. $\dot x = \dot\lambda = 0$. Thus, $(\hat x,\hat u,\hat\lambda)$ satisfies the following conditions

$$\left.\begin{aligned} & f(\hat x,\hat u) = 0 \\ & r\hat\lambda - H_x(\hat x,\hat u,\hat\lambda) = 0 \\ & H(\hat x,\hat u,\hat\lambda) \ge H(\hat x,u,\hat\lambda) \quad \text{for all } u\in\Omega. \end{aligned}\right\} \qquad (2.87)$$

Clearly, if the initial condition $x_o = \hat{x}$, then the optimal control $u^*(t) = \hat{u}$ for all t.

To state the necessary optimality conditions for infinite-time control problems we need the Hamiltonian

$$H = \lambda_o F(x,u,t) + \lambda f(x,u,t) \tag{2.88}$$

(see also section 2.6).

<u>*Theorem 2.4 (Maximum principle for infinite time problems).*</u> *Necessary for the optimality of a feasible pair (x^*,u^*) is the existence of a constant $\lambda_o \geq 0$ and a continuous costate function $\lambda(t)$ such that*

$$(\lambda_o, \lambda(t)) \neq 0 \quad \text{for each} \quad t \in [0,\infty).$$

Moreover, with the Hamiltonian (2.88) the optimality conditions (2.4, 5) of theorem 2.1 are satisfied.

It is remarkable that for $T = \infty$ there is generally *no* transversality condition among the *necessary* optimality conditions.

The following example shows that the "limiting transversality" condition

$$\lim_{t\to\infty} e^{-rt}\lambda(t) = 0 \tag{2.89}$$

representing the pendant to (2.6) is *no* necessary optimality condition.

<u>Example 2.8.</u>

$$\max \{J = \int_0^\infty (1-x)u\,dt\}$$

s.t. $\dot{x} = (1-x)u,\ x(0) = 0;\ 0 \leq u \leq 1.$

Since $F = f$ we have

$$J = \int_0^\infty \dot{x}\,dt = \lim_{t\to\infty} x(t) - x(0) = x(\infty).$$

Therefore each feasible solution maximizing $x(\infty)$, i.e. for which $x(\infty) = 1$, is optimal. Thus, e.g.,

$$u^* = 1/2 \quad \text{for} \quad t \in [0,\infty) \tag{2.90}$$

is an optimal solution. It holds that $0 < x^*(t) < 1$ for $t \in (0,\infty)$, and $x(\infty) = 1$.

The Hamiltonian (2.88) is given as

$$H = (\lambda_o + \lambda)(1-x)u.$$

For

$$H_u = (\lambda_o + \lambda)(1-x)$$

follows

$$u = \begin{Bmatrix} 0 \\ \text{determined} \\ 1 \end{Bmatrix} \quad \text{for} \quad H_u \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0.$$

For u^* in (2.90) we have $H_u = 0$ for all t, and hence

$$\lambda(t) = -\lambda_o \quad \text{for} \quad t \in [0,\infty). \tag{2.91}$$

Since $(\lambda_o, \lambda) \neq 0$ the costate $\lambda(t)$ is a *negative* constant, and the transversality condition (2.89) is not valid (note that $r = 0$).

The present example illustrates the necessity of considering $\lambda_o \geq 0$.

<u>Example 2.9.</u>

$$\max \{J = \int_0^\infty (u-x)\,dt\}$$

s.t. $\dot{x} = u^2 + x,\ x(0) = 0;\ 0 \leq u \leq 1.$

If $u \neq 0$ in an interval of positive length, $x(t)$ diverges exponentially to $+\infty$. Since $u \leq 1$, we then have $J = -\infty$. Thus, the optimal solution is clearly

$$u^*(t) = 0 \quad \text{for} \quad t \in [0,\infty).$$

The necessary optimality conditions are

$$H = \lambda_o(u-x) + \lambda(u^2+x)$$

$$u = \begin{Bmatrix} 0 \\ \text{undetermined} \\ 1 \end{Bmatrix} \quad \text{for} \quad H_u = \lambda_o + 2\lambda u \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \tag{2.92}$$

$$\dot{\lambda} = \lambda_o - \lambda.$$

Assume that $\lambda_o > 0$. From $u^* = 0$ and (2.92) follows $u^* = 1$ which is a contradiction. Thus, the abnormal case $\lambda_o = 0$ prevails.

The following theorem shows that a concavity assumption of the (maximized) Hamiltonian with $\lambda_o = 1$ together with a limiting transversality condition provide a set of sufficient optimality conditions.

Theorem 2.5 (Sufficient optimality conditions for infinite time problems). *Let $u^*(t)$ be a feasible solution and $x^*(t)$ the corresponding state path of the optimal control problem (2.1) with $T = \infty$. Assume that there are functions $\lambda(t) \in \mathbb{R}^n$ such that additionally to (2.4, 5) the limiting transversality condition*

$$\lim_{t\to\infty} e^{-rt}\lambda(t)[x(t) - x^*(t)] \geq 0 \qquad (2.93)$$

is satisfied for any feasible trajectory $x(t)$. Then $u^*(t)$ is an optimal solution provided that $H^0(x,\lambda,t)$ is concave in x for all $(t,\lambda(t))$.

The limiting transversality condition is satisfied if

$$\left.\begin{array}{l}\text{each feasible path } x(t) \text{ is bounded or nonnegative,}\\ x^*(t) \text{ is bounded,}\\ \lambda(t) \text{ is nonnegative and satisfies (2.89).}\end{array}\right\} \qquad (2.94)$$

Note that (2.89) can replaced by the requirement that $\lambda(t)$ is bounded. If any feasible trajectory is bounded then each solution which converges to an equilibrium satisfies the limiting transversality condition (2.93). This property is used in section 4.

2.8. Free Terminal Time

Some optimal control problems require a simulataneous determination of optimal control and terminal time. This problems are usually termed free terminal time problems. Such problems occur most frequently in the area of machine maintenance and replacement, forest management, simulataneous optimization of training and retirement age, etc. The necessary optimality conditions in these problems include an *additional transversality condition* which is expressed in terms of the Hamiltonian and the terminal salvage value function.

The free terminal time optimal control problem is

$$\max_{u,T} \int_0^T e^{-rt}F(x,u,t)dt + e^{-rt}S(x(T),T) \qquad (2.95a)$$

subject to

$$\dot{x} = f(x,u,t), \; x(0) = x_o \qquad (2.95b)$$

$$u \in \Omega. \qquad (2.95c)$$

The following theorem states the necessary conditions. Here and in section 5 we use the following notation

$$H^*[T^*] = H(x^*(T^*),u^*(T^*),\lambda(T^*),T^*) \qquad (2.96a)$$

$$S^*[T^*] = S(x^*(T^*),T^*). \qquad (2.96b)$$

Theorem 2.6. Let u^* and x^* be an optimal solution of the control problem (2.94) in the optimal time interval $[0,T^*]$. Then there exists a continuous and piecewise continuously differentiable costate function $\lambda(t) \in \mathbb{R}^n$ such that with the Hamiltonian (2.2) the necessary optimality conditions (2.4-8) are valid. Moreover, the following additional transversality condition for T to be optimal terminal time can be stated as

$$H^*[T^*] = rS^*[T^*] + S_T^*[T^*]. \qquad (2.97)$$

We now provide an economic interpretation of (2.97). Consider for instance the machine maintenance problem. If T is the optimal sale date[1], then one is indifferent between keeping the machine for an additional dT units of time or selling it dT units of time earlier. The Hamiltonian represents the profit F plus the (current) value

$$\lambda(T)\dot{x}(T) = S_x(x(T),T)\dot{x}(T)$$

of the change in the state variable (quality of the machine). The term $S_T(x(T),T)$ denotes the change in the salvage price. The total of these two quantities equals the interest on the serage value. Thus, the marginal return from keeping the machine longer is equal to its opportunity cost.

Example 2.10. Consider the following resource extraction problem

$$\max_{u,T} \int_0^T e^{-rt}[p(t)u(t) - C(u(t),t)]dt \qquad (2.98a)$$

$$\dot{x}(t) = -u(t) \qquad (2.98b)$$

---

[1] To simplify the notation we suppress the stars indicating optimality.

$x(0) = x_o > 0,\ x(T) \geq 0$ (2.98c)

$u \geq 0.$ (2.98d)

Note that the terminal condition is equivalent to the path restriction $x(t) \geq 0$ for all $t \in [0,T]$.

$x(t)$ denotes the stock of a nonrenewable resource at time t, $u(t)$ the extraction rate, $p(t)$ the given market price for the resource, $C(u,t)$ the cost of extraction. Let us restrict ourselves to the case of a constant selling price, $p = \bar{p}$, and to quadratic, time-independent costs, $C = au^2/2$.

Since (2.98) is a problem with terminal condition $x(T) \geq 0$, the Hamiltonian must include λ_o (see (2.88)):

$$H = \lambda_o(\bar{p}u - au^2/2) - \lambda u.$$

We first show that $\lambda_o = 1$. Assume $\lambda_o = 0$. Then by the non-degeneracy condition (2.77) we see that

$$\lambda \neq 0. \quad (2.99)$$

The adjoint equation is

$$\dot{\lambda} = r\lambda. \quad (2.100a)$$

From (2.80c) follows for the transversality condition

$$\lambda(T) \geq 0,\ \lambda(T)x(T) = 0. \quad (2.100b)$$

(2.99-100) yield

$$\lambda(t) > 0 \quad \text{for all} \quad t \in [0,T]. \quad (2.101)$$

For $\lambda_o = 0$ the Hamiltonian is $H = -\lambda u$. Thus, from (2.98d, 101) we get

$$u^*(t) = 0. \quad (2.102)$$

From (2.100b,101) follows

$$x(T) = 0. \quad (2.103)$$

On the other hand, we obtain from (2.98b, 102) $x(T) = x_o > 0$ which provides a contradiction. Hence $\lambda_o = 1$.

The maximizing condition $H_u = 0$ yields

$$u(t) = [\bar{p} - \lambda(t)]/a. \quad (2.104)$$

The first-order condition for determining the optimal exploitation duration $T = T^*$, (2.97), has the form (the star is omitted)

$$H[T] = u(T)[\bar{p} - au(T)/2 - \lambda(T)] = 0. \quad (2.105)$$

Substituting $\lambda(T)$ from (2.104) into (2.105) yields

$$H[T] = au(T)^2/2 = 0$$

and so

$$u^*(T^*) = 0. \quad (2.106)$$

This together with (2.104) provides

$$\lambda(T^*) = \bar{p}. \quad (2.107)$$

Solving (2.100,107) yields

$$\lambda(t) = \bar{p}e^{-r(T-t)}. \quad (2.108)$$

From (2.98bc, 103) follows

$$x_o = \int_0^T u(t)\,dt. \quad (2.109)$$

By using (2.108) and integrating (2.104) we obtain the following equation to determine the optimal planning period T^*

$$\bar{p}(rT^* - 1 + e^{-rT^*}) = arx_o. \quad (2.110)$$

2.9. Exercices for Section 2

2.1. Show that the Bolza, Lagrange and Mayer form of an optimal control problem are equivalent (Hint: Introduce an additional state variable).

2.2. Solve the following optimal control problem

a) $\max_u \int_0^T (x - u^2/2)\,dt$

subject to $\dot{x} = u,\ x(0) = x_o$.

b) What happens if we impose additionally $u(t) \in \Omega = [0,1]$?

.3. Solve $\max \int_0^T (x-u)\,dt$, subject to $\dot{x} = u,\ x(0) = x_o,\ 0 \leq u \leq 1$.

.4. Using (2.11c) show the equivalence of the optimality conditions in present-value and current-value notation.

2.5. Derive the maximum principle by using the Hamilton-Jacobi-Bellman equation for a positive discount rate r.

2.6. Prove the relation

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + r\lambda f.$$

What happens in the autonomous case for r = 0? (Hint: Calculate the total derivative and use the optimality conditions).

2.7. Derive Euler's differential equation

$$F_x(x,\dot{x},t) - \frac{d}{dt}F_{\dot{x}}(x,\dot{x},t) = 0$$

and the transversality condition

$$F_{\dot{x}}(x(T),\dot{x}(T),T) = 0$$

from the conditions of the maximum principle.

2.8. Show that for the exercices 1.2 and 1.3 the sufficient optimality conditions are satisfied.

2.9. The following example shows that the concavity of H^0 in x can be satisfied but that H must not be concave in (x,u)

$$\max \int_0^T (u^2-x)\,dt$$

$$\dot{x} = u,\ x(0) = x_0$$

$$0 \le u \le 1.$$

Prove the above statement, and show that

$$u = \begin{cases} 0 & \text{for } [0,T-1] \\ 1 & \text{for } [T-1,T] \end{cases}$$

is the optimal solution.

2.10. Show that in the cases (2.82b) and (2.82c) the control problem of example 2.7 is normal (i.e. that $\lambda_0 = 0$ leads to a contradiction).

2.11. Solve

$$\max \int_0^2 (-x)\,dt$$

s.t. $\dot{x} = u$, $x(0) = 1$, $x(2) \ge 0$, $-1 \le u \le 1$.

2.12. Solve the linear maintenance problem (section 3.1) with the additional terminal condition $x(T) = x_T$ and $x(T) \ge x_T$, respectively.

2.13. Prove Theorem 2.5.

2.14. Determine the optimal extraction policy and the optimal terminal time T* for example 2.10. Prove that $\lambda_0 = 1$ and that $x(T^*) = 0$.

3. LINEAR OPTIMAL CONTROL

A control model is linear, if the Hamiltonian H is linear in the control variable u. Since H_u does not depend on u, we have

$$u(t) = \begin{cases} \underline{u} \\ \text{undefined} \\ \bar{u} \end{cases} \text{ if } \sigma(t) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0. \quad (3.1)$$

Here $u \in \Omega = [\underline{u}, \bar{u}]$, and $\sigma(t) = H_u$ is the switching function. A control which jumps from $\underline{u}$ to $\bar{u}$ or vice versa is called "bang-bang-control".

If the switching function

$$\sigma(t) = 0 \quad \text{for} \quad t \in [\tau_1, \tau_2]$$

then u can take values in the interior of $[\underline{u}, \bar{u}]$, and such a solution is denoted as *singular solution*. In this case we have

$$\sigma(t) = \dot{\sigma}(t) = 0 \quad \text{for} \quad t \in [\tau_1, \tau_2]. \quad (3.2)$$

3.1. A Linear Maintenance Model

The problem of determining the lifetime of an asset or an activity simultaneously with its management during that life is an important problem in practice. A typical example is the problem of optimal *maintenance* and *replacement* for a machine.

Consider a single machine whose resale value gradually declines over time. Its output is assumed to be proportional to its resale value. By applying preventive maintenance it is possible to slow down the rate of decline of resale value. The control problem consists of *simultaneously* determining the optimal rate of preventive maintenance and the sale date of the machine. This is an optimal control problem with unspecified terminal time.

A machine will be characterized by a state variable representing its quality of service. We assume that this quality can be adequately measured in monetary units. Let us denote its value at time t by x(t). The evolution of x(t) is described by the following differential equation

$$\dot{x} = -\delta x + u, \; x(0) = x_o, \quad (3.3)$$

where

$$u \in [0, \bar{u}] \quad (3.4)$$

is the preventive maintenance rate (i.e. expenditures for maintenance). It is assumed that for u = 0 the resale value of the machine decreases exponentially.

Furthermore, let π be the constant production rate in dollars per unit resale value, and Sx(T) the resale value of the machine at the end of the planning period T.

The present value of the machine is the sum of two terms, the discounted production stream during its life, plus the discounted resale value at T:

$$J = \int_0^T e^{-rt}(\pi x - u)\,dt + e^{-rT} S x(T). \quad (3.5)$$

For the model parameters we assume that

$$S < 1 < \pi/(r+\delta). \quad (3.6)$$

Considering (3.3) we see that the units are chosen such that the dollar value of maintenance gets directly transformed into quality. It seems not unreasonable to assume that one does not get back the full value of maintenance expenses when an old machine is sold. The constant π measures how a unit of quality gets transformed into operating receipts per unit of time. The costs, regardless of maintenance, are the loss in salvage value due to deterioration $S\delta$, and the interest on the value of the machine, which is Sr per unit of quality. Thus $S < \pi/(r+\delta)$ is a reasonable assumption, and $1 < \pi/(r+\delta)$ is not an unreasonable one.

To solve the linear optimal control problem we define the Hamiltonian

$$H = \pi x - u + \lambda(-\delta x + u).$$

According to (2.4) and (3.1) the maximizing condition is

$$u(t) = \begin{Bmatrix} 0 \\ \text{undefined} \\ \bar{u} \end{Bmatrix} \text{ if } \lambda(t) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 1. \quad (3.7)$$

Note that the switching function is $\sigma = H_u = \lambda - 1$. The shadow price of the state satisfies the adjoint equation

$$\dot{\lambda} = r\lambda - H_x = (r+\delta)\lambda - \pi \quad (3.8)$$

with the transversality condition

$$\lambda(T) = S. \tag{3.9}$$

Since the Hamiltonian is concave in (x,u) the necessary optimality conditions are also sufficient.

The solution of the linear inhomogenous differential equation (3.8) with condition (3.9) is given as

$$\lambda(t) = \left(S - \frac{\pi}{r+\delta}\right)e^{(t-T)(r+\delta)} + \frac{\pi}{r+\delta}. \tag{3.10}$$

From (3.10) and assumption (3.6) follows that $\dot{\lambda} < 0$. We can therefore conclude that there will not be any singular control for any finite interval of time, i.e. $\lambda = 1$ can occur at most at a single time τ. According to (3.7) the decision maker switches from $\bar{u}$ to 0. Moreover, the switching time τ is determined by $\lambda(\tau) = 1$. Thus, from (3.10) follows

$$\pi - (r+\delta) = [\pi - S(r+\delta)]e^{(\tau-T)(r+\delta)}. \tag{3.11}$$

Defining $\theta = T - \tau$, we see that θ does not depend in T and x_o. θ is given by

$$\theta = \frac{1}{r+\delta}\ln\frac{\pi - S(r+\delta)}{\pi-(r+\delta)} \tag{3.12}$$

From (3.6) follows that $\theta > 0$. For $T > \theta$ we conclude from $\dot{\lambda} < 0$

$$\lambda(t) \begin{Bmatrix} > 1 \\ < 1 \end{Bmatrix} \text{ for } \begin{cases} 0 \le t < T - \theta \\ T - \theta < t \le T. \end{cases}$$

This together with (3.7) and the state equation (3.3) gives

time interval	u(t)	x(t)
$[0,T-\theta)$	$\bar{u}$	$\left(x_o - \frac{\bar{u}}{\delta}\right)e^{-\delta t} + \frac{\bar{u}}{\delta}$
$[T-\theta,T]$	0	$[x_o + \frac{\bar{u}}{\delta}(e^{\delta(T-\theta)}-1)]e^{-\delta t}$

Note that for $x_o > \bar{u}/\delta$ the quality of the machine always decreases, i.e. $\dot{x}(t) \le 0$ for $t\epsilon[0,T]$.

