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ON THE DEPENDENCE OF SOLUTIONS UPON THE RIGHT HAND SIDE OF AN ORDINARY DIFFERENTIAL EQUATION

H.W. KNOBLOCH
Mathematisches Institut
der Universitat
Am Hubland
87 Wurzburg
Federal Republic Germany

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H.W.Knobloch

1. Introduction.

We consider differential equations of the form

$$\dot{x} = f(t,x;u) \quad , \quad \dot{} = d/dt, \tag{1.1}$$

where it is assumed that f is a n-dimensional column vector and that its components are C^∞-functions of t,x,u. The variable t ("time") is scalar, $x = (x_1,\dots,x_n)^T$ ("state") n-dimensional and $u = (u_1,\dots,u_m^T)$ ("control") m-dimensional. The name control has been chosen in order to indicate that u has n o t to be regarded just as a parameter. In fact by writing down a differential equation (1.1) we wish to consider simultaneously the whole class of ordinary differential equations $\dot{x} = f(t,x;u(t))$ which arise from (1.1) by substituting for the variable u a function of t ("input"). There is an obvious restriction for the choice of the input function: Existence and uniqueness of solutions for the resulting differential equation should be guaranteed. This is clearly the case if one assumes – as we will do – that the input $u(\cdot)$ is a piecewise C^∞-function of t ("admissible control function"), though the reader will realize that all our results remain valid if one allows specialization of u to a more general type of function. The solution of the initial value problem

$$\dot{x} = f(t,x;u(t)), \quad x(t_o) = x^{(o)}, \tag{1.2}$$

is denoted by $x(t;t_o,x^{(o)},u(\cdot))$. Its dependence on the initial data is well understood; the relevant results are a common place in modern calculus. On the other hand certain basic facts concerning the dependence on the input are not mentioned at all in any work dealing with foundations of the theory of differential equations. The present

paper is intended to close the gap. We are interested in the so-called input/output behaviour of a finite dimensional control system whose dynamics is modelled by the differential equation (1.1). In addition to these data a certain (vector-valued) function of time and state ("output") has been specified. As it is customary in control theory the letter y is used in order to denote the output of a system:

$$y = c(t,x) = (c_1(t,x),\dots,c_k(t,x))^T. \tag{1.3}$$

The value of the output in a given moment t depends upon initial time and state $t_o,x^{(o)}$ and upon the values which the input assumes in the time interval (t_o,t). Accordingly we write

$$y(t;t_o,x^{(o)},u(\cdot)) := c(t,x(t;t_o,x^{(o)},u(\cdot)) \ . \tag{1.4}$$

We altogether present in this paper three results concerning the dependence – or rather the independence – of the output from the input (Theorem 1 and the corollary stated below, Proposition 1 in Sec. 5). They may be viewed upon as background material from the theory of differential equations which enters into the foundation of modern control theory. In fact these results prepare the stage for introducing ideas and notions from differential geometry into control theory. The reader is referred to [2] for more thorough information. We also wish to mention examples of relevant literature in which one finds ample opportunity for applying the results of this paper: [6] – [8].

Before we are in a position to state our main result we have to introduce some notation. In the sequel we use small Latin letters, preferably $a,b,f,$ in order to denote vectors of dimension n which have as components functions of t,x. These functions are supposed to be of class C^∞ on some open subset $\mathfrak{X}$ of the (t,x)-space. We then define

$$ad_a^0 b := b, \quad ad_a^1 b := ad_a b := \partial b/\partial t + (\partial b/\partial x)a - (\partial a/\partial x)b,$$

$$ad_a^{\rho+1} b := ad_a(ad_a^\rho b), \ \rho=0,1,\dots \ . \tag{1.5}$$

$\partial b/\partial x$ etc. of course has to be understood as the Jacobian matrix of b, that is as the $n\times n$ matrix with the columns $\partial b/\partial x_i$.

This definition is the customary one if the vector fields a,b depend upon x only (see e.g. [8],Sec. 2). The extra role which time seems to play in the general case could be removed by choosing a different formal framework. However we prefer to work with the above definition since it allows to assign two interpretations to the ad-operator which serve our purposes best and which can be verified by inspection. The first interpretation establishes a link with the Hamiltonian formalism accompanying the standard formulation of Pontryagin's maximum principle in calculus of variations and goes as follows.

Let $x(t)$ be a solution of the differential equation $\dot{x} = a(t,x)$ and let $y(t)$ be a solution of the corresponding adjoint variational equation $\dot{y} = -(\partial a/\partial x)(t,x(t))^T y$. Then

$$\frac{d}{dt}\Big(y(t)^T b(t,x(t))\Big) = y(t)^T (\mathrm{ad}_a b)(t,x(t)).$$

Our main theorem concerns so-called affine control systems. This terminology is used when ever the control variable u enters linearly into the dynamics:

$$\dot{x} = a(t,x) + B(t,x)u = a(t,x) + \sum_{\mu=1}^{m} b_\mu(t,x)u_\mu, \quad y = c(t,x). \qquad (1.6)$$

We remind the reader, that a, b_μ, c are supposed to be of class C^∞ on some set $\mathcal{X}$. The n-dimensional row vectors

$$\mathrm{ad}_a^\rho b_\mu =: b_{\mu,\rho}, \quad \mu=1,\ldots,m, \quad \rho=0,1,\ldots \qquad (1.7)$$

are therefore well defined on the set $\mathcal{X}$.

<u>Theorem 1.</u> Let a,B,c be analytic functions of (t,x) and assume that these relations hold identically on $\mathcal{X}$:

$$(\partial c/\partial x)b_{\mu,\rho} = 0, \quad \mu=1,\ldots,m, \quad \rho=0,1,\ldots \qquad (1.8)$$

Then the output y is independent from $u(\cdot)$. Using our previously introduced notation (cf. (1.4)) this statement means precisely the following.

We have

$$y(t;t_o,x^{(o)},u(\cdot)) = y(t;t_o,x^{(o)},\tilde{u}(\cdot)) \qquad (1.9)$$

whenever these conditions are satisfied:

$$u(\cdot),\tilde{u}(\cdot) \text{ admissible}, \quad (t_o,x^{(o)}) \in \mathcal{X}, \quad t \in \mathcal{J}, \qquad (1.10)$$

where $\mathcal{J}$ is an open time interval which is such that $t_o \in \mathcal{J}$, $(t,x) \in \mathcal{X}$ if $t \in \mathcal{J}$ and if

$$x = x(t;t_o,x^{(o)},u(\cdot)), x(t;t_o,x^{(o)},\tilde{u}(\cdot)). \qquad (1.11)$$

Conversely: Assume that a,B,c are of class C^∞ on $\mathcal{X}$ and that (1.9)-(1.11) hold true. Then condition (1.8) must be satisfied.

