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SOLAR MAGNETOHYDRODYNAMICS

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1. INTRODUCTION

Magnetohydrodynamics is the study of the motion of an electrically conducting fluid in the presence of a magnetic field. Fluid motions affect and are affected by electromagnetic forces, there being a complex interaction between the two. This is largely brought about by the Lorenz (or $\underline{j} \times \underline{B}$ force), Faraday's Law of induction, and Ohm's law. Solar magnetohydrodynamics is the study of magnetohydrodynamics as it applies to phenomena observed in the Sun's atmosphere and inferred to operate within the solar interior. These Lecture Notes are an attempt to summarize a number of aspects of solar magnetohydrodynamics, selection simply being a reflection of the author's personal preferences and interests. No attempt will be made to be exhaustive in discussing our subject; that would be impossible. Rather, a tutorial style outline of the various topics will be attempted, indicating here and there where the reader may explore further should he or she so desire. Nonetheless, a broad listing of references will be given, including original articles as well as reviews and monographs; in this way the reader may be suitably guided through our subject and protected from the author's short-comings!

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It is convenient to begin with a brief survey (a "Cook's tour") of the solar observations, turning thereafter to the basic equations of magnetohydrodynamics. This ordering of material is entirely appropriate, because it is the observations which stimulate and guide the efforts of the theorist. Indeed, as E N Parker has observed, left to his or her own devices the theorist, though fully armed with all the relevant equations and laws, would almost certainly fail to predict the complex array of phenomena that the Sun in fact displays. "The fundamental equations of Physics may contain all knowledge, but they are close-mouthed and do not volunteer that knowledge" (Parker 1979; Preface).

Within the framework of stellar physics the Sun is a rather ordinary star. Its magnetic field is weak - too weak to be detected at stellar distances - and its phenomena relatively placid. Nonetheless, the Sun provides us with a "Rosetta stone", the clue to understanding stellar magnetism. This is because the Sun is the only star we can view in detail, and in detail the Sun reveals a marked departure from placidity. In this detail we observe the physics of stellar structures in operation, and so we may test and amend our theoretical concepts and understandings. An understanding of solar physics, then, will ultimately lead to a greater understanding of stellar physics, especially in our conception of the physical processes that underlie the basic physics.

A study of the Sun, then, provides a guide to the basic physical processes that are likely to operate in stars also. In turn, a study of the stars aids our understanding of those very basic physical concepts. For it is only in the stars that we are likely to see such concepts tested in physical ranges beyond the narrow confines of the solar plasma, and only by testing *in extremis* are we to gain any confidence in our understanding of the basics of solar and stellar physics. A study of both solar and stellar physics, then, is necessary if we are to be confident in our understanding of hydromagnetism. To this may be added our knowledge of hydromagnetics under laboratory conditions - very different from the stellar extreme - and in the solar system (planetary, magnetospheric, cometary, etc.) plasma, all coupled with numerical simulations in accessible paramter regimes. We have remarked above that, viewed as a star, the Sun is somewhat mediocre and placid in its displays, yet viewed in closer detail a remarkably complex and intriguing demonstration of physical phenomena becomes apparent. The clue to an understanding of this intricacy undoubtedly resides with the solar magnetic field. Devoid of a magnetic field the Sun would indeed be placid. It would no longer possess sunspots in its visible layers; it would possess no hot corona (or, at least, the corona would be a pale shadow of its actual self), no flares, no prominences, and no high-speed solar wind. The perfect sphere, unblemished by magnetism, of the medieval and ancient philosophers would be the result!

Magnetic fields, then, are the cause of a majority of solar phenomena. The presence of the field leads to a complex interaction with the plasma motions, a diversion of energy from one source to another and with it a display of "solar fireworks"!

2. SOLAR PHENOMENA

The Sun may be viewed as a sphere (slightly oblate, in fact) of radius $R_{\odot} = 6.96 \times 10^5$km, within which nuclear energy is converted into radiation and then, in an outer layer, into convective motions, where it is finally radiated away or briefly stored in the atmosphere. We will consider the various regions in turn, beginning with the convection zone. Excellent summaries of the various solar properties and phenomena may be found in Eddy (1979), Noyes (1982), Priest (1982,1983) and Giovanelli (1984).

2.1 The Convection Zone

The convection zone extends to a depth of about 2×10^5km below the surface of the Sun. A precise value for the depth is, of course, uncertain and, until recently, rested solely on a parameterized model of convection, namely the "mixing length theory". However, recent advances in helioseismology have provided an independent assessment of the depth, and the result is broadly in agreement with mixing length estimates. (Aspects of helioseismology will be discussed in Section .)

Within the convection zone motions are ordered on a number of distinct scales. The smallest scale is that of the *granules*. Granules reside at the top of the convection zone, in the layer where the atmosphere is most strongly superadiabatic. They have a typical scale of 10^3km, velocities of 1-3 kms^{-1}, and a characteristic lifetime of some 8 minutes. These values, however, provide only a rough guide to granules. Indeed, very rece t results from Space Lab 2 (Title 1987) reveal a range of sizes for granules stretching from 2×10^3km to a few $\times 10^2$km. Granules are observed to "explode", i.e. to expand outwards and ultimately fade from view. Such "exploding" granules influence their neighbours, possibly terminating a neighbouring granule's life. Shock waves and sound waves are generated by such processes.

The next convective pattern in scale up from the granule is the *mesogranulation*. Mesogranules have smaller velocities (~ 0.1 kms^{-1}) and a dimension of typically $5\text{-}10 \times 10^3$km. Beyond this scale is the supergranulation. *Supergranules* have a scale of some 3×10^4km, velocities of $\frac{1}{2}$ kms^{-1} and lifetimes of about 1 day. The reason for emergence of these particular scales is believed to reside in the various depths at which H and He, the two principal cases in the solar plasma, are ionised. Hydrogen becomes almost fully ionised within about 10^3km below the solar surface; helium becomes 90% singly- and doubly-ionised at depths of $5\text{-}10 \times 10^3$km and 3×10^4km, respectively.

A fourth scale of convection is the *giant cell*, with a typical size of 3×10^5km and velocity of 0.05 kms^{-1} and a long lifetime (about one year). The existence of such giant cells is not firmly established.

It is amusing to contrast these scales with terrestrial ones. Whereas velocities of the order 1 kms^{-1} or less are small in solar terms (the sound speed in the photosphere is about 10 kms^{-1} - see Section 3.1), they are large in human experiences; 1 kms^{-1} is about 2000 miles per hour, which is almost a factor of ten larger than the greatest wind speeds ever recorded (some 240 mph on Mount Washington, USA). Similarly, the Earth could comfortably be lost in a supergranule and a city like London or Trieste in a granule!

2.2 Global Modes of Oscillation

It is a remarkable fact that the Sun oscillates globally; the amplitude of these global modes are very low but, encouragingly for solar physics, it has proved possible to measure the spectrum of the Sun's oscillations to a high degree of accuracy. The modes of oscillation are of intrinsic interest but gain considerably in importance when one considers their abilities as seismic indicators. As T.G. Cowling remarked in 1953, the solar physicist would give much to be able to see beneath the photospheric layers; this is now all but possible, thanks to the progress in the new field of helioseismology (the "shaking sun"). Recent advances have lead to a determination of the run of sound speed with depth in the Sun, and also a measure of the solar rotation. Such results promise much for our future understanding of the internal workings of the Sun.

The modes of oscillation are principally classified into p-modes or g-modes. The p-modes are sound waves trapped within the Sun and are driven by pressure forces. The g-modes are driven by buoyancy forces. What mechanism generates the waves and why p-modes have a peak in their energy spectrum at about 5 minutes is not yet clear. A precise identification of g-modes is also presently uncertain, and a well established oscillation with a period of 160.01 min is of uncertain origin (though presumably a g-mode).

A convenient representation of p-modes is to plot their frequency against horizontal wavenumber. The result is a parabolic dependence, a natural consequence of wave trapping within a stratified atmosphere.

We return to a more detailed description of helioseismology in Section 8. Full discussions are available in a number of recent reviews, including Deubner and Gough (1984), Christensen-Dalsgaard *et al* (1985) and Brown, Mihalas and Rhodes (1986).

2.3 The Solar Atmosphere

Above the convection zone is the solar atmosphere, which extends far out into space and ultimately interacts with the Earth. The solar atmosphere is traditionally divided into three regions. The first is the *photosphere*, which is the layer in the Sun from which the bulk of the visible light emanates. The photosphere is a thin layer of some 500 km thickness. Its temperature is about 6000°K, declining at its top to some 4000°K; this is the *temperature minimum*, since beyond this layer the temperature climbs higher. The gas density in the photosphere is some 10^{-4}kg m^{-3} (= 10^{-7}gm cm^{-3}).

The region immediately above the temperature minimum is the *chromosphere*. Within the chromosphere, the temperature rapidly rises through the *transition region* (a thin boundary layer with steep temperature gradients) and finally reaches some 2 × 10^{6}°K. We are now in the *corona*. The corona is an extremely tenuous though very hot region. The cause of such a high temperature is not yet clear, though it seems likely to be intimately connected with the presence of a magnetic field (see discussion in Section).

While it is a convenient and common practice to divide the solar atmosphere into the various layers mentioned above we should stress that such a division is in fact only broadly appropriate: the solar atmosphere is a very dynamic one and furthermore is highly structured as well as stratified. The cause of much of this inhomogeneity is the presence of magnetic field, to which we now turn.

2.4 Magnetic Fields in the Sun

At one time it was commonly thought that magnetic fields were of secondary interest, except perhaps in such locations as sunspots. It is now increasingly apparent that, on the contrary, the presence of magnetic field is crucial to an understanding of so many solar phenomena. The fact that the Sun has a magnetic field was established by Hale (1908), who first succeeded in measuring the field strength in sunspots by using the Zeeman effect. *Sunspots*, then, are regions of strong magnetic field. They possess field strengths in the range 2000-4000 G. Sunspots appear as dark patches in the visible light from the Sun and in fact correspond to cool regions of the photosphere, their temperatures at 4000°K being some 30% cooler than their surroundings. Spot diameters vary from a few × 10^3km to a few × 10^4km. Sunspots consist of an umbra, which is the inner and coolest part of the spot, surrounded by a penumbra (which is at a temperature intermediate between the umbra and the photosphere). Though often considered to be in a somewhat quiet state, recent high-resolution observations (Title 1987) reveal sunspots to be rather dynamical in form exhibiting a variety of oscillations and waves.

The distribution and number of sunspots present in the photosphere at any one time varies on an approximately 22 year magnetic cycle (the *Hale cycle*), with sunspots occurring in bipolar pairs. If in the northern solar hemisphere, the leading spot in such a bipolar pair is almost always of the same polarity, the following spot having the opposite polarity. Bipolar pairs in the southern hemisphere have the opposite ordering of polarity to their northern cousins. This general polarity ordering is reversed half-way through the 22 year cycle.

In addition to the observed cycle of polarity in bipolar pairs, the distribution of sunspots is found to form a "butterfly diagram" when the latitude of formation of spots is displayed as a function of year: for example, at the beginning of a cycle spot pairs form at an average latitude of 28°, this falling to about 12° after 6 years and to about 7° after 11 years - see Figure 1. It is an aim of hydromagnetic theory to explain such intriguing observations.

The 22 year magnetic cycle recorded over this century suggests a clock-like mechanism driving the solar cycle. In fact, this is not the case. The Sun departs radically from such cyclic behaviour in an unpredictable fashion. By a careful analysis of historical records of sunspots - which in fact go back to Galileo's famous sighting of sunspots with his newly developed telescope, and back to pre-Christian times with Chinese records - it is now clear that in a number of periods in the past the Sun has been virtually devoid of sunspots (Eddy 1976). A notable example is the so-called *Maunder Minimum* (1645-1715), which incidentally coincided with some very cold winters in Europe and North America - with the River Thames freezing over and permitting ice fairs to be held. The connection between solar activity and terrestial weather patterns is not yet clear, though intriguing clues are emerging.

Sunspots and pores are not the only sites of magnetic field in the photosphere. However, until recently it was believed that magnetic fields outside of spots were very much weaker than those within spots. There are good reasons for expecting this to be so but, remarkably, the Sun has chosen instead to form highly concentrated magnetic fields in regions devoid of spots. These are the *intense magnetic flux tubes*, which have field strengths of about 1.5kG (compared with about 3kG in sunspots). Thus, in this respect intense tubes and spots are similar. Intense flux tubes, however, are confined to diameters of only 100-200 km, and are therefore to be listed as one of the smallest structures in the solar atmosphere. There importance, however, is not to be under-estimated. This is because intense flux tubes provide a natural channel - electrodynamic link - between the photosphere and chromosphere/corona. Flux tubes are generally located in the downdraughts of supergranules, the regions where several supergranules come together. The tubes are buffeted and generally shuffled around by granules which border and press in on them. The result, then, is one of a complex interact between intense tubes and the sea of supergranules and granules in which they reside.