If, however, the planning period is not "long enough", i.e. $T \le \theta$, then $\lambda(t) < 1$ for all $t\epsilon[0,T]$. In this case is does not pay at all to invest in preventive maintenance:

time interval	u(t)	x(t)
$[0,T]$	0	$x_o e^{-\delta t}$

The above calculations were made on the assumption that T was fixed. To determine the optimal sale date we use the corresponding transversality condition (2.97) for the Hamiltonian:

$$H - rSx(T) = \pi x(T) - u(T) + \lambda(T)[u(T) - \delta x(T)] - rSx(T) = [\pi - (r+\delta)S]x(T) > 0. \tag{3.12}$$

Here we have used $u(T) = 0$, (3.9), and (3.6). Thus, condition (3.12) can not be satisfied for any finite $T^* > 0$. Since $T^* = 0$ is also suboptimal, the optimal planning period is infinite: $T^* = \infty$.

3.2. Most Rapid Approach Paths

In the maintenance model of section 3.1 the Hamiltonian was linear in the control u. The coefficient of u in the Hamiltonian could be equal to zero only for isolated instants. In other problems, the coefficient of u in H is equal to zero over some period of time. During such periods, the control does not affect the Hamiltonian, and therefore, the choice of u is not determined in the usual way. In these cases the value of u is said *singular*.

There is a class of autonomous problems with one state variable and one control in which the optimal solution is to approach the singular level of the state variable *as fast as possible*. This is called a *most rapid approach path* (MRAP). Such behaviour is typical for many optimal control models being linear in the control but nonlinear in the state.

We consider the following class of one-dimensional autonomous control models

$$\max_u \int_0^\infty e^{-rt}F(x,u)\,dt \tag{3.13a}$$

$$\dot{x} = f(x,u), x(0) = x_o \tag{3.13b}$$

$$u\epsilon[u_1(x),u_2(x)] \tag{3.13c}$$

$$x\epsilon[\underline{x},\bar{x}] \tag{3.13d}$$

with

$$F(x,u) = A(x) + B(x)\Phi(x,u) \quad (3.14a)$$

$$f(x,u) = a(x) + b(x)\Phi(x,u), \quad (3.14b)$$

where all functions are assumed continuously differentiable and $b(x) \neq 0$. The state domain (3.13d) can be infinite, i.e. $\underline{x} = -\infty$ and/or $\bar{x} = +\infty$.

To prove the optimality of the MRAP by using Green's theorem the above problem is transformed as follows.

Lemma 3.1. *The class of optimal control models defined by (3.13, 14) may be written equivalently as*

$$\max_u \int_0^\infty e^{-rt}[M(x) + N(x)\dot{x}]dt \quad (3.15a)$$

$$x(0) = x_0, \ \dot{x}\in\Omega(x), \ x\in[\underline{x},\bar{x}], \quad (3.15b)$$

where M(x) and N(x) are continuously differentiable and the domain Ω depends continuously from x.

Proof. From (3.13b) and (3.14b) follows

$$\Phi(x,u) = \frac{1}{b(x)}[\dot{x} - a(x)].$$

Substituting this into (3.14a) we obtain (3.15a) with

$$M(x) = A(x) - a(x)B(x)/b(x)$$

$$N(x) = B(x)/b(x).$$

Moreover we have

$$\Omega(x) = \{a(x) + b(x)\Phi(x,u) \mid u\in[u_1(x),u_2(x)]\}.$$ □

(3.38) represents a problem of variational calculus. Defining $\dot{x} = v$ as new control variable, according to (3.2) a singular solution can be calculated as follows:

$$H = M(x) + N(x)v + \lambda v$$

$$\sigma = H_v = N(x) + \lambda = 0$$

$$\dot{\sigma} = N'(x)v + \dot{\lambda} = 0$$

$$\dot{\lambda} = r\lambda - H_x = r\lambda - M'(x) - N'(x)v.$$

From the last three equations we obtain the following relation to determine the singular solution $x = \hat{x}$:

$$I(\hat{x}) = rN(\hat{x}) + M'(\hat{x}) = 0. \quad (3.16)$$

Definition 3.1. A most rapid approach path (MRAP) $x^*(t)$ to a given trajectory $\tilde{x}(t)$ has the property

$$|x^*(t) - \tilde{x}(t)| \le |x(t) - \tilde{x}(t)| \quad \text{for} \quad t\in[0,\infty)$$

for all feasible state trajectories $x(t)$.

The following result shows the optimality of the MRAP to the singular solution or to $\underline{x}$, $\bar{x}$, respectively.

Theorem 3.1 (MRAP theorem). *Let $\hat{x}$ be the unique solution of (3.16). Assume that singular level $\hat{x}$ is feasible, i.e.*

$$0\in\Omega(\hat{x}), \ \underline{x} < \hat{x} < \bar{x}.$$

Moreover,

$$I(x)\begin{Bmatrix} > \\ < \end{Bmatrix} 0 \quad \text{for} \quad \begin{matrix} \underline{x} \le x < \hat{x} \\ \hat{x} < x \le \bar{x}. \end{matrix} \quad (3.17)$$

Finally, we assume that for any admissible state trajectory x(t) it holds that

$$\lim_{t\to\infty} e^{-rt}\int_{x(t)}^{\hat{x}} N(\xi)d\xi \ge 0 \quad (3.18)$$

If there is a MRAP starting in x_0, then this path is an optimal solution.

If (3.17) has no admissible solution $\hat{x}$ and $I(x) < 0$ for all $x\in[\underline{x},\bar{x}]$, then the MRAP to $\underline{x}$ is optimal. On the other hand, if $I(x) > 0$ for any $x\in[\underline{x},\bar{x}]$, then the MRAP to $\bar{x}$ is optimal.

Remark. Condition (3.18) is satisfied if N is a negative constant and if any admissible solution $x(t) \ge 0$. For a bounded state domain $[\underline{x},\bar{x}]$ it suffices to suppose that N(x) is bounded. The proof uses that (3.15a) may be written as curve integral and applies Green's theorem.

The philosophy behind the MRAP theorem is that it is optimal to reach the singular arc (the "turnpike") as fast as possible. For an optimal control problem (3.13,14) with *finite* time horizon and prescribed terminal state an analogous MRAP result is valid. If T is sufficiently large, then $\hat{x}$ is reached as fast as possible. Moreover, the turnpike $\hat{x}$ is left as late as possible to meet the terminal condition $x(T) = x_T$. However, if T is too short that the singular level $\hat{x}$ can be reached in time, the optimal path has still the property that it is the nearest feasible path to $\hat{x}$ under all feasible state trajectories.

The MRAP theorem is illustrated by the following two examples.

Example 3.1: The Nerlove-Arrow model

The belief that advertising expenditures by a firm affect its present and future sales has led to treat advertising as an instrument in building up some sort of *advertising capital* usually called *goodwill*. Furthermore, the stock of goodwill depreciates over time because consumers "drift" to other brands, as a result of advertising by competing firms etc. Let A(t) denote the stock of goodwill at time t. Assume that the cost of producting one unit of goodwill is one dollar so that a dollar spent on current advertising increases goodwill by one unit. It is assumed that the goodwill stock depreciates over time at a constant proportional rate δ, so that

$$\dot{A} = a - \delta A, \; A(0) = A_0, \tag{3.19}$$

where $a(t) \geq 0$ is the current advertising effort measured in dollars.

To formulate the optimal control problem for a *monopolistic* firm, we assume that the rate of sales s(t) depends not only on price p(t), but also on the stock of goodwill A(t). Thus, $s = s(p,A)$. Assuming the rate of total production costs is c(s), the total revenue net of production costs is

$$R(p,A) = ps(p,A) - c(s(p,A)). \tag{3.20}$$

The firm wants to maximize the present value of net revenue streams discounted at a fixed rate r, i.e.

$$\max_{a\geq 0, p\geq 0} \{J = \int_0^\infty e^{-rt}[R(p,A) - a]dt\}. \tag{3.21}$$

Since the price p does not occur in the state equation (3.19), we can maximize J by first maximizing R with respect to price p holding A fixed, and then maximize the result with respect to a. Thus, the optimal pricing policy is determined by static maximization, i.e. by $\max_p R(p,A)$. The first-order necessary optimality condition is

$$R_p = s + (p-c')s_p = 0 \tag{3.22}$$

yielding the optimal price p(t) as function of A(t):

$$p = p(A). \tag{3.23}$$

Note that a sufficient optimality condition for (3.23) is the following second-order condition

$$s_{pp} < 2s_p^2/s, \tag{3.24}$$

if we additionally assume a convex cost function c(s).

To calculate the optimal advertising strategy we consider the Hamiltonian

$$H = ps(p,A) - c(s(p,A)) - a + \lambda(a-\delta A).$$

The adjoint equation is

$$\dot{\lambda} = (r+\delta)\lambda - (p-c')s_A. \tag{3.25}$$

According to (3.2) the singular solution is determined by $H_a = 0$, i.e. $\lambda = 1$, and $\dot{\lambda} = 0$. Substituting this into (3.25) and taking into consideration (3.22) we get for the singular level of goodwill

$$\hat{A} = \frac{\beta}{\eta(r+\delta)}ps, \tag{3.26}$$

where $\beta = s_A A/s$ is the elasticity of demand with respect to goodwill, and $\eta = -s_p p/s$ is the elasticity of demand with respect to price.

The corresponding singular advertising rate is given by $\dot{A} = 0$ as $\hat{a} = \delta\hat{A}$.

Note that for isoelastic demand functions

$$s = \gamma p^{-\eta}A^{\beta}, \quad \gamma, \eta, \beta \text{ constant}. \tag{3.27}$$

(3.26) says that along the optimal stationary solution the goodwill is a fixed percentage of the return ps. Thus, for the optimal advertising expenditures $\hat{a}$ the same is true.

Applying the MRAP theorem 3.1 to this model we obtain the following characterization of the optimal advertising policy.

Theorem 3.2 (Optimal advertising in the Nerlove-Arrow model). *Assume that for the revenue function*

$$\pi(A) = \max_p R(p,A) \tag{3.28}$$

there exists a value $\hat{A}$ such that

$$\pi'(A) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} r + \delta \quad \text{for} \quad A \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \hat{A}. \tag{3.29}$$

Then the optimal advertising policy for the control problem (3.19, 21) is to reach the singular level of goodwill as fast as possible. If we assume (to exclude impulse controls)

$$0 \le a \le \bar{a} \quad \text{with} \quad \bar{a} > \delta\hat{A}$$

then the following combination of bang-bang and singular arcs is optimal:

For $A_0 < \hat{A}$:

$$a^*(t) = \begin{Bmatrix} \bar{a} \\ \hat{a} = \delta\hat{A} \end{Bmatrix} \quad \text{for} \quad A^*(t) \begin{Bmatrix} < \\ = \end{Bmatrix} \hat{A}. \tag{3.30a}$$

For $A_0 \ge \hat{A}$:

$$a^*(t) = \begin{Bmatrix} 0 \\ \hat{a} = \delta\hat{A} \end{Bmatrix} \quad \text{for} \quad A^*(t) \begin{Bmatrix} > \\ = \end{Bmatrix} \hat{A}. \tag{3.30b}$$

Proof. The Nerlove-Arrow model fits into the class of models described by (3.13, 14), where

$$M(A) = \pi(A) - \delta A, \; N(A) = -1; \; \Omega(A) = [-\delta A, \bar{a} - \delta A].$$

Because $I = rN + M' = \pi' - (r+\delta)$, the level $\hat{A}$ defined by (3.29) is the singular solution of (3.16). Since (3.30) is equivalent to (3.17) and (3.18) is satisfied, from the MRAP theorem follows the optimality of policy (3.30). See fig. 3.1. □

Remark. The revenue function $\pi(A)$ in (3.28) is obtained by substituting the optimal price $p = p(A)$ from (3.23) into (3.20): $\pi(A) = R(p(A),A)$. A sufficient condition for (3.29) are decreasing marginal returns, $\pi''(A) < 0$. If $R(p,A)$ is jointly concave in p and A, then it holds that

$$\pi''(A) = R_{AA} - R_{pA}^2/R_{pp} < 0.$$

Another application of the MRAP theorem is given in section 6.

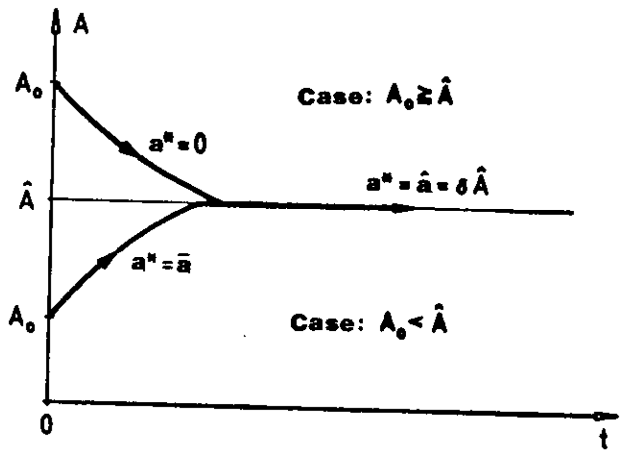


Fig. 3.1: Most rapid approach to the singular level of goodwill in the Nerlove-Arrow model

3.3. Exercices for Section 3

3.1. Optimal allocation between learning and earning (von Weizsäcker, 1967): The stock of human capital, x, can be increased by learning activities u. Assume that x decreases at a constant rate δ for u = 0. Suppose that the productivity w(x) of a worker depends on x as follows:

$$w > 0,\ w' > 0,\ w'' < w'^2/w. \tag{3.31a}$$

The earned income per time unit which is used for working is assumed to be exp(kt)w(x), where k denotes the rate of technical progress. Assume further that

$$w'(0) > (\rho-\delta-k)w(0), \tag{3.31b}$$

where ρ is the discount rate. Furthermore,

$$\rho + \delta > k. \tag{3.32}$$

The problem is to allocate each time unit (e.g. a working day) between learning (investment in the human capital) and earning. The present-value of the earned income stream over the life-cycle is

$$J = \int_0^T e^{(k-\rho)} w(x)(1-u)\,dt. \tag{3.33a}$$

The dynamics of the human capital is as given as

$$\dot{x} = -\delta x + u. \tag{3.33b}$$

Moreover, we must have

$$0 \le u \le 1. \tag{3.33c}$$

Determine the optimal allocation pattern in human capital by using the MRAP approach.

3.2. Solve the Vidale-Wolfe model by the MRAP theorem:

$$\max \int_0^T e^{-rt}(\pi x-a)\,dt$$

$$\dot{x} = \alpha a(1-x) - \delta x,$$

$$x(0) = x_0,\ x(T) = x_T;\ x_0, x_T \in (0,1)$$

$$0 \le a \le \bar{a}.$$

3.3. Prove the MRAP theorem by using Green's theorem.

4. CONCAVE ONE-STATE-MODELS

In many branches of modern application-oriented mathematics linearity does not catch essential features of reality. Recently, in economics *nonlinear models* claim a growing interest. An important class of nonlinear optimal control models are autonomous one-state models with a Hamiltonian being strictly concave in the control u. Furthermore, in many cases it is assumed that the Hamiltonian is concave jointly in x and u. For this class of models it is possible to derive by analytical means *qualitative* insights in the optimal solutions.

It is important to note that the model functions (utility, costs, production function etc.) need not specified fully, but are characterized only by its (first and second-order) derivatives.

4.1. The Model and its Necessary Optimality Conditions

We consider the following one-dimensional autonomous concave control model:

$$\max_u \{J = \int_0^T e^{-rt} F(x,u)\,dt + e^{-rT} S(x(T))\} \tag{4.1}$$

subject to

$$\dot{x} = f(x,u),\ x(0) = x_0. \tag{4.2}$$

There are either no restrictions for the control u and/or the state x or they are no active. Moreover, we also consider the infinite time horizon problem

$$\max_u \{J = \int_0^\infty e^{-rt} F(x,u)\,dt\} \tag{4.3}$$

s.t. (4.2). We assume that the improper integral in (4.3) converges for any feasible solution.

To illustrate this model we interpret the state variable as pollution level, and the control as consumption rate. (A more detailed description and analysis of this pollution-consumption model is given in section 2.)

F(x,u) is the utility implied by the pollution and the consumption rate. It is reasonable to assume that

$$F_x < 0,\ F_{xx} \le 0,\ F_u > 0,\ F_{uu} \le 0,\ F_{xu} \le 0 \tag{4.4a}$$

$$F_{xx}F_{uu} - F_{xu}^2 \ge 0. \tag{4.4b}$$

Thus, the utility function is concave in (x,u).

The efficiency function $f(x,u)$ may be thought as the pollution control function. It is supposed that

$$f_x < 0,\ f_{xx} = 0,\ f_u > 0,\ f_{uu} \ge 0,\ f_{xu} = 0. \tag{4.5}$$

For a discussion of those assumption see section 7.1.

To ensure the nonlinearity of the control model, it is required that

$$F_{uu} + \lambda f_{uu} < 0. \tag{4.6}$$

Using Pontryagin's maximum principle the Hamiltonian is

$$H = F(x,u) + \lambda f(x,u).$$

The necessary optimality conditions are according to theorem 2.1

$$H_u = F_u + \lambda f_u = 0 \tag{4.7}$$

$$\dot\lambda = r\lambda - H_x = (r-f_x)\lambda - F_x \tag{4.8}$$

$$\lambda(T) = S_x(x(T)). \tag{4.9}$$

From (4.7), (4.4a) and (4.5) follows

$$\lambda = -F_u/f_u < 0. \tag{4.10}$$

Clearly, the shadow price λ of the stock of pollution is *negative*.

