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SPRING COLLEGE ON PLASMA PHYSICS

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STOCHASTIC AND NONSTOCHASTIC PROCESSES IN SPACE PLASMAS

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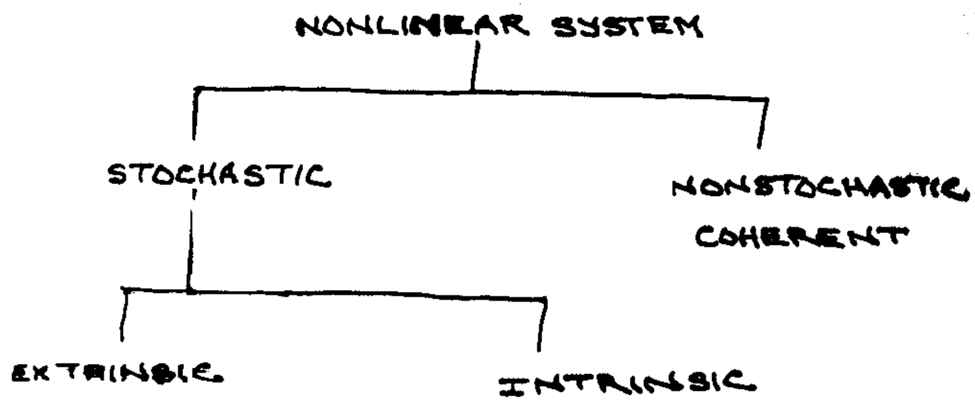
The subject of stochasticity or chaos is, in principle, an old one. Historically, it dates back to the last century when Poincaré (1892) tried to analyse the effect of non-linear perturbations on planetary orbits. But the subject picked up the momentum only in the 1960's partly due to the new mathematical techniques becoming available and partly due to the advent of high-speed computers. Before we go into the discussion of the subject, let us find out what exactly do we mean by stochastic process. What characterises the stochasticity? Why do we want to invoke stochasticity into the study of plasmas- does this study lead to any specific advantages? We will try to find an answer to some of these questions in these lectures. Our stress here would not be on the mathematical techniques but rather on the applications.

When we talk of a stochastic system, we must identify whether the stochasticity is of an intrinsic nature or extrinsic. The latter arises when the system is exposed to an external random noise. On the contrary if the stochasticity is generated by the dynamics itself, it will be strictly intrinsic. For example, even the conservative Hamiltonian systems can be stochastic because of these dynamical effects. This stochasticity ,in turn, can lead to anomalous effects, for example, anomalous diffusion, anomalous heating of plasmas and anomalous particle acceleration. Since we are quite familiar with such anomalous effects in laboratory as well as in natural plasmas which are not stochastic at all, the interesting question then arises; why do we need stochasticity? The simple answer is that stochasticity can lead to even stronger anomalous effects. For example, according to the quasilinear as well as orbit modification theory, the particles can be accelerated by the waves only up to the wave phase velocity whereas if the waves can lead to stochasticity then the particles can be accelerated to velocities much higher than the wave phase velocities. Strong anomalous effects arising because of stochasticity, are indeed essential for the interpretation of the observed phenomena like the energetic heavy ions observed in the vicinity of comets Halley and Giacobini-Zinner by the recent cometary space-missions.



Extrinsic stochasticity — due to random external noise

a) Plasma with random inhomogeneities

b) Uniform plasma with random magnetic field

Intrinsic stochasticity
generated by the system dynamics
Even conservative Hamiltonian systems can become stochastic due to dynamical effects

stochasticity → ① Anomalous ~~effects~~
Diffusion
Heating
Acceleration

② self modulation
self focussing
③ subharmonics

Anomalous effects due to stochasticity can be much stronger compared to, for example wave particle interactions, weak turbulence, soliton turbulence.

According to quasilinear or orbit modification theory, the particles can be accelerated by the waves upto the wave phase velocity whereas stochasticity, if generated, can lead to much larger particle accelerations.

Stochastic Behaviour — when and why?

Integrable System:

The necessary condition for a system, of n degrees of freedom, to be integrable is that it should have n independent constants of motion.