Note that the symbol on the left hand side of (1.8) repesents a k-dimensional row vector since $\partial c/\partial x$ and $b_{\mu,\rho}$ respectively is a matrix of type $k\times n$ and $n\times 1$ respectively.

In accordance with existing custom (see e.g. [2],Ch.III, Sec.3)we speak of output invariance if a control system has property (1.9)-(1.11).

It is not difficult to guess how the above criterion for output invariance can be generalized to systems where f is not necessarily linear in u but merely of class C^∞ on the set

$$\{(t,x,u) : (t,x) \in \mathcal{X}, \ u \text{ arbitrary}\} \qquad (1.12)$$

and hence admits a Taylor expansion

$$f(t,x;u) \simeq f_o(t,x) + \sum_{|\underline{\mu}|>0} \frac{1}{\underline{\mu}!} f_{\underline{\mu}}(t,x)u^{\underline{\mu}}. \qquad (1.13)$$

(The usage of multi-indices $\underline{\mu}$ in writing down Taylor-series is self explanatory (cf. also [1],p.42)) .

<u>Corollary (first version).</u> If f is analytic on the set (1.12) and if the relations

$$(\partial c/\partial x)(ad^{\rho}_{f_o} f_{\underline{\mu}}) = 0, \ |\underline{\mu}|>0, \quad \rho=0,1,\ldots \quad (1.14)$$

are satisfied identically for $(t,x)\in\mathcal{X}$, then we have output invariance.

Conversely: Output invariance - for systems of class C^∞ - implies that (1.14) must hold.

Note that the theorem actually can be viewed upon as a special case of the corollary.

The second version of the corollary is - with respect to the necessary part - slightly sharper than the first one. The main difference however lies in the fact, that now the ad-operator is applied directly to the right hand side of the differential equation (1.1). In doing so one has to regard u as a parameter which is independent from t and x.

<u>Corollary (second version).</u> Let f be an analytic function of t,x,u on the set (1.12). If each of the k-dimensional row vectors

$$(\partial c/\partial x)(ad^{\rho}_{f_o} f), \ \rho=0,1,\ldots, \quad \text{where} \quad f_o(t,x) := f(t,x;0), \qquad (1.15)$$

does not depend upon u then we have output invariance.

Conversely, if f is of class C^∞ on the set (1.12) and if conditions (1.9)-(1.11) are satisfied then the row vectors (1.15) cannot depend upon the control variable u.

The second version brings out more clearly than the first one that the corollary provi des a further example of a familiar type of results in real analysis: A question of f u n c t i o n a l d e p e n d e n c e (and output invariance is nothing else than that) can be reduced to the analogous problem for (countably many) functions of several variables. Invariance of course means then independence from one of these variables. How much one can gain from such an insight will be demonstrated in

Sec. 5 where we discuss briefly an application to a basic problem in control engineering.

In the autonomous case the statement of the theorem can be found - in a more or less disguised form and with a quite different type of proof - in the literature (e.g. in [2],III.3, c.f. in particular Theorem (3.12)); the statement of the corollary however appears to be new. Generally speaking it has not been recognized before that output invariance can well be treated in the simplest context - that of ordinary differential equations. In fact any criterion for output invariance is a local one and therefore - in an appropriate coordinate system - becomes a statement about ordinary differential equations.

The paper is organized as follows. We first establish the only-if-part of the corollary (second version) by demonstrating that it is an almost trivial consequence of Pontryagin's maximum principle. To understand this part of the paper modest prerequisites from calculus of variations are needed; they do not go beyond textbook-level (see e.g. [3], Ch. 7 or [4], Ch.VI, Sec. 7). We then turn to the sufficiency part which requires more subtile considerations. Here we actually have to borrow some material from [1] (Sec. 1-5 and 7) which in this form cannot be found in textbooks though it is rather elementary. It deals with a standard situation - the dependence of solutions from (finitely many) paramters - and uses standard techniques. The only fact which is not "standard" is the meaning which can be assigned to the parameters: They represent the time instants where a switch from one input function to another takes place.

Our terminology is the usual one, occasionally we write c_x etc. instead of $\partial c/\partial x$. What we mean by admissible control has been explained before. An admissible trajectory is simply a solution of $\dot{x} = f(t,x;u(t))$, where $u(\cdot)$ is an admissible control.

2. Necessicity: Proof via the maximum principle.

Given a control system in terms of a differential equation

$$\dot{x} = f(t,x;u) \tag{2.1}$$

and an output function $c(t,x)$. Assume that both f and c are of class C^∞ on a set of the form

$$\{(t,x,u) : (t,x) \in \mathfrak{X}\} \tag{2.2}$$

where $\mathfrak{X}$ is an open set in the (t,x)-space. Assume furthermore that the following statement holds true: Given any $(t_o,x^{(o)}) \in \mathfrak{X}$ and a real number $\tilde{t} > t_o$. Then the value $c(\tilde{t},x(\tilde{t}))$ of the output function is the same for each admissible trajectory which passes at $t=t_o$ through $x^{(o)}$ and satisfies the condition $(t,x(t)) \in \mathfrak{X}$ for all $t \in [t_o,\tilde{t}]$.

Proposition. Let $f_o(t,x) := f(t,x;0)$. Claim: Under the above hypothesis each of the functions

$$c_x(t,x)(\mathrm{ad}^\rho_{f_o} f)(t,x;u), \quad \rho = 0,1,\dots, \tag{2.3}$$

does not depend upon the parameter u.

Proof. The hypothesis and the statement can be rephrased in terms of the components of the output function. Hence we can assume without loss of generality that c is scalar. Let $(\tilde{t},\tilde{x})$ be a fixed point in $\mathfrak{X}$. Consider the solution $\tilde{x}(t)$ of the differential equation $\dot{x} = f_o(t,x)$ which passes at $t=\tilde{t}$ through $\tilde{x}$, and choose some $t_o < \tilde{t}$. Put $x^{(o)} = \tilde{x}(t_o)$. $\tilde{x}(t)$ can be viewed upon as a solution of each of the following two optimization problems:

(i) Minimize $c(\tilde{t},x(\tilde{t}))$ subject to the side conditions
$\dot{x} = f(t,x;u)$, $x(t_o) = x^{(o)}$.

(ii) Minimize $-c(\tilde{t},x(\tilde{t}))$ subject to the same side conditions.

These are free-end-point-problems and therefore the necessary conditions which have to be satisfied by $\tilde{x}(t)$ assume the following simple form . Let $y(t)$ be the solution of the adjoint variational equation with terminal value

$$y(\tilde{t}) = c_x(\tilde{t},\tilde{x})^T. \tag{2.4}$$

Then

$$\pm y(t)^T f(t,\tilde{x}(t);0) = \min_u \{\pm y(t)^T f(t,\tilde{x}(t);u)\}$$

where the + sign has to be taken in case of problem (i), the − sign in case of problem (ii). Hence

$y(t)^T f(t,\tilde{x}(t);0) = y(t)^T f(t,\tilde{x}(t);u)$ for every u and every $t \in [t_o,\tilde{t}]$.

