Confinement and Asymptotic Freedom
In a Simple Quark Model

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1. "Toy-model" for QCD

[Horowitz, Moniz, Negele, Phys. Rev. D31 (1985) 1689]

- N fermions (quarks) in 1-dimensional string of length L
- No internal degrees of freedom
- Hadrons made up of 2 quarks

Hamiltonian: $H = T + V$ (non relativistic)

$$T = -\frac{1}{2}\sum_{i=1}^{N} \partial^2_{x_i} \qquad (m=1\,,\ \hbar=1)$$

$$V = \min_{\{P\}} \sum_{n=1}^{N/2} \mathcal{V}\left(\left|x_{P(2n-1)} - x_{P(2n)}\right|\right)$$

$\{P\}$ = ensemble of all permutations of quark labels which pair the N quarks into N/2 pairs
Confining potential $\mathcal{V}$ acts only inside each pair

$$\mathcal{V}(x) = \frac{1}{2}x^2 \qquad (K=1)$$

$\Rightarrow$ Many-body interaction (i.e. depends upon configurations of all quarks)

Main features of the model:

i) confines quarks into hadrons
ii) no Van der Waals forces
iii) hadron interaction only from exchange of quarks between clusters
iv) low density limit : free hadron gas (?)
v) high density limit : free quark gas

→ In <u>1-Dim</u> : minimum energy configuration via nearest neighbours coupling

<u>Ground state energy</u>

a) Hartree-Fock approximation

$$\frac{\langle T \rangle}{N} = \frac{1}{N} \sum_k \epsilon_k^0 \, \theta(k_F - k) = \frac{\pi^2}{6} \rho^2$$

$$\rho = \frac{N}{L} = \frac{k_F}{\pi} \qquad \text{(homogeneous density)}$$

Hartree energy:

$$\frac{\langle V \rangle_H}{N} = \frac{N!}{N L^N} \int_0^L dx_N \int_0^{x_N} dx_{N-1} \cdots \int_0^{x_2} dx_1 \, \frac{1}{2} \sum_{i=1}^{N/2} (x_{2i} - x_{2i-1})^2$$

$$= \frac{L^2}{2(N+1)(N+2)} \xrightarrow[L\to\infty,\, N\to\infty]{} \frac{1}{2\rho^2}$$

Fock term (only numerical estimate): $\frac{\langle V \rangle_F}{N} \simeq -\frac{1}{4\rho^2}$

Then:

$$\frac{E_{HF}}{N} = \frac{\pi^2}{6} \rho^2 + \frac{1}{4\rho^2}$$

Minimum at $\rho = \rho_0 \simeq 0.62$ (where $E_{HF}/N = 1.28$)

b) path integral Monte Carlo method

$$E(\beta) = \frac{\langle \phi | H e^{-\beta(H-e_N)} | \phi \rangle}{\langle \phi | e^{-\beta(H-e_N)} | \phi \rangle}$$

$|\phi\rangle$ = trial wave function:

i) Fermi gas $\quad |\phi\rangle_{FG} = \prod_{i<j} \sin \frac{\pi}{L}(x_i - x_j)$

ii) Variational $\quad |\phi\rangle_\lambda = e^{-\lambda V} |\phi\rangle_{FG}$

Results for the g.s. energy:

low density : energy of "free" clusters (hadrons)
high density : energy of free quark gas.

Quark-quark correlation function

$$\rho_2(x_1 - x_2) = \frac{1}{N(N-1)} \sum_{i \neq j} \delta(x_i - x_1)\,\delta(x_j - x_2)$$

$$= \frac{1}{(N-1)L} \sum_{i \neq j} \delta\big(|x_1 - x_2| - |x_i - x_j|\big) \quad \text{(infinite system)}$$

$$g(r) \equiv g(|x_1 - x_2|) = \langle \psi | \rho_2(x_1 - x_2) | \psi \rangle$$

$$|\psi\rangle = e^{-\beta(H-e_N)} |\phi\rangle$$

i) Low density: ($\rho = 0.27$) peak at small r → second quark bound in a hadron

ii) Moderate density: ($\rho = 0.5$) free quark gas correlation function

Horowitz, Moniz, Negele
Phys. Rev. D31 (1985) 1689

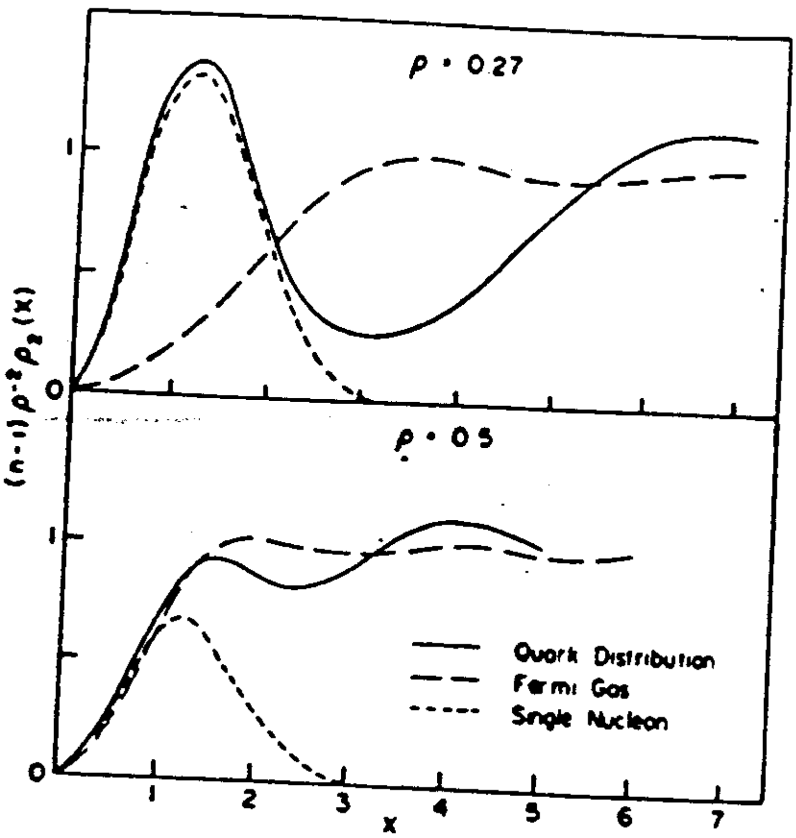


FIG. 3. Quark-quark correlation function ρ_2 at low and moderate densities for an eight-quark system (normalized to one at large distances). Also shown are results for a free Fermi gas and an isolated hadron with the wave function of Eq. (3).

2. Present Model

→ Density-dependent two-body interaction, to make up the N-body force

$$\vartheta(x) = \frac{1}{2} x^2 e^{-c\varrho x} \qquad (c = \text{const.})$$

Ground state energy

$$E_p[\varrho] = \frac{1}{2} \sum_{\mu\nu\gamma\sigma} \langle \mu\nu | \vartheta(\varrho) | \gamma\sigma \rangle \varrho_{\gamma\mu} \varrho_{\sigma\nu} \quad \text{(potential energy}$$

Expansion around the equilibrium density ϱ_0:

$$E_p[\varrho] = E_p[\varrho_0] + \sum_{\alpha\beta} \frac{\delta E_p}{\delta \varrho_{\alpha\beta}} \delta\varrho_{\alpha\beta} + \frac{1}{2} \sum_{\alpha\beta\gamma\sigma} \frac{\delta^2 E_p}{\delta\varrho_{\alpha\beta}\delta\varrho_{\gamma\sigma}} \delta\varrho_{\alpha\beta} \delta\varrho_{\gamma\sigma} + \ldots$$

defines s.p. mean field

$$h_{\alpha\beta} \equiv \frac{\delta E_p}{\delta \varrho_{\alpha\beta}} = \sum_{\sigma\nu} \langle \beta\nu | \vartheta(\varrho) | \alpha\sigma \rangle \varrho_{\sigma\nu} +$$

$$+ \frac{1}{2} \sum_{\mu\nu\gamma\sigma} \langle \mu\nu | \frac{\delta\vartheta}{\delta\varrho_{\alpha\beta}} | \gamma\sigma \rangle \varrho_{\gamma\mu} \varrho_{\sigma\nu}$$