The necessary and sufficient condition for integrability is that these constants of motion should be in involution i.e.,

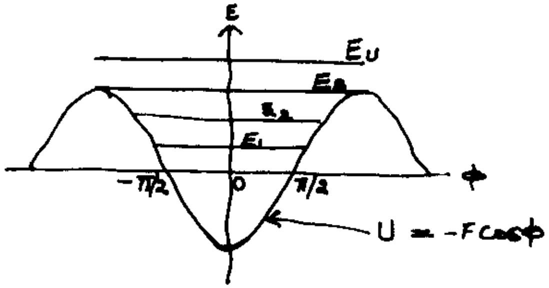
$$\{c_i, c_j\} = 0 \quad , i, j = 1, 2, \ldots n$$

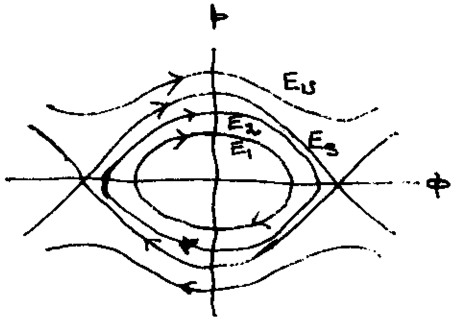
$\{\ \}$ is the Poisson bracket

Integrable system ⇒ closed (trapped) or smooth particle orbits

Some dynamical effects e.g., nonlinear waves can destroy integrability of the system ⇒ adiabatic particle orbits ⇒ stochastic orbits

Integrable System - Example





Particle trajectories for Hamiltonian system with

$$H = \frac{p^2}{2m} + U = E$$

$E = F \equiv E_S \Rightarrow$ Oscillation period $\to \infty$

$\Rightarrow$ separatrix motion

$E > E_S$, orbits are open (non-periodic)

Nonintegrable system — Example

Single particle orbits in magnetic field of the form

$$\underline{B} = \hat{e}_x B_0 \tanh(z/\delta) + \hat{e}_z B_n$$

This field is characteristic of the quasi neutral sheet in the magnetospheric tail with δ as the sheet thickness.

Reconnection models take $B_n = 0$

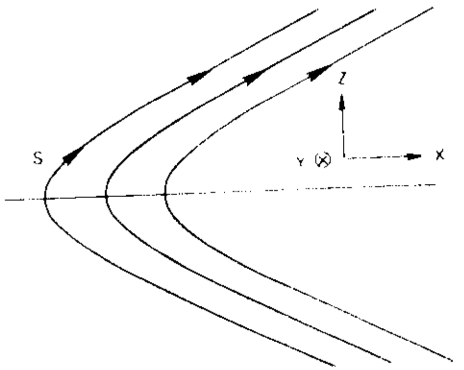
We will show that $B_n \neq 0$ (even weak field) $\Rightarrow$ nonintegrability and stochasticity

Single particle motion:

$$m\frac{d\underline{v}}{dt} = \frac{q}{c}(\underline{v} \times \underline{B})$$

vector equation — 3 degrees of freedom.

Has 3 constants of motion



Schematic drawing of the Harris-type magnetic field (equation (1)) and the coordinate system. The line segment s measures the distance from $Z = 0$ along the magnetic field lines.

1. Hamiltonian $H = \frac{1}{2} m v^2$

2. $C_x = m v_x - \frac{q}{c} B_n y$

3. canonical momentum

$$P_y = m v_y + \frac{q}{c} A_y$$

$A_y(x, z)$ = vector potential

$$= x B_n - \delta B_0 \ln(\cosh z/\delta)$$

Necessary condition for integrability satisfied

How about the sufficient condition for integrability?

$$\{H, C_x\} = 0 \quad , \quad \{H, P_y\} = 0$$

$$\{C_x, P_y\} = -\frac{q}{c} B_n \neq 0$$

sufficient condition <u>not</u> satisfied and hence system is nonintegrable.

What happens to particle orbits? solution (numerical) of eqn. of motion gives the following:

Consequences of finite B_n :

$B_n = 0 \Rightarrow$ no stochasticity

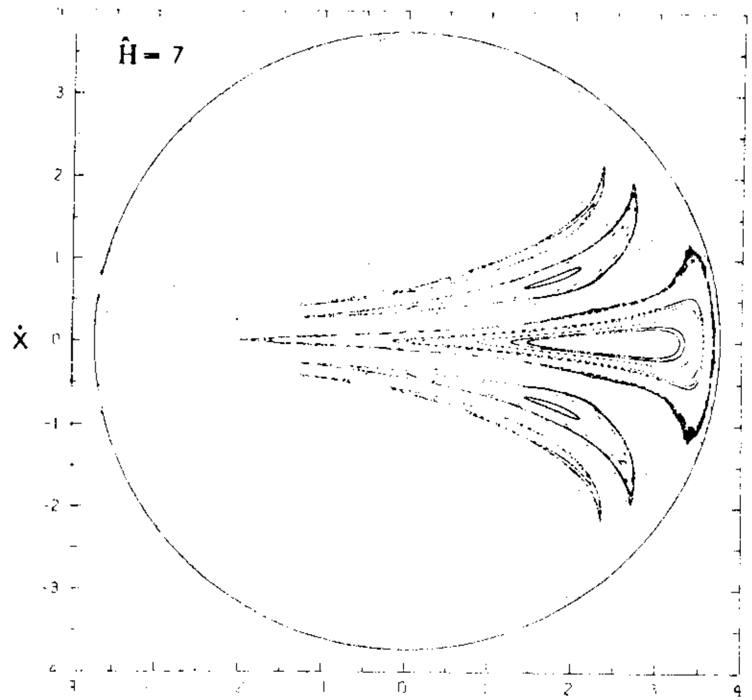
$B_n \neq 0 \Rightarrow$ Three types of orbits