If this identity is differentiated ρ-times with respect to t and if one makes use of the remark following the definition of the ad-operator in Sec. 1 one arrives at this relation

$$y(t)^T(\mathrm{ad}^\rho_{f_o} f)(t,\tilde{x}(t);0) = y(t)^T(\mathrm{ad}^\rho_{f_o} f)(t,\tilde{x}(t);u)$$

which again holds for all u and all $t \in [t_o,\tilde{t}]$. The desired result can then be obtained from (2.4) simply by putting $t=\tilde{t}$.

3. Sufficiency: Reduction to the basic lemma.

We assume now that f and c are analytic on the set (2.2). In this section we wish to demonstrate how the proof of Theorem I (or rather of its corollary) can be reduced to a question concerning formal power series expansions which are associated with the differential equation (2.1). The first step towards this goal is to get rid of the general type of output function by updating the output to a part of the state, a standard procedure in control theory. We introduce an additional state variable ξ

via the differential equation

$$\dot{\xi} = \dot{c}(t,x;u) := (\partial c/\partial t)(t,x)+(\partial c/\partial x)\cdot f(t,x;u) \quad . \tag{3.1}$$

Because of

$$c(t,x(t)) = c(t_o,x(t_o))+\xi(t)-\xi(t_o)$$

it is then clear that the output $y=c(t,x)$ of the original system (2.1) is independent from the input u if the same can be said about the output $y=\xi=C\hat{x}$ of the augmented system $\dot{\hat{x}} = \hat{f}(t,x;u)$ where

$$\hat{x} = (x,\xi)^T, \quad \hat{f}=(f,\dot{c})^T, \; C = (0,I), \; I=(k\times k)\text{-unit matrix.} \tag{3.2}$$

Hence the sufficiency part of the proof of Theorem 1 is completed once we have shown that these statements hold true.

(i) The output $y=C\hat{x}$ of the system $\dot{\hat{x}} = \hat{f}(t,x;u)$ does not depend upon the input if each k-dimensional vector

$$C(ad^{\rho}_{\hat{f}_o}\hat{f})(t,x;u), \quad \rho=0,1,\ldots, \tag{3.3}$$

does not depend upon the parameter u (N.B.: $\hat{f}_o(t,x):=\hat{f}(t,x;0)$).

(ii) For each ρ the two k-dimensional vectors (2.3) and (3.3) differ by a function of (t,x) only.

The second statement is a consequence of the fact that $\hat{f}$ and hence f_o does not depend upon the component ξ of the state variable $\hat{x}$. It can be inferred by induction with respect to ρ based on the following

Lemma. Let h be a n-dimensional vector depending upon t,x (and possibly some parameters as u) but not upon ξ and let $\hat{h} := (h,(\partial c/\partial x)\cdot h)^T$. Then

$$ad_{\hat{f}_o}\hat{h} = (ad_{f_o}h, (\partial c/\partial x)\cdot(ad_{f_o}h))^T. \tag{3.4}$$

In particular: $ad_{\hat{f}_o}\hat{h}$ is a vector of the same type as $\hat{h}$ and does not depend upon ξ.

Proof. Straightforward.

According to the definition of the ad-operator the expression on the left hand side of (3.4) is given in explicit terms as

$$\begin{pmatrix} h_t \\ c_{xt}h + c_x h_t \end{pmatrix} + \begin{pmatrix} h_x & 0 \\ (c_x h)_x & 0 \end{pmatrix}\begin{pmatrix} f_o \\ * \end{pmatrix} - \begin{pmatrix} (f_o)_x & 0 \\ (c_x f_o + c_t)_x & 0 \end{pmatrix}\begin{pmatrix} h \\ * \end{pmatrix} =$$

$$= \left(ad_{f_o}h, c_x h_t + (c_x h)_x f_o - (c_x f_o)_x h\right)^T = (ad_{f_o}h, c_x\cdot ad_{f_o}h)^T.$$

Corollary: $ad^{\rho}_{\hat{f}_o}\hat{f} = (ad^{\rho}_{f_o}f, c_x ad^{\rho}_{f_o}f)^T$ plus a vector which does not depend (upon u.)

What remains to be proved is the sufficiency of the condition stated in the corollary to Theorem 1 under the additional assumption that $\partial c/\partial x$ is a constant matrix C. In other words, what has to be shown is this: If none of the k-dimensional vectors

$$C(ad^{\rho}_{f_o}f)(t,x;u) \quad , \quad \rho=0,1,\ldots$$

actually depends upon the parameter u then the following statement holds true. Given a fixed $(t_o,x^{(o)})\in\mathcal{X}$ and denote by $x(t;u(\cdot))$ the solution of the initial value problem

$$\dot{x} = f(t,x;u(t)) \quad , x(t_o)=x^{(o)} \quad .$$

Let $\tilde{t}>t_o$ and assume that both $x(t;u(\cdot))$ and $\tilde{x}(t) := x(t;0)$ exist on $[t_o,\tilde{t}]$ and satisfy the condition $(t,x)\in\mathcal{X}$. Then the relation

$$C\left(x(t,u(\cdot))-\tilde{x}(t)\right)=0 \tag{3.5}$$

holds for $t = \tilde{t}$.

Now $\|x(\tilde{t},u(\cdot))-\tilde{x}(\tilde{t})\|$ admits an estimate in terms of $\int_{t_o}^{\tilde{t}}\|u(t)\|dt$.

This follows by standard arguments from the Gronwall inequality (or related results, cf. e.g. [4], Hilfssatz 3.1, p. 120). Hence it suffices to establish (3.5) for those $u(\cdot)$ which form a dense subset (in the sense of L^1-norm) within the class of all admissible control functions. This observation together with the assumption that f is an analytic function of all variables allows a step-by-step reduction of our problem to a special case which then is accessible to straight forward treatment by methods which are available in the literature.

Step 1. Clearly one can assume without loss of generality that $u(\cdot)$ is an analytic function of t. Then $x(t,u(\cdot))$ and $\tilde{x}(t)$ are analytic in t, hence it is sufficient to verify (3.5) on an arbitrary small time interval. So we can choose $\tilde{t}$ as close to t_o as we wish.

Step 2. We choose a positive number γ and determine $\tilde{t} = \tilde{t}(\gamma)$ in such a way that the following statement holds true: Whenever an admissible control function $u(\cdot)$ satisfies the condition $\|u(t)\| \le \gamma$ for $t \in [t_o, \tilde{t}]$ then $x(t,u(\cdot))$ exists on $[t_o, \tilde{t}]$ and we have $(t, x(t,u(\cdot))) \in \mathfrak{X}$ for all $t \in [t_o, \tilde{t}]$.