Both intense tubes and sunspots are presumably rooted deep within the convection zone. Indeed, it is the interaction of solar rotation (which, incidently, is differential, being slightly slower near the poles than the equator) with deeply buried magnetic field lines that results in the formation of bipolar pairs. Precisely how the field couples into the deep interior is not clear. Currently, there is much support for the suggestion that a reservoir of magnetic field exists at the base of the convection zone, some 2×10^5km below the photosphere. The manipulation of the solar field to produce the Hale cycle is commonly believed to be a *dynamo* process (see reviews by Moffatt 1978; Parker 1979; Cowling 1982), but no self-consistent numerical simulation is yet able to satisfactorily reproduce the observed magnetic signatures (Gilman 1982). The field strength of magnetic field buried below the convection zone is very uncertain. Estimates range from 10^3G to 10^8G, but we will have to await further developments before this range can be safely narrowed.

Returning to the photospheric level, where the field strength is some 1.5kG in intense tubes and 3kG in sunspots, we follow the field higher into the solar atmosphere. Gravity strongly stratifies the relatively cool photospheric and chromospheric gas, so there is a substantial fall off with height in the gas density and pressure. A magnetic field is in some ways like a balloon; it possesses a mechanical pressure and thus responds to changes in pressure on its boundary (see Section 2 for a more detailed discussion). Consequently, isolated tubes and sunspots expand with height and ultimately fill the whole of space as they interact with neighbouring tubes and spots. This occurs slightly above the temperature minimum. The bulk of the chromosphere and all of the corona is thus filled by magnetic field. It turns out that the magnetic field declines slower than the plasma's gas pressure, so the chromosphere and corona are largely dominated by magnetic forces which shape, guide and control gas motions in these regions of the solar atmosphere. Inhomogeneities in this region of the solar atmosphere are therefore not unexpected, with the presence of magnetic field providing the cause.

Before continuing on with our brief survey of the solar atmosphere it is instructive to compare solar magnetic field strengths with those estimated elsewhere in the cosmos. The galactic field strength is about 10^{-6}G; the field of the Earth is about $\frac{1}{2}$ G; the field of a household bar magnetic is 10-100G; the field in intense tubes and sunspots is 1.5-3kG, which is comparable with the field used in medical applications (nuclear magnetic resonances); the field in a large industrial magnet is 10^5G; and finally the field in a neutron star is 10^{10}-10^{12}G. Magnetic effects generally scale as the square of the magnetic field strength - see Section 3.1 - so this range in field is squared; from 10^{-12} to 10^{24} is vast indeed!

Continuing our survey, we note that the chromosphere is a very dynamic part of the solar atmosphere. Granules generally lie below it - but nonetheless influence it through the sound and shock waves they generate. The dissipation of such waves is an important ingredient in the heating of the chromosphere. Supergranules may actually penetrate to these relatively high levels, with velocities of several kms^{-1} (much higher than the $\frac{1}{2}$ kms^{-1} they possess in the photospheric layers). One of the most intriguing features of the chromosphere is the *spicule*. Spicules have been seen for over a century. They appear as pencil-like jets of gas shooting out of the chromosphere with velocities of about 25 kms^{-1} and rising to heights (above the photosphere) of 10^4km. They give the appearance of a wheatfield blowing in the summer wind. Spicules are evidently guided by the chromospheric magnetic field since the field is sufficiently strong at this level in the atmosphere to resist motions transverse to it; motions are consequently parallel to the field lines. (See Section 2 for a more detailed discussion of the mechanics of magnetic fields and flows). How spicules are generated is not presently clear. It is a prejudice of the author that they are driven from below, in the photospheric intense flux tubes (Roberts 1979; Hollweg 1982).

The inhomogeneous nature of the chromosphere induced by magnetic fields and their concomitant spicular motions is further accentuated by *fibrils*. Fibrils - seen best in H_α wavelengths - are aligned with magnetic fields, forming long sinuous curves; they give the chromosphere a resemblance to the shape of iron filings near bar magnets. H_α fibrils are observed to support wave motions.

Continuing upward into the solar atmosphere we finally come to the corona. The corona was first detected as a faint halo in eclipse photographs (as early as 1851). It is dominated, both mechanically and thermally, by the magnetic field. Where the magnetic field bends back and returns to the solar surface, the coronal gas is trapped and *coronal loops* abound. Such loops may be as long as 10^5km and the gas within them is typically at a temperature of 2-3 $\times 10^6$K. Groupings of such loops, above sunspots visible in the photosphere, are referred to as *active regions*. The field strength in coronal loops is presently uncertain, though estimates of 1-100G are reasonable. Regions where the magnetic field lines are no longer re-entrant but instead stream out into space are called *coronal holes*. They are a little cooler (at about 1.5 $\times 10^6$K) that active regions, and are the source of the high-speed *solar wind*. The high-speed solar wind is born in coronal holes and ultimately flows out past the Earth, at some several hundred km per sec. The acceleration of the solar wind in coronal holes is not yet fully understood.

The agency responsible for the heating of the corona, both in active region loops and coronal holes, is not yet clearly established, though it is becoming increasingly apparent that magnetic effects are crucial. We return to this topic in Section .

One other feature of the corona that we should mention is the *prominence*. Prominences are dense, cool condensations embedded in the corona and supported by magnetic fields. The magnetic field is additionally able to thermally isolate the prominence matter. This is because electrons are tied to the strong magnetic fields. As a consequence thermal conduction across magnetic field lines in coronal conditions is severely reduced, with conduction along the magnetic field dominating.

This concludes our brief survey of the observational aspects of the solar atmosphere and interior. Further details of the various topics mentioned above may be pursued in the general references cited earlier. A number of topics will be discussed in greater detail in the subsequent text, including

(i) MHD waves in uniform media and magnetic flux tubes; solitons

(ii) Causes of fine structure in the photosphere

(iii) Coronal loops, their thermal structure and stability

(iv) Coronal heating

(v) Helioseismology.

To begin, however, it is necessary to introduce the system of equations on which our physical arguments are based. To this we now turn.

3. THE EQUATIONS OF MAGNETOHYDRODYNAMICS

It is not possible in a lecture course this brief to derive the equations of magnetohydrodynamics (MHD, for short). Rather, we will present them as a natural addition to the usual equations of fluid mechanics (gas dynamics) supplemented by Maxwell's equations of electromagnetism. More detailed derivations and discussions are readily available; see, for example, Boyd and Sanderson ( ), Roberts (1967), Cowling (195 ,19), Parker (1979) and Priest (1982).

The solar plasma – regarded as a continuum fluid – obeys the usual equation of continuity (mass conservation) and the ideal gas law. For a fluid of gas density ρ, pressure p, temperature T and velocity $\underline{v}$ we have

$$\frac{\partial\rho}{\partial t} + \text{div}\, \rho\underline{v} = 0, \tag{1}$$

and

$$p = \frac{k_B}{m}\rho T, \tag{2}$$

where k_B (= 1.38×10^{-23} in m.k.s. units, and 1.38×10^{-16} in c.g.s. units) and m is the mean particle mass. We are adopting the m.k.s. system of units. However, it is sometimes convenient to follow the standard solar practice of quoting numerical values in c.g.s. units with magnetic fields in gauss (G).

The equation of conservation of momentum may be written in the form

$$\rho\left[\frac{\partial}{\partial t} + \mathbf{v}.\mathrm{grad}\right]\mathbf{v} = -\mathrm{grad}\,p + \rho\mathbf{g} + \mathbf{j}\times\mathbf{B} + \rho\nu\nabla^2\mathbf{v}, \qquad (3)$$

which differs from the usual fluid equation only in the addition of the Lorenz (or $\mathbf{j} \times \mathbf{B}$) force. This addition has, however, an important consequence because it links the fluid motions $\mathbf{v}$ to the magnetic field. In equation (3), $\mathbf{j}$ is the current density and $\mathbf{B}$ is the magnetic (induction) field; the gravitational force is $\mathbf{g}$ and is assumed constant. Viscous effects are represented through the kinematic viscosity ν.

The current density $\mathbf{j}$ and the magnetic field $\mathbf{B}$ are related by Ampere's law:

$$\mu_0\mathbf{j} = \mathrm{curl}\,\mathbf{B}, \qquad (4)$$

where μ_0 ($= 4\pi \times 10^{-7}$ in mks units) is the permeability of free space. In writing equation (4), we have neglected the usual electromagnetic displacement current, assuming it to be much smaller than curl $\mathbf{B}$; electromagnetic waves are thus ignored in MHD. Equation (4) is supplemented by the requirement that the field $\mathbf{B}$ be solenoidal (no monopoles):

$$\mathrm{div}\,\mathbf{B} = 0. \qquad (5)$$

Faraday's law of induction relates the electric field $\mathbf{E}$ to changes in the magnetic field:

$$\frac{\partial \mathbf{B}}{\partial t} = -\,\mathrm{curl}\,\mathbf{E}. \qquad (6)$$

The electromagnetic part of the system is then completed once we relate $\mathbf{j}$ and $\mathbf{E}$. This is done through Ohm's law, written in the form appropriate for a moving conductor:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \qquad (7)$$

where σ is the electrical conductivity of the gas. This form of Ohm's law is a simplification but for many purposes it is quite adequate.

The system of equations (1)-(7) is completed once we specify how energy exchange takes place. In many examples it is convenient to assume this to be *isentropic*, in which case we have

$$\frac{\partial p}{\partial t} + \mathbf{v}\,.\,\mathrm{grad}\,p = \frac{\gamma p}{\rho}\left[\frac{\partial \rho}{\partial t} + \mathbf{v}_1\,\mathrm{grad}\,\rho\right], \qquad (8)$$

for adiabatic index γ. The *incompressible* extreme, corresponding to large sound speed, follows formally by letting $\gamma \to \infty$. However, in some regions of the Sun, especially in the corona, thermal conduction and radiation are important and equation (8) is modified to allow for such effects. We write (see, for example, Priest 1982)

$$\frac{\partial p}{\partial t} + \mathbf{v}.\mathrm{grad}\,p - \frac{\gamma p}{\rho}\left[\frac{\partial \rho}{\partial t} + \mathbf{v}\,.\mathrm{grad}\,\rho\right]$$
$$= (\gamma-1)\left[\mathrm{div}(\kappa\,\mathrm{grad}\,T) - \rho^2 Q(T) + \frac{1}{\sigma}\,j^2 + H\right], \qquad (9)$$

where κ is the coefficient of thermal conduction along the magnetic field (it being assumed that conduction across the field is negligible). $Q(T)$ is the optically-thin radiative loss function (a known, but relatively complicated, function of temperature T), $\frac{1}{\sigma}\,j^2$ is the usual electromagnetic joule heating, and H represents a mechanical heating term. The precise form of H is uncertain since we do not yet know the agency responsible for heating the corona. Equation (9) will be discussed further in Section .

Equations (1)-(7), with (8) or (9), provide the usual system for MHD. Their implications in a variety of circumstances of interest in solar physics will be outlined in the subsequent text. A number of general features, however, may be discussed immediately.

3.1 The Mechanical Effect of the Magnetic Field

We may eliminate explicit mention of $\mathbf{j}$ in the momentum equation (3) by noting that

$$\mathbf{j}\times\mathbf{B} = -\,\mathrm{grad}\left[\frac{B^2}{2\mu_0}\right] + \frac{1}{\mu_0}\,(\mathbf{B}.\mathrm{grad})\mathbf{B}, \qquad (10)$$

which follows directly from elementary vector identities and use of equations (4) and (5). Thus, the effect of the $\mathbf{j} \times \mathbf{B}$ force may be viewed as made up of two contributions. The first term on the right hand side of equation (10) may be compared directly with the gas pressure gradient, -grad p, which also arises in the momentum equation. Thus, one effect of a magnetic field is to impart to the gas an additional pressure force p_m, namely $B^2/2\mu_0$, which augments the gas pressure. The combined pressure terms may therefore be written in the form

$$p_T = p + p_m = p + \frac{B^2}{2\mu_0}, \tag{11}$$

where p_T denotes the *total* (gas plus magnetic) *pressure*. Of course, the *magnetic pressure* p_m ($= B^2/2\mu_0$) is not related to the gas density ρ in the way the gas pressure is but, nonetheless, relation (11) tells us the magnetic fields have associated mechanical pressures and these pressure forces (strictly, their gradients) interact with the fluid motions. The magnetic pressure $B^2/2\mu_0$ acts *isotropically* (i.e. is the same in all directions).

The second term on the right-hand side of equation (10) is associated with a *tension* force, B^2/μ_0 per unit area, and this acts parallel to the magnetic field $\underline{B}$. In this respect, a magnetic field line is like an elastic string and if "twanged" (bent at some location) it responds like an elastic string and springs back, vibrating as a wave. Waves, then, are to be associated with magnetic fields, both through the tension force and through the magnetic pressure force. Waves are discussed in Section 4.