In the Hartree-Fock basis

$$\varrho_{\sigma\nu} = \delta_{\sigma\nu} \Theta(N-\nu)$$

and single particle HF self-energy reads $[\Sigma(\alpha) = h_{\alpha\alpha}]$

$$\Sigma^{HF}(k) = \sum_{i<N} \langle ki | \vartheta(\varrho) | ki \rangle + \frac{1}{2L} \sum_{i,j<N} \langle ij | \frac{\delta\vartheta}{\delta\varrho} | ij \rangle$$

$$\equiv \Sigma^{HF}_{ord}(k,\varrho) + \Sigma^{HF}_{rearr}(\varrho)$$

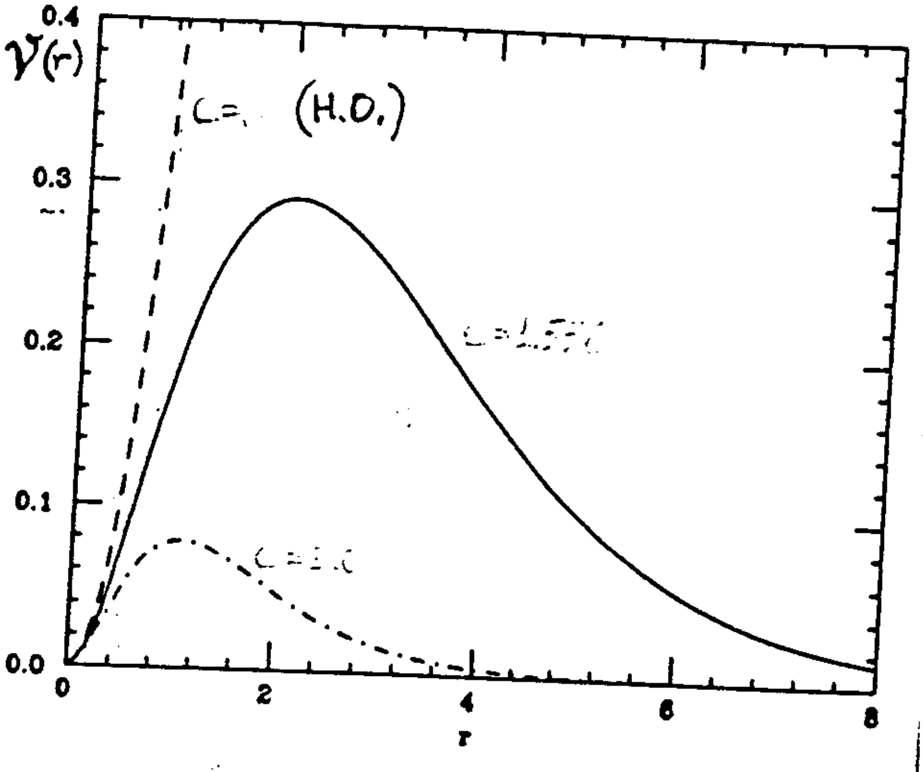


Fig. 1

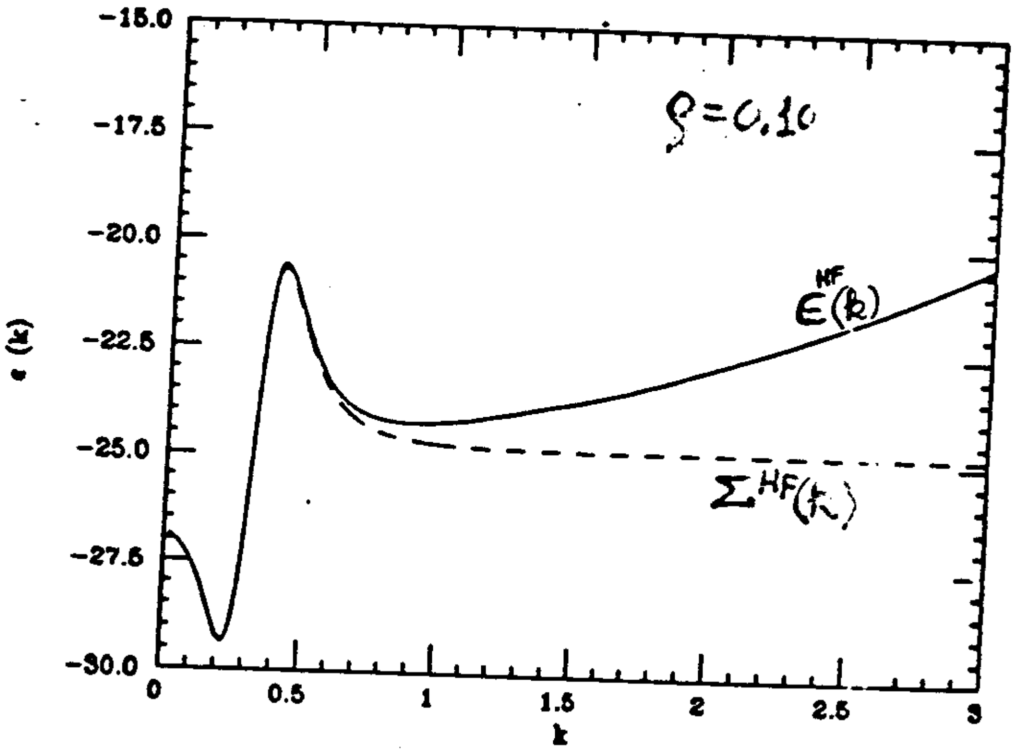


Fig. 2

with our potential:

$$\Sigma^{HF}_{ord}(k,\rho) = \frac{2}{c^3\rho^2} - \frac{c\rho}{\pi}\left\{\frac{k_F+k}{[c^2\rho^2+(k_F+k)^2]^2} + \frac{k_F-k}{[c^2\rho^2+(k_F-k)^2]^2}\right\}$$

$$\Sigma^{HF}_{rearr}(\rho) = -\frac{4\pi^2}{\rho^2}\,\frac{5c^2+12\pi^2}{c^3(4\pi^2+c^2)^2}$$

$\Rightarrow$ HF energy:

$$\frac{\langle V\rangle_{HF}}{N} = \frac{1}{2N}\sum_{k\le k_F}\Sigma^{HF}(k,\rho) = \frac{1}{c^3\rho^2}\,\frac{4\pi^2}{c^2+4\pi^2}$$

NB: same density dependence of "toy model"

Requiring: $\displaystyle \frac{1}{c^3\rho^2}\,\frac{1}{[1+(2\pi/c)^2]} = \frac{1}{4\rho^2} \quad \Rightarrow$

one gets the same HF ground state energy.