Step 3. The piecewise constant control functions which assume the initial value $u(t_o)=0$ form a dense subset within the set of functions as specified in step 2. Hence we are arrived at this problem: Verify the relation (3.5) for those control functions which can be represented with the help of finitely many real numbers $z_1, \ldots, z_N$ and m-dimensional vectors $u_1, \ldots, u_N$ as follows

$$u(t) = \begin{cases} 0 & \text{if } t \le \tilde{t}+z_1, \\ u_i & \text{if } \tilde{t}+z_i < t \le \tilde{t}+z_{i+1}, \ i=1,\ldots,N-1, \\ u_N & \text{if } t > \tilde{t}+z_N . \end{cases} \quad (3.6)$$

Of course, in order that this definition makes sense and yields a control function of the desired type, we have to assume that this inequalities hold

$$t_o \le \tilde{t}+z_1, \ z_1 \le z_2 \le \ldots \le z_N \le 0, \ \|u_i\| \le \gamma \text{ for } i=1,\ldots,N. \quad (3.7)$$

Step 4 Consider in the definition (3.6) the z_i as variable parameters - subject to the side condition (3.7) - and the u_i as fixed. It follows then again by standard arguments (see e.g. [1], proof of Theorem 4.2 on p. 23/24) that $x(\tilde{t},u(\cdot))$ is an analytic function of $z_1, \ldots, z_N$, so $x(\tilde{t},u(\cdot)) - \tilde{x}(\tilde{t})$ admits an expansion in powers of the z_i. So what has to be done is to convince oneself that-as a consequence of the hypothesis of the theorem - this power series vanishes if it is multiplied with the matrix C. This program will be carried through in the next

section and will be based upon some material which can be found in [1]. We conclude this section by introducing certain definitions with which we wish to work in the sequel and by explaining its precise relation with similar definitions given in [1]. From now on N is regarded as a fixed positive integer, z and $\tilde{x}$ respectively are abbreviations for $(z_1, \ldots, z_N)^T$ and $\tilde{x}(\tilde{t}) = x(\tilde{t};0)$ respectively. Regarded as a function of z the terminal value at $t=\tilde{t}$ of $x(t,u(\cdot))$ admits a Taylor-expansion at $z=0$ which is denoted by $\hat{C}(\tilde{t},\tilde{x},z;u_1,\ldots,u_N)$; its coefficients are denoted by $\hat{K}_\nu$. Note that the latter ones depend upon $\tilde{t},\tilde{x}$ and also upon the choice of the u_i which enter into the definition of $u(\cdot)$ (cf. (3.6)). In short: The definition of the symbols $\hat{C}, \hat{K}_{\underline{\nu}}$ can be given as follows

$$\hat{C}(\tilde{t},\tilde{x},z;u_1,\ldots,u_N) :\approx x(\tilde{t};u(\cdot)) \quad (3.8)$$

where $u(\cdot)$ is defined in terms of z, u_i according to (3.6),

$$\hat{C}(t,x,z;u_1,\ldots,u_N) = x + \sum_{|\underline{\nu}|>0} \frac{1}{\underline{\nu}!} \hat{K}_{\underline{\nu}}(t,x;u_1,\ldots,u_N) z^{\underline{\nu}}. \quad (3.9)$$

Again we understand multi-indices in the same way as in [1], p. 42.

All what has to be shown can now be expressed in terms of the $\hat{K}_{\underline{\nu}}$, we repeat the statement for the reader's convenience. Given an analytic control system of the form

$$\dot{x} = f(t,x;u), \quad y=Cx, \ C \text{ a constant matrix}, \quad (3.10)$$

and let us assume that the hypotheses of the corollary of Theorem 1 are satisfied. Then these relations hold

$$C\hat{K}_{\underline{\nu}}(t,x;u_1,\ldots,u_N) = 0 \quad \text{for all } \underline{\nu}. \quad (3.11)$$

It should be noted that the formal power series $\hat{C}$ is a special case of the formal power series introduced in [1] (cf. the definition of a "control variation" as given in [1] on p. 44). The coefficients $K_{\underline{\nu}}$ of the latter one are related with $\hat{K}_{\underline{\nu}}$ as follows

$$\hat{K}_{\underline{\nu}}(t,x;u_1,\dots,u_N) = K_{\underline{\nu}}(t,x;u_0,u_1,\dots,u_N) \qquad (3.12)$$

where $u_0 = (0,0,\dots)$, $u_i = (u_i,0,0,\dots)$, $i=1,\dots N$.

There is one consequence of (3.12) which can be inferred from the definition of $K_{\underline{\nu}}$ ([1],p.42): If f is an analytic function on the set (2.2) then $\hat{K}_{\underline{\nu}}$ is analytic on $\mathfrak{X}$.

4. Proof of the basic lemma.

Given a control system in terms of a differential equation

$$\dot{x} = f(t,x;u) \qquad (4.1)$$

where f is of class C^∞ on a set

$$\{t,x,u : (t,x)\in\mathfrak{X},\ u \text{ arbitrary}\}, \qquad (4.2)$$

$\mathfrak{X}$ being some open set in the (t,x)-space. Unless otherwise stated we will not assume that f is linear in u.

Let us consider a C^∞-mapping which maps some open set $\mathfrak{X}'$ of the (t,x')-space onto $\mathfrak{X}$:

$$(t,x') \to (t,x(t,x')). \qquad (4.3)$$

If we assume that the Jacobian matrix

$$X(t,x') := (\partial x/\partial x')(t,x')$$

is non-singular on $\mathfrak{X}'$ we can consider together with (4.1) the transformed system

$$\dot{x}' = X(t,x')^{-1}\{f(t,x(t,x');u) - \partial x/\partial t)(t,x')\} =: f'(t,x';u). \quad (4.1')$$

Given a positive integer N - which has to be regarded as fixed throughout this section - one can associate with (4.1) and (4.1') respectively the formal power series $\hat{C}(t,x,z;u_1,\dots,u_N)$ and $\hat{C}'(t,x',z;u_1,\dots u_N)$ respectively which have been introduced in Sec. 3. They are well defined whenever $(t,x)\in\mathfrak{X}$ and $(t,x')\in\mathfrak{X}'$ respectively. The obvious question how these two power series are related is answered by

<u>Lemma 1.</u> We have

$$x(t,\hat{C}'(t,x';z;u_1,\dots,u_N)) = \hat{C}(t,x(t,x'),z;u_1,\dots,u_N).$$

The definition of the right hand side of the above relation is clear. The meaning of the left hand side however is not obvious, so a comment seems to be in order.