From (4.4a), (4.5) and (4.6) follows

$$H_{uu} = F_{uu} + \lambda f_{uu} < 0. \tag{4.11}$$

Thus, (4.7) provides a unique maximum of the Hamiltonian H.

Furthermore (4.4), (4.5) and (4.10) imply

$$H_{xx} \le 0,\ H_{xu} \le 0 \tag{4.12a}$$

$$H_{xx}H_{uu} - H_{xu}^2 \ge 0. \tag{4.12b}$$

Let us first assume that there *exists* a *unique* stationary point $(\hat x,\hat u,\hat\lambda)$ which is determined by (2.89). To prove the existence of an equilibrium we need additional informations on the functions F and f.

4.2. Phase Portrait Analysis in the State-Costate Plane

The maximizing condition (4.7) provides a relation for the control (consumption) as an implicit function of the state (level of pollution) and the shadow price. We can visualize the system solved as $u = u(x,\lambda)$ with

$$\frac{\partial u}{\partial x} = -\frac{H_{ux}}{H_{uu}} \le 0,\quad \frac{\partial u}{\partial \lambda} = -\frac{f_u}{H_{uu}} > 0. \tag{4.13}$$

The level of consumption is a nonincreasing function of the pollution rate. Moreover, an increase in the shadow price induces an increase in the consumption.

We now turn to the analysis of the (x,λ) phase plane. By substituting $u = u(x,\lambda)$ into the state and the costate equation we get

$$\begin{aligned} \dot x &= \dot x(x,\lambda) = f(x,u(x,\lambda)) \\ \dot\lambda &= \dot\lambda(x,\lambda) = [r - f_x(x,u(x,\lambda))]\lambda - F_x(x,u(x,\lambda)). \end{aligned} \tag{4.14}$$

To determine the stability properties of the equilibrium of the system (4.14) consider its Jacobian matrix

$$\begin{pmatrix} \frac{\partial \dot x}{\partial x} & \frac{\partial \dot x}{\partial \lambda} \\ \frac{\partial \dot\lambda}{\partial x} & \frac{\partial \dot\lambda}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} f_x + f_u \frac{\partial u}{\partial x} & f_u \frac{\partial u}{\partial \lambda} \\ -H_{xx} - H_{xu}\frac{\partial u}{\partial x} & (r-f_x) - H_{xu}\frac{\partial u}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} - & + \\ + & + \end{pmatrix} \tag{4.15}$$

The signs of the elements in the Jacobian follow from (4.4, 5, 13).

The sign of the determinant Δ of the Jacobian is negative[1]. Thus, every stationary point of the canonical system (4.14) is a *saddle point*. By a theorem of Gale and Nikaido (1965, theorem 7) it is also *unique*. The uniqueness of the equilibrium may also be derived from the shape of the isoclines (see below).

---

[1] Note that this follows *without* the assumptions (4.4b), since the term $f_u H_{xu}(\partial u/\partial x)(\partial u/\partial\lambda)$ cancels out.

It is well known, that, since the Jacobian matrix has a negative determinant (and thus has one positive and one negative eigenvalue), there exists a one-dimensional manifold consisting of all solutions that converge towards the equilibrium. This manifold is called the *stable path* (it consists of two stable branches).

To obtain informations on the shape of the solution trajectories of (4.14) we consider the isoclines $\dot{x} = 0$ and $\dot{\lambda} = 0$. Their slopes can be computed by using again the implicit function theorem.

The $\dot{x} = 0$ curve is upward sloping

$$\left.\frac{d\lambda}{dx}\right|_{\dot{x}=0} = - \frac{\partial\dot{x}/\partial x}{\partial\dot{x}/\partial\lambda} > 0. \tag{4.16}$$

Finally, $\dot{\lambda} = 0$ decreases monotonically:

$$\left.\frac{d\lambda}{dx}\right|_{\dot{\lambda}=0} = - \frac{\partial\dot{\lambda}/\partial x}{\partial\dot{\lambda}/\partial\lambda} \leq 0. \tag{4.17}$$

The four isoclines divide the phase plane in four regions. The orientation of the solution trajectories of the canonical system (4.14) follows from (4.13). Regions I and IV are traps in the sense that if any path enters either of these regions, it remains there forever. Since the *stable path* can only lie in region II and III it must be *downward sloping*; see fig. 4.1.

Note that the saddle point path in the (x,λ) plane is horizontal if and only if in (4.12b) the equality sign holds.

From (4.12) follows that the Hessean matrix

$$D^2H = \begin{pmatrix} H_{uu} & H_{xu} \\ H_{xu} & H_{xx} \end{pmatrix} \tag{4.18}$$

is negative semidefinite. According to remark 2.1 (criterion of principal minors) this implies the concavity of the Hamiltonian H. Thus, the sufficient optimality conditions of theorem 2.2 are satisfied provided that S is also convex in x. For infinite time horizon according to theorem 2.5 the limiting transversality condition must be established. According to the last remark of section 2.7 the saddle point path represents the optimal solution of problem (4.1, 2) for $T = \infty$.

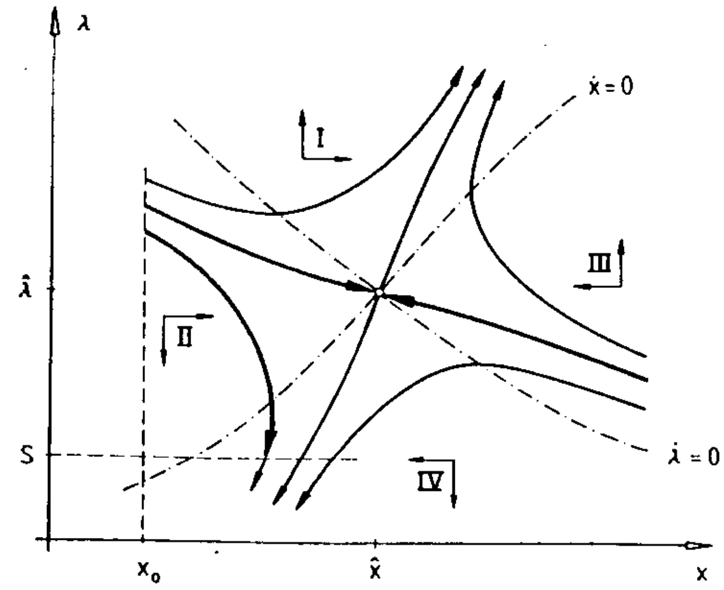


Fig. 4.1: (x,λ)-phase portrait with downward sloping saddle point path

In words the above result in the infinite horizon case has the following economic meaning. If the level of pollution is low, then its shadow price is comparatively high (note that λ is negative). In the case of a high pollution an additional unit of pollution is more detrimental. As time tends to infinity, both the pollution, x, and the shadow price, λ, approach their equilibrium values. Furthermore, this approach is monotonic.

The trajectories in the phase plane which are not identical with the stable path represent optimal solutions for finite horizon problems. The solution starts at the vertical line $x = x_0$ (initial value) and ends at the curve $S_x(x) = \lambda$ (transversality condition) which is a horizontal line if $S(x) = Sx$. The longer the horizon T, the more the solution will approach the equilibrium.

4.3. State-Control Phase Diagram

Differentiating (4.7) locally with respect to time yields

$$H_{xu}\dot{x} + H_{uu}\dot{u} + f_u\dot{\lambda} = 0, \tag{4.19}$$

i.e. the following differential equation for the control u

$$\dot{u} = -(H_{xu}\dot{x}+f_u\dot{\lambda})/H_{uu}, \qquad (4.20)$$

with from (4.2), and $\dot{\lambda}$ from (4.8). We eliminate the adjoint variable λ by (4.10) and get together with (4.2) a system of differential equations for x and u. The elements of the Jacobian determinant are given by

$$\frac{\partial\dot{x}}{\partial x} = f_x < 0 \quad (a), \qquad \frac{\partial\dot{x}}{\partial u} = f_u > 0 \quad (b). \qquad (4.21)$$

The terms $\partial\dot{u}/\partial x$ and $\partial\dot{u}/\partial u$ contain third partial derivatives of F and f. After some calculations we obtain

$$\left.\frac{\partial\dot{u}}{\partial x}\right|_{(\hat{x},\hat{u})} = \frac{(r-2f_x)H_{xu}+f_uH_{xx}}{H_{uu}} \geq 0 \qquad (4.21c)$$

$$\left.\frac{\partial\dot{u}}{\partial u}\right|_{(\hat{x},\hat{u})} = r - f_x > 0. \qquad (4.21d)$$

By multiplying the terms in (4.21) on can see that the determinant of the Jacobian, i.e.

$$(\partial\dot{x}/\partial x)(\partial\dot{u}/\partial u) - (\partial\dot{u}/\partial x)(\partial\dot{x}/\partial u)$$

evaluated in equilibrium $(\hat{x},\hat{u})$ is equal to Δ (see section 4.2).

This shows, that the stationary point $(\hat{x},\hat{u})$ is a *saddle point* in the (x,u) phase plane. The stable path providing the optimal solution for $T = \infty$ *decreases* monotonically in a neighbourhood of $(\hat{x},\hat{u})$.

Note that the saddle point path is horizontal if and only if

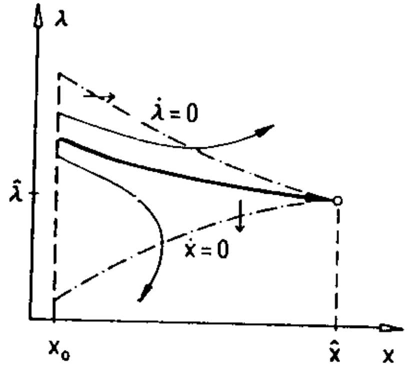
$$H_{xu} = 0 \quad \text{(at least for } H_u = 0\text{), and} \quad H_{xx} = 0. \qquad (4.22)$$

A more detailed analysis of the state-control phase diagram is given in section 7.1. There, the question of *existence* of the equilibrium is also considered.

The saddle point property of a non-linear system of differential equations as shown above by determining the sign of the Jacobian determinant is a *local* property. In economics it would be important, however, to derive *global* optimal solutions. This can be done, e.g., by the following

Theorem 4.1

(Global saddle point theorem). Let $(\hat{x},\hat{\lambda})$ be a saddle point of the canonical differential equation system and x_0 a given initial state such that the line $x = x_0$ crosses both isoclines $\dot{x} = 0$ and $\dot{\lambda} = 0$. Assume that the region bounded by $(\hat{x},\hat{\lambda})$, $x = x_0$, and the both isoclines has the following 'triangular' shape (i.e., the isoclines do not intersect in $(x_0,\hat{x})$):



Then there exists a unique stable path leading from $x = x_0$ to $(\hat{x},\hat{\lambda})$.

4.4. Sensitivity Analysis of the Equilibrium

We now turn to a consideration of the effects on the stationary point $(\hat{x},\hat{u})$ of different values of the model parameters. We restrict ourselves to the discount rate r as parameter.

The stationary point $(\hat{x},\hat{u},\hat{\lambda})$ is defined by

$$f(\hat{x},\hat{u}) = 0 \qquad (4.23a)$$

$$[r - f_x(\hat{x},\hat{u})]\hat{\lambda} = F_x(\hat{x},\hat{u}) \qquad (4.23b)$$

$$F_u(\hat{x},\hat{u}) + \hat{\lambda}f_u(\hat{x},\hat{u}) = 0. \qquad (4.23c)$$

By the implicit function theorem the equations (4.23) determine three functions $\hat{x} = \hat{x}(r)$, $\hat{u} = \hat{u}(r)$, and $\hat{\lambda} = \hat{\lambda}(r)$, and for the derivatives it holds that

$$\begin{pmatrix} 0 & f_x & f_u \\ f_x - r & H_{xx} & H_{xu} \\ f_u & H_{xu} & H_{uu} \end{pmatrix} \begin{pmatrix} \partial\hat{\lambda}/\partial r \\ \partial\hat{x}/\partial r \\ \partial\hat{u}/\partial r \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\lambda} \\ 0 \end{pmatrix} \qquad (4.24)$$

By calculating the determinant Δ_1 of the matrix of system (4.24) we see that

$$\Delta_1 = H_{uu}\Delta > 0,$$

where Δ is the determinant of the Jacobian evaluated in $(\hat{x},\hat{\lambda})$ which is negative (see section 4.2).

Since $\Delta_1 \neq 0$, system (4.24) has a unique solution which can be calculated by using Cramer's rule as follows:

$$\frac{\partial\hat{\lambda}}{\partial r} = \frac{1}{\Delta_1} \begin{vmatrix} 0 & f_x & f_u \\ \hat{\lambda} & H_{xx} & H_{xu} \\ 0 & H_{xu} & H_{uu} \end{vmatrix} = -\frac{\hat{\lambda}}{H_{uu}\Delta} \begin{vmatrix} f_x & f_u \\ H_{xu} & H_{uu} \end{vmatrix} = -\frac{\hat{\lambda}}{\Delta}\frac{\partial\dot{x}}{\partial x} > 0 \qquad (4.25)$$

$$\frac{\partial\hat{x}}{\partial r} = \frac{1}{\Delta_1} \begin{vmatrix} 0 & 0 & f_u \\ f_x - r & \hat{\lambda} & H_{xu} \\ f_u & 0 & H_{uu} \end{vmatrix} = \frac{\hat{\lambda}}{H_{uu}\Delta} \begin{vmatrix} 0 & f_u \\ f_u & H_{uu} \end{vmatrix} = \frac{\hat{\lambda}}{\Delta}\frac{\partial\dot{x}}{\partial\lambda} > 0. \qquad (4.26)$$

The sign of $\partial\hat{u}/\partial r$ is obtained more simple by calculating the total derivative of $f(\hat{x}(r),\hat{u}(r)) = 0$. This yields

$$\frac{\partial\hat{u}}{\partial r} = -\frac{f_x}{f_u}\frac{\partial\hat{x}}{\partial r} > 0.$$

Thus, both consumption, pollution, and its shadow price move in the same direction as the discount rate. This makes economic sense.

Finally, let us try an economic interpretation of the equilibrium relations (4.23) (see also section 7.1 for more economic details). For this transform (4.23b) by (4.23c) to

$$F_u = -f_u F_x/(r-f_x). \qquad (4.28)$$

Consider a stationary pollution level $\hat{x}$ and the corresponding equilibrium consumption $\hat{u}$. If we increase the consumption marginally for a time unit, then $F_u(\hat{x},\hat{u})$ measures the instantaneous gain in utility. On the other hand, the momentary additional consumption causes an increment in pollution by $f_u(\hat{x},\hat{u})$ units. This additional pollution leads to a loss in utility, $-F_x(\hat{x},\hat{u})f_u(\hat{x},\hat{u})$.

The trajectory $\hat{x} + \delta x$ perturbated at $t = 0$ satisfies the differential equation $(\hat{x}+\delta x)^{\cdot} = f(\hat{x}+\delta x,\hat{u})$. By substracting $0 = \dot{\hat{x}} = f(\hat{x},\hat{u})$ we obtain as first approximation

$$(\delta x)^{\cdot} = f_x(\hat{x},\hat{u})\,\delta x, \quad \delta x(0) = f_u(\hat{x},\hat{u}). \qquad (4.29)$$

The solution of this linear homogenous differential equation is

$$\delta x(t) = f_u(\hat{x},\hat{u})\exp(f_x(\hat{x},\hat{u})t).$$

Thus, the loss in utility stream is given (in first approximation) as

$$-\int_0^\infty e^{-rt}[F(\hat{x}+\delta x,\hat{u}) - F(\hat{x},\hat{u})]dt = -\int_0^\infty e^{-rt}F_x(\hat{x},\hat{u})\,\delta x(t)dt = -\frac{F_x(\hat{x},\hat{u})f_u(\hat{x},\hat{u})}{r-f_x(\hat{x},\hat{u})}. \qquad (4.30)$$

Thus, (4.28) says that for a 'small' deviation of the stationary solution the *instantaneous* utility is equal to the present value of the loss in utility.

4.4. Excercices for Section 4

4.1. Solve the 'quadratic maintenance' model (compare section 3.1)

$$\max_u \{\int_0^T e^{-rT}(\pi x - u^2/2)dt + e^{-rT}Sx(T)\} \qquad (4.31a)$$

$$\dot{x} = -\delta x + u, \; x(0) = x_o \qquad (4.31b)$$

$$u \geq 0. \qquad (4.31c)$$

Determine the optimal time paths. Moreover, carry out a phase portrait analysis.

4.2. Carry out a phase diagram analysis for the concave one-state, one-control model for (4.1-6) with the following changes:

$F_x > 0,\; F_u < 0,\; f_{xx} \leq 0,\; f_{uu} \leq 0,\; f_{xu} \leq 0$

instead of

$F_x < 0,\ F_u > 0,\ f_{xx} = 0,\ f_{uu} \geq 0,\ f_{xu} = 0.$

Interpret x as goodwill stock and u as advertising rate (see also example 3.1). This yields a simple non-linear advertising model (Gould model).

4.3. Rework the von Weizsäcker model for a non-linear learning efficiency function g(x,u) with

$g \geq 0,\ g_u > 0,\ g_x < 0$

$g_{uu} < 0,\ g_{xx} \leq 0,\ g_{ux} \leq 0$

such that

$\dot{x} = -\delta x + g(x,u).$

4.4. Prove (4.21). Sketch the (x,u) phase diagram.

4.5. Carry out a sensitivity analysis in the sense of section 4.4 for the model described in exercise 4.2. Interpret the results for the non-linear advertising model mentioned there.

5. DYNAMIC SYSTEMS WITH PATH CONSTRAINTS

5.1. Mixed Inequality Constraints

In optimal control problems that arise in the areas of management science and economics inequality constraints between the constrols and/or the state variables are frequently encountered. Extending our standard model by including these path restrictions as well as various terminal conditions we consider the following optimal control problem:

$$\max_u \int_0^T e^{-rt} F(x,u,t)\,dt + e^{-rT} S(x(T),T) \tag{5.1a}$$

$$\dot{x} = f(x,u,t),\ x(0) = x_o \tag{5.1b}$$

$$g(x,u,t) \geq 0 \tag{5.1c}$$

$$a(x(T),T) \geq 0 \tag{5.1d}$$

$$b(x(T),T) = 0. \tag{5.1e}$$

The functions $F: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$, $g: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^s$, $a: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^\ell$ and $b: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{\ell'}$ are assumed to be continuously differentiable.