1. bound orbits – only for small B_n
2. Unbound stochastic orbits
3. Unbound transit orbits

Unbound orbits start from infinity and end at infinity
The stochastic region (region B) is covered by infinitely many orbits – each one crossing the plane $z = 0$ an infinite or finite number of times.

In region B, two neighbouring orbits diverge rapidly with time before escaping to infinity

These orbits could be viewed as forming two flux tubes – one from $+\infty$ and the other from $-\infty$



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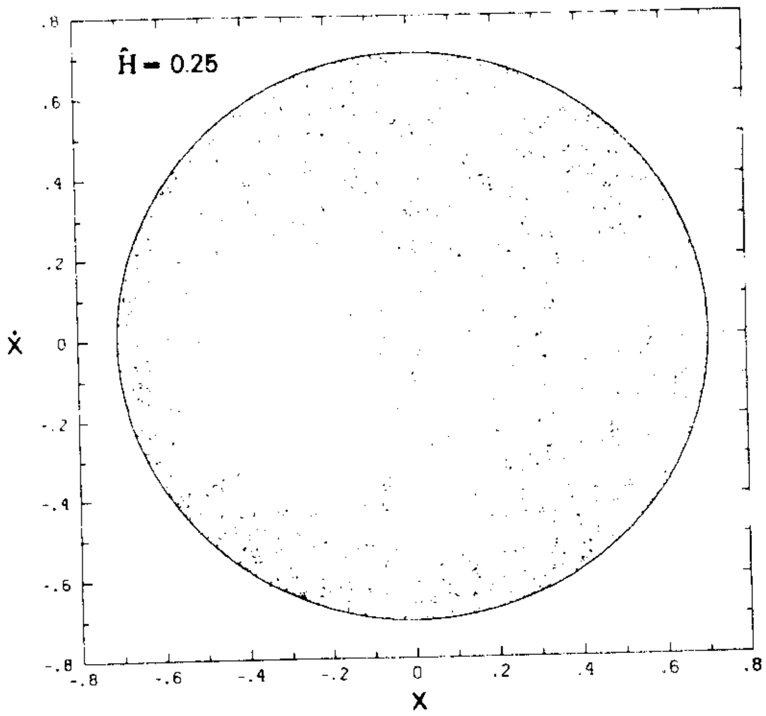
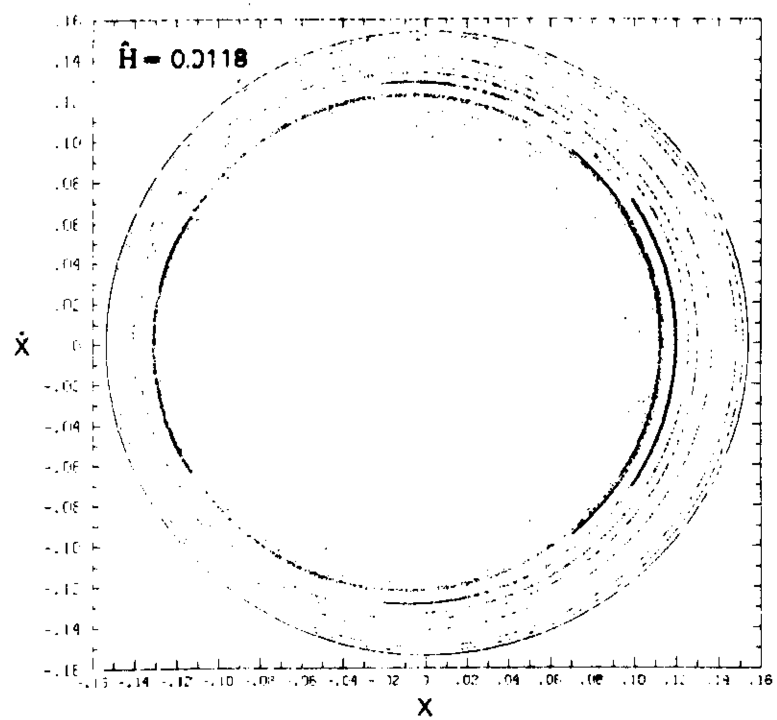
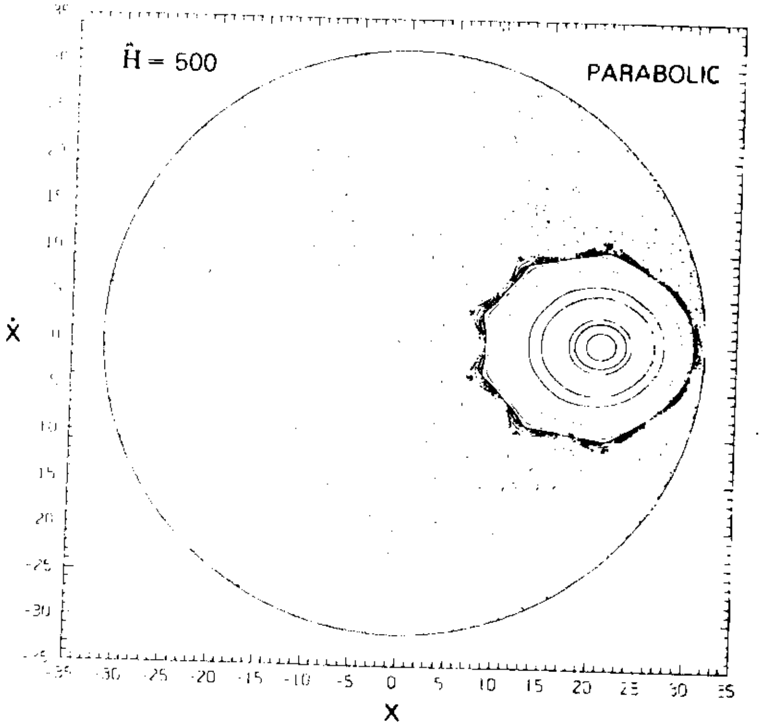