Since the formal series $\hat{C}(t,x';z;u_1,\ldots,u_N)-x'$ starts with first order terms in z one can substitute it for $\xi=(\xi_1,\ldots,\xi_n)^T$ in any formal power series in ξ; the result is a well defined power series in z. If this procedure is carried out with the Taylor expansion (at $\xi=0$) of $x(t,x'+\xi)$ one has an explanation for the left hand side.

Proof of Lemma 1. Fix $t=\tilde{t}$, $x'=\tilde{x}'$. Define $u_z(t)$ as in Sec. 1:

$$u_z(t) = \begin{cases} 0 & \text{if } t\le\tilde{t}+z_1, \\ u_i & \text{if } \tilde{t}+z_i<t\le\tilde{t}+z_{i+1},\ i=1,\ldots,N-1, \\ u_N & \text{if } t>\tilde{t}+z_N, \end{cases}$$

and let $x_z(t)$ be the solution of the initial value problem

$$\dot{x} = f(t,x;u_z(t)), \quad x(t_o) = \tilde{x}(t_o). \qquad (4.4)$$

Here t_o is some number $<\tilde{t}$ and $\tilde{x}(t)$ the solution of the initial value problem (N.B.: $u_o(t) = u_z(t)$ for $z=0$).

$$\dot{x} = f(t,x;u_o(t)), \quad x(\tilde{t}) = x(\tilde{t},\tilde{x}'). \qquad (4.5)$$

As has been explained in detail in [1], pp. 42-44, $x_z(\tilde{t})$ does not depend upon the choice of t_o and is a C^∞-function of z whose Taylor-expansion at $z=0$ is given by

$$\hat{C}(\tilde{t},x(\tilde{t},\tilde{x}'),z;u_1,\ldots,u_N). \qquad (4.6)$$

Now let f' be given according to (4.1) and let $x_z'(t)$ be defined in an analogous way, namely in terms of the relations

$$\dot{x}' = f'(t,x';u_z(t)),\ x'(t_o) = \tilde{x}'(t_o), \qquad (4.4')$$

$$\dot{x}' = f'(t,x';u_o(t)),\ x'(\tilde{t}) = \tilde{x}'. \qquad (4.5')$$

It follows then by inspection that

$$x_z(\tilde{t}) = x(\tilde{t},x_z'(\tilde{t})). \qquad (4.7)$$

This relation holds for all $z=(z_1,z_2,\ldots,z_N)$ with $z_1\le z_2\le\ldots\le z_N\le 0$ and sufficiently small $\|z\|$. Therefore the functions on the left and right hand side of (4.7) must have the same Taylor expansion at $z=0$. The expansion of $x_z(\tilde{t})$ is given by (4.6) as we have remarked above. The expansion of the expression on the right hand side is given by

$$x(\tilde{t},\hat{C}'(\tilde{t},\tilde{x}',z;u_1,\ldots,u_N)).$$

(cf. the remark following the statement of the lemma).

The proof of our main result rests upon two interpretations which can be given to the ad-operator introduced in Sec. 1. One is related to the maximum principle and was demonstrated in Sec. 1 . The other is provided by

Lemma 2. Let $f(t,x),g(t,x)$ be of class C^∞ on some open neighborhood $\mathfrak{X}$ of $(\tilde{t},\tilde{x})$ and let $x(t,x')$ be the solution of the initial value problem.

$$\dot{x} = f(t,x), \quad x(\tilde{t}) = x'. \qquad (4.8)$$

Put

$$K(t,x'):=(\partial x/\partial x')(t,x'), \quad g'(t,x'):=X(t,x')^{-1}g(t,x(t,x')). \qquad (4.9)$$

Then we have for $\rho = 0,1,\ldots$

$$\frac{\partial^\rho g'}{\partial t^\rho}(t,x)\Big|_{t=\tilde t} = (\mathrm{ad}_f^\rho g)(\tilde t,x) \quad \text{identically in } x.$$

Proof. From (4.9) one obtains - by straightforward computation- an analogous formula for the partial derivative of g' with respct to t, namely

$$\frac{\partial g'}{\partial t}(t,x') = X(t,x')^{-1}(\mathrm{ad}_f g)(t,x(t,x')) \quad . \qquad (4.9')$$

One simply has to make use of the identities

$$\frac{\partial x}{\partial t}(t,x') = f(t,x(t,x'))$$

and

$$\frac{\partial}{\partial t}(X(t,x')^{-1}) = -X(t,x')^{-1}(\partial f/\partial x)(t,x(t,x')).$$

Comparing (4.9) and (4.9') one arrives at a formula for the higher t-derivatives of g' which yields the desired result if t and x' is replaced by $\tilde t, x$. Note that we have

$$x(\tilde t,x') = x', \quad X(\tilde t,x') = I$$

identically in x'. $\square$

We now pr oceed to the main result of this section.

Lemma 3. Given an affine system of the form (1.6) and assume that $a(t,x)$, $b_\mu(t,x)$ are of class C^∞ on some open subset of the (t,x)-space. Then each coefficient $\hat K_\nu$ of the associated formal power series (3.9) can be written as a linear combination of finitely many $b_{\mu,\rho}(t,x)$ (cf. (1.7)) and their partial derivatives with respect to x up to a certain order (depending upon ν). The coefficients in this representation depend upon $t,x,u_1,\ldots,u_N$.

Proof. We fix a point $\tilde t,\tilde x$ and use the solution $x(t,x')$ of the initial value problem

$$\dot x = a(t,x), \qquad x(\tilde t) = x' \qquad (4.10)$$

in order to define a mapping of the form (4.3). The sets $\mathcal{X}$ and $\mathcal{X}'$ introduced in connection with (4.3) can be considered as appropriate neighborhoods of $(\tilde t,\tilde x)$. The special choice of the transformation has two consequences: We have

$$f'(t,x';u) = \sum_{\mu=1}^{m} b'_\mu(t,x')u_\mu \quad \text{with} \quad b'_\mu(t,x') := X(t,x')^{-1} b_\mu(t,x(t,x')) \qquad (4.11)$$

and

$$\hat C'(\tilde t,x,z;u_1,\ldots,u_N) = \hat C(\tilde t,x,z;u_1,\ldots,u_N) \qquad (4.12)$$

identically in $x,z,u_1,\ldots,u_N$.

f' is the right hand side of the transformed differential equation (4.1'). Note that the inhomogenous term in f' disappears since $x(t,x')$ is a solution of (4.10). $\hat C'$ is the formal power series associated with the differential equation $\dot x' = f'(t,x';u)$; the relation (4.12) follows from Lemma 1 and from the identity $x(\tilde t,x') = x'$.