A pair (x(t),u(t)) for $t \in [0,T]$ is called a *feasible* solution of the problem (6.1), if u(t) is piecewise continuous in $[0,T]$ and both trajectories satisfy the conditions (5.1b-e).

To assure that (5.1c) provides a (state-dependent) restriction for the control(s) (and not a *pure* state constraint; see section 5.2), we need the following *constraint qualification:*

The matrix

$$\begin{pmatrix} \partial g_1/\partial u & g_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \partial g_s/\partial u & 0 & \cdots & g_s \end{pmatrix} \text{ has full rank } s, \tag{5.2}$$

i.e. the gradients $\partial g_i/\partial u$ of all *active* inequalities $g_i = 0$ $(i = 1, \ldots, s)$ are linear independent. Note that (5.2) is the usual constraint qualification of the Kuhn-Tucker theory.

To state the maximum principle for the control problem (5.1) let us define the Hamiltonian H, the Lagrangean L, and the control region Ω as follows:

$$H(x,u,\lambda_o,\lambda,t) = \lambda_o F(x,u,t) + \lambda f(x,u,t), \tag{5.3}$$

$$L(x,u,\lambda_o,\lambda,\mu,t) = H(x,u,\lambda_o,\lambda,t) + \mu g(x,u,t) \tag{5.4}$$

$$\Omega(x,t) = \{u \mid g(x,u,t) \geq 0\}. \tag{5.5}$$

<u>*Theorem 5.1.*</u> *Let $u^*(t)$ and the corresponding state trajectory $x^*(t)$ be an optimal solution of the problem (5.1). The regularity condition (5.2) is assumed to be valid for all $t \in [0,T]$, $x = x^*(t)$, $u = \Omega(x^*(t),t)$. Then there exist a constant $\lambda_o \geq 0$, a continuous costate trajectory $\lambda(t) \in \mathbb{R}^n$, piecewise continuous multiplicator functions $\mu(t) \in \mathbb{R}^s$, and constant multipliers $\alpha \in \mathbb{R}^{\ell}$, $\beta \in \mathbb{R}^{\ell'}$, such that*

$$(\lambda_o,\lambda,\mu,\alpha,\beta) \neq 0 \quad \text{for all} \quad t.$$

Moreover, the following necessary optimality conditions are satisfied for any point of continuity of $u^*(t)$:

$$H(x^*(t),u^*(t),\lambda_o,\lambda(t),t) = \max_{u \in \Omega(x^*(t),t)} H(x^*(t),u,\lambda_o,\lambda(t),t), \tag{5.6}$$

$$L_u(x^*(t),u^*(t),\lambda_o,\lambda(t),\mu(t),t) = 0, \tag{5.7}$$

$$\dot{\lambda}(t) = r\lambda(t) - L_x(x^*(t),u^*(t),\lambda_o,\lambda(t),\mu(t),t), \tag{5.8}$$

$$\mu(t) \geq 0, \ \mu(t)g(x^*(t),u^*(t),t) = 0 \tag{5.9}$$

$$\lambda(T) = \lambda_o S_x(x^*(T),T) + \alpha a_x(x^*(T),T) + \beta b_x(x^*(T),T) \tag{5.10}$$

$$\alpha \geq 0, \ \alpha a(x^*(T),T) = 0. \tag{5.11}$$

<u>*Remarks.*</u> For the control problem (5.1a-d) with nonactive terminal condition (5.1d) (i.e. for free terminal state) it holds that $\lambda_o = 1$. The abnormal case, however, can not be excluded generally. The conditions (5.7), (5.9) are known as *Kuhn-Tucker* conditions for the static maximization problem (5.6). The Lagrange multiplicator μ_i admits the usual interpretation as shadow price in the sense that μ_i measures the marginal increase in the total profit rate H

if the inequality constraint $g_i \geq 0$ is infinitesimally relaxed. Since this relaxation can not decrease H, μ_i is nonnegative. The second part of (5.9) i.e. the complementary slackness condition $\mu_i g_i$ follows from the fact, that a nonactive restriction has no influence to the objective H. The interpretation of the adjoint equation

$$r\lambda_j - \dot{\lambda}_j = H_{x_j} + \mu g_{x_j}, \qquad j = 1, \ldots, n \tag{5.12}$$

is analogous to (2.59). The additional term μg_{x_j} evaluates the change in the control region induced by an infinitesimal change in the state variable. However, it should be stressed that in the case of a terminal constraint (5.1d) the interpretation of λ_j as shadow price is only valid if the value function is defined in a whole neighbourhood around the state trajectory.

<u>5.2. Pure State Constraints</u>

Consider the optimal control problem (5.1) with the additional pure state variable inequality constraint

$$h(x,t) \geq 0, \tag{5.13}$$

where the function $h: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^q$ is two times differentiable. Let us assume that (5.13) is of *first order*, i.e. the first total derivative of h with respect to time,

$$k(x,u,t) = h_x(x,t)f(x,u,t) + h_t(x,t) \tag{5.14}$$

contains a control. Extending (5.2) we define the following *constraint qualification:*

$$\begin{pmatrix} \partial g_1/\partial u & g_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ \partial g_s/\partial u & 0 & \cdots & g_s & 0 & \cdots & 0 \\ \partial k_1/\partial u & 0 & \cdots & 0 & h_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \vdots \\ \partial k_q/\partial u & 0 & \cdots & 0 & 0 & \cdots & h_q \end{pmatrix} \text{ has full rank } q + s. \tag{5.15}$$

Whereas the weaker regularity assumption (5.2) is always supposed to be valid, the constraint qualification (5.15) is needed only for some of the following results.

We need the following definition. If for $\epsilon > 0$ and at least one $i = 1, \ldots, q$ $h_i(x(\tau_1-\epsilon),\tau_1-\epsilon) > 0$, and $h_i(x(\tau_1+\epsilon),\tau_1+\epsilon) = 0$, then τ_1 is called *entry time*. τ_2 is called *exit time*, if $h_i(x(\tau_2-\epsilon),\tau_2-\epsilon) = 0$, and $h_i(x(\tau_2+\epsilon),\tau_2+\epsilon) > 0$. If $h_i(x(\tau),\tau) = 0$, $h_i(x(\tau-\epsilon),\tau-\epsilon) > 0$, $h_i(x(\tau+\epsilon),\tau+\epsilon) > 0$, then τ is denoted as *contact time*. Together, τ_1, τ_2, and τ are called *junction times*.

Although there exist various forms of optimality conditions, we restrict ourselves in the following to present a brief introduction to the so-called *direct adjoining approach*.

Define the Hamiltonian, the Lagrangean, and the control region as follows:

$$H = \lambda_o F + \lambda f, \tag{5.16}$$

$$L = H + \mu g + \nu h \tag{5.17}$$

$$\Omega(x,t) = \{u \in \mathbb{R}^m | g(x,u,t) \geq 0\}. \tag{5.18}$$

The necessary optimality conditions of the direct method for pure state constraints are contained in the following

Theorem 5.2. *Let $(x^*(t),u^*(t))$ be an optimal pair for the control problem (5.1, 2.5). The regularity condition (5.2) is assumed for all $t \in [0,T]$, $x = x^*(t)$, $u \in \Omega(x^*(t),t)$. Then there exist a constant $\lambda_o \geq 0$, a piecewise continuously differentiable adjoint function $\lambda(t) \in \mathbb{R}^n$, and piecewise continuous multiplicator functions $\mu(t) \in \mathbb{R}^n$ and $\nu(t) \in \mathbb{R}^d$. Moreover, for each point of discountinuity $\tau_j \in [0,T]$ of $\lambda(t)$ there is a vector $\eta(\tau_j) \in \mathbb{R}^q$ of jump sizes. Finally, there exists constant multipliers $\alpha \in \mathbb{R}^\ell$, $\beta \in \mathbb{R}^{\ell'}$, $\gamma \in \mathbb{R}^q$ such that*

$$(\lambda_o,\lambda,\mu,\nu,\alpha,\beta,\gamma,\eta(\tau_1),\eta(\tau_2),\ldots) \neq 0 \quad \text{for all } t.$$

Then the following optimality conditions hold for all t except at points of discontinuity of u^* and junction times

$$u^*(t) = \arg \max_{u \in \Omega(x^*(t),t)} H(x^*(t),u,\lambda_o,\lambda(t),t) \tag{5.19}$$

$$L_u = 0 \tag{5.20}$$

$$\dot{\lambda} = r\lambda - L_x \tag{5.21}$$

$$\mu \geq 0, \ \mu g = 0 \tag{5.22}$$

$$\nu \geq 0, \ \nu h = 0 \tag{5.23}$$

$$\lambda(T) = \lambda_o S^*[T] + \gamma h_x^*[T] + \alpha a_x^*[T] + \beta b_x^*[T] \tag{5.24}$$

$$\gamma \geq 0, \ \gamma h^*[T] = 0 \tag{5.25}$$

$$\alpha \geq 0, \ \alpha a^*[T] = 0. \tag{5.26}$$

In (5.19 - 5.23) the arguments $x^*(t)$, $u^*(t)$, λ_o, $\lambda(t)$, $\mu(t)$, $\nu(t)$, and t are to be substituted. The notation $S_x^*[\,]$, $h_x^*[\,]$ etc. is analogously defined to (2.96).

λ and H are piecewise continuous functions of time. At points in which (5.13) is active jumps of the following form may occur:

$$\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau)h_x^*[\tau] \tag{5.27}$$

$$H[\tau^-] = H^*[\tau^+] - \eta(\tau)h_t^*(x^*[\tau] \tag{5.28}$$

$$\text{with } \eta(\tau) \geq 0 \text{ and } \eta(\tau)h^*[\tau] = 0. \tag{5.29}$$

Here $\tau^-(\tau^+)$ denotes the left-hand (right-hand) limit.

In the transversality condition (5.24) it has been taken into consideration that the path restriction (5.13) must hold also at the end point. This generates a terminal condition (5.1d), namely $h(x(T),T) \geq 0$.

Remark 5.1. Some further results are available for *regular problems* where the maximization (5.19) with respect to u is *unique*. (A sufficient condition for the regularity of H is the negative definiteness of H_{uu}.) In this case it turns out that the control function u is *continuous* even at the junction times.

Consider pure state constraints of first order and continuous controls. Assume further that the constraint qualifications (5.15) is satisfied. Then the adjoint variable λ is *continuous*.

Remark 5.2. In section 2.5 it has been proved that for a concave maximized Hamiltonian and a concave salvage value the necessary optimality conditions are also sufficient. This result remains also valid for optimal control problems (5.1, 13) with path restrictions and terminal conditions, if we additionally assume that the path restrictions g, h, a are concave in x and b is linear in x.

In the case of infinite time horizon the limiting transversality condition (2.93) provides an additional sufficient condition.

If H is not regular but *linear*, then λ is continuous at junction times provided that the entry and/or exit is *nontangential*, i.e., if $k[\tau^-] < 0$ and/or $k[\tau^+] > 0$.

5.3. An Example

We now give two simple illustrations of theorem 5.2. Further applications of the theory provided in section 5.1 and 2 can be found in section 6, 7 and 8.

Example 5.1 (see also example 2.7)

$$\max_u \int_0^4 (-x)dt \tag{5.30a}$$

$$\dot{x} = u, \; x(0) = 1 \tag{5.30b}$$

$$g = (1+u, 1-u) \geq 0 \tag{5.30c}$$

$$h = x \geq 0 \tag{5.30d}$$

$$b = x(4) - 1 \geq 0. \tag{5.30e}$$

Because of the simple structure of this problem, its optimal solution may be easily stated as follows (see also fig. 5.1):

time interval	u	x	active constraint
[0,1)	-1	1-t	$g_1 = 0$
[1,3]	0	0	$h = 0$
(3,4]	1	t-3	$g_2 = 0$

$\tau_1 = 1$ is an entry time, while $\tau_2 = 3$ provides an exit time. Clearly, (5.30) is a first-order state constraint, since $k = h_x\dot{x} = u$ contains the control u.

$$H = -x + \lambda u; \quad L = -x + \lambda u + \mu_1(1+u) + \mu_2(1-u) + \nu x \tag{5.31}$$

The necessary optimality conditions are as follows

$$L_u = \lambda + \mu_1 - \mu_2 = 0, \tag{5.32}$$

$$\dot{\lambda} = -L_x = 1 - \nu, \tag{5.33}$$

$$\mu_i \geq 0, \; \mu_1(1+u) = 0, \; \mu_2(1-u) = 0, \tag{5.34}$$

$$\nu \geq 0, \; \nu x = 0, \tag{5.35}$$

$$\lambda(4) = \beta b_x + \gamma h_x = \beta + \gamma; \; \gamma \geq 0, \; \gamma h = 0. \tag{5.36}$$

Since the entry and exit is nontangential[1] the costate is continuous also at the junction times $\tau_1 = 1$, $\tau_2 = 3$.

Consider first the boundary solution interval [1,3]. From u = 0 and (5.34) follows $\mu_1 = \mu_2 = 0$. Hence, from (5.32) follows $\lambda = 0$, and from (5.33) $\nu = 1$.

Because of the continuity of λ we have

$$\lambda(1^-) = \lambda(3^+) = 0. \tag{5.37}$$

(5.35) implies that because of $x > 0$ it holds that $\nu = 0$ for [0,1). Thus, according to (5.33) $\dot{\lambda} = 1$. This together with (5.37) yields $\lambda = t - 1$. Since $u = -1$ we have $\mu_2 = 0$. Hence, from (5.32) follows $\mu_1 = 1 - t$.

Analogously we obtain for (3,4] $\lambda = t - 3$, $\mu_1 = 0$, $\mu_2 = t - 3$, $\nu = 0$.

Since for t = 4 the relation h > 0 is satisfied, from (5.36) we obtain $\gamma = 0$. Thus, $\beta = \lambda(4) = 1$.

The multiplicator functions are collected in the following table (see fig. 5.1):

time interval	λ	ν	μ_1	μ_2
[0,1)	t-1	0	1-t	0
[1,3]	0	1	0	0
(3,4]	t-3	0	0	t-3

1) Nontangential means that $\dot{k} = h_x\dot{x} = u$ is discontinuous with respect to τ.

Since H^o is linear x, the sufficient optimality conditions are satisfied and the sketched solution is optimal.

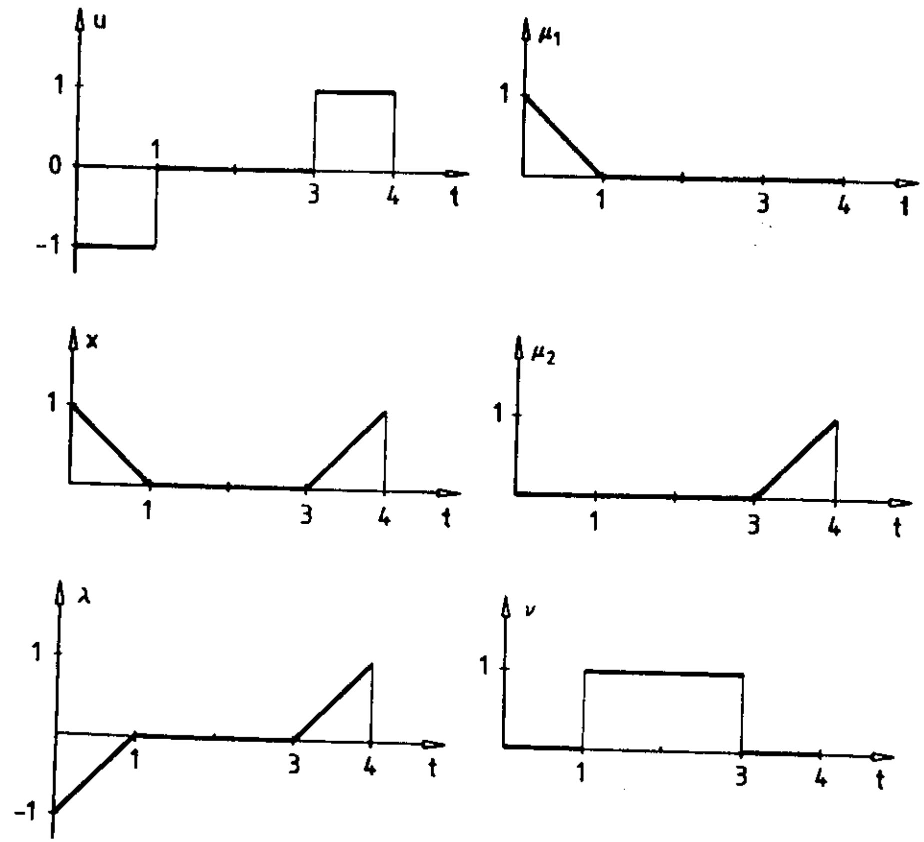


Fig. 5.1: Optimal control, state trajectory, costate and multipliers for example 5.1

Whereas in the preceding example no jumps in λ occur, this is no more true in the following example.

Example 5.2.

$$\min_u \int_0^3 e^{-rt} u\,dt \tag{5.38a}$$

$$\dot{x} = u, \quad x(0) = 0 \tag{5.38b}$$

$$0 \le u \le 3 \tag{5.38c}$$

$$h = x - 1 + (t-2)^2 \ge 0. \tag{5.38d}$$

First we note that for $r = 0$ any feasible solution satisfying $x(3) = 1$ is optimal. For $r \ge 0$ the following solution turns out to be optimal.

time interval	u	x
[0,1)	0	0
[1,2]	2(2-t)	$1-(t-2)^2$
(2,3]	0	1

(5.14) has now the form

$$k = u + 2(t-2). \tag{5.39}$$

In entry time $\tau_1 = 1$ both u and k are discontinuous, i.e. this entry is *nontangential*. On the other hand, u and hence k are continuous at exit time $\tau_2 = 2$, i.e. this exit is tangential (see fig. 5.2).