$\hat{H} = 0.25$
$\dot{X}$
X
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8


$\hat{H} = 0.0118$
$\dot{X}$
X
.16
.14
0



Orbital size in the flux tube is typically

$$d \sim (2\rho S)^{1/2} \quad \text{near } z = 0$$

$$\rho = \text{Larmor radius} = (2mH)^{1/2}\left(\frac{c}{eB_0}\right)$$

stochastic orbits can extend to large distances — say upto magnetopause or even ionosphere.

$\Rightarrow$ stochastic diffusion

Particle dynamics Vs Plasma dynamics

start with a plasma in thermal equilibrium. The integrable orbits carry the memory of this initial distribution indefinitely as long as magnetic field remains unchanged. For stochastic orbits, this memory lasts for time scales corresponding to several crossings of the equatorial plane $z = 0$.

Differential in memory $\Rightarrow$ non-Maxwellian particle distribution

Anisotropic distribution $\Rightarrow$ much larger growth rates (one or two orders) for tearing instability which is supposed to be a strong candidate for magnetic reconnection process

Anisotropy can generate some new plasma instabilities

These instabilities may in turn lead to stochastic heating and/or acceleration of particles.

Stochastic Heating by electrostatic Waves

consider a charged particle in electric and magnetic fields

$$\underline{E} = E_0 \hat{e}_x \cos(kx - \omega t) \qquad ; \; k_\parallel = 0$$

$$\underline{B} = B_0 \hat{e}_z$$

with corresponding potentials

$$\phi = -(E_0/k) \sin(kx - \omega t)$$

$$\underline{A} = -\hat{e}_y B_0 x$$

Hamiltonian

$$H = \frac{1}{2m}\left[p_x^2 + \left(p_y + \frac{eB_0}{c} x\right)^2\right] + e\phi$$

e = charge of particle
m = mass " "

Normalized Hamiltonian

$$H = \frac{1}{2}\left[p_x^2 + (p_y + x)^2\right] - \alpha \sin(x - \nu t)$$

$$\alpha = \frac{E_0}{B_0}\frac{ck}{\Omega} \quad , \quad \nu = \frac{\omega}{\Omega} \quad , \quad \Omega = \frac{eB_0}{mc}$$

p normalized to $(m\Omega/k)$
t " to Ω^{-1}

$\dot{p}_y = \frac{\partial H}{\partial y} = 0 \implies p_y = \text{const}$

Take $p_y = 0$

Use canonical transformations to remove explicit time dependance of H

$X = x$, $P_x = p_x$, $Y = y + p_x$, $P_y = \nu t$

with the new Hamiltonian

$$\bar{H} = H - \nu Y$$
$$= \frac{1}{2}\left(X^2 + P_x^2\right) - \nu Y - \alpha \sin(X - P_y)$$

These are obtained from the generating function

small

$$g_1 = (P_y - \nu t)y + P_x(\cancel{x} + P_y - \nu t)$$

$\frac{\partial g_1}{\partial y} = p_y = 0 \implies P_y = \nu t$

$\frac{\partial g_1}{\partial P_y} = (y + P_x) = Y$; $\frac{\partial g_1}{\partial x} = p_x = P_x$