A further consequence of (4.11) and (4.12) can be derived with the help of Lemma 2: All what we need in order to establish the statement in question is to demonstrate that every coefficient of the formal power series

$$\hat{C}'(\tilde{t},x,z;u_1,\dots,u_N)$$

can be expressed in terms of the quantities

$$b_{\mu,\rho}(\tilde{t},x) := \frac{\partial^\rho b'_\mu}{\partial t^\rho}(t,x)\Big|_{t=\tilde{t}}$$

and their partial derivatives with respect to x. Thereby we have reduced the proof of our lemma to the consideration of a special case which will be then treated in a proper algebraic setting in the last lemma.

Given an open set $\mathfrak{X}$ in the (t,x)-space we denote by $\mathcal{R}$ the ring of all functions of $t,x,u_1,\dots,u_N$ which are of class C^∞ whenever $(t,x)\in\mathfrak{X}$. Accordingly we denote by $\mathcal{R}^n$ the $\mathcal{R}$-module of all n-dimensional vectors $a=(a_1,\dots,a_n)^T, a_i\in\mathcal{R}$.

<u>Lemma 4.</u> Let the formal power series (3.9) be associated with a homogeneous affine system

$$\dot{x} = \sum_{\mu=1}^{m} b_\mu(t,x)u_\mu =: B(t,x)u \qquad (4.13)$$

and let the b_μ be of class C^∞ on $\mathfrak{X}$. Then every coefficient $\hat{K}_{\underline{\nu}}$ of $\hat{C}$ is contained in the module generated over $\mathcal{R}$ by the b_μ and all their partial derivatives with respect to t and x.

<u>Proof.</u> We denote this $\mathcal{R}$-module by $\mathcal{M}$. Next we introduce a module $\mathcal{M}'$ which is simply obtained from $\mathcal{M}$ by admitting a larger coefficient set $\mathcal{R}'$. $\mathcal{R}'$ is the ring of all functions in $t,x,u_1,\dots,u_N$ and further variables $x_1,x_2,\dots,$ each x_ν being a n-dimensional vector. The elements of $\mathcal{R}'$ are supposed to be of class C^∞ whenever $(t,x)\in\mathfrak{X}$.

The essential work one has to do in order to prove the lemma is to convince oneself - by inspection - that the following three statements hold true:

(i) $\mathcal{M}\subseteq\mathcal{M}'$, both modules are closed under differentiation with respect to all variables.

(ii) Specialization of each $x_\nu, \nu=1,2,\dots,$ to an element of $\mathcal{R}^n$ is a mapping of $\mathcal{M}'$ on $\mathcal{M}$. (4.14)

(iii) If $g(t,x,u_1,\dots,u_N)\in\mathcal{M}$ then $\mathcal{D}_x^\nu g(t,x,x_1,\dots,x_\nu,u_1,\dots,u_N)\in\mathcal{M}'$.

Here we have used the symbol $\mathcal{D}_x^\nu$ in the sense as explained in [1],pp. 15/16. See also the recursion formula preceding the proof of Lemma 3.1 in [1].

The proof of the present lemma is now easily completed in view of Theorem 7.1 ([1],p.39) and the formula for h^ν ([1],p.38). Roughly speaking the statement of this theorem can be summarized as follows: $\hat{K}_{\underline{\nu}}$ can be obtained from the vectors $f(t,x;u_i), i=1,\dots,N,$ by repeated application of the operations (i),(iii) followed by a substitution $x_\nu \to a_\nu\in\mathcal{R}^n$. Since $f(t,x;u)$ is linear and homogeneous in u (cf. (4.13)) we have - according to the definition of $\mathcal{M}$ -

$$f(t,x;u_i) = B(t,x)u_i \in \mathcal{M}$$

and hence $\hat{K}_{\underline{\nu}}\in\mathcal{M}$.

The statement of Lemma 3 can easily be generalized to the case of a control system $\dot{x} = f(t,x;u)$ where f is not necessarily linear in u, but merely a C^∞-function of all variables on the set (4.2).

Corollary to Lemma 3. Let (1.13) be the Taylor-expansion of f at $u=0$, in particular let $f_o, f_{\underline{\mu}}$ be its coefficients. Consider then the Taylor-expansion at $u_1 = \dots u_N = 0$ of $\hat{K}_{\underline{\nu}}$, where $\hat{K}_{\underline{\nu}}$ is defined in terms of the formal power series (3.9) associated with the control system $\dot{x} = f(t,x;u)$. Claim: Each coefficient of this Taylor-expansion can be written as a smooth linear combination of finitely many vectors which arise from

$$(ad^\rho_{f_o} f_{\underline{\mu}})(t,x), \quad \rho=0,1,\dots, \qquad (4.15)$$

by taking partial derivatives with respect to x.

Remark. That $\hat{K}_{\underline{\nu}}$ is a C^∞-function of all its arguments has been stated earlier (cf. Sec. 3).

Hence all partial derivatives of $\hat{K}_{\underline{\nu}}$ with respect to $u_1,\dots,u_N$ are well defined whenever $(t,x)\in\mathcal{X}$. Note also that the statement of the lemma itself is a special case of the corollary since for an affine control system the coefficient $\hat{K}_{\underline{\nu}}$ is a polynomial in the components of $u_1,\dots,u_N$, as can be seen again from Theorem 7.1 in [1], p.39.

Proof. We use the same transformation as in the proof Lemma 3 taking

$$a(t,x) := f_o(t,x) \quad (= f(t,x;0)). \qquad (4.16)$$

Again we denote the transformed system equation by

$$\dot{x}' = f'(t,x';u) := X(t,x')^{-1}(f(t,x(t,x');u)-(\partial x/\partial t)(t,x')). \quad (4.17)$$

We have then

$$f'(t,x';0)=0, \quad D_u f'(t,x';u)=D_u(X(t,x')^{-1}f(t,x(t,x');u)) \quad (4.18)$$

if D_u is a differential operator acting on u only. If we specialize t to $\tilde{t}$ and write x instead of x' we obtain from Lemma 2 these relations

$$\frac{\partial^\rho}{\partial t^\rho} D_u f'(t,x;u)\Big|_{t=\tilde{t}} = (ad^\rho_{f_o} D_u f)(\tilde{t},x;u). \qquad (4.18')$$

Note that (4.18') holds under the proviso that D_u is not trivial i.e. involves actual differentiation with respect to a component of u.

Again we use the fact that the given and the transformed control system generate the same formal power series if $t=\tilde{t}$ (cf.(4.12)). Now one observes that for $u=0$ the right hand side of (4.18') can be written in the form (4.15) for a suitable multi-index $\underline{\mu}$. Hence it is clear that the general statement of the corollary is a consequence of a certain special case which can be phrased as follows.