The association of a magnetic pressure p_m with a magnetic field leads naturally to a comparison of that pressure with the ambient gas pressure p. The ratio

$$\beta = \frac{\text{gas pressure}}{\text{magnetic pressure}} = \frac{p}{B^2/2\mu_0} \tag{12}$$

is called the *plasma beta*. It provides a convenient guide as to whether magnetic effects are weak (corresponding to β large) or strong (if β is small).

The value of β varies from region to region in the Sun. For example, in an intense flux tube embedded in the photosphere we have B = 1.5kG (= 0.15 tesla, since 1G = 10^{-4} tesla), which gives a magnetic pressure of $(0.15)^2/(8\pi \times 10^{-7})$ newtons m^{-2}; in c.g.s. units, this is simply

$B^2/8\pi$ dynes.cm^{-2}, with B in gauss.

Thus, a field of 1500 G gives 9×10^4 dynes.cm^{-2}. This is roughly comparable with the ambient gas pressure in the photosphere, some 2×10^5 dynes cm^{-2}. Thus, in an intense flux tube the plasma beta is of order unity. In other words, the tube is neither completely flacid ($\beta >> 1$) nor rigid ($\beta << 1$) but instead is in rough balance with the gas pressure; both the magnetic pressure and the gas pressure are important in an intense flux tube.

Outside of an intense flux tube or sunspot the magnetic field is very much weaker (perhaps a few gauss) and so β is very large; the magnetic field is consequently at the mercy of any gas pressure variations and so is largely manipulated by the hydrodynamics. By contrast, high in the corona β is generally small. For example, for a coronal field of B = 10G we get a magnetic pressure of about 4 dyne cm^{-2}, to be compared with a coronal gas pressure of 0.1-1.0 dyne cm^{-2}.

Another way of looking at these aspects of the magnetic field is to introduce the *sound* and *Alfven* speeds, defined by

$$c_s = \left[\frac{\gamma p}{\rho}\right]^{\frac{1}{2}} = \left[\frac{\gamma k_B T}{m}\right]^{\frac{1}{2}}, \qquad v_A = \frac{B}{(\mu_0 \rho)^{\frac{1}{2}}}. \tag{13}$$

In the absence of a magnetic field the sound speed c_s is the speed with which compressions or rarefactions in the gas are propagated. The Alfven speed v_A is the speed with which a distorted magnetic field line propagates to distant parts. Roughly, the Alfven speed is that speed which we would expect from the formula for the sound speed if we simply replaced p by the magnetic pressure $B^2/2\mu_0$ and choose an effective "γ" of 2. Alternatively, thinking in terms of an elastic string for which the speed of propagation is $(\text{tension/density})^{1/2}$ we have a speed $((B^2/\mu_0)/\rho_0)^{1/2}$ for a tension force of B^2/μ_0.

The sound and Alfven speeds are related to the plasma beta through

$$\beta = \frac{2}{\gamma}\frac{c_s^2}{v_A^2},$$

since the adiabatic index γ is generally close to 5/3, β provides an estimate of the ratio of the sound speed squared to the Alfven speed squared.

To illustrate the magnitude of c_s and v_A, we note that in the photosphere $\rho \approx 2 \times 10^{-7}$ gm cm^{-3} so with $p = 2 \times 10^5$ dynes cm^{-2} and $\gamma = 5/3$ we obtain $c_s \approx 1.3 \times 10^6$ cm s^{-1} $\approx$ 10 km s^{-1}; since $\beta \approx 1$ in an intense tube, we have $v_A \approx 10$ km s^{-1}. In the corona, at a temperature of 2×10^6K, the sound speed climbs to about 200 km s^{-1} and the Alfven speed to roughly 10^3 km s^{-1}. These values are far in excess of those expected under terrestrial or laboratory conditions. Estimates of the Alfven speed in the Earth's core give $v_A \approx 1$ cm s^{-1}; in mercury at room temperature in a field of 1 kG we get $v_A \approx 76$ cm s^{-1}, to be compared with a sound speed of $c_s \approx 1$ km s^{-1}. Generally, then, the sound speed exceeds the Alfven speed in laboratory and terrestrial circumstances (i.e. $\beta \gg 1$), whereas in the solar atmosphere the ordering is generally reversed (i.e. $\beta \leq 1$).

3.2 The Induction Equation

Just as the current density $\underset{\sim}{j}$ can be eliminated from the momentum equation by invoking Ampere's law, the electric field $\underset{\sim}{E}$ can be eliminated by combining Ohm's law (equation (7)) with Faraday's law (equation (6)). The result is

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \mathrm{curl}(\underset{\sim}{v} \times \underset{\sim}{B}) - \mathrm{curl}(\eta\, \mathrm{curl}\, \underset{\sim}{B}), \tag{14}$$

where $\eta = (\mu_o\sigma)^{-1}$ is the *resistivity* (units: m^{2}s^{-1}). Assuming η to be a constant, we may further reduce equation (14) with the solenoidal constraint imposed on $\underset{\sim}{B}$, the result being

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \mathrm{curl}(\underset{\sim}{v} \times \underset{\sim}{B}) + \eta\nabla^2\underset{\sim}{B}. \tag{15}$$

Equation (15) is generally referred to as the Induction Equation. It represents an important summary of the electromagnetic interaction of a flow $\underset{\sim}{v}$ and a magnetic field $\underset{\sim}{B}$.

3.2.1 Decay of magnetic field

To understand the induction equation suppose first that $\underset{\sim}{v} = 0$. Then, in the absence of any flow, the magnetic field in a conducting fluid satisfies

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \eta\nabla^2\underset{\sim}{B}, \tag{16}$$

which we recognise to be the diffusion (or heat conduction) equation. It follows, then, by analogy with the diffusion of heat, that each component of the magnetic field $\underset{\sim}{B}$ diffuses through the plasma. Any irregularities in $\underset{\sim}{B}$ are smoothed out through the action of diffusion; the magnitude of $\underset{\sim}{B}$ at any given location will decline in time.

The characteristic time-scale, τ_{diff}, by which irregularities in the field are removed is readily estimated from (16):

$$\frac{B}{\tau_{diff}} \sim \frac{\eta B}{L^2}, \quad \text{i.e.} \quad \tau_{diff} \sim \frac{L^2}{\eta} = \mu_o\sigma L^2, \tag{17}$$

where L is the scale of the irregularity in the magnetic field. In solar conditions L is expected to be rather large and so τ_{diff} is rather long and magnetic fields can only leak away very slowly in the solar plasma. This contrasts sharply with laboratory conditions, where L is small and τ_{diff} correspondingly short. For example, in a 3cm radius sphere of mercury (for which $\eta \sim 8 \times 10^3$cm^{2}s^{-1}) the diffusion (decay)time is about 1 second. In a sunspot, however, with

$$\eta \sim 10^{13}T^{-3/2}\mathrm{cm}^2\mathrm{s}^{-1} \tag{18}$$

(the molecular value in a fully ionized plasma with temperature T; see, for example, Spitzer 1962), estimates of τ_{diff} give several hundred years, as first pointed out by Cowling (19). The diffusion time for the "global" field of the Sun is much longer, about 10^9 years, and that of the galaxy is longer still. By contrast, the decay time for the Earth is about 5×10^4 years, much shorter than for the larger astrophysical bodies and much shorter than the age of the Earth (some 4.5×10^9 years). Consequently, the Earth's magentic field cannot be promordial but must be regerneated by electromagnetic forces (dynamo action); any initial field would have long ago decayed away, unless maintained by flows ($\mathbf{v} \neq 0$). In the Sun's case this is not so clear; a solar primordial field would still be around. Other aspects, however, suggest that a dynamo operates in the interior of the Sun also.

The above estimates suggest that for most purposes we may safely ignore diffusion, regarding the plasma as *perfectly conducting* (i.e. $\eta \to 0$, $\sigma \to \infty$). However there are important exceptions, notably where $\mathbf{B}$ changes rapidly, and in such circumstances *magnetic reconnection* may occur, a process that depends upon τ_{diff} being (loosely) of order unity, rather than very large, at least in some part of the plasma. Magnetic reconnection is treated elsewhere in these Lecture Notes - see the contribution by Sonnerup.

We should note that the process of field diffusion is not an entirely passive one, even if there is no flow ($\mathbf{v} = 0$). Diffusion of the field results in ohmic heating j^2/σ, which is of order $\eta B^2/\mu_0 L^2$. However, this is generally small (because L is large), except in regions undergoing magnetic reconnection.

<u>3.2.2 Solution of the diffusion equation</u> The general estimate of diffusion provided by equation (17) is adequate for most purposes. However, it is of interest to examine the diffusion process in greater detail for a region of rapid change in magnetic field, such as in a current sheet or magnetic flux tube.

<u>Example 1 : Diffusion of a Current Sheet</u>

Consider an initial distribution of magnetic field $B_0(x)\hat{z}$, in a cartesian coordinate system xyz. We take

$$B_0(x) = \begin{cases} B_0, & x > 0, \\ -B_0, & x < 0, \end{cases} \tag{19}$$

giving a step function representation of a reversing field (such as the 'tanh' function). We may suppose that $\mathbf{B}$ remains in the z-direction for all time. Writing $\mathbf{B} = B(x,t)\hat{z}$, we see from equation (16) that $B(x,t)$ satisfies the one-dimensional diffusion equation, namely

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2} . \tag{20}$$

Our problem is to solve equation (20) subject to

$$B(x, t = 0) = B_0(x). \tag{21}$$

An equation such as (20) is most conveniently solved by use of its *Green's function*, given by (Morse and Feshback 19)

$$G(x-s,t) = \frac{1}{(4\eta\pi t)^{1/2}} \exp\left[-\frac{(x-s)^2}{4\eta t}\right]. \tag{22}$$

The Green's function satisfies

$$\frac{\partial G}{\partial t} - \eta \frac{\partial^2 G}{\partial x^2} = \delta(x-s), \tag{23}$$

where δ is the Dirac delta function.

The general solution of equation (20) now follows as

$$B(x,t) = \int_{-\infty}^{\infty} ds\, G(x-s,t)B(s,0). \tag{24}$$

Applying equation (24) to the initial profile (19) gives (after evaluation of some integrals)

$$B(x,t) = B_0 \operatorname{erf}\left[\frac{x}{(4\eta t)^{1/2}}\right], \tag{25}$$

where erf is the *error function* (Abramowitz and Stegun 1967) defined by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds. \tag{26}$$

The error function has the properties that $\operatorname{erf}(0) = 0$ and $\operatorname{erf}(\pm\infty) = \pm 1$; while for x small, $\operatorname{erf}(x) \sim 2/\sqrt{\pi}\, x$. These properties readily allow us to sketch the field strength $B(x,t)$, showing how the initial step function (19) is diffused away.

The timescale for diffusion follows immediately from equation (25) as

$$\tau_{diff} = \frac{x^2}{4\eta}, \tag{27}$$

showing a *variable* timescale; comparing with relation (17), we see that the effective lengthscale is also variable, $L \sim \frac{1}{2} x$. Thus diffusion is very rapid near $x = 0$, where τ_{diff} is small, and very slow at large $\pm x$. The reason for a variable lengthscale is that there is no natural lengthscale in the initial field configuration. In other words, there is no imposed means of measuring a unit of length save that defined, instantaneously, by diffusion.

__Example 2 : Diffusion of a Flux Tube__

Consider now the diffusion of a *slab* of magnetic field, in which

$$B_o(x) = \begin{cases} B_o, & |x| < a, \\ 0, & |x| > a. \end{cases} \tag{28}$$

The slab (or "top hat") provides a one-dimensional cartesian version of a flux tube. Equation (22) provides the appropriate Green's function and equation (24) gives the evolution of $B(x,t=0) = B_o(x)$. Evaluating the integrals, we find

$$B(x,t) = \frac{B_o}{\sqrt{\pi}} \left\{ \mathrm{erf}\left[\frac{a-x}{(4\eta t)^{1/2}}\right] + \mathrm{erf}\left[\frac{a+x}{(4\eta t)^{1/2}}\right] \right\}, \tag{29}$$

so that the slab's sharp corners at $x = \pm a$ are rapidly smoothed out and spatial symmetry is maintained (i.e. $B(x,t) = B(-x,t)$).

Considering the field at $x = 0$ we see that the central field $B(0,t)$ declines like $t^{-1/2}$ on a diffusion timescale of $\tau_{diff} = a^2/4\eta$:

$$B(x=0,t) = \frac{2B_o}{\sqrt{\pi}} \mathrm{erf}\left[\frac{a}{(4\eta t)^{1/2}}\right] \sim \frac{2aB_o}{\pi(\eta t)^{1/2}} \quad \text{as } t \to \infty. \tag{30}$$

The effective lengthscale is $L = \frac{1}{2} a$, i.e. $\frac{1}{4}$ of the slab width.