$\frac{\partial g_1}{\partial P_x} = X = x$

Now we change to action-angle variables $I_1, I_2, \omega_1, \omega_2$

generating fn. $= g_2 = \frac{1}{2} Y^2 \cot\omega_1 + X\omega_2$

$$P_x = (2I_1)^{1/2}\cos\omega_1$$
$$X = (2I_1)^{1/2}\sin\omega_1$$
$$P_y = \omega_2$$
$$Y = -I_2$$

$\implies$

$$\bar{H} = I_1 + \nu I_2 - \alpha \sin\left[(2I_1)^{1/2}\sin\omega_1 - \omega_2\right]$$

This is like coupled harmonic oscillators

Hamilton's equations

$$\dot{\omega}_1 = \frac{\partial \bar{H}}{\partial I_1} = 1 - \frac{\alpha \sin\omega_1}{(2I_1)^{1/2}}\cos\left[(2I_1)^{1/2}\sin\omega_1 - \omega_2\right]$$

$$\dot{\omega}_2 = \frac{\partial \bar{H}}{\partial I_2} = \nu$$

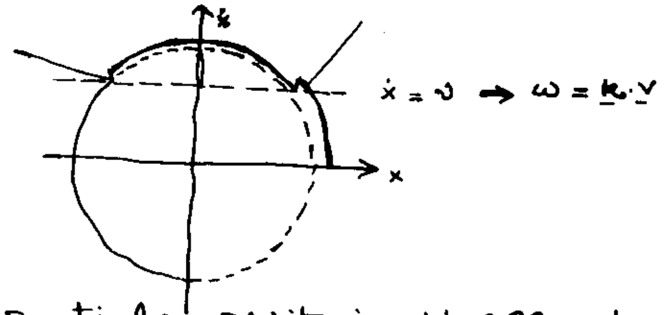
Moreover

$$x = X = (2I_1)^{1/2}\sin\omega_1$$
$$y = Y - P_x = -I_2 - (2I_1)^{1/2}\cos\omega_1$$

$\implies$

$$\dot{y} = x$$

$$\ddot{x} + x = \alpha\cos(y - \nu t)$$



Particle orbit in phase space

Note the kicks at the resonance (wave – particle) level.

From the form of $\overline{H}$, we can make the following observations:

The particle in a magnetic field, described by (I_1, ω_1), behaves like an oscillator with frequency unity i.e. Ω in unnormalized units.

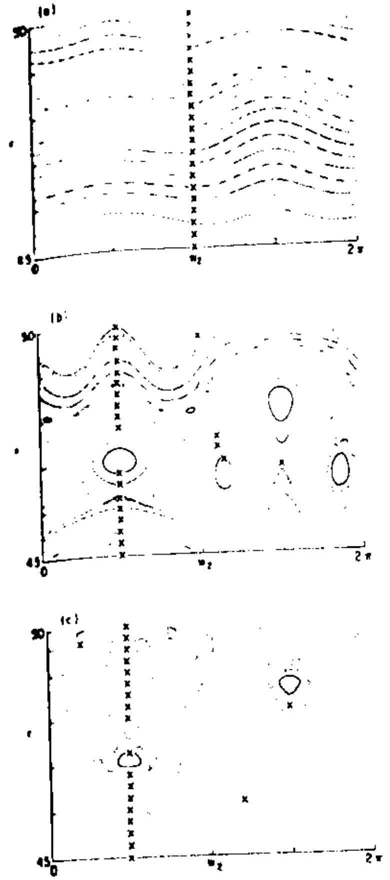
Larmor radius of particle $r_L = (2I_1)^{1/2}$

Larmor angle $= \omega_1$

* The wave, described by (I_2, ω_2), behaves like another oscillator with frequency ν i.e., ω

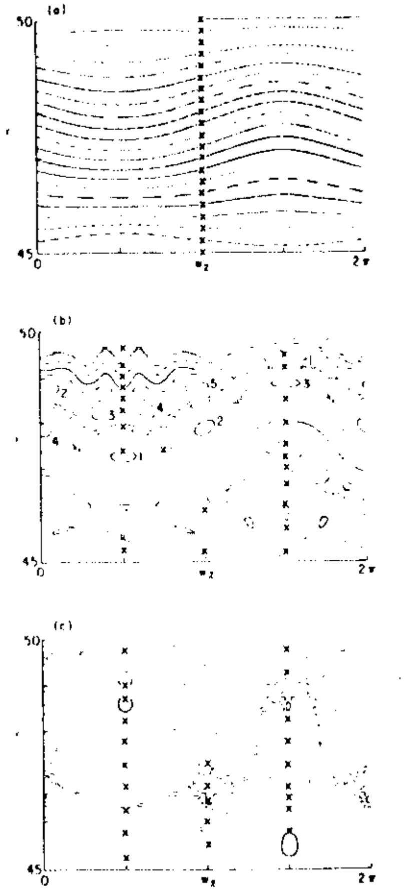
* coupling depends on the amplitude of the wave i.e., E_0.