Assume that f satisfies the additional condition

$$f(t,x;0) = 0 \quad \text{identically in } t,x. \qquad (4.19)$$

If $\hat{K}_{\underline{\nu}}$ is then expanded in a Taylor-series at $u_1=\dots=u_N=0$ each coefficient turns out to be a finite smooth linear combination of

$$Df(t,x;u)\Big|_{u=0} \qquad (4.20)$$

where D is an arbitrary differential operator acting on t,x,u Note that - because of (4.19) the span of the vectors (4.20) would remain the same if D is restricted to operators which involve de-facto differentiation with respect to u and therefore (cf. (4.18')!) can be expressed in terms of partial derivatives with respect to x of vectors of the form (4.15).

We are thus arrived at a question which easily can be settled within the formal framework developed in the proof of Lemma 4. Let $\mathcal{R}, \mathcal{R}^n, \mathcal{R}'$ have the same meaning as before and denote by $\mathcal{M}, \mathcal{M}'$ respectively the module which is generated over $\mathcal{R}$, $\mathcal{R}'$ respectively by all vectors of the form

$$Df(t,x;u)\Big|_{u=u_i}, \quad i=1,\ldots,N,$$

D being an arbitrary differential operator acting on all variables. Then the statement (4.14) holds true and the remainder of the proof is now identical with the arguments employed in the last paragraph of the proof of Lemma 4.

5. Application: Input/output decoupling, an example.

In this section we wish to demonstrate how the ideas which have been brought forward in this paper can be used to approach a fundamental question in control theory. We discuss two problems where the mathematical treatment aims at the construction of feedback-control laws which achieve one and the same purpose: To make certain outputs simultaneously independent from certain inputs. The analysis of the first example is rather elementary and uses nothing else than some arguments discussed in Sec. 3 plus a suitable notation. Given a n-dimensional vector $a(t,x)$ and a scalar C^1-function $\alpha(t,x)$ we put

$$L_a\alpha = \frac{\partial}{\partial t}\alpha + \left(\frac{\partial}{\partial x}\alpha\right)\cdot a .$$

This function is sometimes called the Lie-derivative of α along a (cf. [2], p. 282). Lie-derivatives of vectors are defined componentwise. L_α^0 and the iterates L_α^ρ have the obvious meaning.

Lemma . Let $\gamma(t,x)$ be a smooth function of t,x. Then

$$\left(\frac{d}{dt}\gamma\right)(t,x) = (L_a\gamma)(t,x) + \left(\left(\frac{\partial}{\partial x}\gamma\right)B\right)(t,x)\cdot u$$

where d/dt means differentiation with respect to the differential equation

$$\dot{x} = a(t,x) + B(t,x)u =: f(t,x;u). \tag{5.1}$$

Proof. By inspection.

Given now a control system in terms of a differential equation (5.1) where u is m-dimensional. Accordingly B is a matrix of type $n\times m$. We assume that there are given in addition k scalar functions $\gamma_i(t,x)$ which play the role of outputs. a, B, γ_i are supposed to be of class C^∞ on some open set $\mathfrak{X}$. For each $i=1,\ldots,k$ we denote by ρ_i the smallest non-negative integer ρ such that the m-dimensional row vector

$$\left(\frac{\partial}{\partial x}(L_a^\rho\gamma_i)\right)B \tag{5.2}$$

does not vanish identically. We assume that such a number exists. ρ_i is sometimes called the characterist number of the output $y_i = \gamma_i(t,x)$ (cf. [2], IV.3, where the question of existence is also discussed). We introduce the m-dimensional row vectors

$$c_i := \left(\frac{\partial}{\partial x}(L_a^{\rho_i}\gamma_i)\right)B, \quad i=1,\ldots,k, \tag{5.3}$$

whose elements are of class C^∞ on $\mathfrak{X}$. Our basic hypotheses runs now as follows:

> The vectors $c_i(t,x)$, $i=1,\ldots,k$, are linearly independent for every $(t,x)\in\mathfrak{X}$. (5.4)

It follows then by standard arguments that there exist matrices F,G respectively whose elements are C^∞-functions on $\mathfrak{X}$ and which are of type $m\times 1$ and $m\times m$ respectively such that these relations hold for $i=1,\ldots,k$:

$$c_iF = L_a^{\rho_i+1}\gamma_i, \quad c_iG = e_i \quad . \qquad (5.5)$$

Here $e_i = (0,\dots,0,1,0,\dots)^T$ has the usual meaning as standard basis vector of the $\mathbb{R}^m$.

<u>Proposition 1</u>. Let F,G be matrices which satisfy the conditions (5.3), (5.5) on $\mathfrak{X}$. Let then x(t) be a solution of the differential equation which arises from (5.1) by means of the following procedure. First carry out the substitution

$$u = -F(t,x) + G(t,x)u', \qquad (5.6)$$

then specialize u' to a C^∞-function

$$u'(t) = (u_1'(t),\dots,u_i'(t),\dots,u_m'(t))^T, \qquad (5.7)$$

of t. In other words: x(t) is solution of

$$\dot{x} = f(t,x, - F(t,x) + G(t,x)u') \qquad (5.8)$$

for $u' = u'(t)$ on some interval I and satisfies the condition $(t,x(t)\in\mathfrak{X}$.

<u>Claim:</u> If u'(t) is given componentwise as in (5.7) and if $y_i(t) = \gamma_i(t,x(t))$ then

$$\frac{d^{\rho_{i+1}}}{dt^{\rho_{i+1}}}\, y_i(t) = u_i'(t) \quad \text{for } t\in I,\ i=1,\dots,k. \qquad (5.9)$$

<u>Proof.</u> Straightforward, using the definition of the characteristic number (cf. (5.2),(5.3)). If the scalar function $y_i = \rho_i(t,x)$ is differentiated repeatedly with respect to the differential equation (5.8) one obtains - by induction with respect to ρ - from the Lemma

$$\frac{d^\rho y_i}{dt^\rho} = \begin{cases} L_a^\rho\gamma_i & \text{if } 0\le\rho\le\rho_i , \\ L_a^{\rho_i+1}\gamma_i + \left(\frac{\partial}{\partial x}\,(L^{\rho_i}\gamma_i)\right)Bu & \text{if } \rho = \rho_i+1 . \end{cases}$$

In the last line u has to be replaced by - F+Gu'. The statement in question follows then immediately from (5.3), (5.5).