One-dimensional Polarization Propagator

A) Free case (Lindhard function):

$$\Pi^0(q,q';\omega) =$$

$$= \delta_{qq'}\frac{L}{2\pi}\int_{-\infty}^{+\infty}dk\left\{\frac{\theta(|q+k|-k_F)\,\theta(k_F-|k|)}{\omega-(\epsilon^0_{q+k}-\epsilon^0_k)+i\eta} - \frac{\theta(k_F-|q+k|)\,\theta(|k|-k_F)}{\omega-(\epsilon^0_{q+k}-\epsilon^0_k)-i\eta}\right\}$$

Introduce dimensionless variables:

$$Q = q/k_F$$
$$\nu = \omega/2\epsilon_F$$

$$\epsilon_F = \frac{1}{2}k_F^2 \quad \text{(Fermi energy)}$$

Then

$$\mathrm{Im}\,\Pi^{\circ}(q,q';\omega) = -\delta_{Q,Q'}\,\frac{L}{2k_F|Q|}\,\theta\left(\left|\frac{Q}{2}+\frac{\nu}{Q}\right|-1\right)\theta\left(1-\left|\frac{\nu}{Q}-\frac{Q}{2}\right|\right)$$

only for

$$\left|\frac{Q^2}{2}-Q\right| \leq \nu \leq \frac{Q^2}{2}+Q$$

otherwise $\mathrm{Im}\,\Pi^{\circ}=0$.

$$\mathrm{Re}\,\Pi^{\circ}(q,q';\omega) = -\delta_{Q,Q'}\,\frac{L}{2\pi k_F|Q|}\,\ln\left|\frac{(Q/2+1)^2-(\nu/Q)^2}{(Q/2-1)^2-(\nu/Q)^2}\right|$$

B) Hartree-Fock

$$\mathrm{Im}\,\Pi^{HF}(q,q';\omega) = -\delta_{qq'}\,\frac{L}{2}\int_{-\infty}^{\infty} dk\,\theta(|q+k|-k_F)\,\theta(k_F-|k|)\times$$
$$\times\left[\delta\left(\omega-\epsilon^{HF}_{q+k}+\epsilon^{HF}_{k}\right)+\delta\left(\omega+\epsilon^{HF}_{q+k}-\epsilon^{HF}_{k}\right)\right]$$

where

$$\epsilon^{HF}_k = \frac{1}{2}k^2+\Sigma^{HF}(k,\rho)$$

Explicitly:

$$\mathrm{Im}\,\Pi^{HF}(q,q';\omega) = -\delta_{qq'}\,\frac{L}{2}\int_{-\infty}^{+\infty} dk\,\theta(|q+k|-k_F)\,\theta(k_F-|k|)\times$$
$$\times\sum_i\left\{\frac{\delta[k-k^i(\omega)]}{\left|\frac{\partial}{\partial k}\left(\epsilon^{HF}_{q+k}-\epsilon^{HF}_{k}\right)\right|_{k=k^i}}+(\omega\to-\omega)\right\}$$

with $k^i(\omega)$ satisfying the equation: $\omega=\epsilon^{HF}_{q+k}-\epsilon^{HF}_{k}$.

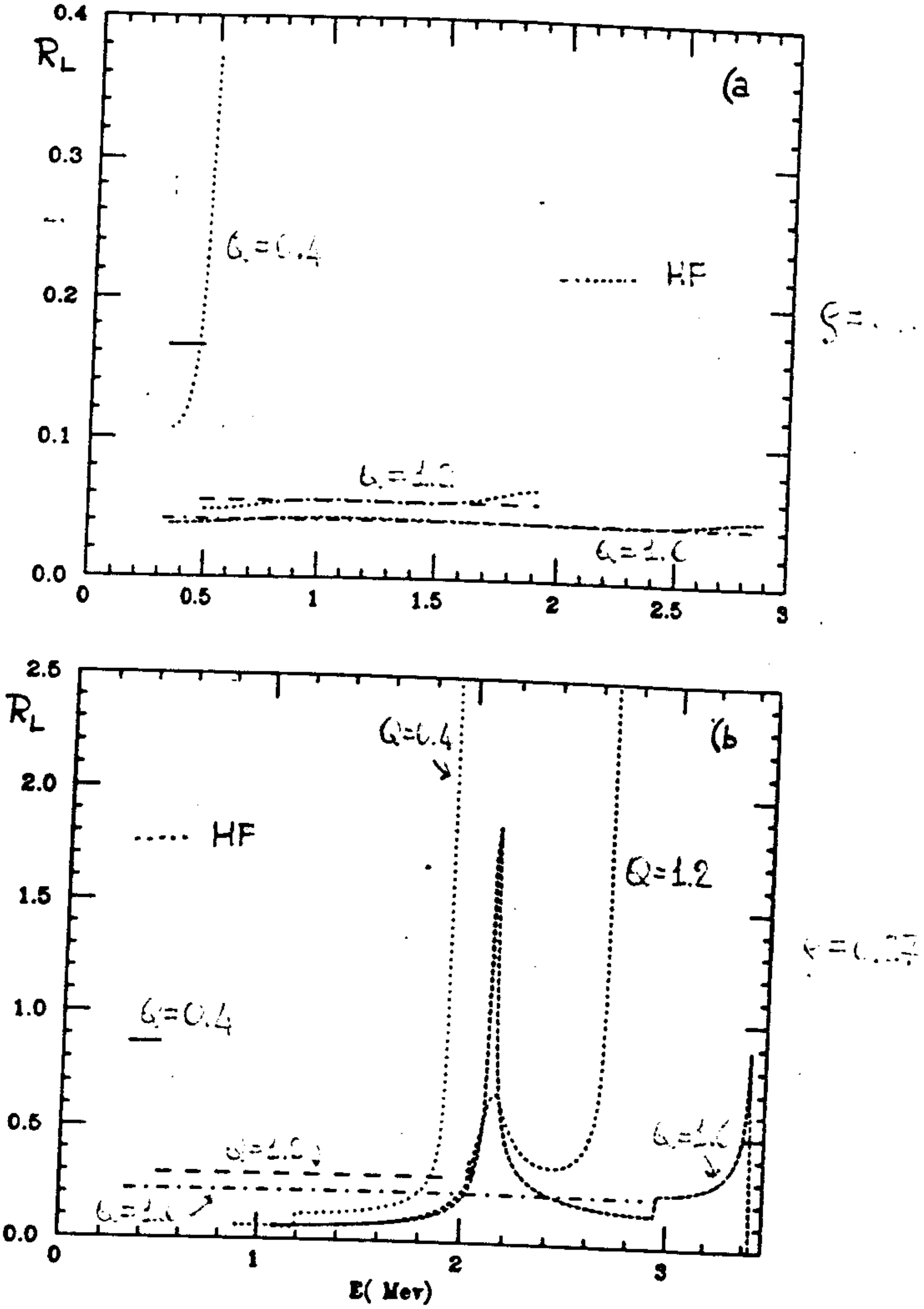


Fig. 3

Pn and HF Responses

Response function

Longitudinal response function per hadron (with two constituents of, e.g., charge $e = 1/2$)

$$\mathcal{R}_L(q,\omega) = -\frac{2e^2}{\pi N}\, \mathrm{Im}\, \Pi(q,q;\omega)$$

$$\Downarrow$$

Sum Rule

[illegible]

Example : free sum rule

$$S^0(Q) = \begin{cases} Q/2 & |Q| < 2 \\ 1 & |Q| \geq 2 \end{cases}$$

NB: HF does not alter the sum rule

$$\int_0^\infty d\omega\, \mathcal{R}_L^0(q,\omega) = \int_0^\infty d\omega\, \mathcal{R}_L^{HF}(q,\omega)$$

$$\Downarrow$$

Correlation function

$$g(r) = 1 + \frac{1}{\rho}\int_{-\infty}^{+\infty} \frac{dq}{2\pi}\, e^{-iqr}\left[S(q) - 1\right]$$

Free quark gas:

$$g^0(r) = 1 - \frac{1}{2(k_F r)^2}\left[1 - \cos(2k_F r)\right]$$

Random Phase Approximation

A) Correlation energy

Particle-hole interaction (for density dependent forces)

$$U_{ph} = \frac{\delta^2 E_P}{\delta \rho_{\alpha\beta}\, \delta \rho_{\gamma\sigma}}$$

Explicitly (and neglecting exchange)

$$U^D_{ph}(q,\rho) = \frac{2c\rho}{L} \; \frac{(c\rho)^6 - 31(c\rho)^4 q^2 + 55 (c\rho)^2 q^4 - 9q^6}{(c^2\rho^2 + q^2)^5}$$