* stronger coupling $\Longrightarrow$ stochastic orbits (see numerical results)



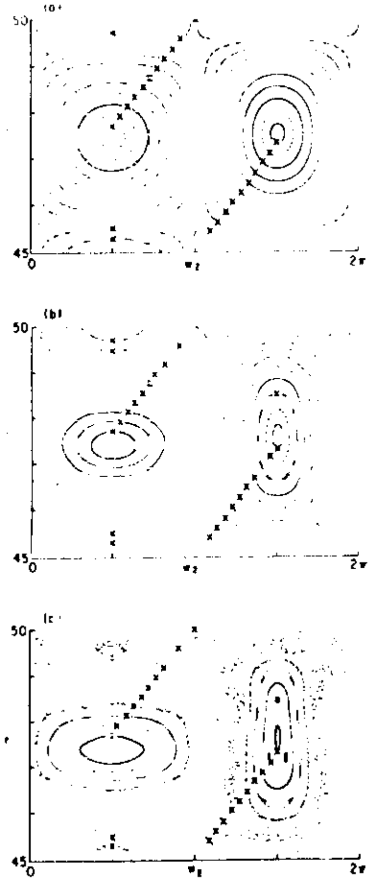
Σ for $\nu = 30.47$ and (a) $\alpha = 1$, (b) $\alpha = 2.5$, (c) $\alpha = 3.5$.

× — initial position



Σ for ν 30, 23 and (a) $\alpha = 1$, (b) $\alpha = 2.2$, (c) $\alpha = 4$. Crosses denote the initial positions of the particles and dots subsequent crossings The particles are followed for 300 crossings of the w_1 π plane.

x – initial position



Σ for $\nu = 30$ and (a) $\alpha = 1$, (b) $\alpha = 3$, (c) $\alpha = 4.1$.

x – initial position

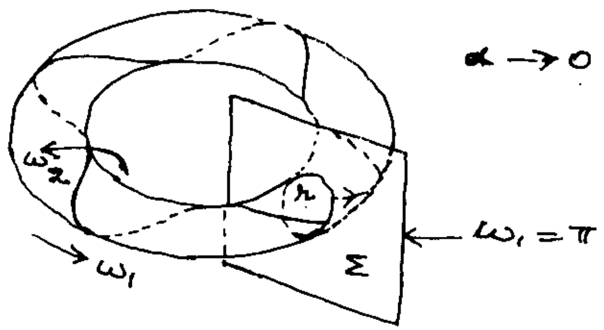
Why does strong coupling (large E_0) lead to chaos ?

One could ask a related question – how does this happen ? How can we interpret the numerical results ?

The system governed by $\bar{H}$ can be described by 3 variables ($\bar{H}$ = const)

$r \equiv r_L$, ω_1 and ω_2 = wave phase

The particle trajectory for $\alpha = 0$ is simply a spiral around the surface of a torus with const minor radius r



Σ = surface of section

For $\alpha \neq 0$, 3-D trajectories are quite complex

Surface of section plots:

Numerical results show that for small α, particle trajectories (r, ω_2 plots) are smooth curves

$\Rightarrow$ integrable system

Crosses represent initial positions of particles

Dots represent particle crossings ($\omega_1 = \pi$ plane)

Increase α (but not large) $\Rightarrow$ particle orbits coherent and also stochastic

This happens because the action I_1 is no more constant of motion

$$\dot{I}_1 = \partial \bar{H} / \partial \omega_1 \neq 0$$

$$\dot{\omega}_1 = \partial \bar{H} / \partial I_1 \neq 1$$

Invariant curves (circles for $\alpha \to 0$) are governed by perturbed action variable I_1

They are invariant under the mapping T, namely

$$r \to r \quad , \quad \omega_2 \to \omega_2 + 2\pi\nu / \langle \dot{\omega}_1 \rangle$$

(rotation)

If $(\nu / \langle \omega_1 \rangle)$ is a rational number say p/q
then to lowest order in α, the invariant curve will map onto itself after q rotations.
All the points on these invariant curves are called q^{th} order fixed points

Such curves of fixed points very sensitive to perturbations.
Due to perturbations, only $(2q)$ number of points remain fixed points