Some comments seem to be in order. (5.6) can be viewed upon as a formal change of control variables, the original one being u, the new one being u'. Note that the first of the relations (5.5) implies that the matrix G is invertible, hence (5.6) actually defines a so-called feedback-transformation. Such a transformation preserves the degree of freedom in the control. Note however that - from a systemtheoretic point of view - it makes a difference whether the decision over the control action in each moment is expressed in terms of u or u'. In the second case the actual steering action depends upon the state, which explains the occurence of the word "feedback". Furthermore the various components of the original input u may get mixed up if the steering action is defined in terms of a control law (5.6). This can make the feedback-transformation meaningless from a physical point of view (see the example at the end of this section). In the present situation we do not bother about such possibilities and assume that F,G can be chosen in such a way that (5.5) holds true. We know then that y_i depends upon the i-th component of the new input u' only. To be more precise: What can be concluded immediately from proposition 1 is the following statement. Given any time interval $(t_o,t_1) = I$. $y_i(t_1)$ depends then upon $u_i'(t), t\in I$, and upon $x(t_o,)\dot{x}(t_o)$, etc. Now the derivatives of x(t) at $t=t_o$ depend upon $x(t_o)$ and the initial values (at $t=t_o$) of $u_j', \dot{u}_j', \ddot{u}_j'$, $\dots, j=1,\dots,m$. That the latter quantities for $j \neq i$ actually do not enter into the value of $y_i(t) = \gamma_i(t,x(t))$ at $t=t_1$ can be seen by an argument which is familiar to us from Section 3. We state it for the original control system (5.1). Given any admissible control function u(t) and an arbitrary sequence $u^{(\nu)}(t)$ which converges in L^1-norm to u(t) on I. The sequence $x^{(\nu)}(t_1)$ converges then to $x(t_1)$ where $x^{(\nu)}(t)$, x(t) respectively are solutions of the differential equation

$$\dot{x} = f(t,x;u) \quad , \text{ for } \quad u=u^{(v)}(t) \text{ and } \quad u=u(t)$$

respectively, having a fixed initial value at $t=t_o$. Now one can choose the $u^{(v)}(t)$ always in such a way that their initial values have nothing to do with those of $u(t)$, say we can have $u^{(v)}(t_o) = \dot{u}^{(v)}(t_o) = \ldots = ((d^k/dt^k)u^{(v)})(t_o)=0$ for some k together with L^1-convergence to $u(t)$ on I. If this argument is applied to the transformed control system (5.8) one arrives at this conclusion.

Corollary to proposition 1. If F,G satisfy the condition (5.5) then the feedback transformation (5.6) serves the following purpose. For every $i=1,\ldots,k$ and every $t_1>t_o$ the value $y_i(t_1,x(t_1))$ of the i-th component of the output depends solely upon the initial time t_o, the initial state $x_o=x(t_o)$ and the values which the i-th component of u' assumes in the interval (t_o,t_1).

Hence the hypothesis (5.4) is a sufficient condition in order that a problem can be solved which in the literature is sometimes called the single-outputs noninteracting control problem. For further information the reader is referred to [2],IV.4, where also the question of necessity is discussed.

We wish to conclude this section with an application of Theorem 1. It concerns a type of problem to which the foregoing analysis does not apply. We assume that the whole input of the system is divided into two parts which have to be kept strictly apart. The symbol u ("control") is used from now on for the first part only, the second one is denoted by d ("external disturbance"). Accordingly we write the system equation in the form

$$\dot{x} = a(x) + B(x)u + g(x)d, \quad y=c(x). \qquad (5.10)$$

The disturbance decoupling problem (or rather a simplified version of it, for a more thorough discussion cf. [2],IV.2) runs then as follows: Specify u as a function of x ("state-feedback") in such a way that y becomes independent from d. Let us assume for simplicitiy that d is scalar (hence $g(x)$ is a column vector) and that B is constant and equal to $(I,0)^T$, I being the m-dimensional unit matrix. If a is decomposed in the form $a = (a_1,a_2)^T$, a_1 being m-dimensional, and if a new control variable $\tilde{u}$ is introduced via

$$u = \tilde{u} - a_1(x) \qquad (5.11)$$

then the disturbance decoupling problem assumes- in the analytic case - the following form, according to Theorem 1: Find an analytic function $\tilde{u}(x)$ such that these identities hold true

$$c_x(ad_f^\rho g) = 0, \ \rho = 0,1,\ldots, \ f(x):=(\tilde{u}(x),a_2(x)). \ (5.12)$$

If one wants to have a criterion for disturbance decoupling which involves finitely many quantities only one arrives at the following obvious modification of the above stated problem: (5.12) has to be satisfied for $\rho=0,\ldots,k-1$ and addition this statement holds true:

$$ad_f^k g \text{ is contained in the span of } ad_f^\rho g \text{ for } \rho<k. \quad (5.13)$$

Thereby we have reached the definite limit of our presentation of the disturbance decoupling problem. Even if this problem is understood in the elementary form (5.12),(5.13) and if one is looking for local solutions only - there exists presently no other approach except the one which is based on notions and results of the geometric control theory (cf. [2], (Ch.IV,Sec. 1,2).

On the other hand if one wants to solve the problem (5.12)(5.13) in a concrete situation and globally (i.e. on a prescribed are $\mathcal{X}$ of the state space) the nature of the problem may help to guess a suitable $\tilde{u}(x)$ for which (5.12), (5.13) is satisfied with a low number k. The following example - which is taken from [5] - serves as an illustration. The state-space is of dimension 6, hence $x=(x_1,\dots,x_6)^T$, the control and the disturbance respectively act on the fourth, fifth and sixth component respectively of the state:

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T, \quad g = (0,\dots,0,1)^T. \qquad (5.14)$$

The uncontrolled and undisturbed system equation $\dot{x} = a(x$ describes the motion of a rigid body in $\mathbb{R}^3$ with respect to an inertial coordinate system (see [5] for details). The control should be used in order to make the motion of $x_3 =: c(x)$ independent from the disturbance regardless of the body's position. In our setting this problem can be formulated as follows: Find two functions $\tilde{u}_4(x)$, $\tilde{u}_5(x)$ which are defined and analytic everywhere and an integer $k \geq 0$ such that (5.13) holds and we have

$$(0,0,1,0,0,0) \cdot ad_f^\rho g = 0, \quad \rho = 0,\dots,k-1.$$

Here g is given as in (5.14) and

$$f = (x_6x_2 - x_5x_3,\ x_4x_3 - x_1x_6,\ x_5x_1 - x_4x_2,\ \tilde{u}_4(x), \tilde{u}_5(x), \beta x_4x_5\ ^T.$$

If f,g are of this special form then

$$ad_f\, g\ -(\partial f/\partial x)g = (-x_2, x_1, 0,\dots)^T .$$

Hence conditions (5.12),(5.13) cannot be met if $k=1$. For $k=2$ however one is succesful if one tries to construct $\tilde{u}_4$, $\tilde{u}_5$ as quadratic forms in x (this choice is suggested by the fact that the a-priori given components of f are functions of this type). Indeed it is not difficult to see that there is a unique sclution to the problem within this class of functions namely

$$\tilde{u}_4 = x_5x_6\ , \ \tilde{u}_5 = -x_4x_6\ ,$$

$ad_f^2 g$ then becomes a multiple of g.

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author's address: Mathematisches Institut, Am Hubland, D-87 Würzburg.