A cylindrical *flux tube* can be dealt with in much the same fashion, though a Green's function appropriate to polar coordinates is required. Consider, then, the diffusion of a tube $B_o(r)\hat{z}$, where

$$B_o(r) = \begin{cases} B_o, & r < a, \\ 0, & r > a. \end{cases} \tag{31}$$

The diffusion equation in polar coordinates (r,θ,z), with $\partial/\partial\theta = \partial/\partial z = 0$, is

$$\frac{\partial B}{\partial t} = \eta\left[\frac{\partial^2 B}{\partial r^2} + \frac{1}{r}\frac{\partial B}{\partial r}\right], \tag{32}$$

which we need to solve subject to $B(r,t=0) = B_o(r)$. The solution of equation (32), equivalent to equation (24), is

$$B(r,t) = \int_0^\infty ds\, G(r,s,t)B(s,0), \tag{33}$$

where the Green's function $G(r,s,t)$ for equation (32) is (see Carslaw and Jaeger 19 , p.260)

$$G(r,s,t) = \frac{s}{2\eta t} \exp\left[-\frac{(r^2+s^2)}{4\eta t}\right] I_o\left[\frac{rs}{2\eta t}\right], \tag{34}$$

for modified Bessel function I_o.

It follows, then, that the field strength $B(0,t)$ on the central axis of the tube is

$$B(0,t) = \frac{B_o}{2\eta t} \int_0^a s e^{-s^2/4\eta t}\, ds$$

$$= B_o\left[1 - \exp\left[-\frac{a^2}{4\eta t}\right]\right]. \tag{35}$$

Thus, the central field declines like t^{-1} (compared with $t^{-1/2}$ in a slab) on a time-scale of $\tau_{diff} = a^2/4\eta$, the characteristic length L being half the radius of the tube.

__3.2.3 The frozen flux theorem__ Returning to the induction equation

(15), we consider the effect of the flow on the magnetic field. Ignoring diffusion of the field by setting $\eta = 0$, we have

$$\frac{\partial B}{\partial t} = \text{curl}(\underline{v} \times \underline{B}). \quad (36)$$

Additionally, we recall that div $\underline{B} = 0$. Equation (36) is precisely in the form of Helmholtz's theory for the behaviour of vorticity $\underline{\omega}$ = curl $\underline{v}$ in a flow, and an analogous result holds for the behaviour of $\underline{B}$ and $\underline{v}$. In the form expressed by Alfven (1943), we may state that in a perfectly conducting fluid (for which $\eta = 0$) magnetic field lines move with the fluid; the field lines are *frozen* into the fluid. In other words, motions along the field do not change it, motions transverse to the field carry the field with them. A proof of this theorem may be found in any of the cited general texts on MHD.

In general terms we may assume that frozen flux conditions pertain in the solar atmosphere, with the exception of in the vicinity of the site of reconnection. By contrast, in laboratory or terrestrial circumstances this is not normally the case. An estimate of the relative sizes of the terms in the induction equation (15) may be given as follows. The timescale τ_{adv} for *advection of field* by the flow, as described by equation (36), is

$$\tau_{adv} = \frac{L}{V}, \quad (37)$$

for a medium with characteristic scale L and velocity V. This is to be contrasted with the timescale τ_{diff} for diffusion of magnetic field (see equation (17)). The ratio of the diffusion time to the advection time defines the *magnetic Reynolds number*, R_m, of the flow:

$$R_m = \frac{\tau_{diff}}{\tau_{adv}} = \frac{\mu_o \sigma L^2}{LV^{-1}} = \mu_o \sigma LV = \frac{LV}{\eta}. \quad (38)$$

In astrophysical circumstances the general largeness of L and V ensure that the magnetic Reynolds number R_m is large. For example, in the accretion disk around the suspected black hole of Cyg X-1 the magnetic Reynolds number is at least 10^{10}; in the galactic disk it is of order 10^8; in the solar convection zone $R_m \sim 10^8$; in Jupiter's core $R_m \sim 10^6$, and in the Earth's core $R_m \sim 5 \times 10^2$. (See also Zeldovich, Ruzmaikin and Sokoloff 1983.) Only in laboratory circumstances do we generally expect to find R_m less than unity, though at the site of reconnection it is also small. In numerical simulations R_m is generally constrained to be moderate, 10^2-10^3 say.

3.3 Advection of Magnetic Field

The general conclusion to emerge from the above estimates of R_m is that in astrophysical plasmas the field is "frozen into the flow". To a high degree of accuracy, diffusive effects are of secondary importance to the overall structure of a field; only in the removal of small-scale magnetic structures (where the effective L is very much reduced) embedded within a flow is the effect of diffusion important.

To explore further how a magnetic field is manipulated by a flow we consider the kinematical solution of the induction equation for a perfect conductor ($\eta = 0$).

__Example 3 : Magnetic Flux Concentration__

Suppose at time t = 0 the flow

$$\underline{v} = (v_o \sin kx, 0, -v_o kz \cos kx), \quad z \le 0, \quad -\infty < x < \infty, \quad (39)$$

is "switched on" in the presence of a magnetic field $B_o\hat{\underline{z}}$. The question is: what happens to the magnetic field if the flow (39) manipulates it according to equation (36)?

We note that the flow (39) is incompressible, i.e. div $\underline{v} = 0$, and has a spatial scale of order k^{-1}. The flow rises at x = 0 (the cell centre), flows to the left and right and finally descends at $x = \pm \pi/k$. The pattern is $2\pi/k$ periodic.

To solve equation (36) suppose that $\underline{B} = B(x,z,t)\hat{\underline{z}}$, i.e. suppose that the field remains vertical at all times. Then the solenoidal constraint (div $\underline{B} = 0$) implies that $B = B(x,t)$. Substituting in the induction equation yields

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial x}(B v_o \sin kx) = 0. \quad (40)$$

This equation may be solved by the method of characteristics. The solution satisfying the condition that $B(x,t = 0) = B_o$ is (Parker 1963)

$$B(x,t) = \frac{B_o e^{-kv_o t}}{\cos^2 \frac{1}{2}kx + e^{-2kv_o t}\sin^2 \frac{1}{2}kx}. \tag{41}$$

This example conveniently illustrates the behaviour of the convective flow (39) in *expelling magnetic field in regions that are updraughts, and concentrating magnetic field in regions where there are downdraughts.* To see this more clearly, set $x = 0$ in equation (41); in an updraft, we have

$$B(x,t) = B_o e^{-kv_o t}, \tag{42}$$

indicating an exponential decay of the magnetic field. By contrast, at $x = \pi/k$ (a downdraught) we have

$$B(x,t) = B_o e^{kv_o t}, \tag{43}$$

and the field is concentrated exponential fast. The timescale for the process of concentration is evidentally $(kv_o)^{-1}$, i.e. the advection timescale (37) with $L = k^{-1}$.

As a numerical illustration, consider supergranules (see Section 2.1) with $v_o = \frac{1}{2}$ km s^{-1} and $2\pi/k = 3 \times 10^4$km. We find

$$\tau_{adv} \sim 10^4 s,$$

indicating a concentration of magnetic field in the downdraughts between supergranules on a timescale of about 3 hours. Since supergranules have lifetimes much longer than 3 hours, they are likely to be efficient at concentrating magnetic field in their downdraughts. Granules, too, are able to concentrate field and do so on a timescale of 160 s (for $v_o = 1$ km s^{-1}, $2\pi/k = 10^3$km).

The above considerations suggest that the magnetic field at the surface of the Sun will be arranged not in a uniform distribution but in the form of local concentrations. Indeed, this is precisely the case as determined from the observations which indicate a large number of isolated intense flux tubes with kilogauss fields (see Section 2.4). However, an explanation of this observational fact cannot rest solely on the above demonstration of advective concentration. This is because the concentration of magnetic field by advection is limited by:

(i) the back-reaction of the field on the flow, which ultimately modifies the flow and its ability to concentrate the field;

(ii) the requirement of transverse pressure balance which limits the magnetic pressure to the confining hydrostatic pressure.

In practice, then, we may expect that the process of concentration of field by advection operates up to a rough equipartion with the dynamical pressure of the convective flow, i.e. we expect a concentrated field B with strength given by

$$\frac{B^2}{2\mu_c} \sim \frac{1}{2}\rho v^2. \tag{44}$$

For $\rho = 10^{-7}$ gm cm^{-3} (the gas density in the photosphere) and $v \sim 10^5$ cm s^{-1} we get $B \approx 10^2$gauss. This is well below the observed 1.5 kG field strengths and so other effects must be sought for explaining the concentrated field in the photosphere. We return to this topic in Section 6.

4. MHD WAVES

We have pointed out in Section 3.1 that there are two characteristic speeds in MHD, the sound speed c_s and the Alfven speed v_A. In a medium with gas density ρ_o and pressure p_o embedded in a magnetic field of strength B_o these speeds are

$$c_s = \left[\frac{\gamma p_o}{\rho_o}\right]^{1/2}, \quad v_A = \left[\frac{B_o^2}{\mu_o \rho_o}\right]^{1/2} \tag{45}$$

for adiabatic index γ. (We shall usually take $\gamma = 5/3$.) Precisely how these speeds enter into a description of the ability of a hydromagnetic plasma to support waves is the topic of this section.

There are a number of reasons for studying the properties of MHD waves. Firstly, there is increasing evidence that hydromagnetic waves occur in a variety of astrophysical plasmas, including the magnetosphere and magnetotail (see the review by Southwood and Hughes 1983), in cometary tails, and in the solar atmosphere. Indeed, bearing in mind that a magnetic flux tube is essentially an *elastic* tube, it would be surprising if the many examples of such structures in the solar atmosphere - the sunspot, the intense flux tube, the H_α fibril, the coronal loop, etc. - were not capable of supporting MHD waves, though we should caution that no unambiguous identification is yet possible. There is, then, an intrinsic interest in the study of the properties of MHD waves, especially their properties in a magnetically structured medium such as a flux tube provides.

Added to an intrinsic interest in the study of hydromagnetic waves is an interest in their possible contribution to the heating of the corona. Sound waves are now ruled out as an explanation of the high temperature corona, leaving MHD waves as a possible mechanism; we return to this aspect in Section 7. We should add, too, that MHD waves are of much current interest in the heating of laboratory plasmas (see Appert *et al* 1984).

4.1 Waves in a Uniform Medium

To begin our investigation of MHD waves it is convenient to consider first their properties in a uniform medium. Though of limited direct application, the study of a uniform medium is a necessary preliminary to an understanding of waves in magnetic flux tubes and other structured media. Consider, then, the equations of *ideal* MHD, which follow from equations (1)-(8) with viscous and diffusive effects neglected (i.e. $\nu = \eta = 0$); we will also ignore gravitational effects to begin with. Our equations are simply

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, \rho \underline{v} = 0,$$

$$\rho\left[\frac{\partial}{\partial t} + \underline{v}.\mathrm{grad}\right]\underline{v} = -\mathrm{grad}\left[p + \frac{B^2}{2\mu_o}\right] + (\underline{B}.\mathrm{grad})\frac{\underline{B}}{\mu_o}$$

$$\left[\frac{\partial}{\partial t} + \underline{v}.\mathrm{grad}\right]\left[\frac{p}{\rho^\gamma}\right] = 0,$$

$$\frac{\partial \underline{B}}{\partial t} = \mathrm{curl}(\underline{v} \times \underline{B}),\ \mathrm{div}\, \underline{B} = 0. \qquad (46)$$

We consider the *equilibrium state*

$$\underline{B} = B_o\hat{\underline{z}}, \quad p = p_o, \quad \rho = \rho_o, \qquad (47)$$

representing a uniform gas (constant p_o and ρ_o) within which is embedded a uniform magnetic field $B_o\hat{\underline{z}}$. The cartesian coordinate system xyz is oriented with the z-axis aligned along the equilibrium field.

It may be noticed at this earlier stage a significant difference between the problem of waves in a magnetic atmosphere and that of sound waves (the non-magnetic case): the presence of a magnetic field introduces a preferred sense of direction, namely that of the field itself. Thus, we may anticipate that magnetic waves are *anisotropic* in the manner in which they propagate, a property in contrast to sound waves in a uniform medium which are isotropic.

In investigative wave motions we perturb the equilibrium state (47) and consider how such perturbations behave. Restricting attention to very small amplitude disturbances, we may linearize equations (46) about the equilibrium state (47). Write

$$p = p_o + p_1, \quad \rho = \rho_o + \rho_1, \quad \underline{B} = B_o\hat{\underline{z}} + \underline{B}_1, \quad \underline{v} = \underline{v}_1, \qquad (48)$$

for perturbations p_1, ρ_1, etc. assumed small (squares of perturbations are neglected). Then, discarding the suffix "1" on the perturbations, we have

$$\frac{\partial \rho}{\partial t} + \rho_o\Delta = 0, \qquad (49)$$

for the linearized equation of continuity; we have written

$$\Delta = \mathrm{div}\, \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}, \qquad (50)$$

for velocity $\underline{v} = (v_x, v_y, v_z)$.