$(2q)$ fixed points

q - Elliptic fixed points

T^q mapping stable
$\Rightarrow$
elliptic particle orbits – Islands
$\Rightarrow$
coherent.

q - hyperbolic fixed point

T^q mapping unstable
$\Rightarrow$
hyperbolic particle orbit with separatrix as an asymptote $\Rightarrow$
* stochastic

* A band around the separatrix becomes stochastic i.e., a small change in initial conditions can lead to very big changes to the ensuing orbit. For example a particle near the separatrix cab be thrown into or out of the region of trapped orbits by an arbitrarily small perturbation

: Large $\alpha \Rightarrow$ completely chaotic
Disappearance of coherent structures
This happens because larger perturbations lead to
1. larger number of islands
2. growth of islands
3 growth of stochastic band around separatrix
4 Overlapping of islands because of (1) and (2)

Stochasticity threshold

To find the field amplitude E_0 or α at which system becomes globally stochastic (all coherent structures destroyed), one has to use both analytical and numerical results. This gives (Karney, Phys Fluids, 21, 1584, 1978):

$$\alpha_t \approx \frac{1}{4}\,\nu^{2/3}$$

or

$$E_t = \frac{B_0}{4}\,\frac{\omega}{ck}\left(\frac{\Omega}{\omega}\right)^{1/3}$$

$E_0 > E_t$ — stochastic orbits

$\Longrightarrow$

particle motion not constrained i.e. particles can have a random walk throughout the phase space

$\Longrightarrow$

strong particle heating
-> resonance heating

Stochasticity can also lead to very high particle acceleration

Partitioning between heating and acceleration ??

Nonlinear Polarized Alfvén Waves

One dimensional Two-fluid Eqs. for waves propagating obliquely to the external magnetic field are:

$$\frac{d\rho}{dt} + \rho\,\frac{\partial V_x}{\partial x} = 0$$

$$\frac{dV_x}{dt} + \frac{1}{\rho}\,\frac{\partial}{\partial x}\left(P + \frac{B_\perp^2}{8\pi}\right) = 0$$

$$P\rho^{-\gamma} = \text{const}$$

$$\frac{d\vec{V}_\perp}{dt} = \frac{B_x}{4\pi\rho}\,\frac{\partial \vec{B}_\perp}{\partial x}$$

$$\boxed{\frac{d\vec{B}_\perp}{dt} = B_x\,\frac{\partial \vec{V}_\perp}{\partial x} - \vec{B}_\perp\,\frac{\partial V_x}{\partial x} \underbrace{- \frac{\partial}{\partial x}\left[\frac{cB_x}{4\pi n e}\,\frac{\partial}{\partial x}\left(\hat{e}_x \times \vec{B}_\perp\right)\right]}_{\text{ion-inertia term}}}$$

ρ = fluid mass density
n = particle density
$\vec{V}$ = fluid velocity

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V_x\,\frac{\partial}{\partial x}$$

Electron inertia neglected.

Use normalized variables

Governing Eqs. in Lagrange variables (t_o, x_o):

$$\frac{\partial^2 U}{\partial \tau^2} + \frac{\partial^2}{\partial \xi^2}\left(P_o U^{-\gamma} + \frac{B_\perp^2}{2}\right) = 0$$

$$\frac{\partial^2}{\partial \tau^2}\left(U \vec{B}_\perp\right) - \frac{\partial^2 B_\perp}{\partial \xi^2} + \lambda \frac{\partial^3 (\hat{e}_x \times \vec{B}_\perp)}{\partial \xi^2 \partial \tau} = 0$$

$$U = \rho_o/\rho \quad , \quad \xi = \int dx_o = \frac{1}{\rho_o}\int \rho \, dx$$

$$\tau = V_A t_o \cos\theta \quad , \quad V_A = B_o/(4\pi\rho_o)^{1/2}$$

$$P_o = \left[K(T_e + T_i)/m_i\right] / V_A^2 \cos^2\theta$$

$$\vec{B}_o = B_o(\cos\theta, \sin\theta, 0); \lambda = V_A/\Omega_i$$

Nonlinear coupled equations give evolution of finite amplitude elliptically polarized Alfvén waves

Look for localized stationary solutions of the form

$$f(\xi - \tau ; \epsilon\tau)$$

Use boundary conditions at $\eta = \xi - \tau = \infty$

$$U = 1, \quad B_y = B_o \sin\theta, \quad B_z = \frac{\partial U}{\partial \eta} = \frac{\partial \vec{B}_\perp}{\partial \eta} = 0$$