⇒ ring approximation for polarization propagator

$$\Pi^{ring}(q,\omega) = \frac{\Pi^0(q,\omega)}{1 - U^D_{ph}(q,\rho)\,\Pi^0(q,\omega)}$$

and for the ground state energy

$$\frac{E^{ring}}{N} = \frac{\epsilon_F}{\pi}\int_0^\infty dQ \int_0^\infty d\nu \left\{ V(Q)\, \mathrm{Im}\,\Pi^0(Q,\nu) + \right.$$

$$\left. + \frac{V(Q)}{U^D_{ph}(Q)} \tan^{-1}\left[\frac{-U^D_{ph}(Q)\,\mathrm{Im}\,\Pi^0(Q,\nu)}{1 - U^D_{ph}(Q)\,\mathrm{Re}\,\Pi^0(Q,\nu)}\right]\right\}$$

where

$$V(Q) = -\frac{2c}{\pi k_F^3}\,\frac{3Q^2 - (c/\pi)^2}{[Q^2 + (c/\pi)^2]^3}$$

particle-particle interaction in momentum space

$$\frac{E^{ring}}{N} = \underbrace{\frac{E^{ring}_{(1)}}{N}}_{\substack{\text{first term}\\ \text{(analytical)}}} + \underbrace{\frac{E^{ring}_{(2,ph)}}{N}}_{\substack{\text{from ph}\\ \text{continuum}}} + \underbrace{\frac{E^{ring}_{(2,coll)}}{N}}_{\substack{\text{from truly}\\ \text{collective modes}}}$$

p-h interaction

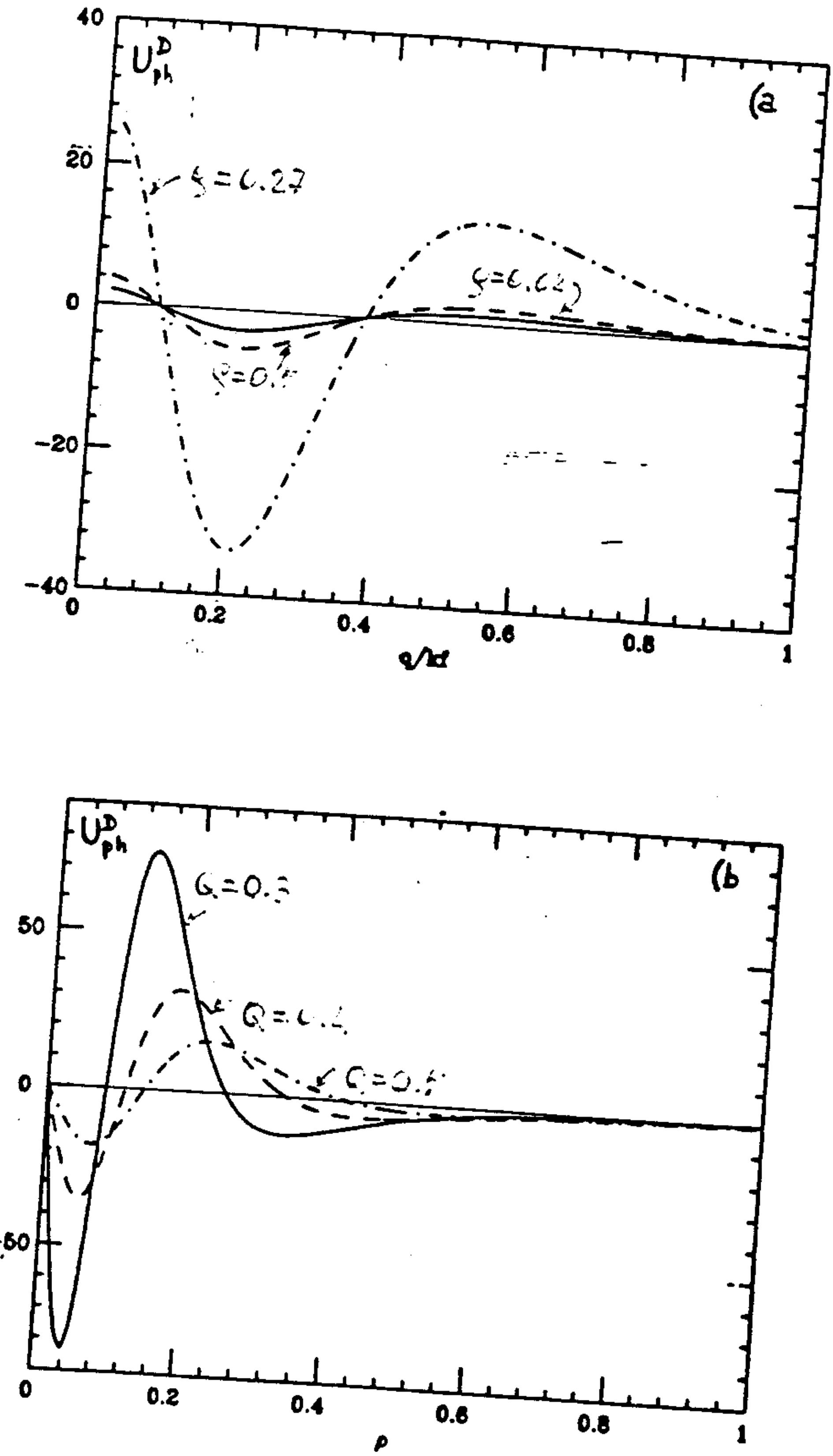
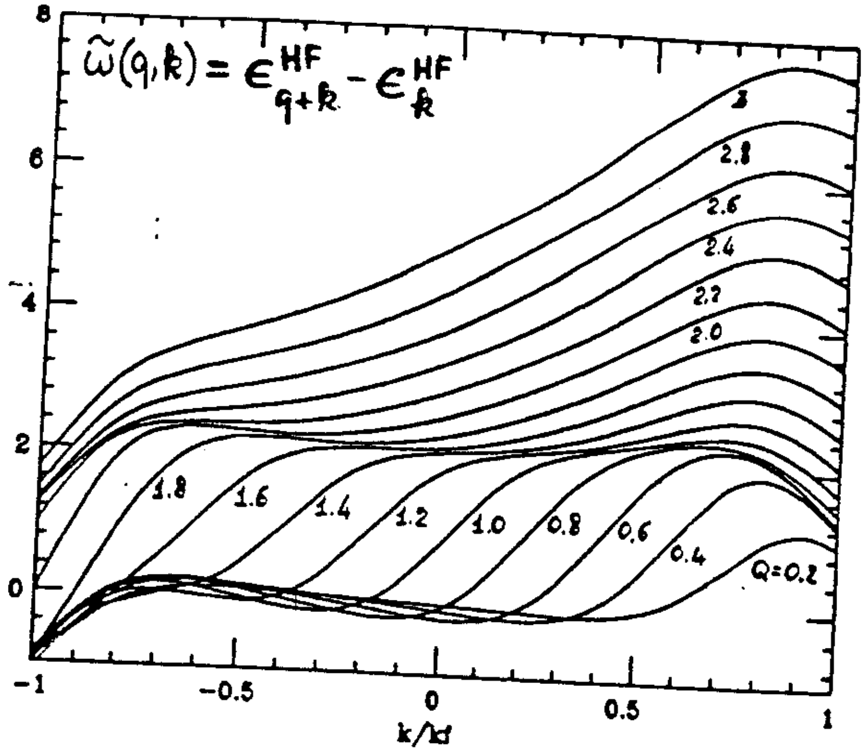
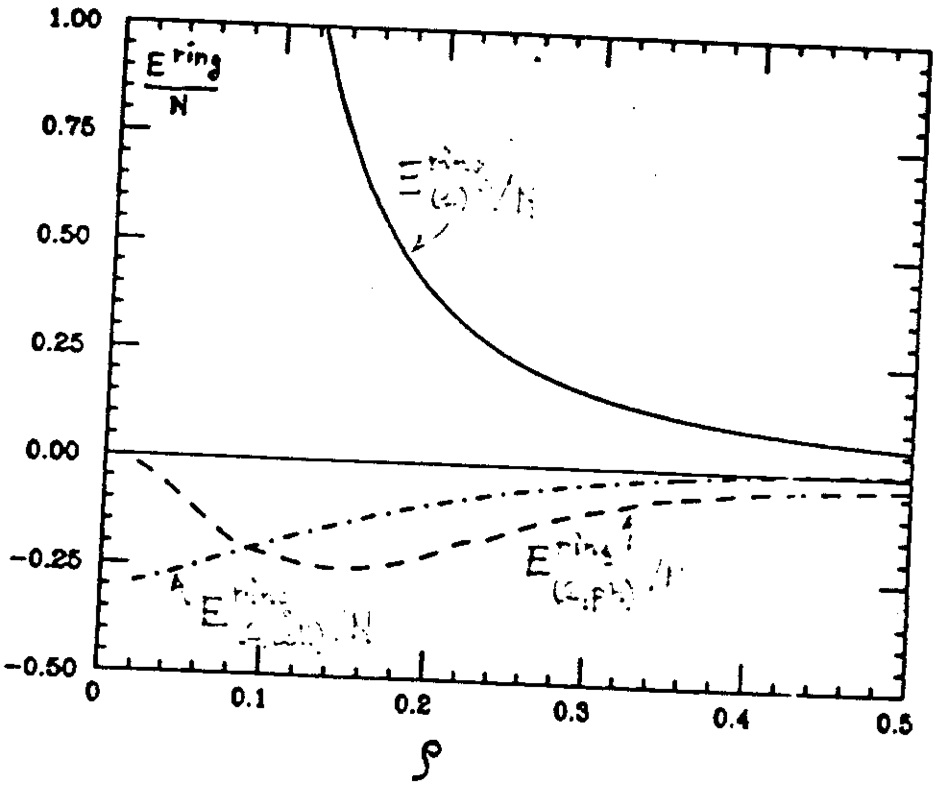