The momentum equation yields

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = - \text{grad } p_T + \frac{B_0}{\mu_0} \frac{\partial \underline{B}}{\partial z}, \tag{51}$$

where

$$p_T = p + \frac{B_0}{\mu_0} B_z \tag{52}$$

is the perturbation in the total pressure $(p + B^2/2\mu_0)$. Notice the anisotropy arising in the momentum equation.

The isentropic (adiabatic) energy equation yields

$$p = c_s^2 \rho, \tag{53}$$

just as for the usual sound waves. The sound speed (in the equilibrium state) is defined in equation (45).

Finally, the induction equation gives

$$\frac{\partial \underline{B}}{\partial t} = -B_0 \Delta \hat{\underline{z}} + B_0 \frac{\partial \underline{v}}{\partial z}, \tag{54}$$

on noting that div $\underline{B} = 0$.

Following Lighthill (1960), we may introduce the variable

$$\Gamma = \frac{\partial v_z}{\partial z} . \tag{55}$$

Then, after some manipulations, we may recast our equations entirely in terms of Γ and Δ:

$$\frac{\partial^2 \Delta}{\partial t^2} = (c_s^2 + v_A^2)\nabla^2 \Delta - v_A^2 \nabla^2 \Gamma, \tag{56}$$

$$\frac{\partial^2 \Gamma}{\partial t^2} = c_s^2 \frac{\partial^2 \Delta}{\partial z^2} . \tag{57}$$

Added to equations (56) and (57), we may show that the z-component of vorticity (i.e., the component of vorticity in the direction of the applied magnetic field) satisfies a wave equation, namely

$$\frac{\partial^2 \omega_z}{\partial t^2} = v_A^2 \frac{\partial^2 \omega_z}{\partial z^2} , \tag{58}$$

where

$$\omega_z = \hat{\underline{z}}.\text{curl } \underline{v} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} . \tag{59}$$

Equations (56)-(59) completely describe the perturbation in that the three components of $\underline{v}$, namely v_x, v_y and v_z, may be deduced from Δ, Γ and ω_z. It is apparent from the above equations that the system (56)-(58) splits into two independent classes: disturbances described by (56) and (57), and those described by (58). Hence, in the presence of an applied magnetic field the component of vorticity in the direction of that field is propagated one-dimensionally with the Alfven speed v_A; this is independent of the behaviour of Δ and Γ. The property of one-dimensional propagation of the z-component of vorticity is in contrast with that of sound waves in the absence of a field: for sound waves (with $B_0 = 0$) vorticity is conserved, not propagated.

Turning now to equations (56) and (57), we may eliminate Γ to obtain

$$\frac{\partial^4 \Delta}{\partial t^4} - (c_s^2 + v_A^2)\nabla^2 \frac{\partial^2 \Delta}{\partial t^2} + c_s^2 v_A^2 \nabla^2 \left[\frac{\partial^2 \Delta}{\partial z^2}\right] = 0. \tag{60}$$

<u>4.1.1 Alfven Waves</u> One solution of equation (60) is $\Delta = 0$, i.e. div $\underline{v} = 0$. This is not a trivial solution but in fact corresponds to the *Alfven wave*. An examination of equations (49) to (54) reveals that $p = p_T = \rho = \Gamma = v_z = B_z = 0$. Thus, in an Alfven wave there is no perturbation in density, in gas or magnetic pressure, and there is no flow along the applied magnetic field. The perturbation in the field and flow are perpendicular to the applied field and, in fact, are propagated one-dimensionally with speed v_A. In an Alfven wave, the components v_x, v_y, B_x and B_y each satisfy the one-dimensional wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = v_A^2 \frac{\partial^2 \phi}{\partial z^2} , \tag{61}$$

for ϕ any one of v_x, v_y, B_x or B_y.

The Alfven wave, then, is simply like a wave on an elastic string. It involves no compressions of the plasma and, since $p_T = 0$, the wave is driven purely by *tension* forces.

If we introduce a Fourier representation for wave motions by writing

$$\phi = \phi_o e^{i(\omega t - \underline{k}.\underline{r})} = \phi_o e^{i(\omega t - k_x x - k_y y - k_z z)}, \quad (62)$$

for frequency ω, wavenumber $\underline{k} = (k_x, k_y, k_z)$ and constant wave amplitude ϕ_o, we see from equation (61) that

$$\omega^2 = k_z^2 v_A^2 = k^2 v_A^2 \cos^2\theta, \quad (63)$$

where θ is the angle between the wave vector $\underline{k}$ (of magnitude $k = (k_x^2 + k_y^2 + k_z^2)^{1/2}$) and the applied field $\underline{B}_o$. Equation (63) is the *dispersion relation for Alfven waves*. Comparing equation (63) with the dispersion relation for a sound wave in the absence of a magnetic field, namely

$$\omega^2 = k^2 c_s^2, \quad (64)$$

we see the highly anisotropic nature of the Alfven wave.

4.1.2 Magnetoacoustic waves Consider now $\Delta \neq 0$. Introducing the Fourier representation

$$\Delta = \Delta_o e^{i(\omega t - \underline{k}.\underline{r})}$$

into equation (60) we obtain (on noting that $\partial^2/\partial t^2 \to -\omega^2$, $\nabla^2 \to -k^2$) the *dispersion relation for magnetoacoustic waves*:

$$\omega^4 - (c_s^2 + v_A^2)\omega^2 k^2 + c_s^2 v_A^2 k^2 k_z^2 = 0. \quad (65)$$

Again, we observe the anisotropic nature of this equation, with solutions evidently depending upon the angle of propagation θ. In the absence of a magnetic field (when $v_A = 0$) equation (65) yields the usual dispersion relation (64) for sound waves (as well as $\omega^2 = 0$).

Equation (65) is evidently quadratic in ω^2. Consider its solution for fixed k but varying angle θ. When $\theta = 0$, we obtain

$$\omega^2 = k^2 c_s^2 \quad \text{and} \quad \omega^2 = k^2 v_A^2; \quad (66)$$

when $\theta = \pi/2$, we obtain

$$\omega^2 = 0 \quad \text{and} \quad \omega^2 = k^2(c_s^2 + v_A^2). \quad (67)$$

The two solutions of equation (65), ordered in the magnitude of ω/k, are referred to as the *slow* magnetoacoustic wave and *fast* magneto-acoustic wave, respectively. They are *compressive* (i.e. $\Delta \neq 0$, $p \neq 0$, $\rho \neq 0$). The slow wave is highly anisotropic; like the Alfven wave, it is unable to propagate across the magnetic field lines, i.e. $\omega^2 = 0$ for $\theta = \pi/2$ ($k_z = 0$) in both the slow wave and the Alfven wave.

The fast wave is only mildly anistropic. It propagates at all angles θ, being fastest at right angles to the field (at $\theta = \pi/2$) when its *phase-speed* ω/k reaches $(c_s^2 + v_A^2)^{1/2}$. For propagation along the magnetic field (i.e. at $\theta = 0$) the fast wave has a phase-speed equal to the larger of c_s and v_A.

The distinctive features between the slow and fast waves may be traced to the behaviour of the gas pressure p and magnetic pressure p_m, which together make up p_T. In the slow wave the gas pressure is out of phase with the magnetic pressure (i.e. $pp_m < 0$), becoming exactly out of phase (and therefore nullifying one another) when $\theta = \pi/2$. In the fast wave, however, the gas and magnetic pressures are in phase (i.e. $pp_m > 0$) and p_T reaches a relative maximum at $\theta = \pi/2$. The fast wave at an angle $\theta = \pi/2$ corresponds to compressions and rarefactions of the magnetic field with no bending of the field lines (i.e. $B_x = B_y = 0$, $B_z \neq 0$), and therefore no tension forces; the wave is purely a response to pressure forces.

Finally, it is of general interest to examine the behaviour of the magnetoacoustic waves in the low-β extreme. This is of particular interest for coronal applications. Examining the dispersion relation (65) in the circumstances $v_A >> c_s$, we see that the *fast* wave has

$$\omega^2 \approx k^2 v_A^2, \qquad (68)$$

and the *slow* wave gives

$$\omega^2 \approx k_z^2 c_T^2 \approx k_z^2 c_s^2, \qquad (69)$$

where the *magnetoacoustic cusp speed* c_T is defined by

$$c_T^2 = \frac{c_s^2 v_A^2}{c_s^2 + v_A^2}. \qquad (70)$$

(The cusp speed c_T arises in magnetic flux tubes too. We note that it is both sub-sonic and sub-Alfvenic.) Hence, in a low-β plasma the fast wave is essentially isotropic, propagating with the Alfven speed. Notice that relation (68) is quite distinct from that for an Alfven wave, relation (63), which is highly anistropic. The slow wave, however, gives essentially one-dimensional propagation of sound waves. The magnetic field lines are so strong (if $\beta \ll 1$) that sound waves are forced to propagate one-dimensionally along the almost rigid field lines, a result in nice contrast to the usual three-dimensional behaviour of sound waves.

The generally anisotropic nature of MHD waves poses an amusing question for music lovers listening to their favourite orchestra playing in the solar atmosphere! In a solar amphitheatre with a vertical magnetic field, where should one sit in order to most enjoy the music? Presumably on a line perpendicular to the orchestra (regarded as a point source), for then one would receive only the fast wave. But those with unusual musical tastes might prefer the seats in "the gods", and receive the double benefit of both fast and slow sounds! Alfven waves, of course, are of no concern since they are inaudible.

4.2 Waves in a Magnetically Structured Medium

Since the solar atmosphere - and for that matter many other astrophysical plasmas - is structured by magnetic fields it is important to understand the behaviour of MHD waves in a non-uniform medium. Unfortunately, magnetic structuring introduces a number of complexities some of which is still under active investigation. So an exhaustive discussion of the various modes and their properties is not possible. Instead, we will be content to outline the general equations and present their solution in a number of simple circumstances.

We consider an equilibrium state of a unidirectional magnetic field $B_0(x)\hat{z}$, the magnitude of which varies in a direction perpendicular to the field. We assume that the plasma's temperature, pressure and density are similarly structured; this stucturing in plasma properties may be as a result of the magnetic field or quite independent of it. The equilibrium state, then, is one of constancy of total pressure:

$$p_0(x) + \frac{B_0^2(x)}{2\mu_0} = \text{constant}. \qquad (71)$$

In general, then, the sound speed c_s and Alfven speed v_A are functions of x:

$$c_s(x) = \left[\frac{\gamma p_0(x)}{\rho_0(x)}\right]^{1/2}, \quad v_A(x) = \left[\frac{B_0^2(x)}{\mu_0 \rho_0(x)}\right]^{1/2}. \qquad (72)$$

Consider isentropic (adiabatic) perturbations of the equilibrium (71). For simplicity we restrict attention to motions in the x-z plane, taking $v_y = 0$ and $\partial/\partial y = 0$. This eliminates the Alfven wave from consideration. It is convenient to work in terms of v_x, the component of the velocity $\underline{v} = (v_x, 0, v_z)$ perpendicular to the applied magnetic field, and the perturbation in the total pressure, p_T. The linearized equations of ideal MHD may be written down from system (46), much as was done for a uniform medium but taking account of the non-uniformity in $\rho_0(x)$, $p_0(x)$ and $B_0(x)$. After some manipulation we obtain

$$\frac{\partial p_T}{\partial t} = \rho_0 v_A^2 \Gamma - \rho_0 (c_s^2 + v_A^2)\Delta, \qquad (73)$$

$$\rho_0\left[\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2}\right] v_x = -\frac{\partial^2 p_T}{\partial x \partial t}, \tag{74}$$

and

$$\frac{\partial^2 v_z}{\partial t^2} = c_s^2 \frac{\partial \Delta}{\partial z}, \tag{75}$$

with p_T, Γ and Δ defined as previously.

Introducing the Fourier representation

$$v_x(x,z,t) = v_x(x) e^{i(\omega t - k_z z)}, \tag{76}$$

for frequency ω and wavenumber k_z along the applied field, we may eliminate v_z, say, from equations (73)-(75) and obtain a pair of ordinary differential equations for $p_T(x)$ and $v_x(x)$:

$$\frac{dp_T}{dx} = \frac{i\rho_0(x)}{\omega} \cdot (k_z^2 v_A^2(x) - \omega^2) v_x, \tag{77}$$

$$(k_z^2 c_T^2 - \omega^2)\frac{dv_x}{dx} = -\frac{i\omega}{\rho_0(\omega)} \frac{(k_z^2 c_s^2(x) - \omega^2)}{(c_s^2(x) + v_A^2(x))} p_T. \tag{78}$$

We notice immediately an unusual feature of equation (78), namely that it is *singular* at $\omega^2 = k_z^2 c_T^2(x)$. Recalling the definition of the cusp speed c_T (see equation (70)), we identify this singularity with the *slow* magnetoacoustic wave. In fact, the presence of a singularity in equation (78) is associated with the existence of a *continuous spectrum*, the so-called *cusp continuum*. The existence of such a singularity implies that in ideal MHD the slow magnetoacoustic wave in a non-uniform medium exhibits a complicated temporal behaviour which is not describable in terms of simple normal modes.