The regularity condition (5.15) is *not* valid, since for $g_1 = u$, $g_2 = 3 - u$ the matrix

$$\begin{pmatrix} \partial g_1/\partial u & g_1 & 0 & 0 \\ \partial g_2/\partial u & 0 & g_2 & 0 \\ \partial k/\partial u & 0 & 0 & h \end{pmatrix} = \begin{pmatrix} 1 & u & 0 & 0 \\ -1 & 0 & 3-u & 0 \\ 1 & 0 & 0 & x-1+(t-2)^2 \end{pmatrix}$$

has for $t = 2$ the rank 2 (the first and the third row are linear dependent).

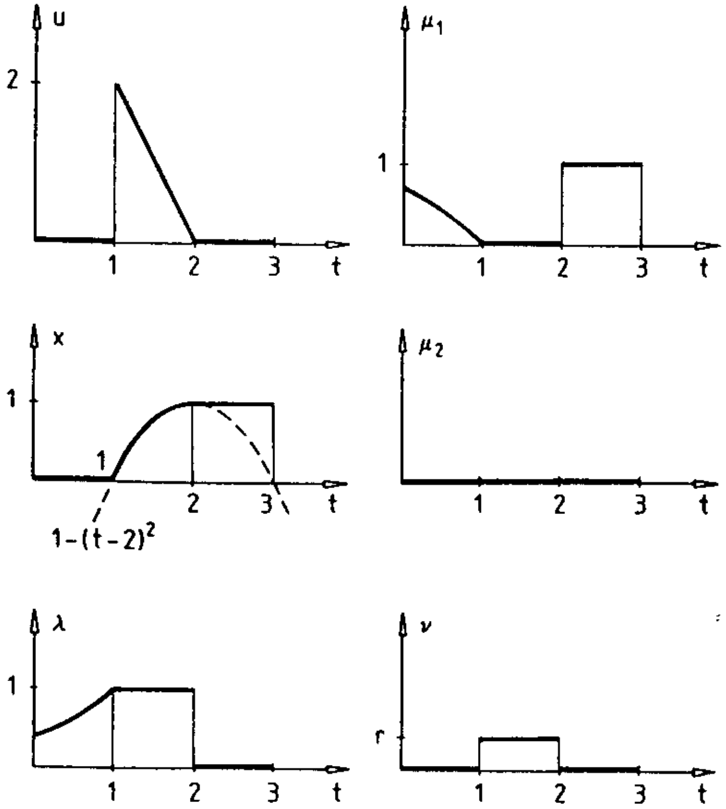


Fig. 5.2: Optimal solution paths for example 5.2

To apply theorem 5.2 we consider the Lagrangean

$$L = -u + \lambda u + \mu_1 u + \mu_2(3-u) + \nu[x - 1 + (t-2)^2].$$

The necessary optimality conditions are

$$L_u = -1 + \lambda + \mu_1 - \mu_2 = 0 \qquad (5.40)$$

$$\dot{\lambda} = r\lambda - L_x = r\lambda - \nu \qquad (5.41)$$

$$\mu_i \geq 0,\ \mu_1 u = 0,\ \mu_2(3-u) = 0 \qquad (5.42)$$

$$\nu \geq 0,\ \nu[x - 1 + (t-2)^2] = 0 \qquad (5.43)$$

$$\lambda(3) = 0 \qquad (5.44)$$

$$\lambda(2^-) = \lambda(2^+) + \eta,\ \eta \geq 0. \qquad (5.45)$$

Note that according to the remarks at the end of section 5.2 the costate λ is continuous at entry time $\tau_1 = 1$. On the other hand, λ can have a jump of the form (5.45) at exit time $\tau_2 = 2$.

For $t\epsilon(2,3]$ from the complementary condition (5.43) follows $\nu = 0$. Hence, because of (5.41, 44), $\lambda = 0$. From (5.42) follows $\mu_2 = 0$ in $(2,3]$. Thus, from (5.40) we get $\mu_1 = 1$.

In the boundary solution interval $[1,2]$ we have $0 < u < 3$ and so $\mu_1 = \mu_2 = 0$. Thus, according to (5.40) $\lambda = 1$. Hence, (5.41) yields $r = \nu$. For $t = 2$ we obtain $\eta = 1$. This shows that (5.45) is satisfied.

$[0,1]$: (5.43) implies $\nu = 0$. Solving (5.41) backwards with $\lambda(1) = 1$ we obtain $\lambda = \exp\{r(t-1)\}$. Note that λ is continuous at $t = 1$. From $\mu_2 = 0$ and (5.40) follows $\mu_1 = 1 - \exp\{r(t-1)\} \geq 0$. In the following table the multipliers are summarized:

time interval	λ	ν	μ_1	μ_2
$[0,1)$	$e^{r(t-1)}$	0	$1-e^{r(t-1)}$	0
$[1,2]$	1	r	0	0
$(2,3]$	0	0	1	0

5.4. Exercices for section 5

5.1. Solve

$$\max_u \int_0^1 u\,dt$$

subject to $\dot{x} = u$, $x(0) = 1$, $0 \leq u \leq x$.

5.2. $\max_u \int_0^2 (-x)\,dt$ (5.46a)

s.t. $\dot{x} = u,\ x(0) = 1$ (5.46b)

$u + 1 \geq 0,\ 1 - u \geq 0,\ x \geq 0.$ (5.46c)

5.3. $\max_u \int_0^2 (-u)\,dt$ (5.47)

s.t. (5.46bc).

Show that the solution is not unique.

5.4. Rework example 5.2 will terminal time $T = 1/2$.

5.5. Solve example 5.2 with *negative* parameter r.

5.6. Solve the following linear optimal control problem

$\min_u 2\int_0^3 x_1\,dt$ (5.48a)

$(\dot{x}_1,\dot{x}_2) = (x_2,u),(x_1(0),x_2(0)) = (2,0)$ (5.48b)

$-2 \leq u \leq 2$ (5.48c)

$x_1 \geq 0.$ (5.48d)

This problem admits an interpretation as 'weak landing' of a rocket (omitting the gravitation). x_1 measures the position of a particle ('the rocket'), x_2 its velocity.

(Note that the constraint (5.48d) is of order 2, i.e. the second derivative of x_1 with respect to time, i.e. $\ddot{x}_1 = \dot{x}_2 = u$ is the first which contains the control variable.)

6. Optimal Economic Growth

6.1. The Neoclassical Growth Model

In any economy choices must be made between provision for the present (*consumption*) and provision for the future (*capital accumulation*). While more consumption is preferable to less at any moment to time, more consumption means less capital accumulation - and the smaller the capital accumulation, the smaller the future output, hence the smaller the future potential consumption. There are two extreme consumption policies: "Live today, for tomorrow we die", and the Stalinist policy of consuming as little as possible today so as to increase capital and the potential for future consumption.

The neoclassical growth model characterizes economic growth in an aggregate closed economy. Aggregative means that the economy produces a single homogeneous good, the output of which at time t is Y(t), using two factor inputs, labor L(t) and capital K(t), where t is assumed to vary continuously; closed means that neither output nor input is imported or exported: all output is either consumed or invested. If consumption at time t is C(t) and investment at time t is I(t), then the income identity states

$$Y(t) = C(t) + I(t). \quad (6.1)$$

Assuming that the existing capital stock K(t) depreciates at the constant proportionate rate α, the capital accumulation is governed by

$$\dot{K}(t) = I(t) - \alpha K(t). \quad (6.2)$$

Output is determined by the aggregative production function

$$Y = F(K,L).$$

The production function is assumed to be concave in both production factors:

$$F_K > 0,\ F_L > 0;\ F_{KK} < 0,\ F_{LL} < 0. \quad (6.3)$$

Moreover, F exhibits constant returns to scale, so, for any positive scale factor ζ:

$$F(\zeta K,\zeta L) = \zeta F(K,L) = \zeta Y. \quad (6.4)$$

In particular, choosing $\zeta = 1/L$:

$$y = \frac{Y}{L} = F\left(\frac{K}{L},1\right) = f\left(\frac{K}{L}\right) = f(k), \qquad (6.5)$$

where the per-worker quantities are denoted by the corresponding lower-case letters, and the function $f(\cdot)$ gives output per worker as a function of capital oer worker:

$$k = \frac{K}{L},\ c = \frac{C}{L},\ i = \frac{I}{L},\ y = \frac{Y}{L}.$$

From (6.3) follows

$$f'(k) > 0,\ f''(k) < 0 \quad \text{for} \quad k > 0. \qquad (6.6)$$

If we assume that the labor force grows at given exponential rate ρ, $\dot{L}/L = \rho$, the investment identity can be rewritten as

$$\dot{k} = \frac{d}{dt}\left(\frac{K}{L}\right) = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = i - \alpha k - k\rho. \qquad (6.7)$$

From (6.1, 5) we get the following differential equation of economic growth:

$$\dot{k} = f(k) - \delta k - c,\ k(0) = k_o, \qquad (6.8)$$

where $\alpha + \rho = \delta$, and k_o is the initial level of capital per worker.

The economic objective of a central planner is assumed to be based on standards of living as measured by consumption per worker. In particular, it is assumed that the control planner has a utility function, giving utility, U, at any instant of time as a function of consumption per worker at that time: $U = U(c(t))$.

6.2. The Ramsey Model

We first consider a strictly concave utility function with

$$U'(c) > 0,\ U''(c) < 0. \qquad (6.9)$$

Thus, the central planner is faced to the following infinite time optimal control problem (Ramsey model):

$$\max_c \int_0^\infty e^{-rt} U(c)\,dt \qquad (6.10a)$$

subject to the state equation (6.8) and the (mixed) path restriction

$$0 \le c \le f(k). \qquad (6.10b)$$

In addition to (6.6) we suppose that

$$f(0) = 0,\ f'(0) > r + \delta,\ f'(\infty) < \delta. \qquad (6.11)$$

To apply theorem 5.1 we define the Hamiltonian and Lagrangean as follows:

$$H = U(c) + \lambda[f(k) - \delta k - c],$$
$$L = H + \mu_1 c + \mu_2[f(k) - c].$$

The necessary optimality conditions are:

$$L_c = U'(c) - \lambda + \mu_1 - \mu_2 = 0, \qquad (6.12)$$

$$\dot{\lambda} = r\lambda - L_k = [r + \delta - f'(k)]\lambda - \mu_2 f'(k) \qquad (6.13)$$

$$\mu_1 \ge 0,\ \mu_1 c = 0, \qquad (6.14)$$

$$\mu_2 \ge 0,\ \mu_2[f(k) - c] = 0. \qquad (6.15)$$

We first consider the *interior* of the control region (6.10b). From the complementary slackness conditions (6.14, 15) follows $\mu_1 = \mu_2 = 0$, and by (6.12)

$$\lambda = U'(c). \qquad (6.16)$$

Differentiating with respect to time we obtain

$$\dot{c} = \frac{U'(c)}{U''(c)}[r + \delta - f'(k)]. \qquad (6.17)$$

We now analyze the canonical system of differential equations (6.8, 17) by means of a phase portrait. Because of (6.11) there exists is unique equilibrium point $(\hat{k},\hat{c})$ defined by

$$f'(\hat{k}) = r + \delta \qquad (6.18a)$$

$$\hat{c} = f(\hat{k}) - \delta\hat{k}. \qquad (6.18b)$$

Note that $(\hat{k},\hat{c})$ lies in the interior of the domain (6.10b).

Defining

$$f'(k_m) = \delta,\ f(\tilde{k}) = \delta\tilde{k}$$

we have $0 < \hat{k} < k_m < \tilde{k} < \infty$ (see fig. 6.1). Note that k_m is those stock of capital which can be sustained in the long-run. It corresponds to the maximum sustainable consumption rate.

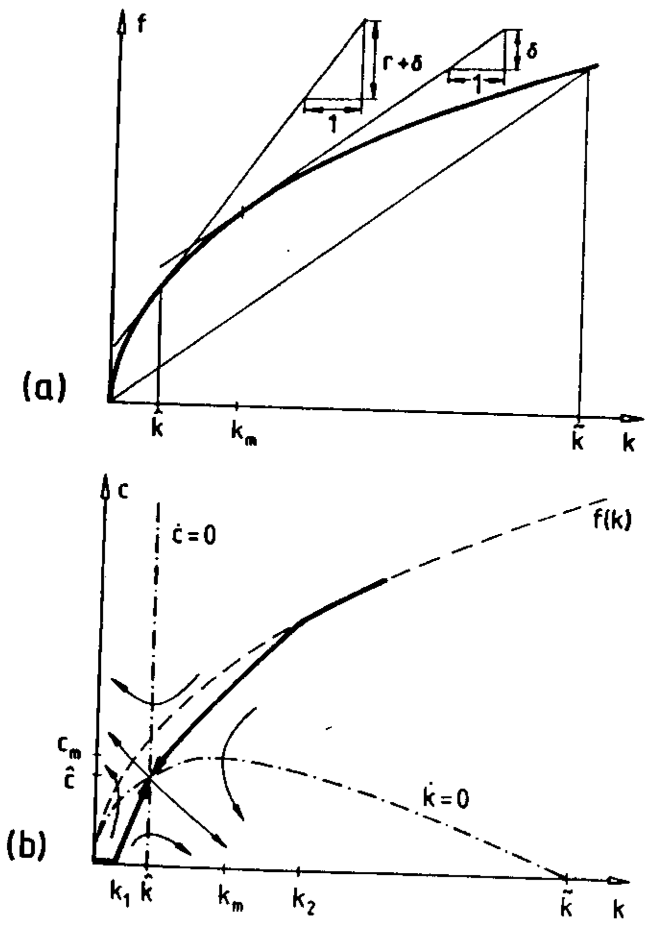


Fig. 6.1: Phase portrait for capital and consumption in Ramsey's model

The isocline $\dot k = 0$ is a concave curve crossing the origin. It reaches its maximum at $k = k_m$, and meets the abscissa again for $k = \tilde k$. The $\dot c = 0$ isocline is the vertical line $k = \hat k$.

It hold that

$$\frac{\partial \dot k}{\partial k} = f'(k) - \delta > 0 \quad \text{für} \quad k < k_m, \quad \frac{\partial \dot k}{\partial c} = -1 < 0$$

$$\frac{\partial \dot c}{\partial k} = -\frac{U'}{U''} f'' < 0, \qquad \left.\frac{\partial \dot c}{\partial c}\right|_{\dot c = 0} = 0.$$

Since the determinant of the Jacobian matrix is negative, $(\hat k, \hat c)$ is a saddle point. The saddle point path is upward sloping. We denote by (k_1, k_2) the interval in which the consumption path lies in the interior of (6.10b).

Let us now deal with the possibility that the optimal trajectory is on the boundary of the admissible control region (6.10b).

It seems reasonable to choose

$c = 0$ for $k \geq k_1$, and $c = f(k)$ for $k \geq k_2$.

To show that this path satisfies the necessary optimality conditions we consider an initial stock of capital $k(0) < k_1$ and set $c(t) = 0$. From (6.8) follows $\dot k > 0$. Let τ_1 be the time at which $k(\tau_1) = k_1$. We assume that for $t \geq \tau_1$ the saddle point path is choosen.

To show that the necessary optimality conditions are satisfied also for $t < \tau_1$ we first state

$$\lambda(\tau_1) = U'(c(\tau_1)) = U'(0). \tag{6.19}$$

From (8.39) follows $\mu_2 = 0$. Hence the adjoint equation (6.13) yields

$$\dot\lambda = [r + \delta - f'(k)]\lambda < 0.$$

Thus, $\lambda(t) > \lambda(\tau_1)$ for $t \in [0, \tau_1)$, and from (6.12) follows

$$\mu_1 = \lambda - U'(0) > 0.$$

So the optimality condition (6.12-15) are satisfied. Note that in the case $U'(0) = \infty$ the boundary arc $c = 0$ can not occur, i.e. $k_1 = 0$.

For $k(0) > k_2$ we set $c = f(k)$, and $\dot k = -\delta k < 0$. Denote by τ_2 the time at which $k(\tau_2) = k_2$. Again, for $t \geq \tau_2$ the stable path to equilibrium is choosen.

To show that the necessary optimality conditions are satisfied we note that for $t > \tau_2$ the differential equation (6.17) is satisfied. Moreover, for $k = k_2$ the equilibrium path is steeper than the boundary $c = f(k)$, i.e.

$$\left.\frac{dc}{dk}\right|_{\substack{\text{saddle}\\ \text{point path}}} = \frac{\dot{c}}{\dot{k}} > f'(k) = \left.\frac{dc}{dk}\right|_{c=f(k)} . \tag{6.20}$$

Substituting (6.17) into (6.20) we get for $t = \sigma_2$

$$U'(c)[r + \delta - f'(k)] > U''(c)f'(k)\dot{k}. \tag{6.21}$$

From (6.12) follows for $t \in [\tau_2 - \tau_1, \tau_2]$

$$\mu_2 = U'(c) - \lambda. \tag{6.22}$$

Differentiation with respect to t yields for left-side limit τ_2^-

$$\dot{\mu}_2(\tau_2^-) = U''(c(\tau_2))\dot{c}(\tau_2^-) - \dot{\lambda}(\tau_2).$$

From $c = f(k)$ for $t \le \tau_2$ follows $\dot{c} = f'(k)\dot{k}$. From this, (6.13) and (6.22) we obtain

$$\begin{aligned}\dot{\mu}_2(\tau_2^-) &= U''f'\dot{k} - (r+\delta-f')\lambda + \mu_2 f' \\ &= U''f'\dot{k} - (r+\delta-f')U' + (r+\delta)\mu_2.\end{aligned} \tag{6.23}$$

From (6.21) and $\mu_2(\tau_2) = 0$ we get $\dot{\mu}_2(\tau_2^-) < 0$. Thus, $\mu_2 > 0$ for $(\tau_2-\epsilon, \tau_2)$. This shows that the optimality conditions (6.12-15) are satisfied.

Since at least for $k_o \le k_2$ from (6.12) follows $\lambda \ge 0$, the Hamiltonian is concave. Since the function $f(k) - c$ in (6.10b) is concave in (k,c) too, remark 5.2 can be applicated. The limiting transversality condition (2.93)

$$\lim e^{-rt}\lambda(t)[k(t) - k^*(t)] \ge 0$$

is for the saddle point path $(k^*(t), \lambda(t)) \to (\hat{k}, \hat{\lambda})$ satisfied. (Note that any feasible solution $k(t)$ is because of (6.8, 10b) nonnegative. Thus all sufficient conditions are satisfied, and the solution constructed above is indeed optimal.