Fig. 5



$\tilde{\omega}(q,k) = \epsilon^{HF}_{q+k} - \epsilon^{HF}_{k}$
k/kf
Q=0.2

Fig. 4



$\frac{E^{ring}}{N}$
ρ

Fig. 6

About collective modes.

In regions where $\mathrm{Im}\,\Pi^0 = 0$, they are found as solutions of

$$1 - U^D_{ph}(Q)\,\mathrm{Re}\,\Pi^0(Q,\nu) = 0$$

namely

$$\ln\left|\frac{(Q/2+1)^2 - (\nu/Q)^2}{(Q/2-1)^2 - (\nu/Q)^2}\right| = -\frac{2\pi k_F Q}{U^D_{ph}(Q)}$$

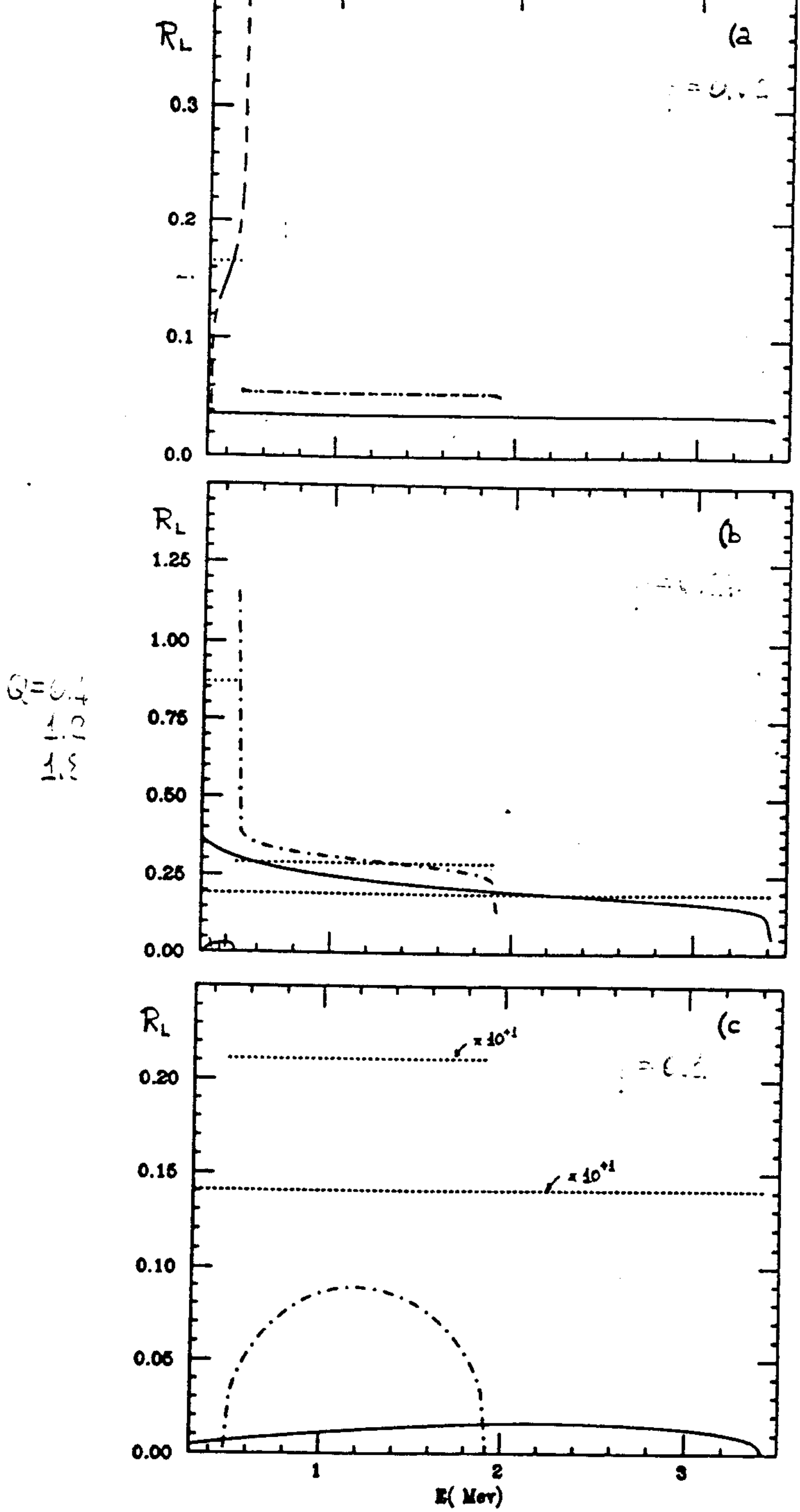
$$\Rightarrow \quad \nu = \nu_{coll} = |Q| \left\{ \frac{(Q/2+1)^2 - (Q/2-1)^2 \exp\{-[2\pi k_F Q/U^D_{ph}(Q)]\}}{1 - \exp\{-[2\pi k_F Q/U^D_{ph}(Q)]\}} \right\}^{1/2}$$

two possibilities:

a) hard modes, if $U_{ph} > 0$

b) soft modes, if

$$0 > U^D_{ph} > -\frac{1}{4\pi k_F Q \ln|(Q/2+1)/(Q/2-1)|}$$



R_L
(a
0.3
0.2
0.1
0.0
R_L
(b
1.25
1.00
0.75
0.50
0.25
0.00
R_L
(c
× 10⁺¹
0.20
0.15
× 10⁺¹
0.10
0.05
0.00
1
2
3
E(Mev)