If we now eliminate p_T between equations (77) and (78) we obtain a single ordinary differential equation for v_x:

$$\frac{d}{dx}\left[\frac{\rho_0(x)(c_s^2(x) + v_A^2(x))(k_z^2 c_T^2(x) - \omega^2)}{(k_z^2 c_s^2(x) - \omega^2)} \frac{dv_x}{dx}\right]$$

$$= \rho_0(x)(k_z^2 v_A^2(x) - \omega^2) v_x. \tag{79}$$

Equation (79) is the basic equation we require. It has been discussed by a number of authors (e.g. Goedbloed 1971, 1981, 1984; Roberts 1981a, 1984, 1985; Rae and Roberts 1982). If three-dimensional wave propagation is allowed for (i.e. if $k_y \neq 0$, $v_y \neq 0$), then we still obtain a second order differential equation but its structure is more complicated than that in equation (79). In particular, in addition to the cusp continuum the three-dimensional equation possesses an *Alfven continuum* associated with a singularity at $\omega^2 = k_z^2 v_A^2(x)$. This continuum, too, is associated with the non-existence of normal modes attached to the Alfven wave. The two waves that possess continuous spectra, namely the slow wave and the Alfven wave, are the waves that are unable to propagate across the field in a uniform medium. The fast wave, which propagates more or less isotropically in a uniform medium, does not possess a continuous spectrum and instead discrete frequencies – normal modes – arise.

The significance of a continuous spectrum is most readily brought out in the *incompressible* case. With $c_s \to \infty$, equation (79) reduces to

$$\frac{d}{dx}\left[\rho_0(k_z^2 v_A^2 - \omega^2)\frac{dv_x}{dx}\right] = \rho_0 k_z^2 (k_z^2 v_A^2 - \omega^2) v_x, \tag{80}$$

which possesses an Alfven continuum. This equation is analogous to that arising in a cold plasma (Uberoi 1972) and the significance of a continuous spectrum was made clear by Barston (1964) and Sedlacek (1973). Applications in magnetospheric physics have been made by Southwood (1974) and Chen and Hasegawa (1974). The topic has been of special interest to solar coronal physics because of the possibility of coronal heating, as stressed by Ionson (1978). An analysis of equation (80) from a Laplace transform viewpoint has been recently given by Van Abels-Maanen (1980), Rae and Roberts (1981) and Lee and Roberts (1985), following on from the earlier analysis of the cold plasma case by Sedlacek (1971). The general topic of continuous spectra in mathematical physics, with applications to waves in fluid and MHD systems, has been discussed by Adam (1986). Finally, we should mention the extensive interest in laboratory plasma applications; see the recent reviews by Goedbloed (1984) and Appert et al. (1984).

4.3 Surface Waves

As a first application of the general equations for MHD waves in a structured magnetic medium, we consider the incompressible case governed by equation (80). Suppose the equilibrium consists of a step function in $B_o(x)$, defined by

$$B_o(x) = \begin{cases} B_e, & x > 0, \\ B_o, & x < 0, \end{cases} \tag{81}$$

representing a region of rapid change in the magnetic field. The density $\rho_o(x)$ and the Alfven speed $v_A(x)$ may also be in the form of step functions:

$$\rho_o(x) = \begin{cases} \rho_e, & x > 0, \\ \rho_o, & x < 0, \end{cases} \qquad v_A^2(x) = \begin{cases} v_{Ae}^2, & x > 0, \\ v_{Ao}^2, & x < 0. \end{cases} \tag{82}$$

On either side of $x = 0$ the medium is uniform, so equation (80) applies with $v_A^2(x)$ constant in $x > 0$ and in $x < 0$. Thus, in either $x > 0$ or $x < 0$ we have

$$\frac{d^2 v_x}{dx^2} - k_z^2 v_x = 0, \tag{82}$$

in addition to $\omega^2 = k^2 v_A^2(x)$ in each of $x > 0$ and $x < 0$.

In a uniform ($\rho_o = \rho_e$, $B_o = B_e$) and unbounded medium, as discussed in Section 4.1, the only acceptable solution of equation (82) is $v_x = 0$, a trivial solution. In a uniform medium the appropriate solution is, in fact, $\omega^2 = k_z^2 v_A^2$, with $v_x \neq 0$; this is the limiting behaviour of the slow magnetoacoustic wave when $c_{\parallel} \to \infty$ (see equation (65)).

Returning to equation (82) we note that an appropriate solution is

$$v_x(x) = \begin{cases} \alpha_e e^{-k_z x}, & x > 0, \\ \alpha_o e^{k_z x}, & x < 0, \end{cases} \tag{83}$$

where we have supposed that k_z is positive and imposed the requirement that $v_x(x)$ is bounded at $\pm\infty$. The condition of boundedness rules out the exponentially growing solutions of equation (82). In equation (83), α_e and α_o are arbitrary constants.

To complete the solution we require matching conditions across the interface $x = 0$. It is necessary that v_x (the normal component of velocity) be continuous across $x = 0$, so $\alpha_e = \alpha_o$. A second boundary condition is that p_T, the total pressure perturbation, be continuous across $x = 0$, for otherwise there would be unbalanced forces at the interface. We may see this condition from a mathematical viewpoint too if we observe from equation (80) that the term in { } brackets must be continuous; for if it were discontinuous, then its derivative (formed on the left-handside of (80)) would involve Dirac delta functions, which are evidently not present on the right-handside of (80). Hence, in the incompressible case we require that

$$\rho_o(k_z^2 v_A^2(x) - \omega^2)\frac{dv_x}{dx} \text{ be continuous across an interface.} \tag{84}$$

Substitution of equation (83) into the boundary condition (84) yields

$$\omega^2 = k_z^2 \left[\frac{\rho_o v_A^2 + \rho_e v_{Ae}^2}{\rho_o + \rho_e}\right] = k_z^2 \frac{(B_o^2 + B_e^2)}{\mu_o(\rho_o + \rho_e)}. \tag{85}$$

This is the dispersion relation for *hydromagnetic surface waves*. They are *non-dispersive*, the phase speed ω/k_z of the wave being independent of wavelength or frequency.

The surface wave on a magnetic interface is quite distinct from the Alfven wave of a uniform medium. A surface wave is confined to the interface on which it propagates; it propagates along the interface, disturbing the surrounding medium only slightly (it penetrates a distance k_z^{-1} either side of the interface). Its phase speed, ω/k_z, is intermediate between the Alfven speeds of the two regions.

Surface waves are also possible in a *compressible* medium though their structure is more involved. However, the phase-speed of a surface wave is still intermediate between the two Alfven speeds, though in the compressible case two surface waves are possible (Roberts 1981).

4.4 Waves in a Magnetic Slab

As a second application of equation (80) we consider the waves in a magnetic slab, extending from x = -a to x = a, and defined by

$$B_o(x) = \begin{cases} 0\,, & |x| > a, \\ B_o, & |x| < a, \end{cases} \qquad \rho_o(x) = \begin{cases} \rho_e, & |x| > a, \\ \rho_o, & |x| < a. \end{cases} \tag{86}$$

Again, equation (82) is applicable though now one cannot discard any solutions within the slab. Thus, we take

$$v_x(x) = \begin{cases} \alpha_e e^{-k_z x}, & x > 0, \\ \alpha_o \cosh k_z x + \alpha_o' \sinh k_z x, & |x| < a, \\ \alpha_e' e^{k_z x}, & x < 0, \end{cases} \tag{87}$$

for arbitrary constants α_o, α_o', α_e and α_e'. The appropriate boundary conditions are (again) that v_x and p_T (equivalently (84)) are continuous across the interfaces at $x = \pm a$.

It is evident on geometrical grounds that we may discuss separately the cases $\alpha_o = 0$ and $\alpha_o' = 0$. (If this simplification is not observed it in any case emerges from the full analysis.) Setting $\alpha_o = 0$ is equivalent to looking for modes of oscillation that do not disturb the centre of the slab: $v_x = 0$ at $x = 0$. Such modes are *sausage waves*. See Figure . Correspondingly, setting $\alpha_o' = 0$ (for $\alpha_o \neq 0$) gives modes that disturb the central axis of the slab in a serpentine fashion (Figure); these are the *kink* waves.

Application of the appropriate boundary conditions on x = a gives, for the *sausage* modes, the dispersion relation

$$\omega^2 = \frac{k_z^2 v_A^2}{1 + \left(\frac{\rho_e}{\rho_o}\right) \tanh k_z a}, \tag{88}$$

where v_A is the Alfven speed within the slab. The dispersion relation for the *kink* modes is

$$\omega^2 = \frac{k_z^2 v_A^2}{1 + \left(\frac{\rho_e}{\rho_o}\right) \coth k_z a}. \tag{89}$$

These relations are sketched in Figure .

We see from the dispersion relations (88) and (89) that the wave's phase-speed, ω/k_z, is reduced below the Alfven speed within the slab as a consequence of the inertia of the surrounding field-free atmosphere. Wave propagation is rendered *dispersive*, with the phase-speed ω/k_z of the wave depending upon the wavelength and frequency of the wave. In the limit of $k_z a \gg 1$, corresponding to a wide slab (or, equivalently, wavelengths much shorter than the slab width), both the sausage and kink modes have the same phase-speed, namely

$$\frac{\omega}{k_z} = \left(\frac{\rho_o v_A^2}{\rho_o + \rho_e}\right)^{1/2}. \tag{90}$$

We recognise this to be the phase-speed of a surface wave at a single magnetic interface one side of which is field-free (cf. equation (85)). Physically, this is as one would expect; for in a wide slab the two interfaces (at $x = \pm a$) are so far separated as not to have any influence on one another, and therefore the phase-speed of the wave is that for an interface one side of which is field-free.

In the opposite limit of $k_z a \ll 1$, corresponding to a slender slab of field (or, equivalently, to wavelengths that are much longer than the width of the slab) the sausage mode has phase-speed given approximately by

$$\frac{\omega}{k_z} = v_A - \frac{1}{2}\left[\frac{\rho_e}{\rho_o}\right] v_A|k_z|a, \qquad |k_za| \ll 1. \tag{91}$$

In writing equation (91) we have selected the wave propagating in the direction of the magnetic field (the positive z-axis), corresponding to choosing the positive root for ω/k_z. Also, we have allowed for the possibility that k_z may in general be negative, which requires that k_z be replaced by $|k_z|$ in dispersion relations (88) and (89). We return to the approximation (91) in Section 4.7.

Finally, we note that all of the above formulation may be generalized to include the effects of compressibility. The main conclusions that emerge from such an analysis (see Roberts 1981b; Edwin and Roberts 1982, 1983) are: (a) the sausage mode's maximum speed, v_A, in an incompressible medium is reduced to the cusp speed c_T (see equation (70)); (b) in addition to the surface modes, body waves may also propagate along the slab which acts as a wave guide for them. Their speed is between c_T and c_s, the sound speed in the slab. (c) waves may also leak from the slab and propagate away as sound waves (see Spruit 1982; Davilla 1985; Cally 1985, 1986).

4.5 Guided Waves in a Coronal Field

As our last application of equations (79) and (80) we turn away from the incompressible case (eqn (80)) and consider modes given by equation (79). We are particularly interested in applications to the corona, which is generally a low-β plasma. We therefore consider the extreme of $c_s \to 0$ (a cold plasma).

Setting $p_o = 0$, $c_s = 0$ and $c_T = 0$, equation (79) reduces to

$$\frac{d}{dx}\left[\rho_o(x)\, v_A^2 \frac{dv_x}{dx}\right] = \rho_o(x)(k_z^2 v_A^2 - \omega^2)v_x. \tag{92}$$

Furthermore, since the equilibrium is one of constancy of total pressure (see equation (71)) and we have set $p_o = 0$ we are considering a *uniform* magnetic field $B_o = B_o\hat{z}$. The density, however, may vary from place to place and therefore the Alfven speed $v_A(x) = B_o/(\mu_o\rho_o(x))^{1/2}$ is a function of location x. The produce $\rho_o(x)v_A^2(x)$ is a constant and so equation (92) reduces further to

$$\frac{d^2v_x}{dx^2} + \left[\frac{\omega^2}{v_A^2(x)} - k_z^2\right]v_x = 0. \tag{93}$$

In a *uniform* medium (i.e. a medium with $\rho_o(x)$ constant) equation (93) immediately yields the dispersion relation

$$\omega^2 = (k_x^2 + k_z^2)v_A^2, \tag{94}$$

for $v_x(x) \propto \exp(-ik_xx)$. This is simply the low β limit of the general dispersion relation (65) for magnetoacoustic waves in a uniform medium, specialized to two-dimensional modes with $k_y = 0$, $k^2 = k_x^2 + k_z^2$. If follows, then, that equation (93) describes the behaviour of *fast magnetoacoustic* waves in the low β limit.