Let us briefly interpret economically the optimal interior solution. The equilibrium $(\hat{k}, \hat{c})$ characterized by (6.18) represents a balanced growth path. Along it capital per worker and consumption per worker are constant; hence total capital $(K = kL)$, and total output $(Y = Lf(k))$ all grow at the same rate - namely, the rate δ of gwoth of the labor force.

If $k_o \neq \hat{k}$, then $c^*(0)$ is optimaly choosen at the saddle point path. For increasing t both $k^*(t)$ and $c^*(t)$ converge to $(\hat{k}, \hat{c})$, where the optimal consumption rate increases with the capital stock. Above we have shown that for $U'(0) < \infty$ for a small stock of capital $(k \le k_1)$ it is optimal to consume nothing to let increase k as fast as possible.

6.3. A Prescribed Minimum Stock of Capital

Let us now add the pure state to the Ramsey model (6.8, 10) constraint

$$k \ge \underline{k}. \tag{6.24}$$

Clearly, $k_o \ge \underline{k}$.

Define

$$\begin{aligned} H &= U(c) + \lambda[f(k) - \delta k - c] \\ L &= H + \mu_1 c + \mu_2[f(k) - c] + \nu(k-\underline{k}). \end{aligned} \tag{6.25}$$

The optimality conditions of theorem 5.2 are

$$L_c = U'(c) - \lambda + \mu_1 - \mu_2 = 0 \tag{6.26}$$

$$\dot{\lambda} = r\lambda - L_k = [r + \delta - f'(k)]\lambda - \mu_2 f'(k) - \nu \tag{6.27}$$

$$\mu_i \ge 0,\ \mu_1 c = 0,\ \mu_2[f(k) - c] = 0 \tag{6.28}$$

$$\nu \ge 0,\ \nu(k-\underline{k}) = 0. \tag{6.29}$$

Since the Hamiltonian is strictly concave in c, it is regular. Thus, according to remark 5.1 no jumps in c and λ can occur. Note that the constraint qualification is always satisfied, especially at the entry time. Note further that for the solution stated below none of the mixed control restrictions $c \ge 0$, $c \le f(k)$ and the pure state constraint $k \ge \underline{k}$ become simultaneously active. For $k \ge \underline{k}$ the statements in section 6.2 remain valid $(\nu = 0)$.

We proceed as follows. We try to find a path starting in $k = k_o$ which remains in the feasible domain and satisfies the necessary optimality conditions. To this end we intersect the $k = \underline{k}$ with the $\dot{k} = 0$ curve and consider the trajectory leading to $(\underline{k}, \underline{c})$ (see fig. 6.2).

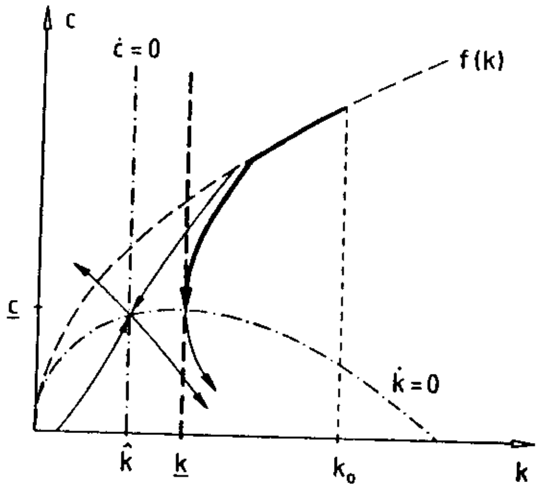


Fig. 6.2: Solution trajectory for the Ramsey model with prescribed minimal stock of capital

For $k = k_o$ we choose the consumption rate at this trajectory. Proceeding at this path the point $(\underline{k},\underline{c})$ is reached in finite time. Furthermore, for $t \in [\tau,\infty)$ we set $c(t) = \underline{c}$.

For $t < \tau$ the necessary optimality condition are clearly satisfied. For $t \geq \tau$ from (6.26) we conclude $\lambda = U'(\underline{c})$ (since $\mu_1 = \mu_2 = 0$). We set $\dot{\lambda} = 0$ in (6.27) and obtain because of $\underline{k} > \hat{k}$ and (6.18a)

$$\nu = [r + \delta - f'(\underline{k})]U'(\underline{c}) > 0.$$

Note that at time τ the adjoint variable λ is continuous. Thus, the necessary conditions are satisfied for $t \geq \tau$ too.

Since k and λ are bounded for the trajectory so constructed, the sufficient conditions are satisfied and the solution is optimal. Note that all other trajectories are suboptimal or infeasible.

6.4. Linear Consumption Utility

Consider the case of constant marginal utility of per-capita consumption, $U(c) = c$. This yield the following control model

$$\max_c \int_0^\infty e^{-rt} c\,dt \tag{6.30a}$$

$$\dot{k} = f(k) - \delta k - c,\ k(0) = k_o \tag{6.30b}$$

$$\underline{c} \leq c \leq f(k). \tag{6.30c}$$

In (6.30c) we suppose a minimum consumption rate $\underline{c} > 0$.

To determine the optimal solution we use the MRAP-theorem. It is easily seen that problem (6.30) has the form (3.13, 14, 15)

$$M(k) = f(k) - \delta k,\ N(k) = -1 \tag{6.31a}$$

$$\Omega(k) = [-\delta k, f(k) - \delta k - \underline{c}],\ k \geq 0. \tag{6.31b}$$

The singular solution $\hat{k}$ is solution of (3.16)

$$I(\hat{k}) = rN(\hat{k}) + M'(\hat{k}) = f'(\hat{k}) - (r+\delta) = 0. \tag{6.32}$$

This shows that the equilibrium in the linear case (i.e. the singular solution) coincides with the stationary solution (6.18)) of the nonlinear model. To apply the MRAP-theorem 3.1 we first observe that (3.16) is satisfied because of $f'' < 0$. Moreover, (3.18) is also true.

Denote the stationary consumption rate as $\hat{c} = f(\hat{k}) - \delta\hat{k}$. Clearly, we assume $\underline{c} \leq \hat{c}$. Thus, $0 \in \Omega(\hat{k})$, i.e. the equilibrium can be sustained by an admissible control c. Hence, theorem 3.1 provides the optimality of the following most rapid approach trajectory:

$$c(t) = \begin{Bmatrix} \underline{c} \\ \hat{c} \\ f(k(t)) \end{Bmatrix} \text{ for } \begin{cases} k \in [\underline{k}, \hat{k}) & (6.33a) \\ k = \hat{k} & (6.33b) \\ k > \hat{k}, & (6.33c) \end{cases}$$

where $\underline{k}$ denotes the intersection of $\dot{k} = 0$ and $c = \underline{c}$:

$$f(\underline{k}) - \delta\underline{k} = \underline{c}. \tag{6.34}$$

In (6.33a) it has been taken into consideration that for $k < \underline{k}$ no feasible solution, i.e. no MRAP, do exist. Even by choosing the minimum consumption rate $\underline{c}$ the stock of capital is exhausted in finite time. After this time nothing can be consumed which contradicts to (6.30c). The optimal solution is sketched in fig. 6.3.

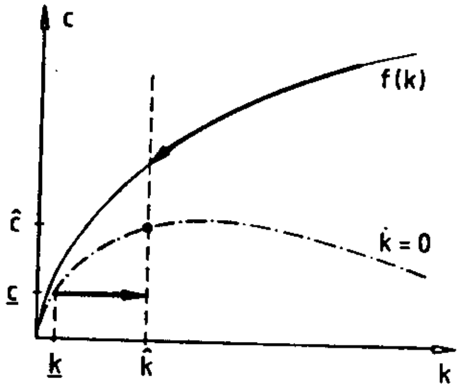


Fig. 6.3: The optimal consumption pattern for the linear Ramsey-model with minimum consumption rate

It turns out that the long-run equilibrium is the same in the linear and in the nonlinear model. However, in the linear case the stationary point is reached in finite time, while for a strictly concave utility function the convergence is asymtotic.

6.5. Exercices for Section 6

6.1. Solve the Ramsey model for a minimum consumption rate $\bar{c} > 0$.

6.2. Analyze the Ramsey model with a convex-concave utility function.

6.3. Formulate and solve the 'vampire problem' (Hartl and Mehlmann, 1982).

6.4. *Rich Rentier* plans to retire at age 65 with a lump sum pension of k_o dollars. Rich estimates his remaining life span to be T years and he wants to consume his wealth during these T years in a way that will maximize his total utility of consumption. Since he does not want to take investment

risks, he plans to put his money into an insured savings account that pays interest at a continuously compounded rate of ρ. If we let the state variable be the capital stock k at time t and the control the consumption rate formulate an optimal control model using an utility functions,

a) $U(c) = \sqrt{c}$, b) $U(c) = \ln c$.

Solve these models. Discuss possible modifications (bequest functions, terminal constraint $k(t) \geq k_T$, maximal consumption rate $\bar{c} \geq c(t)$, etc.).

6.5. Capital accumulation and environmental control. Consider the following extension of Ramsey's model (see section 6.1)

$$F(K) = C + I + A,$$

where A are the abatement expenditures. Capital produces not only output but also pollution P (in form of emissions). Pollution may be controled by abatement activities. Thus

$$P = P(K,A) \quad \text{with} \quad P_K > 0,\ P_{KK} > 0;\ P_A < 0,\ P_{AA} > 0.$$

Assume a utility function U(C,P) of form (7.7) (discussed below in more detail). Then the following control problem arises:

$$\max_{C,A} \int_0^\infty e^{-rt} U(C,P(K,A))\,dt$$

$$\dot{K} = F(K) - \delta K - C - A$$

$$C \geq 0,\ A \geq 0,\ F(K) - C - A \geq 0.$$

Carry out a phase portrait analyse. Use theorem 5.1 for a complete solution. Compare the optimal consumption pattern with the Ramsey model.

7. POLLUTION CONTROL

The improvement of *environmental quality* has become an important objective in the framework of economic and social policy not only in the industrialized world. It is often claimed that some of the goals of economic policy are in a *trade-off* relation, implying that better levels of one goal variable must, cet. par., be "paid for" by lower levels of another. In this section a central issue in pollution control and economic activity is studied, namely the pure *consumption-pollution trade-off*. This model can be extended by including also the accumulation of capital. Using the capital stock as (single) production factor the produced output can either be consumed, invested or used for abatement activities, i.e. for emission control (see exercise 7.5).

7.1. Consumption-Pollution Trade-off

In order to focus on the pure *consumption-pollution trade-off*, for the present we assume away problems of capital accumulation, natural resource exhaustion, population growth, and technical progress. Thus, it is supposed that there is a fixed supply of factor inputs which produce a fixed amount of output in each period. We may think that a policy of 'zero' growth in output has been decided upon by the central decision-makers.

The decision dilemma is as follows. The society derives utility from consumption but in concerning output, pollution is generated which yields disutility. By foregoing present consumption the amount of pollution in the future may be reduced.

Forster (1977) considers the following model. The fixed output produced in each time period is denoted by $\bar{Q}$. This output is allocated to consumption L or to pollution control A such that

$$\bar{Q} = C + A. \tag{7.1}$$

The stock of pollution, P, increases as a result of the consumption process. It seems reasonable to assume that the flow increases at an increasing rate with respect to consumption. Thus, we suppose that there is a convex pollution rate of consumption E(C) such that

$$E(0) = 0,\ E'(C) > 0,\ E''(C) > 0 \quad \text{for} \quad C > 0. \tag{7.2}$$

Furthermore, it is possible for the community to slow the accumulation (or hasten the decline) of pollution by devoting some expenditure to anti-pollution activities. The amount of pollution cleaned up will be a function G of the amount of pollution control expenditure A. This function satisfies the following conditions:

$$G(0) = 0,\ G'(A) > 0,\ G''(A) < 0 \quad \text{for} \quad A > 0. \tag{7.3}$$

The net contribution to the flow of pollution is measured by the pollution control function

$$Z(C) = E(C) - G(\bar{Q}-C)$$

being convex in the consumption rate C, i.e.

$$Z'(C) > 0,\ Z''(C) > 0. \tag{7.4}$$

Thus, the society gains in a *two-fold* manner by pollution control. First, with more antipollution control expenditure more pollution can be cleaned up. Secondly, since the consumption level is lower, less pollution is being generated.

(7.2, 3) imply that there exists a (unique) level of consumption, C_o, which will just sustain a clean environment. It holds that $0 < C_o < \bar{Q}$, and

$$Z(C) \lesseqgtr 0 \quad \text{for} \quad C \lesseqgtr C_o. \tag{7.5}$$

Moreover, we assume that the stock of pollution is subject to natural decay at a constant exponential rate α.

Thus, the dynamics of waste is governed by the differential equation

$$\dot{P} = Z(C) - \alpha P. \tag{7.6}$$

Furthermore, it is assumed that socio-economic welfare is measured by a strictly concave utility function of current consumption and the current stock of pollution. For simplicity, we restrict ourselves to separable (additive) utility functions U(C) + D(P) such that

$$U'(C) > 0,\ U''(C) < 0 \tag{7.7a}$$

$$D'(P) < 0,\ D''(P) < 0 \tag{7.7b}$$

(U is the consumption utility, while -D is the disutility due to pollution).

The aim of the central planning authority is to maximize the discounted flow of utility. In order to give consideration to generations not yet born, the time horizon is left infinite. This yields the following optimal control problem with P as state and C as control variable:

$$\max_C \int_0^\infty e^{-rt}[U(C) + D(P)]dt \tag{7.8a}$$

$$\dot{P} = Z(C) - \alpha P, \; P(0) = P_0 \tag{7.8b}$$

$$0 \le C \le \bar{Q} \tag{7.8c}$$

$$P \ge 0. \tag{7.8d}$$

To solve problem (7.8) we consider the Hamiltonian and the Lagrangean

$$H = U(C) + D(P) + \lambda[Z(C) - \alpha P]$$

$$L = H + \mu_1 C + \mu_2(\bar{Q}-C) + \nu P.$$

The necessary optimality conditions are (see theorem 5.2)

$$L_C = U' + \lambda Z' + \mu_1 - \mu_2 = 0 \tag{7.9}$$

$$\dot{\lambda} = r\lambda - H_P = (r+\alpha)\lambda - D' - \nu \tag{7.10}$$

$$\mu_i \ge 0, \; \mu_1 C = \mu_2(\bar{Q}-C) = 0 \tag{7.11}$$

$$\nu \ge 0, \; \nu P = 0. \tag{7.12}$$

It turns out that the limiting transversality condition

$$\lim_{t\to\infty} e^{-rt}\lambda(t) = 0 \tag{7.13}$$

is satisfied. This together woth (7.4, 7) establishes the sufficiency of the optimality conditions.

Clearly, the costate λ of the pollution is (as shadow price) negative.

Note that a phase portrait analysis in the (P,λ) plane has been carried out for this model in section 4.2.

Furthermore, the constraint qualification (5.15) is satisfied (provided that the control constraint (7.8c) and the state constraint (7.8d) do not become active simultaneously), i.e. the matrix

$$\begin{pmatrix} 1 & C & 0 & 0 \\ -1 & 0 & \bar{Q}-C & 0 \\ Z' & 0 & 0 & P \end{pmatrix} \tag{7.14}$$

has the rank 3.

Since the Hamiltonian is regular, the shadow price λ is continuous also at junction times (see remark 5.1).

Let us consider at first interior solutions with respect to (7.8c), i.e. $\mu_1 = \mu_2 = 0$. To carry out a phase portrait analysis we differentiate (7.9) with respect to t and eliminate $\dot{\lambda}$ and λ by using (7.10) and (7.9). This yields

$$\dot{C} = Z'[(r+\alpha)U'/Z' + D' + \nu]/(U''-U'Z''/Z'). \tag{7.15}$$

For $P > 0$ we have $\nu = 0$. By using standard phase plane techniques we obtain the state-control diagram as sketched in fig. 7.1.

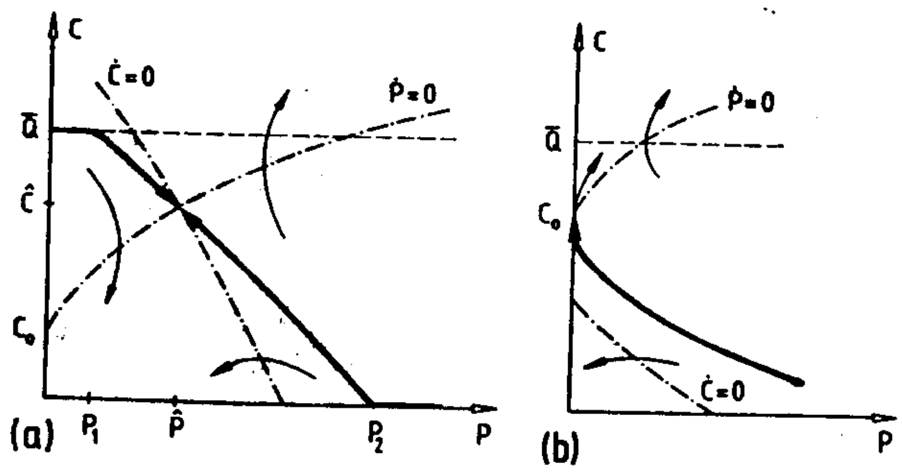


Fig. 7.1: Phase portrait for Forster's model (a) for a positive long-run stock of pollution, (b) for a clean stationary environment

The equilibrium $(\hat{P},\hat{C})$ defined by $\dot{P} = \dot{C} = 0$ characterizes the long-run behaviour of the optimal solution. The conditions on C and P in equilibrium are (compare also section 4.4):

$$Z(\hat{C}) = \alpha\hat{P} \tag{7.16}$$

$$U'(\hat{C}) = -\frac{Z(\hat{C})D(\hat{P})}{r+\alpha}. \tag{7.17}$$

(7.16) says that in equilibrium the amount of new pollution generaled, $Z(\hat{C})$, is just equal to the amount which decays away, $\alpha\hat{P}$.