Fig. 7

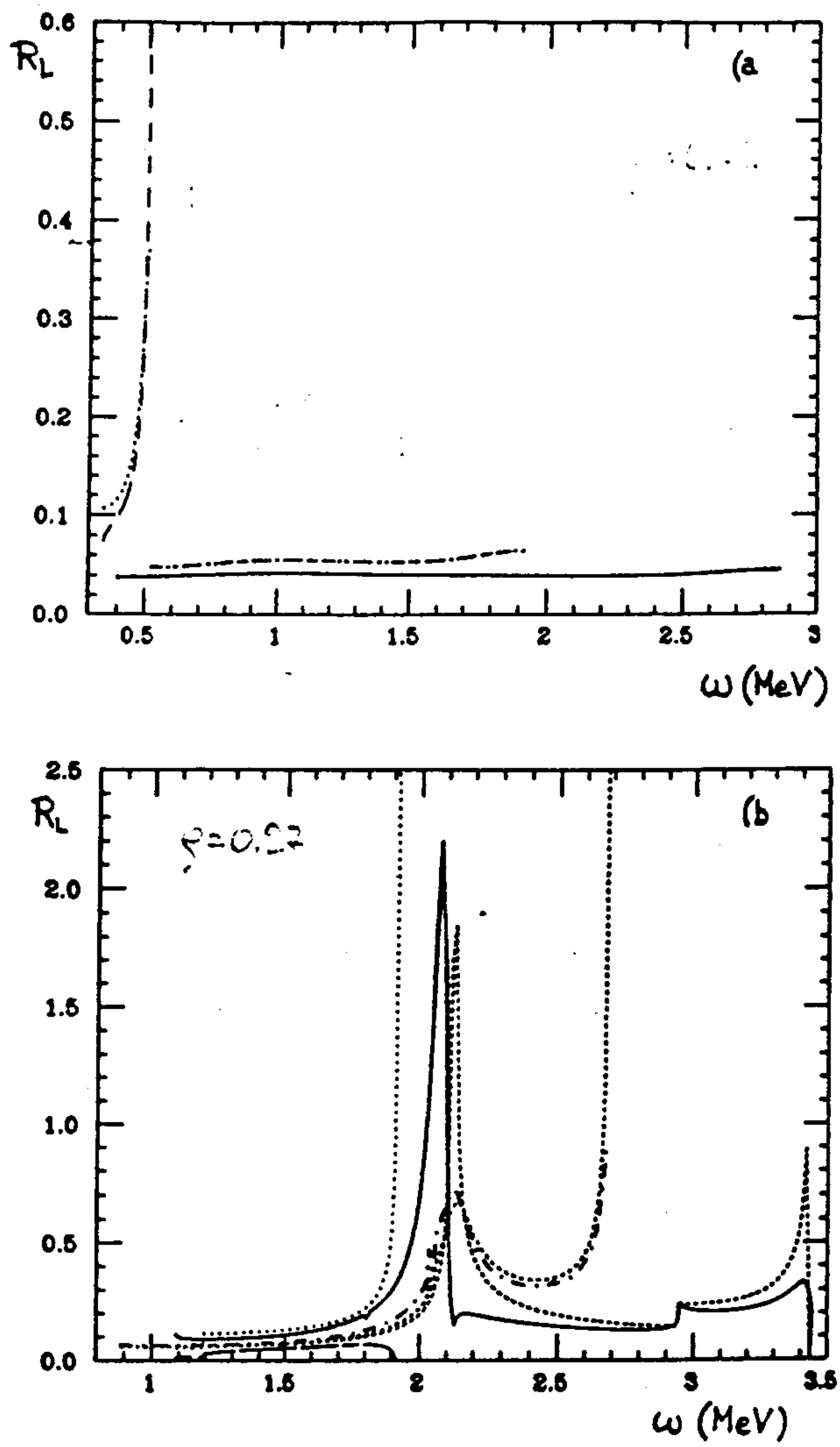


Fig. 8

HF HF-RPA

$$\Pi = \frac{\Pi^{HF}}{1-U^{D}_{ph}\Pi^{HF}}$$

F) RPA: Sum rule and correlation function

$$S^{RPA}(q) = -\frac{1}{\pi N}\int_0^\infty d\omega \, \mathrm{Im}\,\Pi^{ring}(q,\omega)$$

$$= -\frac{1}{\pi N}\int_0^\infty d\omega \frac{\mathrm{Im}\,\Pi^0(q,\omega)}{\left[1-U^D_{ph}(q)\,\mathrm{Re}\,\Pi^0(q,\omega)\right]^2+\left[U^D_{ph}(q)\,\mathrm{Im}\,\Pi^0(q,\omega)\right]^2}$$

$$= S^{RPA}_{ph}(q) + S^{RPA}_{coll}(q)$$

with

$$S^{RPA}_{coll}(q) = \frac{1}{2|\nu_{coll}|}\left[\frac{\pi k_F}{U^D_{ph}(Q)\,Q}\right]^2\left\{\nu^2_{coll}-\left(\frac{Q^2}{2}-Q\right)^2\right\}\left\{\nu^2_{coll}-\left(\frac{Q^2}{2}+Q\right)^2\right\}$$

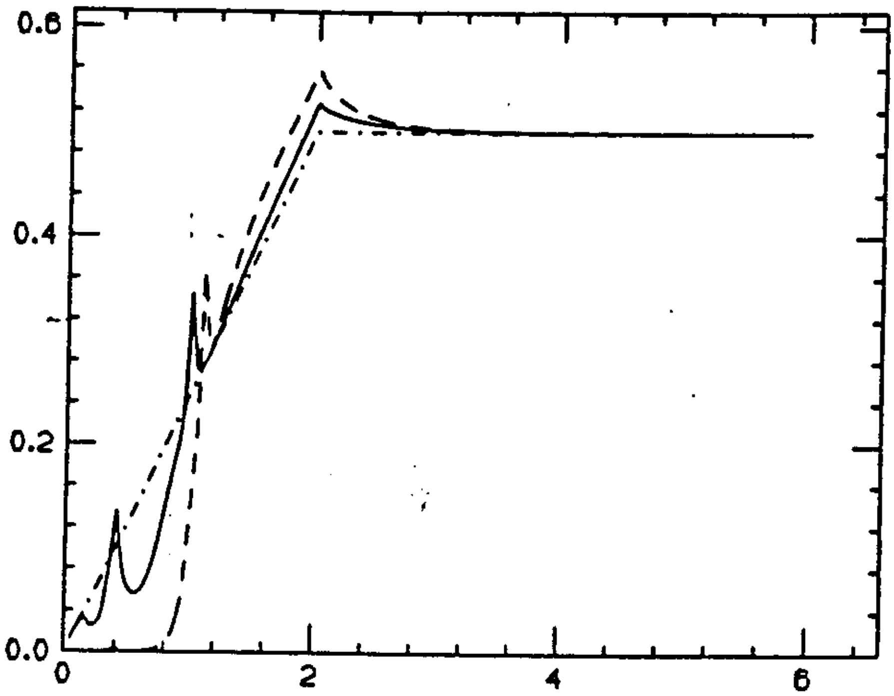
$$g^{ring}(r) = 1+\frac{1}{\rho}\int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{-iqr}\left[S^{RPA}(q)-1\right]$$

Results

- For moderate and high densities $(0.27 \div 0.62)$: small deviations from free quark gas
- For low densities (< 0.20): $g(r)$ develops a peak at $r \simeq 5$, much less that first maximum of $g^0(r)$ [at $r = 1/\rho$]

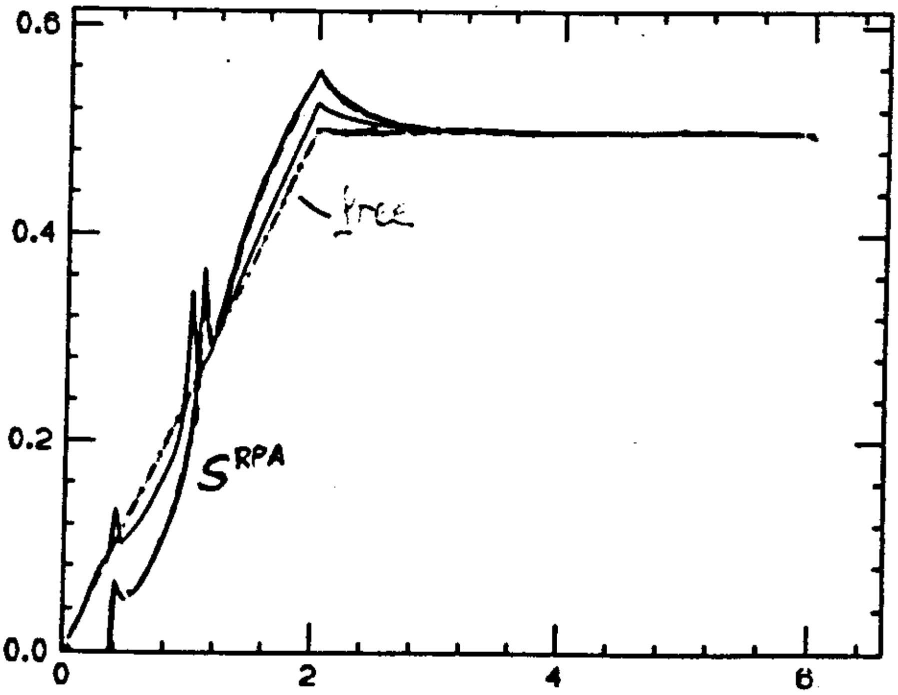
$\Rightarrow$ indication for a confined phase