For a *non-uniform* equilibrium, consider the density distribution

$$\rho_o(x) = \begin{cases}\rho_o, & |x| < a,\\ \rho_e, & |x| > a,\end{cases} \tag{95}$$

representing a region of constant density ρ_o, contained in $-a < x < a$, surrounding by a gas of constant density ρ_e. Just as with waves in an isolated magnetic slab (Section 4.4), we may discuss sausage and kink modes separately. Consider first the *kink* mode, which satisfies equation (93) with

$$v_x(x) = \alpha_o \cos nx, \tag{96}$$

where

$$n^2 = \frac{\omega^2}{v_A^2} - k_z^2, \tag{97}$$

and v_A is the Alfven speed within the density inhomogeneity ($|x| < a$). The constant α_o is arbitrary. The transverse wavenumber n is still to be determined (by relating ω to k_z) and so n^2 may be negative (i.e. na may be imaginary).

In the external region ($|x| > a$), the solution satisfying the imposed requirement that v_x tend to zero as $|x| \to \infty$ is

$$v_x(x) = \alpha_e e^{-mx}, \tag{98}$$

where

$$m^2 = k_z^2 - \frac{\omega^2}{v_{Ae}^2}. \tag{99}$$

The Alfven speed in the environment is $v_{Ae} = B_o/(\mu_o \rho_e)^{1/2}$. Notice that

$$\rho_o v_A^2 = \rho_e v_{Ae}^2. \tag{100}$$

In selecting the solution (98) we require $m > 0$, and so the frequency ω is restricted to lie in the range

$$\omega^2 < k_z^2 v_{Ae}^2. \tag{101}$$

In imposing condition (101) we are demanding that the waves within the coronal inhomogeneity ($|x| < a$) are *evanescent* outside that inhomogeneity. It is possible, of course, that wave propagate outside the inhomogeneity, and then the solution (98) is inappropriate; instead, we would consider outwardly propagating waves, with the coronal inhomogeneity acting as a generator of such modes. In such a circumstance the frequency ω or the wavenumber k_z would be complex; these are *leaky* waves, discussed in detail by Spruit (1982), Pavilla (1985) and Cally (1986).

Continuing with our discussion of guided (non-leaky) waves, we now impose the boundary conditions across the sides ($x = \pm a$) of the slab. As with an isolated slab or a single magnetic interface we require that v_x be continuous and also, from equation (92), that dv_x/dx be continuous. These two conditions imply the dispersion relation governing fast *kink* modes in a coronal magnetic slab (Edwin and Roberts 1982):

$$\tan na = \frac{m}{n}. \tag{102}$$

The restriction $m > 0$, imposed by the requirement of evanescence in the environment of the coronal inhomogeneity, limits the frequency ω to a maximum of $k_z v_{Ae}$ (for $k_z > 0$). In fact there is also a lower limited to ω, imposed by the dispersion relation itself. This is because equation (102) implies that $n^2 > 0$ (for real ω, k_z). To see this suppose that, on the contrary, $n^2 < 0$; write $n^2 = -N^2$, for $N > 0$. Then we may take $n = iN$. But, according to equation (102), $iN \tan iNa = m > 0$; i.e. $-N \tanh Na > 0$, which is a contradiction. Therefore, $n^2 > 0$. Hence, solutions of equation (102) lie in the range

$$k_z v_A < \omega < k_z v_{Ae}. \tag{103}$$

It follows, then, that for guided waves to occur we must have $v_A < v_{Ae}$. Since the field is uniform, this corresponds to $\rho_o > \rho_e$. Thus, *regions of high density (low Alfven speed) provide wave guides for the propagation of fast magnetoacoustic waves under coronal (low β) conditions.*

It is an interesting observation that relation (102) for kink waves is identical to that found in terrestrial seismic studies where it describes Love waves (see Love 1911). Fast kink waves under coronal conditions may therefore be termed *magnetic Love waves* (Edwin and Roberts 1982).

The behaviour of *sausage* modes is similar to the above, and the governing dispersion relation is

$$\tan na = -\frac{n}{m}, \tag{104}$$

with frequencies again restricted to the range (103). The sausage modes have an analogy, too, this time with sound waves in ocean layers which were extensively investigated by Pekeris (1948). It is convenient, then, to refer to sausage waves in a coronal innomogeneity as *magnetic Pekeris waves*.

The effect of a finite sound speed may also be incorporated into the analysis of magnetic Love and Pekeris waves. The details are considerably more complicated, but may also be worked out for cylindrical tubes. Fast modes are still guided by regions of low

Alfven speed though, additionally, slow waves arise - essentially, the one-dimensional propagation of sound in a strong field pointed out in Section 4.1.2. Details are discussed in Edwin and Roberts (1982, 1983, 1986a), and in Cally (1985, 1986).

The results of this section have a number of applications. For example, they may be applied to the coronal fast puslations that have been recorded in radio wavelengths for many years now (see the review by Aschwander (1987), a detailed application requiring a discussion of the impulsively generated fast waves (see Roberts, Edwin and Benz 1983, 1984). Also, such an application ofcers the possibility of using coronal oscillations as a diagnostic of *in situ* conditions in the atmosphere, in particular a possible determination of the uncertain magnetic field strength in the corona (Roberts 1986; Edwin and Roberts 1986b). Finally, we should mention that similar results apply to *current sheets*, which are also capable of guiding fast magnetoacoustic waves. An application of such properties in current sheets has been given by Edwin, Roberts and Hughes (1986), who advance a new explanation of Pi 2 pulsations based upon the properties of impulsively generated fast waves.

4.6 Slender Flux Tube Theory

The properties of magnetoacoustic waves in magnetic slabs or flux tubes are complicated by the finite size of the slab or tube, which renders wave propagation dispersive. However, in the limit of a thin slab or tube, corresponding to $|k_z| \ll 1$, the dispersion relation is much simplified. For example, in an isolated magnetic slab of width 2a we showed in Section 4.4 that, to lowest order (as $k_z a \to 0$), the frequency of the sausage mode is given by $\omega \sim k_z v_A$; if compressive effects are included this result is changed to

$$\frac{\omega}{k_z} \sim c_T. \qquad (105)$$

It is natural to ask whether these results could not have been derived directly, assuming *ab initio* that the slab (or tube) is thin. This is in fact possible and provides us with the *slender flux tube equations*.

Consider the sausage (surface) mode in a magnetic slab (or tube). We regard the slab as simply an elastic medium of cross-sectional area $A(z,t)$ and we assume that motions within the slab are predominantly longitudinal (i.e. $v_z \gg v_x$). Also, sinc ethe slab is thin, variations across the slab are small, with pressure p, density ρ and velocity v_z being functions of z and t (but independent of x). The equation of continuity for an elastic tube (slab or cylinder) is simply

$$\frac{\partial}{\partial t}\rho A + \frac{\partial}{\partial z}\rho v A = 0, \qquad (106)$$

where we have written $v = v_z$ for the flow velocity along the tube.

The equation of momentum for the longitudinal component is

$$\rho\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z}\right] = -\frac{\partial p}{\partial z}, \qquad (107)$$

we may notice that no contribution here arises from the $\underline{j} \times \underline{B}$ force, since on the axis of the tube the longitudinal component of that force must be zero (the $\underline{j} \times \underline{B}$ force being perpendicular to $\underline{B}$). Across the tube pressure balance is maintained instantaneously, so that the total (gas plus magnetic) pressure equals the external gas pressure:

$$p + \frac{B^2}{2\mu_0} = p_e. \qquad (108)$$

Here $B(z,t)$ is the longitudinal component of the magnetic field; there is also a transverse component but this is very much less than B and in zero on the axis of the tube. The external gas pressure $p_e(z,t)$ is the pressure calculated on the (moving) boundary of the tube.

Disturbances are again assume isentropic (equation (8)), which for a longitudinal motion yields

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial z} = \frac{\gamma p}{\rho}\left[\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial z}\right]. \qquad (109)$$

Finally, we may relate the magnetic field B(z,t) to the cross-sectional area A(z,t) by the condition of flux conservation (esssentially div $\underline{B}$ = 0 combined with the ideal induction equation), viz.

$$BA = B_o A_o, \tag{110}$$

where B_o and A_o are the undisturbed (constant) values of the field and tube area.

The system (106)-(110) provides the slender flux tube equations. They may be derived directly from the full MHD equations by employing a Maclaurin expansion for the fluid and magnetic variables (Roberts and Webb 1978). More intuitive approaches are also possible (Parker 1974; Defouw 1976). We should note that the system is *nonlinear* and furthermore may be readily generalized to include additional effects, such as gravity (Section 5) or non-isentropic behaviour.

Consider firt the equilibrium state. With $\underline{v}$ = 0 and $\partial/\partial t$ = 0, system (106)-(110) reduces to

$$p_o + \frac{B_o^2}{2\mu_o} = p_e, \tag{111}$$

for external uniform gas pressure p_e.

We may readily linearize our equations about this equilibrium, the result being

$$\left.\begin{array}{ll} \rho_o \dfrac{\partial A}{\partial t} + A_o \dfrac{\partial \rho}{\partial t} + \rho_o A_o \dfrac{\partial v}{\partial z} = 0, & \rho_o \dfrac{\partial v}{\partial t} = -\dfrac{\partial p}{\partial z} \\ p + \dfrac{B_o}{\mu_o} B = \pi_e, \quad p = c_s^2 \rho, & BA_o + B_o A = 0, \end{array}\right\} \tag{112}$$

for sound speed $c_s = (\gamma p_o/\rho_o)^{1/2}$. Here p, ρ, B, A, π_e and v denotes the *perturbations*.

In order to solve equations (112) some constraint on π_e must be imposed. This may be done by requiring that π_e satisfy the usual wave equations in the field-free environment of the tube. Alternatively, we may argue that the tube wave disturbs the environment of the tube only slighly (recall that the modes discussed in Section 4.4 are assumed evanescent and therefore decay to zero in the tube's environment) and so to a first approximation we may set π_e = 0.

Adopting the assumption that π_e = 0 (so that the external gas pressure is taken to be the undisturbed pressure, p_e), we may reduce equation s(112) to the wave equation

$$\frac{\partial^2 v}{\partial t^2} = c_T^2 \frac{\partial^2 v}{\partial z^2}, \tag{113}$$

where the cusp speed c_T (defined in equation (70)) may also be expressed through

$$\frac{1}{c_T^2} = \frac{1}{c_s^2} + \frac{1}{v_A^2}. \tag{114}$$

We thus obtain the dispersion relation

$$\omega^2 = k_z^2 c_T^2, \tag{115}$$

in agreement with the value (eqn (105)) deduced from the dispersion relation for a finite slab in the limit $k_z a \to 0$. The incompressible limit ($c_s \to \infty$, ρ = 0) of equations (112) and (115) yields $\omega^2 = k_z^2 v_A^2$.

It is clear from the above that the cusp speed c_T plays an important role in describing the sausage mode in a magnetic flux tube For this reason it is commonly referred to as the *tube speed*. An estimate of its value in the photosphere may be of value. The sound and Alfven speeds in an intense photospheric flux tube are of order 9 km s^{-1} which implies that c_T ~ 6 km s^{-1}.

We should note, too, that the *kink* mode may also be described by a slender tube theory (see Spruit 1981 a,b).

Finally, we point out that dispersive effects, such as displayed in equation (91) and its compressible equivalent, may be incorporated by removing the assumption that π_e = 0. If, instead, we retain π_e then system (112) yields (Roberts and Webb 1979; Roberts 1981b)

$$\frac{\partial^2 v}{\partial t^2} - c_T^2 \frac{\partial^2 v}{\partial z^2} = -\frac{1}{\rho_o} \frac{\partial^2 \pi_e}{\partial z \partial t}. \tag{116}$$

Determining π_e from the gas dynamic equations for the field-free environment, coupled with the assumption of zero disturbance at infinity, finally leads to a dispersion relation of the form (91); in the incompressible limit we obtain equation (91).

4.7 Solitons in a Magnetic Slab

We have seen in the previous sections that waves in a magnetic slab are *dispersive*. The dispersion is such that any initially generated sausage wave tends to spread out as it propagates. Now the theory presented so far is for linear motions, whereas for nonlinear motions it is well known that sound waves tend to grow in amplitude and form shocks. These two effects, the tendency to form grow in amplitude when nonlinearities are accounted for, and the tendency to spread out when dispersion of a linear wave is included, are opposite to one another. The possibility arises, then, that the two effects may achieve a balance and out of that balance a nonlinear wave of permanent form be permitted. If such a wave is sufficiently robust that on interaction with waves of similar form but perhaps different amplitude it is able to emerge essentially unchanged, we refer to such a nonlinear wave as a *soliton*.