The left-hand side of (7.17) is the *instantaneous* gain in utility from additioral consumption. The numerator of the right-hand side is the loss in utility causec by an increment in consumption. This utility is lost because additional pollution has been generated as a result of the momentary increase in consumption. If the pollution does not decay then this loss in utility must be endured *forever*. If however, the pollution does decay at some positive rate, the loss in utility in any period will be less. The present value of the forgoing measure is the right-hand side of (7.17).

Let us now turn to the optimal transitory solution optimal consumption path tha: lies along the saddle point path which is *downward sloping*. fig. 7.1 shows that for pollution levels "small enough" it is optimal to choose maximal consumption rate, $C = \bar{Q}$, whereas for $P \geq P_2$ it turns out that $C = 0$ is optimal (if we assume $U'(0) = \infty$, then $C > 0$ and $P_2 = \infty$).

The optimal consumption pattern is as follows: There is a unique long-run optimal stationary consumption rate $\hat{C}$ which is above the pollution-neutral consumption rate C_o. For $P_o < \hat{P}$ the optimal consumption rate $C_o > \hat{C}$, whereas in a "polluted" environment the optimal consumption is "low". In the first case $C^*(t)$ gradually decreases and $P^*(t)$ increases; for $P_o > \hat{P}$ the optimal paths show an opposite behaviour.

Until now we have silently assumed that in the long-run it is optimal to live in an environment which is polluted to a certain amount $\hat{P} > 0$ (given by (7.16, 17) and sketched in fig. 7.1(a)). From (7.8b) and (7.15) we see that this situation occurs if and only if the following condition holds

$$U'(C_o) > \frac{Z'(C_o)D'(0)}{r+\alpha}. \tag{7.18}$$

Thus, if starting with $P = 0$ and $C = C_o$ the marginal consumption utility exceeds the present value of the disutility stream $Z'(C_o)D'(0))$ induced by the additionally consumed unit, then it is optimal in the long-run to consume $\hat{C} > C_o$. This leads to $\hat{P} = Z(\hat{C})/\alpha > 0$.

Finally, let us consider the case that (7.18) does not hold. Then the isoclines $\dot{P} = 0$, $\dot{C} = 0$ have no intersection for $P > 0$, $C > 0$. Since the stationary pollution level P would be negative, the pure state constraint (7.8d) becomes active for any initial state $P_o > 0$. To construct the optimal solution also in this case we determine the intersection of the isocline $\dot{P} = 0$ and the restriction $P = 0$, i.e. the point $(0,C_o)$. Let us consider the path running into this point (see fig. 7.1b). The optimal solution is given by this trajectory: starting from a given initial pollution level P_o the corresponding consumption rate is to be choosen according to this trajectory. Having reached $(0,C_o)$ at time τ the clean environment is maintained by $C = C_o$ for $t \geq \tau$.

Note that the optimal consumption rate lies *below* the pollution-neutral consumption C_o, which is in this case optimal in the long-run.

The main result is that a certain pollution level or a pristine environment is optimal according as the trade-off relation (7.18) holds true or not.

7.2. Exercices for Section 7

7.1. a) Carry out for Forster's model a phase portrait analysis for the (P,C) plane. (Reconsides also section 4.3, where a brief introduction into the state-control phase plane was given.)

b) Analyze the case that the control constraints (7.8c) becomes active (see fig. 7.1a). (Hint: compare section 6.2.)

7.3. Carry out a sensitivity analysis for the equilibrium with respect to the cleaning rate α. Show that $d\hat{C}/d\alpha > 0$.

7.4. Analyze Forster's model for the case, wher in the long run a clean environment is optimal (compare Fig. 7.1b). Determine the multiplier ν.

7.5. Consider the following extension of Forster's model analyzed by Luptacik and Schubert (1982):

$$\max_{C,A} \int_0^\infty e^{-rt} U(C,P)\,dt \qquad (7.19a)$$

$$\dot{K} = F(K) - \delta K - C - A, \quad K(0) = K_o \qquad (7.19b)$$

$$\dot{P} = \varepsilon_1 F(K) + \varepsilon_2 C + \varepsilon_3 K - G(A) - \alpha P, \quad P(0) = P_o \qquad (7.19c)$$

$$A \geq 0 \qquad (7.19d)$$

$$P \geq 0. \qquad (7.19e)$$

Additionally to the model of Forster (1977) the capital accumulation by investment is included.

$U(C,P)$ is strictly concave utility function in C and P with $U_{CP} \leq 0$ (see also (7.7)).

(7.19c) arises from (7.6) by replacing $Z(C)$ through

$\varepsilon_1 F(K) + \varepsilon_2 C + \varepsilon_3 K - G(A)$.

Here $F(K)$ is the GNP produced by capital K, ε_1 is the emission rate refecting a part of technology which is used to produce $F(K)$. The emissions generated by consumption are described by ε_2. Third, the depreciated capital stock has potentially harmful effects (ε_3). A denotes the abatement expenditures, and $G(A)$ measures the effectiveness of these expenditures ($G' > 0$, $G'' < 0$, $G(0) = 0$).

(7.19b) is the usual capital accumulation dynamics by investment $I = F(K) - C - A$.

8. INVENTORY AND PRODUCTION PLANNING

Many manufacturing enterprises use a production-inventory system to manage *fluctuations* in consumer demand for a product. Once a product is made and put into inventory it incurs inventory holding costs of two kinds: first, the costs of physically storing the product, insuring it, etc.; and second the carrying charges of having storing the firm's money invested in the unsold inventory. The advantages of having products in inventory are: first, they are immediately available to meet demand; second, by using the warehous to store excess production during low demand periods to be available during high demand periods, it is possible to *smooth* the production schedules of the manufacturing plant. This usually permits the use of a smaller manufacturing plant that would otherwise be necessary.

8.1. The Arrow-Karlin Model

We remember to the notation used in example 1.2

$z(t)$ = inventory level at time t (state variable)

$v(t)$ = production rate at time t (control variable)

$d(t)$ = demand (sales) rate at time t (exogenously given, assumed to be positive and continuously differentiable).

We assume *linear* inventory holding costs and *convex* production costs, i.e. the cost function occuring in (1.13) are assumed to be time-independent and are specified as

$$c(v,t) = c(v) \quad \text{with} \quad c(0) = 0,\ c' > 0 \quad \text{for} \quad v > 0,\ c'' > 0 \qquad (8.1)$$

$$h(z,t) = hz. \qquad (8.2)$$

The task of the inventory-production manager is to determine a production path such that the cost functional

$$\int_0^T [c(v) + hz]\,dt \qquad (8.3a)$$

is minimized, where the following stock-flow differential equation is to be satisfied:

$$\dot{z} = v - d, \quad z(0) = z_o \geq 0 \qquad (8.3b)$$

Moreover, we stipulate that no shortage can occur. Clearly, the production rate is nonnegative:

$$z \geq 0,\ v \geq 0. \tag{8.3c}$$

Before we start to analyze this inventory problem by using the maximum principle we state the following result

<u>*Lemma 8.1.*</u> *For* $z_0 < \int_0^T d(t)dt$ *it holds that* $z(T) = 0$. *If the opposite is true, then* $v(t) = 0$ *for* $t\epsilon[0,T]$.

The necessary optimality conditions are according to theorem 5.2

$$H = -c(v) - hz + \lambda(v-d)$$

$$L = H + \mu v + \nu z \tag{8.4}$$

$$v = \arg\max_{v\geq 0} H$$

$$L_v = -c'(v) + \lambda + \mu = 0 \tag{8.5}$$

$$\dot{\lambda} = -L_z = h - \nu \tag{8.6}$$

$$\mu \geq 0,\ \mu v = 0 \tag{8.7}$$

$$\nu \geq 0,\ \nu z = 0 \tag{8.8}$$

$$\lambda(T) = \gamma \geq 0,\ \gamma z(T) = 0. \tag{8.9}$$

Since the Hamiltonian is strictly concave in H, from remark 5.1 follows that both the production rate and the shadow price λ is continuous. Note that the constraint qualification (5.15) is satisfied, i.e. the matrix

$$\begin{pmatrix} 1 & v & 0 \\ 1 & 0 & z \end{pmatrix}$$

has rank 2 provided that v and z are not simultaneously equal to zero. (In Lemma 9.4 it will be shown that this case can not occur.)

Since the Hamiltonian H is jointly concave in (z,v), the conditions (8.4-9) are sufficient for optimality.

We first consider the case of an empty initial inventory:

$$z(0) = 0. \tag{8.10}$$

We now need the following definition. A *boundary solution interval* $[\tau_1,\tau_2]$ is defined by $z(t) = 0$ for $t\epsilon[\tau_1,\tau_2]$, $\tau_1 = 0$ or $z(\tau_1-\epsilon) > 0$, and $\tau_2 = T$ or $z(\tau_2+\epsilon) > 0$ for small $\epsilon > 0$. For an *interior solution* interval (t_1,t_2) we have $z(t) > 0$ for $t\epsilon(t_1,t_2)$, $t_1 = 0$ or $z(t_1) = 0$, and $t_2 = T$ or $z(t_2) = 0$.

Next we study interiory and boundary solution intervals.

<u>*Lemma 8.2.*</u> *In interior solution intervals the optimal production rate v is positive and it holds*

$$v(t) = (c')^{-1}(\lambda_0+ht), \tag{8.11}$$

where λ_0 *is a constant to be determined depending on the interior solution interval.*

<u>Proof.</u> Let (t_1,t_2) be a time interval for which an interior solution path $z > 0$ is optimal. From (8.10) follows $z(t_1) = 0$. To guarantee $z(t) > 0$ for $t = t_1 + \epsilon$ we must have $\dot{z}(t_1^+) \geq 0$, i.e.

$$v(t_1^+) \geq d(t_1) > 0. \tag{8.12}$$

From (8.4, 5, 7) follows the following relation between the optimal production rate and the shadow price λ of the inventory:

$$v = 0 \quad \text{for} \quad \lambda \leq c'(0) \tag{8.13a}$$

$$v > 0,\ \lambda = c'(v) \quad \text{for} \quad \lambda > c'(0). \tag{8.13b}$$

From $z > 0$ and (8.8) follows $\nu = 0$. Hence, because of (8.6), we obtain $\dot{\lambda} = h$, and

$$\lambda = \lambda_0 + ht. \tag{8.14}$$

From (8.12, 13) follows $\lambda(t_1) > c'(0)$. Thus, because (8.14) we conclude $\lambda(t) > c'(0)$ for all $t\epsilon[t_1,t_2)$. From (8.13) we get $v > 0$, and (8.11) is valid. □

<u>*Lemma 8.3.*</u> *In boundary solution intervals the production rate meets the demand:*

$$v(t) = d(t) > 0. \tag{8.15}$$

Moreover, it holds that

$$h \geq \dot{d}(t)c''(d(t)). \tag{8.16}$$

Proof. Let $[\tau_1,\tau_2]$ be a boundary solution interval. Thus, $z(t) = 0$, and $\dot{z}(t) = 0$, i.e. (8.15) is valid. This together with (8.7) yields $\mu = 0$, and from (8.5) we obtain

$$\lambda = c'(d). \tag{8.17}$$

Differentiation with respect to t yields $\dot{\lambda} = c''(d)\dot{d}$. From (8.6) follows

$$v = h - \dot{\lambda} = h - c''(d)\dot{d}. \tag{8.18}$$

Since $v \geq 0$, we obtain (8.16). □

From Lemma 8.2 and 8.3 we get the following result.

Lemma 8.4. *The optimal production rate is continuous and positive at any time. If an interior interval (t_1,t_2) succeeds to a boundary interval, then λ_o in (8.11) may be specified, and it holds that*

$$v(t) = (c')^{-1}(c'(d(t_1)) + h(t-t_1)). \tag{8.19}$$

Proof. From Lemma 8.2 and 8.3 follows $v > 0$ for $t\epsilon[0,T]$. Thus the constraint qualification (5.15) is satisfied. Thus, remark 5.1 is applicable, and v and λ are continuous. Let $t_1 > 0$ be the begin of an interior interval.

Then by a continuity argument we have

$$c'(d(t_1)) = \lambda(t_1). \tag{8.20}$$

Note that $t_1 - \epsilon$ lies in a boundary interval in which (8.17) holds. Using (8.20) in (8.14) we obtain $\lambda_o = c'(d(t_1)) - ht_1$. This provides (8.19). □

In a special case we can give the optimal solution.

Theorem 8.1. *Suppose that (8.16) holds for any $t\epsilon[0,T]$. Then the production meets just the demand, i.e.*

$$v(t) = d(t),\ z(t) = 0 \quad \text{for} \quad t\epsilon[0,T]. \tag{8.21}$$

Proof. Let us consider policy (8.21). Then from (8.7, 5) follows (8.17). From the adjoint equation (8.6) we get (8.18). Thus the necessary and sufficient optimality conditions (8.4-9) are satisfied provided that (8.16) holds. Note that because of (8.17) and $z(T) = 0$ the transversality condition (8.9) is also satisfied. □

The more interesting case that (8.16) does not hold globally is treated in the following theorem.

Theorem 8.2. *Assume that there is an interval (σ_1,σ_2) in which (8.16) does not hold, i.e.*

$$h < \dot{d}(t)c''(d(t)) \quad \text{for all} \quad t\epsilon(\sigma_1,\sigma_2). \tag{8.22}$$

Then there exists an interval $(t_1,t_2) \supseteq (\sigma_1,\sigma_2)$ with interior solution. The boundary points are determined by

$$\int_{t_1}^{t_2} d(t)dt = \int_{t_1}^{t_2} (c')^{-1}(\lambda_o + ht)dt, \tag{8.23}$$

$$\lambda_o = c'(d(t_1)) - ht_1 \quad \text{falls} \quad t_1 > 0, \tag{8.24}$$

$$\lambda_o = c'(d(t_2)) - ht_2 \quad \text{falls} \quad t_2 < T. \tag{8.25}$$

Proof. Because of Lemma 8.3 and (8.22) there is no optimaly boundary solution in (σ_1,σ_2). Thus, their exists (t_1,t_2) including (σ_1,σ_2) with interior solution (see fig. 8.1).

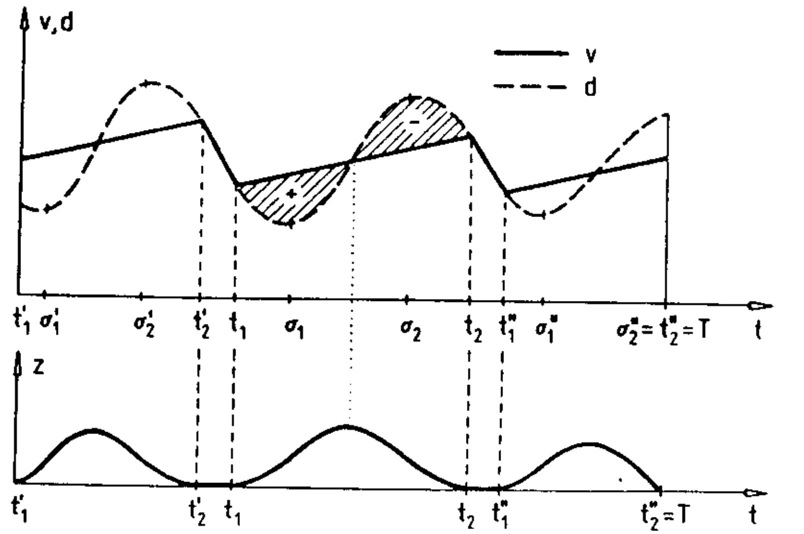


Fig. 8.1: Production smoothing in the Arrow-Karlin model

From $z(t_1) = z(t_2) = 0$ and (8.11) follows (8.23). For $t_1 > 0$ or $t_2 < T$, respectively, from the continuity of v (Lemma 8.4) follows

$$v(t_i) = (c')^{-1}(\lambda_o + ht_i) = d(t_i) \quad \text{for} \quad i = 1,2.$$

This, establishes (8.24, 25). So for any interior interval there are three relations to calculate the three unknown quantities t_1, t_2, λ_o. □

Remarks. Equation (8.23) says that in (t_1,t_2) the areas below the curves d and v are equal, i.e. the sum of the marked areas in fig. 8.1 is equal to zero, if we pay attention to the orientation.

Relations (8.24, 25) mean that both at the begin and the end of an interior interval the production just meets the demand provided that before or after that a boundary interval occurs. So the entry and exit is tangential (see fig. 8.1).

Note that the optimal production policy is a 'smooth' version of the demand trajectory. 'Peaks' of the demand are pulled down, while 'valleys' are filled up. An optimal production/inventory policy is characterized by a trade-off between constant production with large fluctuations of the inventory and *synchronous* a production rate being synchronous to the demand without any inventory. By differentiating (8.11) with respect to time revals how the cost structure influences the productive rate:

$$\dot{v} = h[(c')^{-1}]' = h/c'' > 0.$$

The optimal production rate increases over time along interior intervals. The higher the inventory costs, the more the production follows the demand. On the other hand, the increase of marginal production cost has a smoothing effect to the rate of production. In fig. 8.1 we have considered the case of quadratic production costs $c(v) = av^2/2$. Now (8.11) has the form

$$v(t) = v_o + ht/a \tag{8.26}$$

with $v_o = \lambda_o/a$. Relations (8.23-25) for the calculation of t_1, t_2 are

$$\int_{t_1}^{t_2} d(t)\,dt = v_o(t_2-t_1) + \frac{h}{2a}(t_2^2-t_1^2) \tag{8.27}$$

$$v_o = d(t_i) - ht_i/a \tag{8.28}$$

for $i = 1,2$ provided that $t_1 > 0$, $t_2 < T$. Note that the inventory t in (t_1,t_2) is given by

$$z(t) = d(t_1)(t-t_1) + \frac{h}{2a}(t-t_1)^2 - \int_{t_1}^{t} d(\tau)\,d\tau > 0. \tag{8.29}$$

Finally, we determine the decrease in costs by production smoothing for the interior interval (t_1,t_2) of fig. 8.1. For this we compare the accumulated production and inventory costs in (t_1,t_2), for the curve $v(t)$ in fig. 8.1 with the 'unsmoothed' production rate $\tilde{v}(t) = d(t)$ for $t \in (t_1,t_2)$. The savings in costs are

$$\begin{aligned} \int_{t_1}^{t_2}(ad^2/2 - av^2/2 - hz)\,dt &= \frac{a}{2}\int_{t_1}^{t_2}(d^2 - v^2 - 2\dot{v}z)\,dt \\ &= \frac{a}{2}\int_{t_1}^{t_2}(d^2 - v^2 + 2v\dot{z})\,dt - avz\Big|_{t_1}^{t_2} = \frac{a}{2}\int_{t_1}^{t_2}(v-d)^2\,dt > 0. \end{aligned} \tag{8.30}$$

Here we have used (8.26), partial integration for $\int \dot{v}z\,dt$, $z(t_1) = z(t_2) = 0$ and $\dot{z} = v - d$.