S(q)

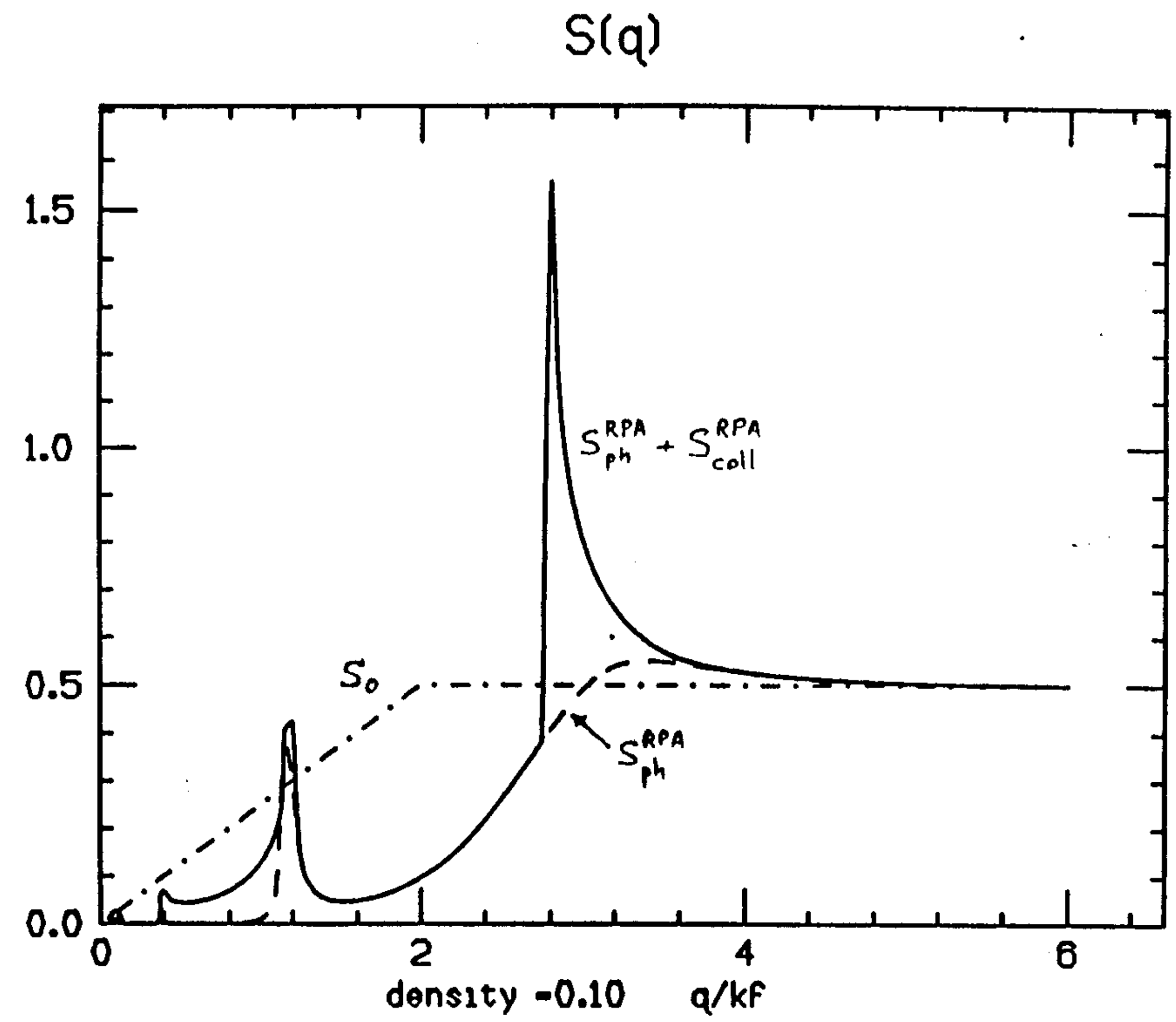


density =0.27

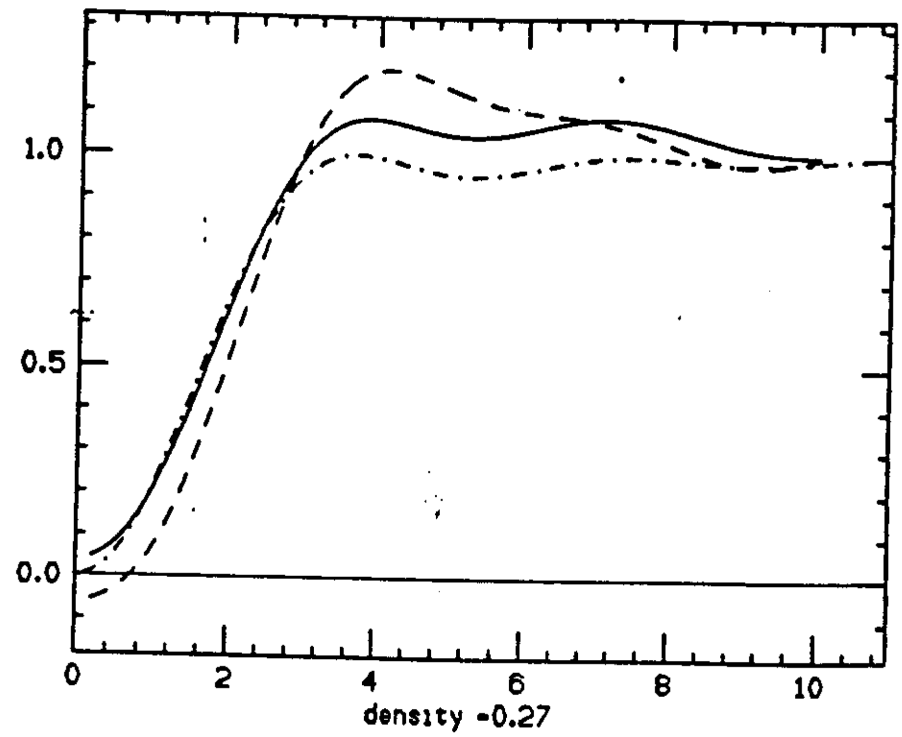
S(q) (with coll.modes)