$\Longrightarrow$

$$U = 1 - \frac{1}{(1-\beta)}\left(\Theta \tilde{B}_y - \frac{1}{2}\tilde{B}_\perp^2\right) \quad ; \beta \neq 1$$

$$\frac{\partial \vec{b}_\perp}{\partial \tau} + \frac{1}{4(1-\beta)}\frac{\partial}{\partial \eta}\left[\vec{b}_\perp\left(b_\perp^2 - \Theta^2\right)\right] + \frac{\lambda}{2}\frac{\partial^2}{\partial \eta^2}\left(\hat{e}_x \times \vec{b}_\perp\right) = 0$$

$$\beta = \gamma P_o \quad , \quad \Theta = \frac{B_{yo}}{B_x} \quad , \quad \vec{b}_\perp = (0, \tilde{B}_y + \Theta, \tilde{B}_z)$$

⊙ Polarized waves

$$\frac{\partial b_\pm}{\partial \tau} + \frac{1}{4(1-\beta)}\frac{\partial}{\partial \eta}\left[|b_\pm|^2 b_\pm - \Theta^2 b_\pm\right] \pm \frac{i\lambda}{2}\frac{\partial^2 b_\pm}{\partial \eta^2} = 0 \qquad \text{DNLS}$$

$b_+ = b_y + i b_z$ LHP

$b_- = b_y - i b_z$ RHP

b_+ modulationally unstable

b_- " stable

$\Longrightarrow$ self modulation

$\Theta = 0 \Rightarrow$ circular polarization

$$\Theta \gg \tilde{B}_y\,,\ \tilde{B}_z$$

$$\tilde{B}_z \approx \frac{\lambda}{\Theta^2}(1-\beta)\,\frac{\partial \tilde{B}_y}{\partial \eta}$$

behaviour confirmed by simulation

$$\frac{\partial \tilde{B}_y}{\partial \tau'} + \frac{3\Theta}{2}\tilde{B}_y\frac{\partial \tilde{B}_y}{\partial \eta} - \frac{\lambda^2(1-\beta)}{2\Theta^2}\frac{\partial^3 \tilde{B}_y}{\partial \eta^3} = 0$$

Complex KdV

$$\frac{\partial}{\partial \tau'} = \left(\frac{\partial}{\partial \tau} + \frac{\Theta^2}{2}\frac{\partial}{\partial \eta}\right)$$

$\Longrightarrow$ Elliptic Polarization

$$\tilde{B}_y(\xi) = \frac{2M}{\Theta}\,\mathrm{sech}^2\left[\frac{\Theta}{\lambda}\left(\frac{M}{2(1-\beta)}\right)^{1/2}\xi + i\delta\right]$$

$$\xi = (\eta - M\tau')$$

δ = arbitrary phase

[Ref: Phys. Scripta 34, 729, 1986]

Ellipticity

$$P \simeq \frac{M}{\Theta}\left[\frac{2(1-\beta)}{M}\right]^{1/2}\tan\delta$$

P independent of dispersion

Note: For $\delta = (4n+1)\pi/2$, solution for $\tilde{B}_y$ is singular i.e.,

$$\tilde{B}_y \sim \mathrm{cosech}^2[\Lambda\xi]$$

Steepened Alfvén waves have been observed upstream of bow shock (earth) and also in the solar-wind comet interaction region.

Alfvénic envelope solitary waves (amplitude modulated) observed in terrestrial magnetosphere (GEOS data)

Stochastic Self Modulation

Nonlinear Waves ⇒ harmonic generation ⇒ self modulation

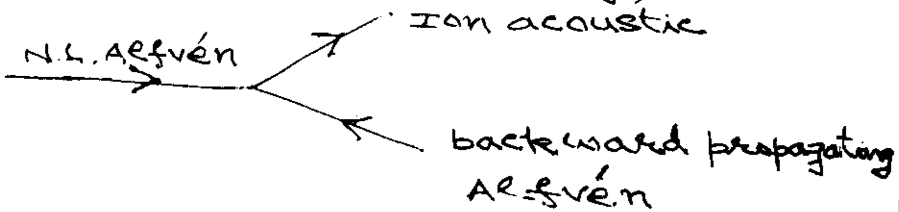
In stochastic systems, well known bifurcation phenomena, period doubling takes place ⇒

subharmonics

Interaction of fundamental wave with subharmonics ⇒ self modulation which occurs only in chaotic systems – stochastic self modulation.

Decay Chaos:

corresponding to decay instability in nonlinear plasmas e.g.,



we can have decay chaos

Physical mechanism:

Mismatching of interacting modes

Physical Mechanism

Frequency mismatch of interacting modes ω and $\omega/2$ (fundamental and subharmonic) due to nonlinear frequency shift is responsible for both stochastic self modulation and decay chaos.

Remember: Modulated envelope solitons appear when nonlinear frequency shift matches with the dispersive frequency shift

Parametric processes require frequency matching conditions

$$\omega = \omega_1 \pm \omega_2$$

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