The results discussed above describe the structure of optimal production patterns. They also allow to construced a *forward algorithm* to calculate optimal production policies for given demand functions. Following theorem 8.2 the basic idea is to choose a boundary solution as long as an interior solution at any time τ would lead to a positive inventory for all $t \in (\tau, T]$.

8.2. Optimal Pricing and Production in an Inventory Model

We consider a monopolist who controls a stock of inventory, denoted by $x(t)$, by choosing the production rate, $u(t)$, and the price, $p(t)$, at every instant $t \in [0,T]$. The sales are given by a demand function $f(p,t)$ for which we assume

$$f_p(p,t) < 0, \tag{8.31a}$$

$$f_{pp}(p,t) < 2f_p(p,t)^2/f(p,t). \tag{8.31b}$$

Condition (8.31b) states that the marginal revenue $MR = p + f/f_p$ [1] is an increasing function of price, i.e. $\partial(MR)/\partial p > 0$.

---

[1] Note that MR is the marginal revenue with respect to the output f, i.e. $MR = (\partial/\partial f)(pf)$, where $p = p(f,t)$ is the inverse demand function for each t.

If the production exceeds the demand then the difference between the supplied output and the demand is added to the inventory. Otherwise, the excess demand is taken from the inventory. Thus, the dynamics of the inventory level being the state variable is governed by the following differential equation:

$$\dot{x} = u - f(p,t). \tag{8.32}$$

We suppose that the monopoly faces a strictly convex nonnegative increasing production cost function $c(u)$ with

$$c'(0) = 0, \; c'(u) > 0 \quad \text{for} \quad u > 0, \; c''(u) > 0. \tag{8.33}$$

While other models impose a nonnegativity constraint for the inventory, we allow for backlogging: If the inventory is exhausted and an excess in demand occurs, the resulting shortage is backlogged and, as usual, penalized with shortage costs. By $h(x)$ we denote the inventory cost (for $x \geq 0$) and shortage cost (for $x < 0$), respectively. We shall consider continuously differentiable and strictly convex cost functions.

More specifically,

$$h(0) = h'(0) = 0, \; h''(x) > 0. \tag{8.34}$$

Note that this implies that $h'(x) \gtreqless 0$ for $x \gtreqless 0$.

The objective of the monopoly is to choose the controls u und p in order to maximize the present value of its net return discounted at the rate r over a given planning period $[0,T]$:

$$\max_{p,u} \{J = \int_0^T e^{-rt}[pf(p,t) - c(u) - h(x)]dt + e^{-rT}Sx(T)\} \tag{8.35}$$

subject to the system dynamics (8.32), a given initial inventory level $x(0) = x_o \geq 0$ and the control constraints $p(t) \geq 0$, $u(t) \geq 0$. The parameter S denotes the salvage value of one unit of inventory. We shall also consider the case $T = \infty$ in which there is no salvage value.

Thus, the firm is faced with a continuous-time nonlinear optimal control problem with one state variable, x, a two controls, p and u.

We now solve this problem by applying the necessary and sufficient optimality conditions of Pontrjagin's maximum principle.

Defining the current value Hamiltonian as

$$H = pf(p,t) - c(u) - h(x) + \lambda[u - f(p,t)] \tag{8.36}$$

the set of necessary optimality conditions is as follows:

$$p = 0 \qquad \text{if } \lambda \leq f(0,t)/f_p(0,t), \tag{8.37a}$$

$$\lambda = p + f/f_p \quad \text{if } \lambda > f(0,t)/f_p(0,t) \tag{8.37b}$$

and

$$u = 0 \qquad \text{if} \quad \lambda \leq 0, \tag{8.38a}$$

$$\lambda = c'(u) \quad \text{if} \quad \lambda > 0, \tag{8.38b}$$

$$\dot{\lambda} = r\lambda + h'(x), \tag{8.39}$$

$$\lambda(T) = S. \tag{8.40}$$

Hence, $\lambda(t)$ is the current value abjoint variable measuring the value of an additional unit of the stock of inventory along the optimal path.

(8.37) and (8.38) say that the Hamiltonian is maximized with respect to p and u. Note that boundary solutions are possible since λ can become negative[1]. If $f(0,t)/f_p(0,t) = -\infty$, then the price is always positive.

It is easily checked that the Hamiltonian is globally maximized with respect to p and u if (8.37) and (8.38) are satisfied, since $H_{pu} = 0$, and

$$H_{uu} = -c''(u) < 0, \tag{8.41a}$$

$$H_{pp} = 2f_p + (p-\lambda)f_{pp} < 0 \quad \text{for} \quad H_p = 0. \tag{8.41b}$$

The values u and p maximizing the Hamiltonian (8.36) are, because of $H_{ux} = H_{px} = 0$, independent of x. Therefore with H^o denoting the Hamiltonian maximized with respect to u and p we obtain:

$$H^o_{xx} = H_{xx} = -h'(x) \leq 0,$$

---

1) This can be seen by economic reasoning. Assume, for instance, that there is an upper bound of the demand function: $f(0,t) \leq M$. Then $x(T) \geq x_o - TM$, and for $x_o > TM$ it is easily seen that the value of the objective functional would improve if x_o would diminish (provided that the terminal value S is not too large). This is because the choice of the time paths of the control variables being optimal for x_o would still give a feasible solution and the inventory cost would decrease.

i.e. the maximized Hamiltonian H^0 is concave in the state x. Thus, the optimality conditions are not only necessary but also sufficient (see theorem 2.2). For the infinite horizon optimal control problem, i.e. for $T = \infty$, the transversality condition (8.40) is replaced by

$$\lim_{t\to\infty} e^{-rt}\lambda(t)\left[x(t) - x^*(t)\right] \geq 0 \tag{8.42}$$

as a part of the sufficiency conditions. In (8.42) the functions λ and x^* refer to an optimal solution and x denotes any other feasible state trajectory. From now onwards we assume that the demand does not depend on time explicitely, i.e. $f_t = 0$, and f_p may be simply denoted by f'.

Equating (8.37b) and (8.38b) we obtain an implicit relationship, $u = u(p)$, for the control variables in the interior of their admissible domain. The derivative is

$$\frac{du}{dp} = (2-ff''/f'^2)/c''(u) > 0.$$

Thus, the two controls show a *synergistic effect* with respect to the change of the inventory level in the sense that an increase in production causes an increase in $\dot{x}$, and this effect is reinforced by an increase in price. In other words, along the optimal path the price and the output rate influence the state variable in the *same* direction.

The next section provides an optimal steady state solution and a phase portrait analysis for the case of strictly convex inventory and shortage costs.

The equilibrium defined by $\dot{x} = \dot{\lambda} = 0$ characterizes the long-run behavior of the optimal solution. Denoting the equilibrium values by hats, the following relationships are easily obtained from (8.32), (8.37)

$$\hat{u} = f(\hat{p}), \quad \hat{\lambda} = f(\hat{p})/f'(\hat{p}) + \hat{p}, \tag{8.43a}$$

$$\hat{\lambda} = c'(\hat{u}), \quad \hat{\lambda} = -h'(\hat{x})/r. \tag{8.43b}$$

Under reasonable assumptions[1] it can be shown that there exists a unique solution of (8.43). If, however, the shortage cost function h(x) is 'almost linear', then $\hat{x}$ might become $-\infty$.

The equilibrium conditions can also be interpreted economically. It is interesting to note that $\hat{x} < 0$, i.e. that a stationary level of shortage occurs. This is in accordance with economic intuition: Assume, on the contrary, that $\hat{x} > 0$. Then it would be better to choose $u = 0$ for some time interval $[0,\bar{t}]$ as long as $x(t) > 0$. For $t\in[0,\bar{t}]$ both h(x) and c(u) would be smaller than in the equilibrium. For $t > \bar{t}$ the zero inventory could be maintained by choosing the old equilibrium controls $\hat{u}$ and $\hat{p}$. Thus, the steady state with $\hat{x} > 0$ could not represent an optimal solution, since the modified solution described above would yield a higher aggregate profit.

In what follows we shall derive the properties of the optimal transitory solution.

By using standard phase plane techniques we obtain the state-costate diagram as sketched in fig. 8.1. The details are omitted here.

The solutions represented by solid lines in fig. 8.1 refer to the case of $S = 0$. By (8.40) we have $\lambda(T) = 0$ implying that the trajectory ends on the abscissa. In most cases the optimal inventory decreases monotonically. For large shortages, however, it may be optimal that x increases initially.

---

1) These assumptions are, e.g., $f(\infty) = 0$, $h'(-\infty) = -\infty$, existence of $\tilde{p}$ maximizing $pf(p)$. Thus, $\tilde{p}$ satisfies $\tilde{p} + f(\tilde{p})/f'(\tilde{p}) = 0$. Since according to (8.31b) $p + f/f'$ is an increasing function and (8.43) implies $\hat{p} + f(\hat{p})/f'(\hat{p}) > 0$ it holds that $\tilde{p} < \hat{p}$. Thus, the equilibrium price exceeds the price maximizing the instantaneous turnover. Since the equilibrium price corresponds to the optimal price in the static profit-maximizing problem, (8.43) is the well-known order relation between the return-maximizing and profit-maximizing prices.

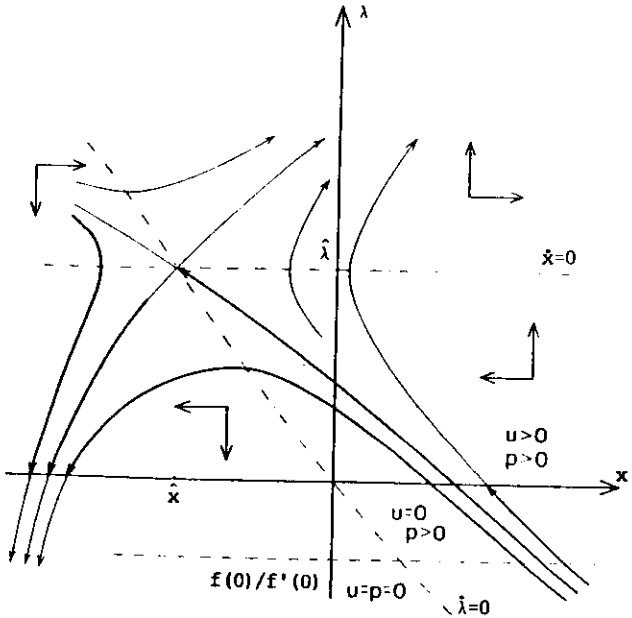


Fig. 8.1: The state-costate phase diagram in the case of strictly convex cost functions

In order to derive the state control phase diagrams we employ (8.37) and (8.38). Thus, for $\lambda > 0$ we have the relationships (8.37b) and (8.38b) which are monotonic increasing. Therefore the saddle point property is carried over from the (x,λ) diagram to each of the state control diagrams, and fig. 8.1 provides also pictures of both phase speces (x,u) and (x,p) for $u > 0$ and $p > 0$. For $\lambda < 0$ and $\lambda < f(0)/f'(0)$ we have, respectively, $u = 0$ and $p = 0$, i.e. the solution lies on the abscissa.

In fig. 8.2 we show how the (x,u) and (x,p) diagrams can be derived from the (x,λ) phase diagram graphically.

For $S = 0$ we have $p(T) = \tilde{p}$ (see footnote at p. 101 and $u(T) = 0$.

Let us now describe the behavior of the optimal solution in the case where x_o and T are sufficiently large.

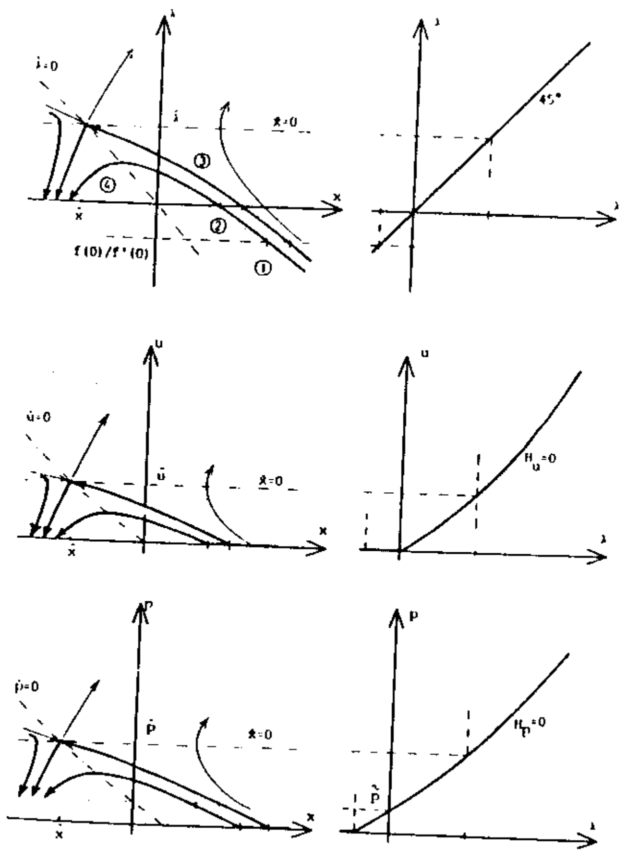


Fig. 8.2: The graphical derivation of the state-control diagrams.

Phase 1. At the beginning, i.e. for $t\in[0,t_1)$, when the inventory is much larger than required, the shadow price λ has a negative value, $\lambda < f(0)/f'(0)$. In this situation the high inventory costs make it optimal to decrease the inventory as fast as possible. This is done by producing nothing, i.e. $u = 0$, and giving away the product, i.e. $p = 0$.

Phase 2. When the inventory has fallen to a 'reasonable' level, i.e. $t \in [t_1, t_2]$, we have $0 < \lambda \leq f(0)/f'(0)$. Then it is optimal to charge a positive but low price in order to deplete the stock of inventory quickly while earning some return. Since x is sufficiently large, it is still not optimal to produce anything.

Phase 3. As soon as the inventory has become sufficiently small, the optimal production becomes positive and increases in order to compensate a part of the demand rate. The optimal price is positive and gradually increasing.

Phase 4. In this phase shadow price λ, production u, and price p decrease again to get a higher instantaneous profit. The induced increase in the shortage cost h(x) in the future can be accepted since the remaining time period is very short.

For the case of an *infinite horizon*, $T = \infty$, phase 4 does not occur, i.e. phase 3 lasts forever. The trajectories $x(t)$, $\lambda(t)$, $p(t)$, $u(t)$ converge to their long-run stationary values. In other words, the solution for $T = \infty$ is given by the saddle point paths in fig. 8.1 and 8.2. It can be shown that the 'transversality condition' (8.42) is met[1].

To conclude this section let us finally consider a case where there exists no equilibrium. Assume that h(x) is 'almost linear' as, for instance

$$h(x) = \begin{cases} h_2[a(e^{x/a}-1) - x] & \text{if } x \leq 0, \\ h_1[b(e^{-x/b}-1) + x] & \text{if } x \geq 0, \end{cases} \tag{8.44}$$

with a, b, h_1, $h_2 > 0$. If, e.g., a = 1 and $h_2 < r\hat{\lambda}$ then for $x < 0$ the $\dot{\lambda} = 0$ isocline is given by

$$\lambda = -h'(x)/r = (1-e^x)h_2/r < \hat{\lambda}.$$

[1] Since the stable branch $(x^*(t), \lambda(t))$ converges to $(\hat{x}, \hat{\lambda})$, it remains to show that (*) $\lim_{t\to\infty} e^{-rt}x(t) \geq 0$ is satisfied for every feasible x(t). Note that (8.32) and (8.31a) imply that $\dot{x}(t) \geq -f(0)$ so that (*) is clearly satisfied if f(0) is finite. It can be shown that if f(0) were infinite, every x(t) violating (*) would yield a value of $-\infty$ for the objective functional.

Thus there is no equilibrium point and for $T = \infty$ the solution satisfies $\lambda(t) < h_2/r$ and $\lambda(t) \to h_2/r$ as $t \to \infty$.

The proof that the transversality condition (8.42) is met in this case, although $x^*(t) \to -\infty$, is almost identical to that of the linear model with $h_2 < r\hat{\lambda}$.

8.3: Excercices to section 8

8.1. Consider the inventory model with linear production costs and linear inventory costs:

$$\max_v \{-\int_0^T (cv+hz)dt\} \tag{8.45a}$$

$$\dot{z} = v - d, \quad z(0) = z_0 \tag{8.45b}$$

$$0 \leq v \leq \bar{v}, \quad z \geq 0. \tag{8.45c}$$

Show that the optimal policy is given as

$$v(t) = \begin{cases} 0 \\ d(t) \end{cases}, \quad z(t) = \begin{cases} z_0 - D(t) \\ 0 \end{cases} \quad \text{for} \begin{cases} 0 \leq t < \tau \\ \tau \leq t \leq T \end{cases} \tag{8.46}$$

where the time τ is defined by

$$D(\tau) = \int_0^\tau d(t)dt = z_0. \tag{8.47}$$

How changes the optimal solution, if we assume that the terminal inventory stock has the unit value S?

8.2. Solve excercise 8.1 for a general inventory cost function $h(z,t) > 0$ for $z > 0$ and $h(0,t) = 0$.

8.3. A firm has received an order for S units of product to be delivered by time T. It seeks a production schedule for filling this order at the specified delivery date at minimum cost, bearing in mind that unit production cost rises linearly with the production rate and that the unit cost of holding inventory per unit time is constant.

8.4. Carry out the phase portrait analysis of the model presented section 8.2.

8.5. Introduce a state constraint $x \geq \underline{x}$ with $\underline{x} > \hat{x}$. Solve the model and discuss the solution.