The nonlinear equations that solitons obey are of particular mathematical elegance, the more so as methods have recently been developed that permit the exact soliton of a number of these equations. Perhaps the best known example is provided by the *Korteweg-de Vries equation*, which is of the form

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial z} + v\frac{\partial v}{\partial z} + \frac{\partial^3 v}{\partial z^3} = 0. \qquad (117)$$

The nonlinearity in equation (117) is in the term $v\,\partial v/\partial z$, while the term $\partial^3 v/\partial z^3$ describes dispersion. The linearized form of equation (117) yields a dispersion relation of the form

$$\omega = k_z - k_z^3. \qquad (118)$$

The Korteweg-de Vries equation occurs in a wide variety of physical systems including water waves in a channel (see, for example, Lamb 1980).

It is natural to enquire whether a similar nonlinear equation exists that describes the (weakly) nonlinear behaviour of the sausage wave in a magnetic slab. The dispersion is somewhat different to the k^3-type possessed by the Korteweg-de Vries equation, being of the form (91). It is necessary to determine what partial differential operators correspond to a dispersion relation of the form (91). In fact, a general recipe exists for answering this question (see Whitham 1974) and gives rise to the *Hilbert transform*. To be specific, in a *compressible* flux tube with sound speed c_s and Alfven speed v_A the dispersion relation that generalizes equation (91) is

$$\omega = k_z c_T - \alpha k_z |k_z|, \qquad |k_z a| \ll 1, \qquad (119)$$

where

$$\alpha = \frac{1}{2}\left(\frac{\rho_e}{\rho_o}\right)\left(\frac{c_T}{v_A}\right)^3 a\, c_T. \qquad (120)$$

(The sound speed in the environment of the slab is also assumed to be c_s.) The coefficient α is a measure of the dispersion in the wave. Equation (119) reduces to (91) in the limit $c_s \to \infty$ (for which $c_T \to v_A$).

Following Whitham's recipe, it is shown in Roberts and Mangeney (1982) that equation (119) corresponds to

$$\frac{\partial v}{\partial t} + c_T\frac{\partial v}{\partial z} + \frac{\alpha}{\pi}\frac{\partial^2}{\partial z^2}\int_{-\infty}^{\infty}\frac{v(s,t)}{s-z}\,ds = 0. \qquad (121)$$

If now we allow for weakly nonlinear terms and employ the slender flux tube equations (Section 4.6) we may obtain (after much algebra) the desired result:

$$\frac{\partial v}{\partial t} + c_T\frac{\partial v}{\partial z} \quad \beta_o b\frac{\partial v}{\partial z} + \frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{v(s,t)}{s-z}\,dx = 0; \qquad (122)$$

here the nonlinear coefficient β_o is given by

$$\beta_o = \frac{1}{2} v_A^2(3c_s^2 + (\gamma+1)v_A^2)/(c_s^2 + v_A^2)^2. \qquad (123)$$

The integro-differential equation (122) is an example of the *Benjamin-Ono* equation (see Benjamin 1967; Ono 1975; Ablowitz and Segur 1981; Matsuno 1984). Its occurrence in a magnetic slab was first determined by Roberts and Mangeney (1982). Details of the calculation are given in Roberts (1985); see also Merzljakov and Ruderman (1985). The case of a cylindrical flux tube is of interest, too, and gives rise to a different integro-differential equation which we record here for reference (see Roberts (1985) for a derivation)

$$\frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \beta_o v \frac{\partial v}{\partial z} + \alpha' \frac{\partial^3}{\partial z^3} \int_{-\infty}^{\infty} \frac{v(s,t)ds}{[\lambda^2 + (s-z)^2]^{1/2}} = 0, \qquad (124)$$

for constants α' and λ.

Returning to the slab geometry we observe that an exact solution of equation (122) is (see Benjamin 1967)

$$v(z,t) = \frac{v_o}{1 + \left[\frac{z - ut}{\ell}\right]^2}, \qquad (125)$$

where v_o is the amplitude of the wave, ℓ is its characteristic length scale, and u the speed of the disturbance. These quantities are related by

$$u = c_T + \frac{1}{4}\beta_o v_o, \qquad \ell = \frac{4\alpha}{v_o \beta_o}. \qquad (126)$$

Solution (125) is the *single soliton* wave; it propagates with speed u (in excess of the tube speed c_T) and resembles a "swollen knee cap", a symmetrical swelling moving without change of shape along the slab.

Whether such solitons can actually arise in solar photospheric flux tubes is not clear and must no doubt await the expected developments in high resolution observations. One possible candidate, however, is the spicule (see Section 2), which is known to propagate, with little change of shape, along magnetic structures in the chromosphere and corona. It would be a nice marriage of nonlinear theory and fine scale solar observations if the spicule turns out to be a soliton!

5. THE SLENDER FLUX TUBE EQUATIONS IN A STRATIFIED ATMOSPHERE

We have discussed in Section 4.6 how longitudinal motions in a slender magnetic flux tube (cylindrical or slab geometries) may be described by a simplified system of equations. Those equations may be generalized to include the effect of stratification, introduced into an atmosphere by the presence of gravity. The main change is in the longitudinal equation of momentum (see eqn (107)) which now acquires a body force $-\rho g$, for constant gravitational acceleration g ($= 0.274\,\mathrm{km\,s^{-2}}$ for the Sun's atmosphere).

The slender flux tube equations for the sausage mode in a stratified medium are thus

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\rho A + \frac{\partial}{\partial z}\rho v A = 0, \quad \rho\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z}\right] = -\frac{\partial p}{\partial z} - \rho g, \\ &p + \frac{B^2}{2\mu_o} = p_e, \; BA = \text{constant}, \; \frac{\partial p}{\partial t} + v\frac{\partial p}{\partial z} = \frac{\gamma p}{\rho}\left[\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial z}\right]. \end{aligned}\right\} \qquad (127)$$

A derivation of these equations direct from the full MHD ones is given in Roberts and Webb (1978).

A full discussion of equations (127) is outside the range of these lectures (see Spruit (1981b), Thomas (1985), and Roberts (1986) for such a discussion). Instead, we shall briefly outline their consequences for wave propagation in an isothermal atmosphere. The instability associated with a superadiabatic atmosphere is described in Section 6.

Consider, first, the equilibrium ($v = 0$, $\partial/\partial t = 0$) form of equations (127). We have

$$p_o'(z) = -\rho_o(z)g, \quad p_o(z) + \frac{B_o^2(z)}{2\mu_o} = p_e(z), \quad B_o(z)A_o(z) = 0, \tag{128}$$

where a dash (′) denotes differentiation with respect to height z. The magnetic field strength $B_o(z)$ is a function of height, the tube expanding in response to the decline of the external gas pressure, $p_e(z)$, with height. Outside the tube we suppose the atmosphere to be in hydrostatic equilibrium, so that

$$p_e'(z) = -\rho_e(z)g \tag{129}$$

for external gas density $\rho_e(z)$.

Introducing the ideal gas law (see equation (2)) into our system finally permits a determination of the run of pressure, density, etc, with height. For an *isothermal* atmosphere, for which the pressure scale-height $\Lambda_o = k_B T_o/gm$ is a constant in the gas of temperature T_o, we obtain

$$\left.\begin{aligned} p_o(z) &= p_o(0)e^{-z/\Lambda_o}, \quad & \rho_o(z) &= \rho_o(0)e^{-z/\Lambda_o}, \\ A_o(z) &= A_o(0)e^{z/2\Lambda_o}, \quad & B_o(z) &= B_o(0)e^{-z/2\Lambda_o}, \end{aligned}\right\} \tag{130}$$

where z = 0 is an arbitrary reference level. Thus, the magnetic field declines half as fast as the gas pressure. It follows from equations (130) that the Alfven speed v_A in the tube is a constant.

We may perform an analysis of the linearized form of equations (127), for equilibrium (130). We assume that $\pi_e = p_e(z)$ for all time t. After some algebra, we obtain (Rae and Roberts 1982)

$$\frac{\partial^2 Q}{\partial t^2} - c_T^2 \frac{\partial^2 Q}{\partial z^2} + \Omega^2 Q = 0, \tag{131}$$

where c_T is the tube speed (a constant) in the isothermal atmosphere of the tube. The quantity Q is given by

$$Q(z,t) = e^{-z/4\Lambda_o} v(z,t), \tag{132}$$

and the frequency Ω is given by (Defouw 1976)

$$\Omega^2 = \left(\frac{9}{4} - \frac{2}{\gamma}\right)\frac{c_s^2}{4\Lambda_o^2} - \left(\frac{3}{2} - \frac{2}{\gamma}\right)^2 \left(\frac{\beta}{\beta + \frac{2}{\gamma}}\right)\left(\frac{c_s^2}{4\Lambda_o^2}\right), \tag{133}$$

for plasma beta defined in equation (13).

Equation (131) is of the *Klein-Gordon* type. It possesses a simple dispersion relation, namely

$$\omega^2 = k_z c_T^2 + \Omega^2, \tag{134}$$

which generalizes relation (115) to which it reduces if g = 0 ($\Lambda_o \to \infty$, $\Omega \to 0$). It follows from relation (134) that sausage waves in an isothermal flux tube possess a *frequency cut-off*: we must have $\omega > \Omega$ for propagation.

The cut-off frequency Ω is close to the usual acoustic cut-off, $c_s/2\Lambda_o$, provided $\gamma = 5/3$. Taking $c_s = v_A = 9$ km s^{-1} and $\Lambda_o = 125$ km as typical of photospheric conditions, we obtain a cyclic frequency $\Omega/2\pi$ of about 4.8 mHz, which corresponds to a period of roughly 210s. However, if conditions in the photosphere cause γ to depart strongly from 5/3, we find that the cut-off is substantially reduced (and the period correspondingly increased).

Sausage waves in a magnetic flux tube may be responsible for the generation of spicules (e.g. Roberts 1979; Hollweg 1982). The presence of a cut-off frequency implies that, if *impulsively generated*, sausage waves will propagate with a wve front moving at the tube speed c_T and trailing behind it a *wake* which oscillates with the period $2\pi/\Omega$ (Rae and Roberts 1982). The effect of such a wake, when shock waves are formed, on the transition region has been explored numerically by Hollweg (1982), who demonstrates its potential for forming spicules.

Finally, we note that *kink* modes in a stratified flux tube may also be investigated by a slender flux tube approach (see Spruit 1981a,b). The kink mode also satisfies a Klein-Gordon equation; the speed of the kink wve, given in equation (90), is slightly slower than the sausage speed. The cut-off frequency, however, is much lower than the adiabatic ($\gamma = 5/3$) value for the sausage mode, being about 1.4 mHz (700s period). This suggests that kink modes are likely to be readily generated in the solar atmosphere (Spruit 1981 a,b).

6. THE CAUSES OF FINE STRUCTURE IN THE PHOTOSPHERE

We have pointed out in Section 2 that magnetic field in the photospheric layers is almost solely concentrated into intense flux tubes or into sunspots. Intense flux tubes are on a scale of 10^2km, at or below the limit of present telescopic resolution. Why does the Sun apparently prefer to arrange the available magnetic flux, supplied from deep within the solar interior, in a concentrated rather than diffuse form? A partial explanation is to be found in the ability of convection motions, such as supergranules and granules, to expel magnetic field from the interiors of cells, forcing the field to find residence in the corners between convective cells where downdraughts exist.

The elementary calculations outlined in Section 3.3 demonstrate such effects. However, such a calculation cannot be pushed too far because it ignores the back-reaction of the magnetic field on the flow. While the field is diffuse such a back-reaction is negligible but as concentrations of field build-up this effect becomes correspondingly more important. Roughly, we may expect that advection is able to concentrate magnetic fields up to an equipartition with the confining dynamical pressure, so that

$$\frac{B^2}{2\mu_0} \sim \frac{1}{2}\rho_0|\underline{v}|^2. \qquad (135)$$

With $\rho_0 = 10^{-7}$gm cm^{-3} and $|\underline{v}| = 10^5$cm s^{-1} ($= 1$ km s^{-1}) we obtain (in cgs units) $B^2/8\pi \sim 500$ G^2, giving $B \sim 10^2$gauss. This is far short of the required 1500 gauss residing in intense flux tubes, and so addition effects must be sought to explain their field strength.

A possible explanation lies in the fact that intense tubes reside in the strongly superadiabatic part of the Sun's atmosphere, the top of the convection zone (Parker 1979). The effect is most conveniently demonstrated using the slender flux tube equations of Section 5, but allowing for a non-isothermal atmosphere.