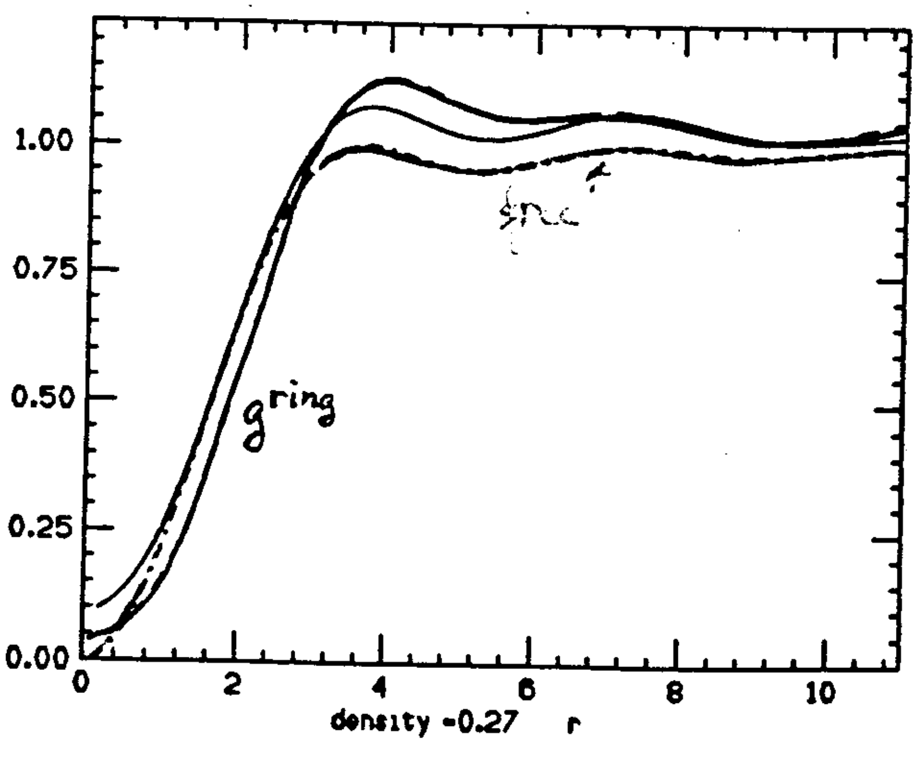
density =0.27 q/kf

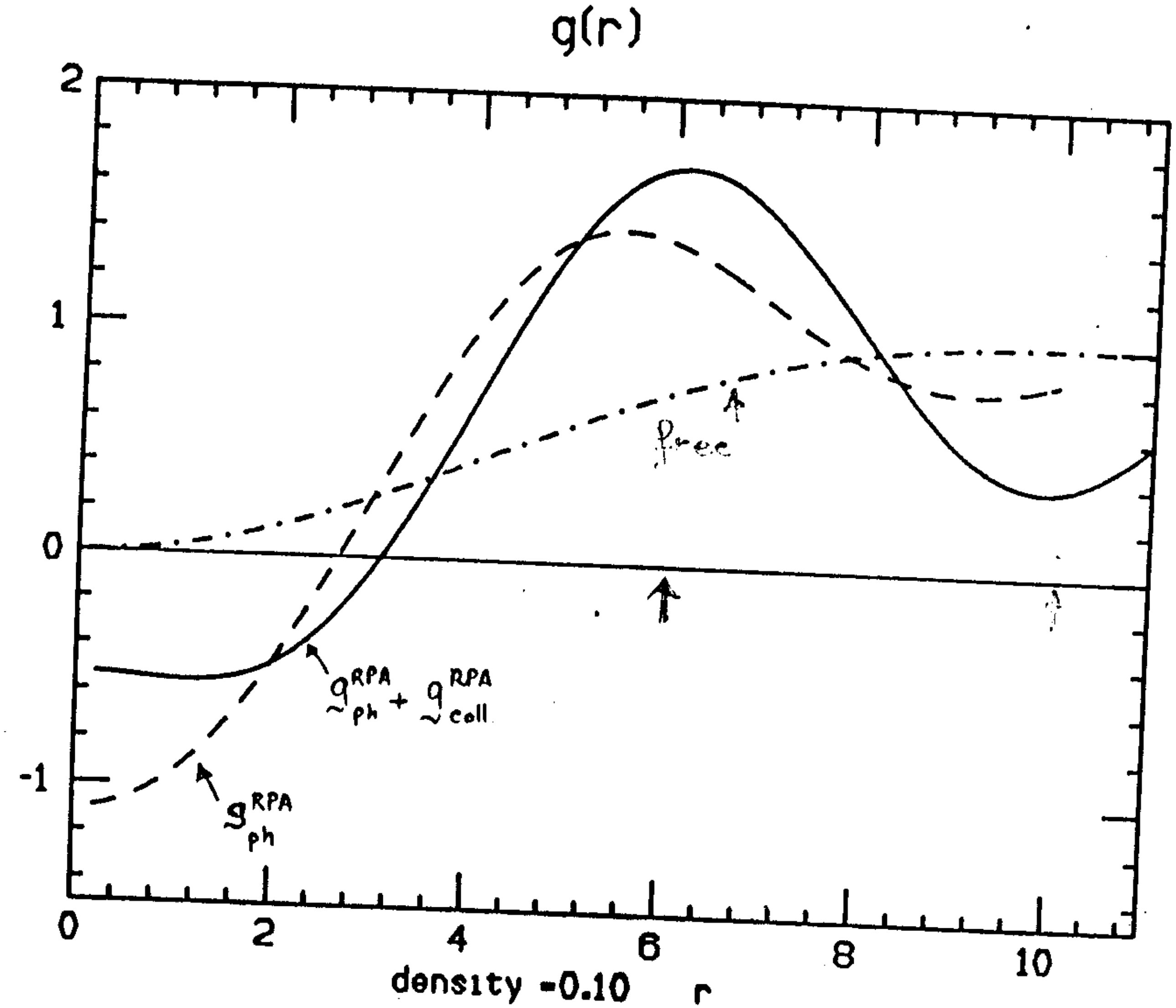

S(q)
1.5
1.0
0.5
0.0
0
2
4
6
$S_{ph}^{RPA} + S_{coll}^{RPA}$
S_o
S_{ph}^{RPA}
density =0.10
q/kf

g(r)



g(r) [with coll.modes]




g(r)
2
1
0
-1
0
2
4
6
8
10
density =0.10
r
$g^{RPA}_{ph} + g^{RPA}_{coll}$
g^{RPA}_{ph}
free

i) Negativity of $g(r)$ at small r

ii) correlation energy at small density

To cure i): account for exchange effects (phenomenological

$$\tilde{\Pi}(q,\omega) = \frac{\Pi^{\circ}(q,\omega)}{1 - U^{D}_{ph}(q)[1 - G(q)]\Pi^{\circ}(q,\omega)}$$

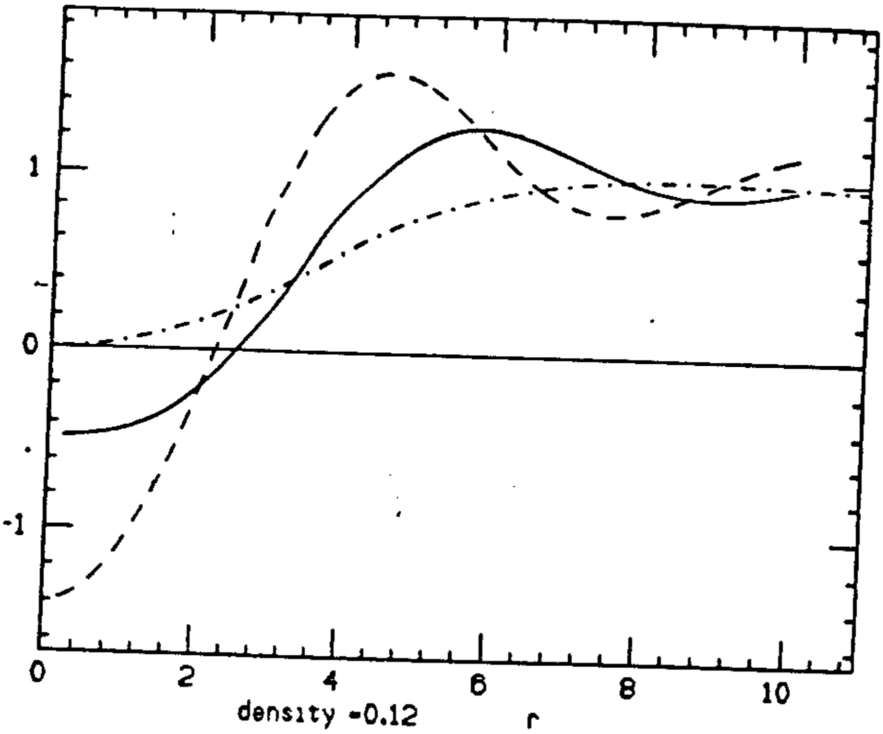
with

$$G(q) = \frac{1}{1 + \varsigma (q/k_F)^2}$$

Future developments

- HF-RPA in E^{ring} and $g(r)$
- Extension to 3-dimensions
- Momentum distribution (relevent to EMC effect)
- Relativistic effects

g(r)



g(r) [with coll.modes]